# III Simplicial Sets

They provide a combinatorial presentation of spaces.

**3.1. Def**: We define a small cat $\Delta$ with

$Ob(\Delta) = \{[n] ;\ n \in \mathbb{N}\}$, $[n] = \{0, 1, \dots, n\}$ (ordered)

$\Delta([m],[n]) = \{ f : [m] \to [n] ;\ f \text{ order-preserving}\}$.

For $n \geq 1$ and $0 \leq i \leq n$, let $\delta_i : [n-1] \to [n]$ be the (unique) morphism of $\Delta$ with $Im(\delta_i) = [n] \setminus \{i\}$.

For $n \geq 0$ and $0 \leq i \leq n$, let $\sigma_i : [n+1] \to [n]$ be the (unique) morphism of $\Delta$ with $\sigma_i(i) = \sigma_i(i+1) = i$.

$$\begin{tikzcd}
\cdots {[n-1]} \arrow[r, shift left=6, "\delta_0"] \arrow[r, shift right=6, "\delta_n"'] & {[n]} \arrow[l, shift right=3, "\sigma_0"'] \arrow[l, shift left=3, "\sigma_{n-1}"] \arrow[l, phantom, "\vdots"] \arrow[r, shift left=6, "\delta_0"] \arrow[r, shift right=6, "\delta_{n+1}"'] & {[n+1]} \cdots \arrow[l, shift right=3, "\sigma_0"'] \arrow[l, shift left=3, "\sigma_n"] \arrow[l, phantom, "\vdots"]
\end{tikzcd}$$

**3.2 Lemma**: when composable, the $\delta_i$ and $\sigma_j$ satisfy the following identities (called cosimplicial identities)

$$\begin{cases} \delta_j \delta_i = \delta_i \delta_{j-1} & \text{if } i<j \\ \sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i<j \\ id & \text{if } i=j, j+1 \\ \delta_{i-1}\sigma_j & \text{if } i>j+1 \end{cases} \\ \sigma_j \sigma_i = \sigma_i \sigma_{j+1} & \text{if } i \leq j \end{cases}$$

Moreover, any $\alpha : [m] \to [n]$ in $\Delta$ can be written in a unique way as $\alpha = \delta_{i_1} \cdots \delta_{i_p} \sigma_{j_1} \cdots \sigma_{j_q}$

were $\{ i_p < \cdots < i_1 \} = [n] \setminus \alpha([m])$ and

$\{ j_1 < \cdots < j_q \} = \{ j \in [m] ;\ \alpha(j) = j+1 \}$

proof: obvious but tedious check. ▨

**3.3 Remark.** In fact the morphisms of $\Delta$ are generated by all $\delta_i$'s and $\sigma_i$'s, and all relations are generated by the relations given in 3.2. This is an easy consequence of 3.2 (unicity of decomposition given).

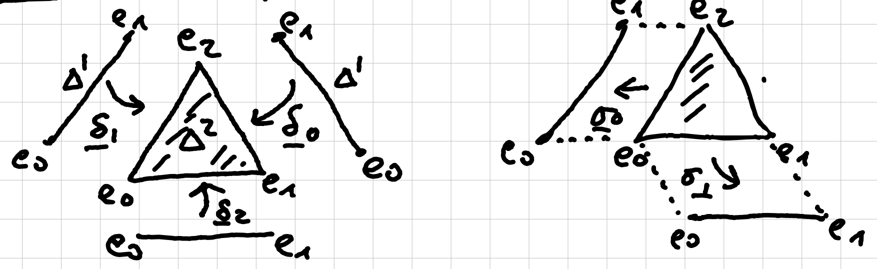
**3.4. Definition:** For $n \in \mathbb{N}$, define the (geometric) standard $n$-simplex $\Delta^n$ as

$$\Delta^n = \{ x \in \mathbb{R}^{n+1} ; x \in [0,1]^{n+1}, \sum_{i=1}^{n+1} x_i = 1 \} .$$

For $0 \leq i \leq n$, define the $i$th vertex $e_i^n = e_i \in \Delta^n$ as $e_i = (\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0)$. Thus $\Delta^n$ is the convex hull of $\{e_0, \dots, e_n\}$.

