II CW-complexes

2.1 Def: let X a space. A <u>CW-decomposition of X</u> consists of

(a) A filtration $X^{(-1)} = \emptyset \subset X^{(0)} \subset X^{(1)} \subset \dots$ indexed by $\mathbb{N} \cup \{-1\}$; Require $X = \bigcup_{n=-1}^{\infty} X^{(n)}$ and X has the weak topology w.r.t. the filtration: $U \subset X$ open $\iff \forall n$, $U \cap X^{(n)}$ open in $X^{(n)}$.

(b) For all $n \in \mathbb{N}$, $X^{(n)}$ is obtained from $X^{(n-1)}$ as a push-out of the form

$$\begin{tikzcd} \coprod_{\sigma \in J_n} S^{n-1} \arrow[r, "\amalg \varphi_\sigma"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X^{(n-1)} \arrow[d] \\ \coprod_{\sigma \in J_n} D^n \arrow[r, "\amalg \chi_\sigma"] & X^{(n)} \end{tikzcd}$$

Here $D^n = \{x \in \mathbb{R}^n ; \|x\| \leq 1\}$
$S^{n-1} = \partial D^n \hookrightarrow D^n$
$(S^{-1} = \emptyset \hookrightarrow D^0 = \{0\})$.

Here J_n is a set.

A <u>CW-complex</u> is a space X <u>equipped</u> with a CW-decomposition (omitted from the notation).

2.2 Terminology: For $n \in \mathbb{N}$ and $\sigma \in J_n$, we call $e_\sigma^n := e_\sigma := \chi_\sigma(\mathring{D}_n)$ an "<u>open n-cell</u> of X" (⚠ not an open subset in general), and $\bar{e}_\sigma^n := \bar{e}_\sigma := \chi_\sigma(D_n)$ a <u>closed n-cell</u> of X.

We call φ_σ the <u>attaching map</u> of e_σ and χ_σ its caracteristic map.

We call $X^{(n)}$ the <u>n-skeleton</u> of X.

We say X is a <u>finite CW-complex</u> if it has finitely many open cells. If $J_n \neq \emptyset$, $J_k = \emptyset$ for $k > n$, we say that X has <u>dimension</u> n.

2.3. Proposition: Let X be a CW-complex.

(a) As a set, X is the disjoint union of its open cells:

$$X = \coprod_{n \in \mathbb{N}} \coprod_{\sigma \in J_n} e_\sigma$$

(b) X is Hausdorff

(c) Each e^n_σ is open in $X^{(n)}$, and $\bar{e}^n_\sigma$ is closed in X; moreover $X^{(n)}$ is closed in X.

(d) If $K \subset X$ is compact, then it is included in a finite number of cells.

(e) X has the weak topology with respect to the closed cells.

proof (a) From the pushout definition, we see that we have homeo $\coprod_{\sigma \in J^n} \mathring{D}^n_\sigma \to X^{(n)} \setminus X^{(n-1)}$, and $X^{(n)} \setminus X^{(n-1)} = \coprod_{\sigma \in J^n} e^n_\sigma$. By induction this proves (a).

(b) First, we show by induction on n that $X^{(n)}$ is Hausdorff. $X^{(0)}$ is discrete, so this is clear. Suppose $n \geq 1$ and $X^{(n-1)}$ Hausdorff. Then we have a quotient map

$$\pi : Z = X^{(n-1)} \amalg \coprod_{\sigma \in J_n} D^n_\sigma \longrightarrow X^{(n)}.$$

Choose $x, y \in X^{(n)}$, and representant $\tilde{x}, \tilde{y}$ in $X^{(n-1)} \amalg \coprod_{\sigma \in J_n} \mathring{D}^n_\sigma$ (these are unique).

(1°) $\tilde{x}, \tilde{y} \in X^{(n-1)}$ Choose U_{n-1}, V_{n-1} open disjoint nbhds of $\tilde{x}, \tilde{y}$ in $X^{(n-1)}$. Then we can take

$$U'_n = U_{n-1} \amalg \coprod_{\sigma \in J_n} \{ w \in D^n_\sigma ; \|w\| > 0, \varphi_\sigma(\tfrac{w}{\|w\|}) \in U_{n-1} \}$$

and similarly V'_n. Then $U_n = \pi(U'_n)$, $V_n = (\pi(V'_n))$ are open, disjoint nbhds of x, y resp. in $X^{(n)}$.

Note also that $U_n \cap X^{(n-1)} = U_{n-1}$.

(2°) $x \in X^{(n-1)}$ and $y \in \mathring{D}^n_\sigma$.

Take $U' = X^{(n-1)} \amalg \coprod_{\tau \in J_n \setminus \{\sigma\}} \mathring{D}_\tau \amalg \{ w \in \mathring{D}_\sigma ; \|w\| \geq \frac{1}{2}(\|x\|+1)\}$

$$V' = \{ w \in \mathring{D}_\sigma ; \|w\| < \tfrac{1}{2}(1+\|x\|)\}.$$

Then $\pi(U') =: U$, $\pi(V') =: V$ are open, disj'a nbhd's of x and y resp.

(3°) $x, y \in \coprod_{\sigma \in J_n} \mathring{D}^n_\sigma$: separate them there.

Now we can prove that X is Hausdorff: take $x, y \in X$ with $x \neq y$. $\exists\, n \in \mathbb{N}$, $x, y \in X^{(n)}$. Thus $\exists\, U_n, V_n$ open nbhd of x and y in $X^{(n)}$, disjoint.
Construct inductively U_m, V_m $(m \geq n+1)$ as in (1°) above, and take $U = \bigcup_{m \geq n+1} U_m$, $V = \bigcup_{m \geq n+1} V_m$.

(c) It is obvious that e^n_σ is open in $X^{(n)}$ since $\pi^{-1}(e^n_\sigma) = \mathring{D}^n_\sigma \subset X^{(n-1)} \amalg \coprod_{\sigma \in J_n} D^n_\sigma$ is open.
Also, $\overline{e^n_\sigma} = \chi_\sigma(D^n)$ is closed in X (the image of the compact $D^n \to X^{(n-1)} \to X$ in X, which is Hausdorff).
By induction, we see that $X^{(n)} \cap X^{(k)}$ is closed in $X^{(k)}$ for all k; thus it is closed in X.

(d) Suppose $A \subset X$ meets each open cell of X in at most one point. Then A is closed and discrete as a subspace.
Show this for $A \cap X^{(n)}$ in $X^{(n)}$ for all n. For $n=0$, it is obvious since $X^{(0)}$ discrete. Suppose $B = A \cap X^{(n)}$ is closed & discrete in $X^{(n)}$. Since $X^{(n)}$ closed in $X^{(n+1)}$, B also closed and discrete in $X^{(n+1)}$. Obviously $C = A \cap (X^{(n+1)} \setminus X^{(n)}) = \bigcup_{\sigma \in J_n} A \cap e^{n+1}_\sigma$ is also closed and discrete in $X^{(n+1)}$.

Thus A is closed and discrete in X.
If $K \subset X$ is compact, form A by choosing a point in each open cell that meets K. Then A is discrete compact, so finite.
(e) It suffices to show that $X^{(n)}$ has the weak topology with respect to $\{ \bar{e}^k_\sigma \; ; \; k \leq n, \; \sigma \in J_k \}$, which is easy.

<u>2.4 Remarks</u> (1) CW stands for "closure finite" (each cell is attached to only finitely many cells, which is obvious from (d) since $\bar{e}$ compact), and "weak topology".
(2) From the definition, $X^{(0)}$ is discrete (homeo to $\coprod_{\sigma \in J_0} D^0_\sigma$).
(3) The requirement that X has the weak topology w.r.t. $\{X^{(n)}, n \in \mathbb{N}\}$ is needed only if X ∞-dim'l.
(4) One can show that the dimension of a CW-cx does not depend on the chosen CW-decomposition.
(5) The topology of a CW-complex X has nice features that we only mention. For proofs, see eg. Hatcher (Appendix):
- X is normal (T4)
- X is paracompact (every open cover has a locally finite refinement)
- X is locally contractible; in particular:
- X locally path connected and semi-locally simply conn'd. In particular, path-components = connected components, and if X is connected, it admits a universal covering space $\tilde{X} \xrightarrow{p} X$
- X is locally compact $\Leftrightarrow$ each point $x \in X$ admits a nbhd that meets only finitely many cells.

2.5 Definition. If $(X, A) \in Top^2$, a CW-decomposition of X relative to A is obtained as in 2.1 by taking $X^{(-1)} = A$. If X is endowed with such a CW-decomp., we say that (X, A) is a relative CW-complex.
If $A = \{x_0\}$, we write $(X, A) = (X, x_0)$: this is a pointed CW-complex. Notation: $(X, A)^{(n)}$ or just $X^{(n)}$.

2.6 Definition A subspace A of a CW-complex X is called a sub-complex if admit a CW-decomposition with each cell of A being a cell of X.

2.7 Remarks (1) If A is a sub-complex of a CW-cx X, then the CW-decomposition of X makes (X, A) a relative CW-cx in an obvious way!
(2) A sub-cx $A \subset X$ is compact if and only if it is finite.

(3) A relative CW (X, A) is Hausdorff $\Leftrightarrow$ A is.

2.8 Def: Let X, Y be (relative) CW-complexes. A map $f: X \to Y$ is called cellular if $\forall n \in \mathbb{N} \cup \{-1\}$, $f(X^{(n)}) \subset Y^{(n)}$.

2.9 Examples: Obviously, the inclusion of a sub-complex $A \hookrightarrow X$ is a cellular map. Thus the examples $S^n \hookrightarrow S^{n+1}$, $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, $\mathbb{F}P^n \hookrightarrow \mathbb{F}P^{n+1}$, etc. are cellular.
The next proposition summarizes standard constructions on CW-complexes.

2.10 Proposition:
(a) If (X, A) is a relative CW complex, then the quotient X/A is a pointed CW-cx with one -1 cell (the class of A), and one cell for each (relative) cell of (X, A).

Moreover, the quotient map $q : (X, A) \to (X/A, *)$ is cellular.

(b) If A is a sub-complex of a CW-cx X, Y is a CW-cx, and $f : A \to Y$ is cellular, then the push-out

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ X \arrow[r, "\bar{f}"] & X \amalg_f Y \end{tikzcd}$$

is a CW-complex having Y as subcomplex with one (relative) cell for each cell of X that is not a cell of A.

Moreover, we have an isomorphism $X/A \longrightarrow X \amalg_f Y / Y$ induced by $\bar{f}$.

(c) If $X_0 \to X_1 \to X_2 \to \dots$ is a sequence of inclusion of subcomplexes, then its colimit inherits a CW-decomposition, for which each X_i is a sub-complex.

(d) If X, Y are CW-complexes, and if $(I^m, \partial I^m) \xrightarrow{\chi_e} (X^{(m)}, X^{(m-1)})$, $(I^n, \partial I^n) \xrightarrow{\chi_f} (Y^{(n)}, Y^{(n-1)})$ are cells, let the product-cell $e \times f$ be defined as

$$I^{m+n} = I^m \times I^n \xrightarrow{\chi_e \times \chi_f} X \times Y.$$

Then, endowed with the weak topology with respect to the (closed) product cells, $X \underset{w}{\times} Y$ inherits a CW-structure. If X or Y is locally compact (e.g. finite), then the weak and product topology agree.

proof: elementary, and left as exercises

We can endow I with the cell structure having two 0-cells and one 1-cell. Then, if X is a CW, we endow $X \times I$ with the product CW-structure. Then $X \times \{0\}$ and $X \times \{1\}$

are obviously sub-complexes of $X \times I$.

<u>2.11 Def</u>: let X, Y be CW-cx and $f, g : X \to Y$ cellular. A <u>cellular</u> homotopy from f to g is a cellular map $H : X \times I \to Y$ that is a homotopy from f to g.

<u>2.12 Def</u>: a map $i : A \to X$ has the <u>homotopy extension property</u> with respect to a space Y, if given a commutative diagram

$$\begin{tikzcd}
 & X \arrow[d, "i_0"] \arrow[dr, bend left, "H_0"] & \\
A \arrow[ur, "i"] \arrow[dr, "i_0"'] & X \times I \arrow[r, dashed, "\exists H"] & Y \\
 & A \times I \arrow[u, "i \times id_I"'] \arrow[ur, "h"'] &
\end{tikzcd}$$

the dotted map H exists*.

adjoined picture:

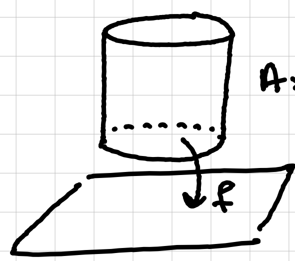
$$\begin{tikzcd}
A \arrow[r, "\hat{h}"] \arrow[d, "i"'] & Y^I \arrow[d, "\varepsilon_0"] \\
X \arrow[r, "H_0"'] \arrow[ur, dashed, "\exists \hat{H}"] & Y
\end{tikzcd}$$

the dotted map $\hat{H}$ exists*.

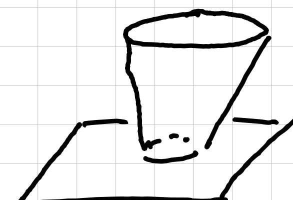
The map $i : A \to X$ is called a <u>cofibration</u> if it has the HEP w.r.t all spaces Y. (* → resulting diagram must commute!)