**3.5 Def** We have a functor $\Delta \to Top$ defined by $([m] \xrightarrow{\alpha} [n]) \mapsto (\Delta^m \xrightarrow{\underline{\alpha}} \Delta^n)$, $\underline{\alpha} : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ is the linear map uniquely determined by $\underline{\alpha}(e_i^m) = e^n_{\alpha(i)}$.

**3.6. Example** $\delta_i : \Delta^1 \to \Delta^2$ and $\sigma_i : \Delta^2 \to \Delta^1$



**3.7 Def:** $D$ a small category, and $\hat{D} = Funct(D^{op}, Sets)$ the cat of presheaves (of sets) on $D$. The functor

$$D \xrightarrow{h} \hat{D}$$
$$d \mapsto h_d = D(-,d)$$

is called the Yoneda embedding.

Thus $X \in \hat{D}$ is a contravariant functor; For $a \xrightarrow{\alpha} b$ in $D$, we denote $X(a \xrightarrow{\alpha} b)$ as $X_b \xrightarrow{\alpha^*} X_a$, where $X_a := X(a)$ is a set. Recall:

**Yoneda's Lemma:** For any $d \in D$ and $X \in \hat{D}$, we have a bijection

$$\begin{aligned} \hat{D}(h_d, X) &\longrightarrow X_d \\ f &\longmapsto f_d(1_d) \end{aligned}$$

In particular, $h : D \to \hat{D}$ is fully faithful.

proof: Note that $f : h_d \to X$ is a natural transf, i.e. for any $a \in \mathrm{Ob}\, D$, $f_a : D(a,d) = (h_d)_a \longrightarrow X_a$

In particular $f_d : D(\underset{\substack{\rotatebox{90}{$\in$} \\ 1_d}}{d},d) \to X_d$, so this explains $f_d(1_d)$

The inverse $X_d \to \hat{D}(h_d, X)$ is given by

$$s \longmapsto \left( f_a : D(a,d) \to X_a\, ,\ \alpha \mapsto \alpha^*(s) \right)_a$$

(Check that these are inverse of each other). Then the id of $D(a,b)$ factorises as $D(a,b) \xrightarrow{h} \hat{D}(h_a, h_b) \xrightarrow{\cong} (h_b)_a = D(a,b)$. ☒

**3.8 Remark** : the category $\hat{D}$ is complete and cocomplete: limits and colimits are constructed point-wise (in sets):

For $J \xrightarrow{F} \hat{D}$, define

$$(\lim_J F)(d \xrightarrow{\alpha} e) = \lim_J \left( F(j)_e \xrightarrow{\alpha^*} F(j)_d \right) \quad \text{(lim in sets)}$$

$$(\operatorname{colim}_J F)\ (d \xrightarrow{\alpha} e) = \operatorname{colim}_J \left( F(j)_e \xrightarrow{\alpha^*} F(j)_d \right) \quad \text{(colim in sets)}$$

Check that these are indeed limits and colimits. ☒

**3.9 Def** : For $X \in \hat{D}$, we form the *category of objects of* $X$, $D/X$, as pairs $(d, s)$, $s \in X_d$, and morphisms $(d,s) \xrightarrow{\alpha} (e,t)$ for any $\alpha \in D(d,e)$ with $\alpha^*(t) = s$.

We have a functor $\varphi_X : D/X \to \hat{X}$, $(d,s) \mapsto h_d$.

**Remark** : can view $D/X$ as having objects $h_d \xrightarrow{s} X$ and morphisms

$$\begin{tikzcd} h_d \arrow[rr, "\alpha"] \arrow[dr, "s"'] & & h_e \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

, and $\varphi_X$ send this morp. to $h_d \xrightarrow{\alpha} h_e$.