<u>2.13. Definition</u>. Suppose given a map $f : A \to X$. The <u>cylinder of f</u>, $cyl(f)$, is given as the push-out

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "k"] \\
A \times I \arrow[r, "j"'] & Cyl(f)
\end{tikzcd}$$


$A \times I$

X


$cyl(f)$

The maps $A \times I \xrightarrow{f \times id_I} X \times I \xleftarrow{i_0} X$ induce a canonical map $s : Cyl(f) \to X \times I$.

<u>2.14 Lemma</u>: Suppose given $A \xrightarrow{i} X$. TFAE:

(a) i is a cofibration

(b) $s : Cyl(i) \to X \times I$ has a retraction:

$\exists\ r : X \times I \to Cyl(i)$ with $r \circ s = id_{Cyl(i)}$.

proof: (a) $\Rightarrow$ (b) r is then given as the solution of the following HE prob:

$$\begin{tikzcd} & X \arrow[d, "i_0"] \arrow[dr, "k"] & \\ A \arrow[ur, "i"] \arrow[dr, "i_0"'] & X\times I \arrow[r, dashed, "\exists r"] & Cyl(i) \\ & A\times I \arrow[u] \arrow[ur, "j"'] & \end{tikzcd}$$

Since $Cyl(i)$ is the p.o., this forces $r\circ s = id_{cyl(i)}$.

(b) $\Rightarrow$ (a) Suppose given such an $r : X\times I \to Cyl(i)$.

Given any HE problem:

$$\begin{tikzcd} & X \arrow[d] \arrow[dr, "k"] \arrow[drr, bend left] & & \\ A\times I \arrow[ur] \arrow[dr] & X\times I \arrow[r, "r"] & Cyl(i) \arrow[r, dashed, "\exists! F"] & Y \\ & A\times I \arrow[u] \arrow[ur, "j"'] \arrow[urr, bend right] & & \end{tikzcd}$$

Since $Cyl(i)$ is the p.o., $\exists!$ F $Cyl(i) \to Y$ making the diag. commute. Therefore $F\circ r$ is a solution to the HEP, and i is a cofibration. ▨

<u>2.15 Exercise</u>: If $A \xrightarrow{i} X$ is a cofibration, then it is an embedding (onto a closed subspace if X Hausdorff).

Therefore, we can assume that given a cofibration $i: A \to X$, it is the inclusion of a subspace (of course, the converse is not true in general).

<u>2.16 Examples</u>: (a) $S^{n-1} \hookrightarrow D^n$ $(\partial I^n \hookrightarrow I^n)$ are cofibration (see the retraction given at 1.18).

(b) Assume $A \xrightarrow{i} X$ is a cofibration, Y a space. A sufficient condition for $Y\times A \xrightarrow{1_Y\times i} Y\times X$ to be a cofibration, is that the induced map (which is a continuous bijection) ϕ be a <u>homeo</u> [Then apply 2.14 with the retraction $\phi^{-1}\circ(1\times r)$].

$$\begin{tikzcd} Y\times A \arrow[r, "1\times i"] \arrow[d, "1\times i_0"'] \arrow[dr, phantom, "\Gamma"] & Y\times X \arrow[d] \arrow[ddr, bend left, "1\times k"] & \\ Y\times A\times I \arrow[r] \arrow[drr, bend right, "1\times j"'] & Cyl(1\times i) \arrow[dr, dashed, "\exists!\,\phi"] & \\ & & Y\times Cyl(i) \arrow[r, shift right, "1\times s"'] & Y\times X\times I \arrow[l, shift right, "1\times r"'] \end{tikzcd}$$

The fol. conditions $\Rightarrow$ ϕ homeo:

(i) A closed (then $Y \times A \times I$ and $Y \times X$ closed in $Y \times Cyl(i)$, so $Y \times Cyl(i)$ is the pushout, and ϕ is a homeo).

(ii) Y is Hausdorff locally compact (use exp. law).

(c) We deduce that $X \times \partial I \to X \times I$ cofib. $\forall X$ (by (i))

(d) "Cobase change preserves cofibrations": given a p.o.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "j"] \\ X \arrow[r] & Y \end{tikzcd}$$

then i cofibration $\Rightarrow$ j cofibration.

(e) We can factorize any map $X \xrightarrow{f} Y$ as $X \xrightarrow{i} Cyl(f) \xrightarrow{q} Y$, where $i_1 : X = X \times \{1\} \hookrightarrow X \times I \xrightarrow{j} Cyl(f)$ and $q : Cyl(f)$ is induced:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "k"] \arrow[ddr, bend left, "1_Y"] & \\ X \times I \arrow[r, "j"] \arrow[drr, bend right, "(x,t) \mapsto f(x)"'] & Cyl \arrow[dr, dashed, "\exists ! q"] & \\ & & Y \end{tikzcd}$$

Then $q \circ k = 1_Y$ and $k \circ q \simeq 1_{cyl(f)}$ (easy exercise!)

Now it is easy to see that i is a cofibration: we take the following variation of the def of $cyl(f)$:

$$\begin{tikzcd} X \times \partial I \arrow[r, "f \amalg 1_X"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\ulcorner"] & Y \amalg X \arrow[d, "k \amalg i"] \\ X \times I \arrow[r] & Cyl(Y) \end{tikzcd}$$

Since α cofibration, by (d) so is $k \amalg i$, $\Rightarrow$ so are k and i.

(f) If $A \xrightarrow{i} X$ is a cofibration, then so is $(X \times \partial I) \cup A \times I \hookrightarrow X \times I$ (easy exercise, as in example (c) above, which is the case $A = \emptyset$).

<u>2.17 Proposition</u>: If (X, A) is a relative CW-cx, then the inclusion $A \hookrightarrow X$ is a cofibration

<u>proof</u>: suppose given:

$$\begin{tikzcd} & X \arrow[d] \arrow[dr, "H_0"] & \\ A \arrow[ur, "i"] \arrow[dr, "i_0"'] & X \times I \arrow[r, dashed, "H"] & Y \\ & A \times I \arrow[u] \arrow[ur, "h"'] & \end{tikzcd}$$

We construct $H^{(n)} : X^{(n)} \times I \to Y$ by induction on n, with $H^{(n)}|_{X^{(n-1)} \times I} = H^{(n-1)}$, with $H^{(-1)} = h$; Suppose $H^{(n-1)}$ given.

We have a pushout diagram

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d, "\coprod i"'] \arrow[dr, phantom, "\ulcorner"] & X^{(n-1)} \arrow[d, "i_{n-1}"] \\ \coprod D^n \arrow[r] & X^{(n)} \end{tikzcd}$$

Obviously, by 2.16 (a), $\coprod i$ is cofibration, thus so is i_{n-1} (2.16 (d))

Thus we can find a solution $H^{(n)}$ to the problem:

$$\begin{tikzcd} & X^{(n)} \arrow[d, "i_0"] \arrow[dr, bend left, "H^{(0)}|_{X^{(n)}}"] & \\ X^{(n-1)} \arrow[ur] \arrow[dr, "i_0"'] & X^{(n)} \times I \arrow[r, dashed, "\exists H^{(n)}"] & Y \\ & X^{(n-1)} \times I \arrow[u] \arrow[ur, "H^{(n-1)}"'] & \end{tikzcd}$$

Thus the collection $\{H^{(n)}\}$ provides $H : X \times I \to Y$ (weak topology!) $\blacksquare$

<u>2.18 Proposition</u> : Suppose given $(Y, B) \in \mathrm{Top}^2$, and $n \in \mathbb{N} \cup \{\infty\}$. TFAE :

(a) (Y, B) is n-connected.

(b) If (X, A) is a rel. CW-cx of dim $\leq n$, for any $f : (X, A) \to (Y, B)$, $\exists\, f \overset{H}{\simeq} g$ rel A with $g(X) \subset B$.
Moreover, if dim $X < n$, any two such g are homotopic rel A (as maps $X \to B$).

<u>proof</u> (a) => (b). Suppose such $f : (X, A) \to (Y, B)$ given. By induction on $k \geq 0$, we construct a homotopy $H^k . X \times I \to Y$ such that: (i) $H^k_0 = f$

(ii) H^k is a homotopy relative $X^{(k-1)}$

(iii) $H^k_1 (X^{(k)}) \subset B$.

For $k = 0$, since (Y, B) is 0-connected, we can choose a path $u_\sigma : I \to Y$ from $f(D^0_\sigma)$ to B for any $\sigma \in J_0$. This defines $h : X^{(0)} \times I = (X^{(-1)} \amalg \coprod_{\sigma \in J_0} D^0_\sigma) \times I \to Y$: take $h_t|_A = f$ and $h_t|_{D^0_\sigma} = u_\sigma$. Using that

$X^{(0)} \hookrightarrow X$ is a cofibration, we extend this homotopy to $H^0 : X \times I \to Y$; obviously it satisfies (i)–(ii)–(iii).

Suppose $k < n$ and H^k constructed; consider

$$\begin{tikzcd}
\coprod_{\sigma \in J_{k+1}} S^k_\sigma \arrow[r, "\varphi"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X^{(k)} \arrow[r, "H^k_1"] \arrow[d] & B \arrow[d, hook] \\
\coprod_{\sigma \in J_{k+1}} D^{k+1}_\sigma \arrow[r, "\chi"] & X^{(k+1)} \arrow[r, "H^k_1"] & Y
\end{tikzcd}$$

Since (Y, B) is n-conn. and $k+1 \leq n$, $f_n \circ \chi \overset{h}{\simeq} \tilde{\chi}$ rel $(\coprod S^k_\sigma)$ with $\tilde{\chi}(\coprod D^{k+1}_\sigma) \subset B$.

$$\begin{tikzcd}
(\coprod S^k_\sigma) \times I \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X^{(k)} \times I \arrow[d] \arrow[dr, bend left, "\ell"] & \\
(\coprod D^{k+1}_\sigma) \times I \arrow[r] \arrow[rr, bend right, "h"'] & X^{(k+1)} \times I \arrow[r, dotted, "\exists \tilde{h}"] & Y
\end{tikzcd}$$

ℓ with $\ell_t = H^k_1 \ \forall t \in I$. The outer diagram commutes since h rel $(\coprod S^k_\sigma)$. We deduce $\exists\, \tilde{h} : X^{(k+1)} \times I \to Y$ rel $X^{(k)}$ with the property that $\tilde{h}_1(X^{(k+1)}) \subset B$.

Using that $X^{(k+1)} \hookrightarrow X$ is a cofibration, we can extend $\tilde{h}$ to $H^{k+1} : X \times I \to Y$.

Obviously H^{k+1} satisfies (i)–(ii)–(iii).

Now we take the concatenation of homotopies $H^0 * \dots * H^n = H$, and H is as required by the proposition. If $n = \infty$, we take an infinite concatenation: $H = \mathop{*}\limits_{i=0}^{\infty} H^i$ by

$$H(x,t) = \begin{cases} H^i(x, 2^{i+1}(t-1+2^{-i})), & 1-2^{-i} \leq t \leq 1-2^{-i-1} \\ H^i(x,1) & x \in X^{(i)},\ t = 1. \end{cases}$$

By the weakness of CW, this is continuous! If $\dim X < n$ and $g \simeq f$, $g' \simeq f$ rel B, reapply the same argument to the pair $(X \times I,\ X \times \partial I \cup A) \longrightarrow (Y, B)$.

(b) $\Rightarrow$ (a) This is clear by taking $(X, A) = (I^k, \partial I^k)$ for all $0 \leq k \leq n$, and using lemma 1.45. ∎

2.19 Thm Whitehead : $f : Y \to Z$ is a n-connected map, and let X be a CW-cx of dimension k, $n, k \in \mathbb{N} \cup \{\infty\}$.

$$f_* : [X, Y] \to [X, Z] \text{ is } \begin{cases} \text{bijective if } \dim k < n \\ \text{surjective if } \dim k = n \end{cases}$$

In the pointed case, the same is true.

proof : We can assume f is the inclusion of a subspace (using the (pointed) cylinder). Then apply 2.18 the the pair $(X, A) := (X, \emptyset)$ for surjectivity and
$(X, A) := (X \times I, X \times \partial I)$ for injectivity.
For the pointed case, use $(X, \{x_0\})$ and
$(X \times I, X \times \partial I \cup \{x_0\} \times I)$ respectively. ☒

2.20 Thm (Whitehead) : Let $f : Y \to Z$ be a map between CW-cx.
(a) f is a homotopy equivalence $\Leftrightarrow$ f is a weak

equivalence (i.e. an n-equivalence $\forall n \in \mathbb{N}$).
(b) If $\dim Y \leq k$ and $\dim Z \leq k$, it suffices that $f_* : \pi_q(Y, y) \to \pi_q(Z, f(y))$ be bijective $\forall y \in Y$, $\forall q \leq k$.

proof. Assume $f : Y \to Z$ is a weak equivalence. Then by 2.19 ($n = \infty$), $[X, Y] \xrightarrow{f_*} [X, Z]$ is an iso for any CW-cx X. Taking $X = Z$, we obtain that $\exists g \in [Z, Y]$ with $[f \circ g] = f_*([g]) = [Id_Z]$; thus g is a left homotopy inverse of g. We deduce that g is also a weak equivalence $[\forall n \in \mathbb{N}, \forall y \in Y, \pi_n(Y, y) \xrightarrow{f_*} \pi_n(Z, f(z))$ is an iso with left inverse g_*, so $g_* : \pi_n(Z, z) \to \pi_n(Y, g(z))$ $\forall n, \forall z \in Z]$
Thus, by the same argument, g has a left inverse

h with $[g]\circ[h]=[id_Y]$. We deduce $[f]=[h]$ and g is a homotopy inverse of f; so f is a homotopy equivalence.

(b) Same argument, using 2.19 with $n<\infty$. ☒

We deduce that a CW complex X is contractible iff $X\neq\emptyset$, $\pi_k(X,x)=0$ $\forall\, x\in X$, $\forall\, k\in\mathbb{N}$.