**3.10 Lemma** We have a canonical isomorphism in $\hat{D}$

$$\operatorname*{colim}_{D/X} \varphi_X = \operatorname*{colim}_{(d,s)\in D/X} h_d \xrightarrow[\cong]{\eta} X \qquad \Bigg| \quad \eta_{(d,s)} = s$$

$$\begin{tikzcd} h_d \arrow[r, "\alpha"] \arrow[d, "s"'] & h_e \arrow[d, "t"] \\ X \arrow[r, "id"] & X \end{tikzcd}$$

induced by $\varphi_X \xrightarrow{\eta} C_X$, $\left[(d,s) \xrightarrow{\alpha} (e,t)\right] \mapsto$ the square above.

proof: Suffices to show that if we have another cone $\varphi_X \xrightarrow{f} C_Y$, $\exists!\ X \xrightarrow{f} Y$ in $\hat{D}$:

$$\begin{tikzcd} h_d \arrow[rr, "\alpha"] \arrow[dr, "s"] \arrow[ddr, bend right, "f_{(d,s)}"'] & & h_e \arrow[dl, "t"'] \arrow[ddl, bend left, "f_{(e,t)}"] \\ & X \arrow[d, dotted, "\exists!"'] & \\ & Y & \end{tikzcd}$$

This is obvious! ▨

**3.11. Theorem** Suppose given a functor $F : D \to C$, where $C$ is cocomplete, $D$ small. For $X \in \hat{D}$, let $F_X : D/X \to C$, $F_X\left((d,s) \xrightarrow{\alpha} (e,t)\right) = F(d) \xrightarrow{\alpha_*} F(e))$.

Then we have a functor $F_! : \hat{D} \to C$, $X \to \operatorname*{colim}_{D/X} F_X$, left adjoint to $F^* : C \to \hat{D}$

$$Y \mapsto (d \mapsto C(F(d), Y))$$

Moreover, we have a unique natural iso $F_!(h_d) \xrightarrow{\eta} F(d)$ for all $d \in Ob(D)$ inducing for any $Y \in Ob(C)$ the comp.

$$C(F(d), Y) =: F^*(Y)_d \cong \hat{D}(h_d, F^*(Y)) \cong C(F_!(h_d), Y)$$

proof: Take $X \in \hat{D}$ and $Y \in C$. Then $C(F_! X, Y)$

$$= C(\operatorname*{colim}_{(d,s)\in D/X} F(d), Y) = \lim_{(d,s)\in D/X} C(F(d), Y)$$

$$= \lim_{(d,s)\in D/X} F^*(Y)_d = \lim_{(d,s)\in D/X} \hat{D}(h_d, F^*(Y)) =$$

$\hat{D}(X, F^*(Y))$. For $\eta$, note that $id : h_d \to h_d$ is a terminal object of $D/h_d$, so $F_!(h_d) = \operatorname*{colim}_{(e,t)\in D/h_d} F(e) = F(d)$.

▨

**3.12 Remark**: From 3.11 we can also deduce that a colim-preserving functor $\hat{D} \xrightarrow{G} C$ is isomorphic to $F_!$ for the functor $F : D \to C$, $d \mapsto G(h_d)$, thus admits a right adjoint (given by $F^*$).

**3.13. Proposition**: Let D be a small category. Then $\widehat{D}$ is cartesian closed; the internal $\underline{\widehat{D}}(X, Y)$ is defined by $\underline{\widehat{D}}(X, Y)_d := \widehat{D}(h_d \times X, Y)$.
We have natural iso's $\underline{\widehat{D}}(X, \underline{\widehat{D}}(Y, Z)) \cong \underline{\widehat{D}}(X \times Y, Z)$.

*proof*: Exercise, using 3.12 and 3.13. ▨

Recall the small cat $\Delta$ defined above in 3.1.