<u>2.21 Lemma</u> Let (X,A) be a relative CW cx. Then the pair $(X,X^{(n)})$ is n-connected.

proof: First, we prove by induction on $\ell\geqslant n$ that the pair $(X^{(\ell)},X^{(n)})$ is n-connected. For $\ell=n$ this is obvious since $(X^{(n)},X^{(n)})$ is ∞-connected. Assume $\ell\geqslant n$ given and $(X^{(\ell)},X^{(n)})$ n-connected.

We first show that $(X^{(\ell+1)},X^{(\ell)})$ is ℓ-connected; suppose $X^{(\ell+1)}$ is obtained from $X^{(\ell)}$ by attaching $\ell+1$-cells along $\{\varphi_\sigma: S^\ell\to X^{(\ell)}\}_{\sigma\in J}$ and $\{\chi_\sigma: D^{\ell+1}\to X^{(\ell+1)}\}_{\sigma\in J}$.

Take $U_0=\coprod_{\sigma\in J} e_\sigma\subset X^{(\ell+1)}$ (the union of open cells) and $U_1=B\cup\bigcup_{\in J}(\bar e_\sigma\setminus\chi_\sigma(0))$. Then obviously U_0,U_1 are open, $U_0\cup U_1=X^{(\ell+1)}$, and $U_{01}=U_0\cap U_1=\coprod_{\sigma\in J}(e_\sigma\setminus\chi_\sigma(0))\neq\emptyset$; can assume (U_1,U_{01}) 0-connected.

Since we know that $(D^{\ell+1},S^\ell)$ is ℓ-connected, so is (U_0,U_{01}). Thus $(U_0,U_{01})\to(Y,U_1)$ is ℓ-connected, thus $(X^{(\ell+1)},U_1)$ is ℓ-connected.

But $U_1\simeq X^{(\ell)}$, so also $(X^{(\ell+1)},X^{(\ell)})$ is ℓ-connected.

To finish the iteration, use Exercise 1.9 and the exact sequence of the triple $(X^{(\ell+1)},X^{(\ell)},X^{(n)})$.

Then use $(X, X^{(n)}) = \operatorname{colim}_k (X^{(n+k)}, X^{(n)})$

and the (relative) version of Exercise 1.10. ▨

<u>2.22 Theorem</u> (cellular approximation): Let X, Y be two CW-cxs, $A \subset X$ a sub-complex (possibly empty), and $f : X \to Y$ a map such that $f|_A : A \to Y$ is cellular. Then $f \simeq g$ rel A where g is cellular.

Moreover, given a htpy between two cellular maps, $f \overset{H}{\simeq} g$ rel $A : X \to Y$, H can be chosen cellular.

<u>proof</u>: as earlier! Construct inductively a sequence of homotopies $H^n : X \times I \to Y$ rel $A \cup X^{(n-1)}$, $n \in \mathbb{N}$, with $H^0_0 = f$, $H^n_1 = H^{n+1}_0$, $H^n_1(X^{(n)}) \subset Y^{(n)}$.

Then the infinite concatenation will do.

Can start with H^{-1} (constant homotopy) for the initialization.

To do the iteration, let us just assume given $n \in \mathbb{N}$, $f : X \to Y$ with $f|_A$ cellular and $f(X^{(n)}) \subset Y^{(n)}$.

We have the attaching of cells: Since $(Y, Y^{(n+1)})$ is $(n+1)$ connected, and $\dim(\amalg D^{n+1}, \amalg S^n) = n+1$,

$$\begin{tikzcd} \amalg S^n \arrow[r, "\varphi"] \arrow[d] & X^{(n)} \arrow[r, "f"] \arrow[d] & Y^{(n)} \arrow[d] \\ \amalg D^{n+1} \arrow[r, "\chi"] & X^{(n+1)} \arrow[r, "f"] & Y \end{tikzcd}$$

we can deform $f \circ \chi$ to a map into $Y^{(n+1)}$ rel $\amalg S^n$.

This is given by a homotopy $H^{(n+1)} : A \cup X^{(n+1)} \times I \to Y$ rel $A \cup X^{(n)}$, with $H_0^{(n+1)} = f$ and $H_1^{(n+1)}(X^{(n+1)}) \subset Y^{(n+1)}$.

Using that $A \cup X^{(n+1)} \hookrightarrow X$ is a cofibration, extend the homotopy to $X \times I$.

This proves: $\exists\, g$ cellular, $f \overset{H}{\simeq} g$ rel (A).

If f and g are both cellular, we can suppose

H cellular also: apply the above proof to the relative CW.

$(X \times I, X \times \partial I \cup A \times I) \xrightarrow{H} Y$. ▨

<u>2.23 Theorem (CW-approximation)</u> Let $\ell \geq -1$, A, Y spaces (not CW) and $f: A \to Y$ a ℓ-conn. map. Suppose $\ell \leq n \leq \infty$ given. Then there exists a relative CW-cx (X, A) and an extension $F: X \to Y$ of f, such that F is n-connected. If A is a relative CW, then X can be chosen so that (X, A) is a relative CW-pair.

<u>proof</u>: We start with $X^{(-1)} = A$, $f^{(-1)} := f : X^{(-1)} \to Y$. We inductively on k define $f^{(k)}: X^{(k)} \to Y$ such that

$$\left.\begin{array}{ll} (i) & X^{(k)} \text{ obtained from } X^{(k-1)} \text{ by attaching } k\text{-cells}, \\ (ii) & f^{(k)}|_{X^{(k-1)}} = f^{(k-1)}, \\ (iii) & f^{(k)} \text{ is } k\text{-connected}. \end{array}\right\} \forall k \geq 0$$

Then take $X = \bigcup_{k \geq -1}^{n} X^{(k)}$ (weak top in $n = \infty$) and $F: X \to Y$ with $F|_{X^{(k)}} = f^{(k)}$.

<u>Note</u>: since at step k we attach only k-cells, if $f^{(k-1)}$ was $(k-1)$-connected, so will be $f^{(k)}$; so we just need to check that $f^{(k)}$ is bijective on π_{k-1} and surjective on π_k.

We assume $\ell = -1$ (otherwise start induction at $k = \ell$).

<u>Case $k = -1$</u>: If $f_*: \pi_0 A \to \pi_0 Y$ surjective, take $X^{(0)} = A$ and $f^{(0)} = f$. Otherwise take $X^{(0)} = A \amalg \coprod_{\sigma \in J_0} D^0_\sigma$ with $J_0 = \pi_0 Y \setminus \pi_0 A$, and $f^{(0)}(D^0_\sigma) \in \sigma \subset Y$.

<u>case $k = 0$</u> Assume $f^{(0)}_*: \pi_0 X^{(0)} \to \pi_0 Y$ surjective.

If U, V are different path components of A mapping to the same comp. of Y, choose $u \in U$, $v \in V$, $\alpha: I \to Y$ a path in Y from $f^{(0)}(u)$ to $f^{(0)}(v)$. Then attach a 1 cell D^1_α to $X^{(0)}$ with end points u, v, and use α to extend

$f^{(0)}$ to $X^{(0)} \cup D^1_\alpha$. Repeat until have a π_0-iso. The resulting map $g: Z = X^{(0)} \cup \bigcup_{\sigma \in J_1} \bar{e}^1_\sigma \to Y$ is not nec. a surjection on π_1. Repeat the procedure by attaching 1-cells to Z (with same end point, so not changing π_0) to ensure that the resulting map $f^{(1)}: X^{(1)} \to Y$ is surjective on π_1.

Assume $k \geq 2$, and $f^{(k-1)}: X^{(k-1)} \to Y$ $(k-1)$-connected constructed. Suppose $f^{(k-1)}$ is the inclusion of a subspace (using mapping cylinder $X^{(k-1)} \hookrightarrow Cyl(f^{(k-1)}) \simeq Y$. Can also assume $X^{(k-1)}$ (and thus Y) are path-connected. Take $* \in X^{(k-1)}$, and choose a family of maps $\{ g_\sigma : (D^k, S^{k-1}, *) \to (Y, X^{(k-1)}, *) \}_{\sigma \in J_k}$ that generate $\pi_k(Y, X^{(k-1)}, *)$ as a $\pi_1(X^{(k-1)}, *)$-module.

Along these maps, attach k-cells to obtain $X^{(k)}$, and let $f^{(k)}$ be the map induced on the push-out $X^{(k)}$ by $f^{(k-1)}$ and $\amalg g_\sigma$

$$\begin{tikzcd}
\coprod_{J_k} S^{k-1} \arrow[r, "g_\sigma|S^{k-1}"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X^{(k-1)} \arrow[d, "f^{(k-1)}"] \arrow[dd, bend left, "f^{(k-1)}"] \\
\coprod_{J_k} D^k_\sigma \arrow[r] \arrow[dr, bend right, "\amalg g_\sigma"'] & X^{(k)} \arrow[d, dashed, "f^{(k)}"] \\
 & Y
\end{tikzcd}$$

Then obviously $f^{(k)}: \pi_k(X^{(k)}, *) \to \pi_k(Y, *)$ is surjective. Consider the diagram with exact rows

$$\begin{tikzcd}
\pi_k(X^{(k-1)}) \arrow[r] \arrow[d, "="] & \pi_k(X^{(k)}) \arrow[r] \arrow[d, "(1)"] & \pi_k(X^{(k)}, X^{(k-1)}) \arrow[r] \arrow[d, "(2)"] & \pi_{k-1}(X^{(k-1)}) \arrow[r] \arrow[d, "="] & \pi_{k-1}(X^{(k)}) \arrow[r] \arrow[d, "(3)"] & 0 \\
\pi_k(X^{(k-1)}) \arrow[r] & \pi_k(Y) \arrow[r] & \pi_k(Y, X^{(k-1)}) \arrow[r] & \pi_{k-1}(X^{(k-1)}) \arrow[r] & \pi_{k-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

(2) surjective $\Rightarrow$ (3) is injective; since it was surjective, it is also bijective: ($\downarrow \; \downarrow \; \downarrow \; \downarrow \; \downarrow$ $\Rightarrow$ middle arrow injective).

(2) surjective now implies (1) surjective ($\downarrow \; \downarrow \; \downarrow \; \downarrow \; \downarrow$ $\Rightarrow$ mid. row surj.)

$\Rightarrow f^{(k)}$ k-connected ! If wished, could assume all g_σ cellular. $\square$

Remark: X obtained from A by attaching only cells of dim k with $\ell+1 \leq k \leq n$. In particular if Y is n-connected, can assume the CW-approx $X \to Y$ chosen so that X has 1 0-cell, and no cells of dim k with $1 \leq k \leq n$.

2.24 Def + Lemma (Naturality) Let Y be a space. If X is a CW-complex and $X \xrightarrow{f} Y$ a weak equivalence, we call f a CW-approximation of Y (It exists thanks to 2.22 with $A = \emptyset$). If $Y_1 \xrightarrow{g} Y_2$ is a map, and $f_i : X_i \to Y_i$ ($i = 1,2$) are CW-approximations, then there exists a map $h : X_1 \to X_2$, unique up to homotopy, such that $g \circ f_1 \simeq f_2 \circ h$. In particular, a CW-approx. is unique up to unique homotopy equivalence.

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, dashed, "\exists!\, h.e."'] & Y_1 \arrow[d, "g"] \\ X_2 \arrow[r] & Y_2 \end{tikzcd}$$

The same holds in the pointed case.

proof: this is just an application of 2.19, since $[X_1, X_2] \xrightarrow{f_{2*}} [X_1, Y_2]$ is bijective. $\boxtimes$

2.25 Theorem (Freudenthal's isomorphism). If (X, x_0) is a pointed CW-cx, and if (X, x_0) is n-connected, then $\Sigma_* : \pi_m(X, x_0) \to \pi_{m+1}(\Sigma X, x_0)$ is an isomorphism for $0 \leq m \leq 2n$, and is surjective for $m = 2n+1$.

2.26 Lemma: Let $A \hookrightarrow X$ be a cofibration, $a \in A$, and $p: (X, A) \to (X/A, a)$ the quotient. If $\pi_i(A, a) = 0$ and $\pi_j(X, A, a) = 0$ for $0 \leq i \leq k$, $1 \leq j \leq \ell$, then

$$p_* : \pi_n(X, A, a) \to \pi_n(X/A, *)$$

is an iso for $1 \leq n \leq k+\ell$, and surj. for $n = k+\ell+1$.

In particular, in the long ex. seq of (X, A, a), we can replace $\pi_n(X, A, a)$ by $\pi_n(X/A, a)$ in these degrees.

proof: we consider the (unreduced) cone of $A \hookrightarrow X$, $C(X,A)$:

$CA = A \times I / (A \times \{1\})$ (unreduced cone of A)

$C(X,A) = (X \amalg CA)/\sim \quad X \ni a \sim (a,0) \in CA$.

We also have the open cones $\mathring{C}A = A \times [0,1)$ and $\mathring{C}(X,A) = (X \amalg \mathring{C}A)/\sim$. Then we have that the quotient map p factorizes as $(X,A) \overset{\alpha}{\hookrightarrow} (\mathring{C}(X,A), \mathring{C}(A)) \overset{\beta}{\hookrightarrow} (C(X,A), C(A)) \overset{r}{\twoheadrightarrow} (C(X,A)/C(A), *) \cong (X/A, *)$. Here

- α is a htpy equivalence (admits a retraction up to htpy)
- β induces an iso (surj) on π_n for $n < k+l$ ($= k+l+1$)
- r is a htpy equivalence (since $C(A) \hookrightarrow C(X,A)$ is also a cofibration, and $C(A) \simeq *$).