**3.14 Def**: We call $\widehat{\Delta} = \mathrm{Funct}(\Delta^{op}, \mathrm{Sets})$ the category of *simplicial sets*. If $X \in \widehat{\Delta}$ is a simp. set, we call $X_n := X_{[n]}$ the set of *n-simplices* of X.
If $\alpha : [m] \to [n]$ is a morphism in $\Delta$, we have $\alpha^* : X_n \to X_m$. We call:
$d_i = (\delta_i)^* : X_n \to X_{n-1}$ the "ith *face* operator" and
$s_i = (\sigma_i)^* : X_n \to X_{n+1}$ the "ith *degeneracy* operator".

**3.15 Remarks** (a) $X \in \widehat{\Delta}$ is thus determined by the family $\{X_n\}_{n \in \mathbb{N}}$ of sets, and the face & deg. operators.
We often picture it as

$$\begin{tikzcd} X_0 \arrow[r, "s_0"] & X_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & X_2 \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] & \cdots \end{tikzcd}$$

Note that because of $d_i s_i = id$, the $s_i$ are injective (i.e the "size" of $X_i$ does not decrease when $i$ increases.
(b) $\widehat{\Delta}$ is complete and cocomplete (3.8) and cartesian closed (3.13). Note that the cartesian product is given by $(X_\cdot \times Y_\cdot)_n = X_n \times Y_n$, with component-wise faces/degeneracies. The internal hom is given by $\underline{\widehat{\Delta}}(X_\cdot, Y_\cdot)_n = \widehat{\Delta}(X_\cdot \times \Delta^n_\cdot, Y_\cdot)$, with faces and degeneracies $\alpha \mapsto (1_{X_\cdot} \times \alpha_*)^*$. Here $\Delta^n_\cdot$ :

**3.16 Def** For $n \in \mathbb{N}$, we denote $\Delta^n_\bullet \in \hat{\Delta}$ the image of $[n] \in \Delta$ by the Yoneda embedding $\Delta \xhookrightarrow{h} \hat{\Delta}$ : $\Delta^n_\bullet = h_{[n]}$.
$\Delta^n_\bullet$ is called the standard n-simplex (in $\hat{\Delta}$).

Thus by definition, $\Delta^n([k] \xrightarrow{\alpha} [\ell]) = \Delta([\ell],[n]) \xrightarrow{\alpha^*} \Delta([k],[n])$

**3.17 Def** : We have the functor $F : \Delta \to Top$, $n \mapsto \Delta^n$ ("geometric" standard n-simplex) described above. By 3.11 we have an adjoint pair of functors $(|\ |, Sing_\bullet)$

$$|-| = F_! : \hat{\Delta} \rightleftarrows Top : F^* = Sing_\bullet(-)$$

$Sing_\bullet : Top \to \hat{\Delta}$ is called the singular functor;
$|\ | : \hat{\Delta} \to Top$, $X \mapsto |X|$ is called the geometric realization

**3.18 Remark** Let's be concrete. By 3.11 we have a preferred, natural homeo $|\Delta^n_\bullet| = F_!(\Delta^n_\bullet) \cong F([n]) = \Delta^n$.

By definition of $F^*$, for $X \in Top$,
$Sing_\bullet(X) = F^*(X)$ is the simplicial set with

$$Sing_n(X) = Top(F([n]), X) = Top(\Delta^n, X)$$

and $\alpha : [m] \to [n]$ induces

$$Top(\Delta^n, X) \xrightarrow{F(\alpha)^*} Top(\Delta^m, X).$$

By definition of $F_!$, for $X_\bullet \in \hat{\Delta}$, we have

$$|X_\bullet| = \operatorname*{colim}_{([n],s)\in \Delta/X_\bullet} \Delta^n = \operatorname*{colim}_{[n]\in\Delta} \operatorname*{Colim}_{s \in X_n} \Delta^n =$$

$$\operatorname*{colim}_{[n]\in\Delta} (X_n \times \Delta^n) = \coprod_{[n]\in\Delta} (X_n \times \Delta^n) \big/ \sim$$

where $\sim$ is given by $(\alpha^*(s), x) \sim (s, \alpha_*(x))$.

**Remark**: use the formula for colim given by $\coprod$ + coequalizer.

We now want to see that $|X_\bullet|$ comes with a natural CW-dec.