To check β, use Blakers-Massey with respect to $U_0 = \mathring{C}(X,A)$ and $U_1 = (A \times (0,1]) / (A \times \{1\})$

This needs moving the base point to a point in U_{01}. ▨

proof of 2.25. We consider the reduced cone of (X, x_0)

$$C(X,x_0) = X \times I / (X \times \{1\}) \cup (\{x_0\} \times I);$$

Then $(C(X,x_0), X)$ is a CW-pair, and $X \hookrightarrow C(X,x_0)$ is a cofib. Since $C(X,x_0) \simeq *$, we see that $\pi_j(C(X,x_0), X, x_0) = 0 \quad \forall j \leq n+1$. Thus P_*, and Σ_*:

$$\begin{tikzcd} \pi_m(X,x_0) \arrow[r, "\partial_{m+1}^{-1}"] \arrow[dr, "\Sigma_*"'] & \pi_{m+1}(C(X,x_0), X, x_0) \arrow[d, "P_*"] \\ & \pi_{m+1}(\Sigma X, x_0) \end{tikzcd}$$

are iso for $m \leq 2n$ and surjective for $m = 2n+1$

2.27 Corollary (a) Let X, Y be CW-complexes with Y n-connected for some $n \geq 0$. Then

$\Sigma_*: [X, Y]_* \to [\Sigma X, \Sigma Y]_*$ is $\begin{cases} \text{bijective if } \dim(X) \leq 2n \\ \text{surjective if } \dim(X) = 2n+1. \end{cases}$

(b) If $k \geq \dim(X) + 2$, and Z is any pointed CW, then $\Sigma_*: [\Sigma^k X, \Sigma^k Z] \to [\Sigma^{k+1} X, \Sigma^{k+1} Z]$ is bijective.

proof: We take the unit $Y \xrightarrow{\sigma} \Omega\Sigma Y$ of the adjunction (Σ, Ω), and obtain a commutative diagram

$$\begin{tikzcd} \pi_m(Y) \arrow[rr, "\Sigma_*"] \arrow[dr, "\sigma_*"'] & & \pi_{m+1}(\Sigma Y) \\ & \pi_m(\Omega\Sigma Y) \arrow[ur, "\cong\ (\text{adj.})"'] & \end{tikzcd}$$

Thus σ_* is a bijection (surj) if $m \leq 2n$ ($m = 2n+1$) by Freudenthal.

Thus σ is a $2n$ connected, and we conclude using

$$\begin{tikzcd} {[X, Y]_*} \arrow[rr, "\Sigma_*"] \arrow[dr, "\sigma_*"'] & & {[\Sigma X, \Sigma Y]} \\ & {[X, \Omega\Sigma Y]} \arrow[ur, "\cong"'] & \end{tikzcd}$$

and Whitehead 2.19.

(b) By induction on k, using that ΣZ is connected, $\Sigma^2 Z$ is 1-connected (Van Kampen), and generally $\Sigma^k Z$ is $k-1$-connected. $\blacksquare$

2.28 Definition: Let C be a category and W be a class of morphisms. A localization of C at W is a functor $Q: C \to C[W^{-1}]$ such that

(a) For any $w \in W$, $Q(w)$ is a iso.

(b) Q is initial with that property

(c) $Q^*: \mathrm{Funct}(C[W^{-1}], A) \to \mathrm{Funct}(C, A)$ is fully faithfull.

2.29 Example: Let C be the category of (left) $\mathbb{Z}$-modules. Say that $f \in W \Leftrightarrow \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} f$ is an iso.

Then $\mathbb{Z}\text{-mod}[W^{-1}] \cong \mathbb{Z}_{(p)}\text{-mod}$, and

$$Q: \mathbb{Z}\text{-mod} \to \mathbb{Z}_{(p)}\text{-mod}$$
$$M \mapsto \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$$

2.30 Remark: let us take W to be the class of weak equivalences on Top. If Top$[W^{-1}]$ exists, then it is not difficult to see that it is equivalent to $Ho(CW)$, the homotopy category of CW-complexes. Indeed, the functor Q : Top $\to$ Top$[W^{-1}]$ will factorize through hoTop:

$$\begin{tikzcd} \mathrm{Top} \arrow[r] & \mathrm{Ho(Top)} \arrow[r] & \mathrm{Top}[W^{-1}] \\ & \mathrm{Ho(CW)} \arrow[u] \arrow[ur, "\cong"'] & \end{tikzcd}$$

A standard construction of Top$[W^{-1}]$ is given by (18)
model categories, using the following classes of morphisms:

(1) Weak equivalence as given above.
(2) Fibrations are Serre fibrations.
(3) Cofibrations are the morphisms having the left lifting property with respect to fibrations that are weak-equivalences.

It can be shown that cofibrations are inclusions of (generalized)-CW or retracts of such inclusions.

└ these are sequential colimit of relative CW-inclus.

This will be done in Homotopy 2.

We now want to make the link to homology.

Recall the axioms defining a (reduced) homology theory:

2.31 Definition: A reduced homology theory on Top$_*$ is a family of functors $\{h_n : \mathrm{HoTop}_* \to \mathcal{A}b\}_{n\in\mathbb{Z}}$ and natural isomorphisms $\sigma_n : h_n \to h_{n+1} \circ \Sigma$ (20)

satisfying :

(1) Exactness: If $A \hookrightarrow X$ is a cofibration, the sequence $h_n(A) \to h_n(X) \to h_n(X/A)$ is exact

(2) Additivity : the canonical maps induce an iso
$$\bigoplus_i h_n(X_i) \longrightarrow h_n(\bigvee_i X_i)$$
for any family $\{X_i\}_{i \in I}$ in $hoTop_*$.

(3) Weak htpy : a weak homotopy induces an iso.

The graded group $h_*(S^0) = \bigoplus_n h_n(S^0)$ is called the (graded) gp of coefficients of (h, σ).

<u>2.32 Example</u>: (unreduced) Singular homology $H_*(-;\mathbb{Z})$ provides a reduced homology theory $\tilde{H}_*(-;\mathbb{Z})$

defined by $\tilde{H}_n((X,x_0);\mathbb{Z}) = \ker(H_n(X,\mathbb{Z}) \to H_n(x_0,\mathbb{Z}))$.
Its coefficient are $\mathbb{Z}$, and $\tilde{H}_k(X;\mathbb{Z}) = 0 \ \forall k < 0$.
We leave it to the reader to deduce the construction of σ and the axioms from the Eilenberg-Steenrod axioms for $H_*(-;\mathbb{Z})$. The following proposition will be detailed in the exercises :

<u>2.33 Proposition</u>: If $f : X \to Y$ is a weak equivalence, then $H_*(X,\mathbb{Z}) \xrightarrow{f_*} H_*(Y,\mathbb{Z})$ is an isomorphism.

<u>2.34 Definition</u> : let $X \in Top_*$, and $n \geq 0$.
Choose a generator $i_0 \in \mathbb{Z} = \tilde{H}_0(S^0,\mathbb{Z})$, and
$i_n = \sigma^n(i_0) \in \tilde{H}_n(S^n,\mathbb{Z}) \cong \mathbb{Z}$.
Define $h_n : \pi_n(X) \to \tilde{H}_n(X,\mathbb{Z})$, $[\alpha] \mapsto \alpha_*(i_n)$

where $(I^n, \partial I^n) \xrightarrow{\alpha} (X, x_0)$. Then h_n is well-def,

$$\begin{tikzcd}(I^n,\partial I^n) \arrow[r, "\alpha"] \arrow[d, two heads, "q"'] & (X,x_0) \\ (S^n,*) \arrow[ur, "\bar\alpha"'] & \end{tikzcd}$$

is a homomorphism if $n \geq 1$, and defines a natural transf. of functors, called the Hurewicz homomorphism.

proof: left as an easy exercise; for the additivity, use the additivity axiom, together with the fact that the addition of $\pi_n(X)$ is induced by $S^n \xrightarrow[\text{pinch}]{} S^n \vee S^n$

2.35 Theorem (Hurewicz) Let $n \geq 2$ and $(X, x_0) \in \mathrm{Top}_*$ be $(n-1)$-connected. Then $h_n : \pi_n(X, x_0) \to \tilde{H}_n(X, x_0)$ is an isomorphism. If (X, x_0) is connected, then h_1 factors as

$$\begin{tikzcd}\pi_1(X,x_0) \arrow[r, "h_1"] \arrow[dr, "q"'] & \tilde{H}_1(X,x_0) \\ & \pi_1(X,x_0)^{ab} \arrow[u, "\cong"'] \end{tikzcd}$$

proof: The case $n=1$ was proven earlier? Assume $n \geq 2$ and X $(n-1)$-connected. Using CW approximation, we can assume that X is a CW complex with one zero cell and no cells of dimension k for $0 < k < n-1$: $X^{(n-1)} = \{x_0\}$. Moreover, the inclusion $X^{(n+1)} \xhookrightarrow{i} X$ induces isos on π_n, $\tilde{H}_n$.

$$\begin{tikzcd}\pi_n(X^{(n+1)}) \arrow[r, "i_*", "\cong"'] \arrow[d, "h_n"] & \pi_n(X) \arrow[d, "h_n"] \\ \tilde{H}_n(X^{(n+1)}) \arrow[r, "i_*", "\cong"'] & \tilde{H}_n(X)\end{tikzcd}$$

Thus we can assume we have a cofibr. seq $\underbrace{\bigvee_{\sigma \in J_{n+1}} S^n}_{A} \to \underbrace{Cyl(\vee \varphi_\sigma)}_{B} \to X = B/A$. We have

$$\begin{tikzcd}\pi_n(A) \arrow[r] \arrow[d, "①"] & \pi_n(B) \arrow[r] \arrow[d, "②"] & \pi_n(X) \arrow[r] \arrow[d, "h_n"] & 0 \\ \tilde{H}_n(A) \arrow[r] & \tilde{H}_n(B) \arrow[r] & \tilde{H}_n(X) \arrow[r] & 0\end{tikzcd}$$

Note that $B \simeq \bigvee_{\sigma \in J_n} S^n$. So ① + ② are isos if

$\pi_n(\bigvee_{i\in I} S^n) \xrightarrow{\alpha} \tilde{H}_n(\bigvee_{i\in I} S^n)$ is an iso for any set I.
We know that $\pi_n(S^n) \longrightarrow \tilde{H}_n(S^n)$ is an iso.
We also have that the can. map

$$\bigoplus_i \pi_k(S^n) \longrightarrow \pi_k(\bigvee_i S^n) \qquad \left(\begin{array}{l}\text{For } k=n=1,\\ \text{replace by } \ast_i\end{array}\right)$$

is an iso (Ex. 4.3) for k in some finite range (including $k=n$). So by additivity, α is an iso. $\square$

We deduce: If X is simply connected and if for some $n \geq 2$, $\tilde{H}_i(X;\mathbb{Z}) = 0$ for all $i<n$, then X is n-connected.

<u>2.36 Remark</u>: There is also a relative version:
For $(X,A) \in Top^2_*$, $n \geq 1$, let

$$h_n : \pi_n(X,A) \longrightarrow H_n(X,A), \quad h_n[\alpha] = \alpha_*(j_n)$$

where $j_n = \partial_n^{-1}(i_n) : H_n(D^n, S^{n-1}) \xrightarrow[\cong]{\partial_n} H_{n-1}(S^{n-1})$

<u>2.37 Proposition</u>: (X,A) a CW pair of 1-conn. spaces. Suppose that we have $n \geq 2$ and $H_i(X,A) = 0$ $\forall i<n$. Then (X,A) is $(n-1)$-connected, and

$$h_n : \pi_n(X,A) \longrightarrow H_n(X,A) \text{ is an iso.}$$

<u>proof</u>: consider the commutative diagram

$$\begin{tikzcd}
\pi_k(X,A) \arrow[r, "q_k^\pi"] \arrow[d, "h'_k"'] & \pi_k(X/A) \arrow[d, "h_k"] \\
H_k(X,A) \arrow[r, "q_*^H"] & \tilde{H}_k(X/A)
\end{tikzcd}$$

Since $A \hookrightarrow X$ cofibration we know that q^H_* is an iso. Then work by induction on k, for $2 \leq k \leq n$, using 2.25:
Since (X,A) 1-connected, q_k^π is an iso for $k=1,2$.
By 2.34, h_2 is an iso, so so is h'_2.
Thus $\pi_2(X,A) = 0$; reapply 2.25, etc. $\square$

2.38 Theorem (Whitehead) : Let $f: X \to Y$ be a map between 1-connected spaces, and $2 \le n \le \infty$. Suppose that $f_*: \tilde{H}_k(X) \to H_k(Y)$ is an iso for all $k < n$ and surjective for $k = n$.
Then $f: X \to Y$ is n-connected.
For $n = \infty$: a homology equivalence of simply connected spaces (CW-cx) is a weak-equivalence (a htpy equivalence).

2.39 Example: Need 1-conn. above! S^3 has a finite perfect subgroup of order 120: the binary Icosaedral group I'. The quotient $P = S^3/_I$ of groups is a smooth, compact, orientable manifold: the Poincaré Sphere. We have a fibration $I' \to S^3 \xrightarrow{q} P$
So $\pi_k(S^3) \xrightarrow{\cong} \pi_k(P)$ if $k \geq 2$, and $\pi_1(P) \cong I'$. Moreover

$q_*: \tilde{H}_*(S^3, \mathbb{Z}) \to \tilde{H}_*(P; \mathbb{Z})$ is an iso for all $*$.
See Exercise Sheet 5 for the details.