I CATEGORIES

1.1 Def: a category C consists of

(i) a class of objects, denoted C or $Ob(C)$

(ii) for any pair $X, Y \in Ob(C)$, a set $C(X,Y)$ (the set of morphisms from X to Y); a morphism $f \in C(X,Y)$ is often denoted $f : X \to Y$

(iii) For any $X, Y, Z \in Ob(C)$, a composition law

$$C(Y,Z) \times C(X,Y) \to C(X,Z)$$
$$(g, f) \mapsto g \circ f$$

This data is supposed to satisfy:

(a) The composition is associative

(b) For all $X \in Ob(C)$, there exists $1_X \in C(X,X)$ (the id. of X) such that for any $Y, Z \in Ob(C)$, any $f \in C(X,Z)$, any $g \in C(Y, \ )$, we have $f \circ 1_X = f$ and $1_X \circ g = g$

1.2 Remark: In general $Ob(C)$ does not form a set.
We call C a small category if $Ob(C)$ forms a set.

1.3 Examples (a) Set, the category of sets and maps of sets.
(b) Gp, the category of groups and group homomorphisms.
(c) R a commutative ring, R-Mod, the category of R-modules and R-linear homomorphisms.
(d) Top, the category of topological spaces.

Note that none of these is small. In all of them, objects are sets with more structure, and the morphisms are maps of set preserving this structure. We have also example of more "abstract" type.

1.4 Def: Let C be a category. Define the opposite category C^{ob} with

(i) $Ob(C^{op}) = Ob(C)$

(ii) $\forall X, Y \in Ob(C^{op})$, $C^{op}(X,Y) = C(Y,X)$

(iii)
$$\begin{tikzcd}
C^{op}(Y,Z) \times C^{op}(X,Y) \arrow[r, equal] \arrow[d, "\circ"] & C(Z,Y) \times C(Y,X) \arrow[d, "\cong"] \\
C^{op}(X,Z) \equiv C(Z,X) & C(Y,X) \times C(Z,Y) \arrow[l, "\circ"']
\end{tikzcd}$$

(Thus for $\begin{array}{ccc} C(Y,X) & \equiv & C^{op}(X,Y) \\ h & \mapsto & h^{op} \end{array}$ we have $f^{op} \circ g^{op} := (g \circ f)^{op}$.

1.5 Examples: (a) $\underline{Set}^{op}$. You can describe $f \in Set^{op}(X,Y)$ as a map of sets $X \to P(Y)$ satisfying a condition (exercise)

(b) If $(G, \cdot, e)$ is a gp, we can view it as a category $\underline{G}$ with one object, $ob(\underline{G}) = \{*\}$, and
$$(\underline{G}(*,*), \circ, 1_*) = (G, \cdot, e)$$

(c) P a set with a reflexive and transitive relation $\leq$ is called a pre-order; it can be viewed as a category with $Ob(P) := P$, and for $a, b \in P$,
$$P(a,b) = \begin{cases} \{a \xrightarrow{\leq} b\} & \text{if } a \leq b \\ \emptyset & \text{otherwise}; \end{cases}$$
obvious composition!

Note that (b) and (c) are small categories.

1.6 Definition: Let D be a category; a subcategory of D is a category C with

(i) $Ob(C) \subset Ob(D)$

(ii) $\forall X, Y \in Ob(C)$, $C(X,Y) \subset D(X,Y)$

(iii) Composition in C is the restriction of the composition in D, and for $X \in C$, $1_X \in C(X,X) \subset D(X,X)$.

We denote it by $C \subset D$

We say that C is a <u>full subcategory of D</u> if for all $X, Y \in C$, $C(X, Y) = D(X, Y)$.

<u>1.7 Examples</u>: (a) <u>Fin</u> $\subset$ <u>Sets</u>, the full subcategory of finite sets.

(b) <u>Ab</u> $\subset$ <u>Gp</u>, the full subcategory of abelian groups.

(c) <u>Inj</u> $\subset$ <u>Sets</u>, the subcategory of injective maps.

<u>1.8 Def</u>: Let C, D be categories. A functor $F : C \to D$ is

(a) A rule assigning $F(X) \in D$ for each $X \in C$

(b) For all $X, Y \in C$, a map $C(X, Y) \to D(F(X), F(Y))$, $f \mapsto F(f)$ such that $F(1_X) = 1_{F(X)}$ $\forall X \in C$, and $F(g \circ f) = F(g) \circ F(f)$ for any composable pair of morphisms g, f in C.

A <u>cofunctor</u> $F : C \to D$ is a functor $F : C^{op} \to D$. (spell it out).

<u>1.9 Examples</u>: (a) The "inclusion" of subcategory $C \subset D$ is a functor

(b) "Forget" functors: $Gr \to Set$, $Ab \to Set$, $R\text{-Mod} \to Set$, $Top \to Set$,

(c) For $Z \in Ob(D)$, the constant functor $C_Z : C \to D$, with $C_Z(X) = Z$ $\forall X \in C$, $C_Z(f) = id_Z$ $\forall f$ morphism in C.

(d) The identity functor $id : C \to C$.

<u>1.10 Def</u>: We can compose functors
$$\begin{tikzcd} C \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & D \arrow[r, "G"] & E \end{tikzcd}$$
in an obvious way.
(Spell it out).

1.10 Def: a functor $F: C \to D$ is called *full* (resp. *faithful*) if for all $X, Y \in \mathrm{Ob}\, C$,

$$F: C(X,Y) \to D(F(X), F(Y))$$

is surjective (resp. injective), and is called *fully faithful* if both full and faithful.

1.11 Examples: (a) The inclusion of a full subcat $C \subset D$ is a fully faithfull functor.
(b) The forgetful functor $Gp \to Sets$ is faithfull but not full.

1.12 Def: Let C be a cat. A morphism $f \in C(X,Y)$ is called an *isomorphism* in C if $\exists\, g \in C(Y,X)$ with $g \circ f = 1_X$ and $f \circ g = 1_Y$.
Then g is unique, and is called the *inverse* f^{-1} of f.

Note that a functor maps isomorphisms to isomorphisms.
For each $X \in \mathrm{Ob}(C)$, we define the group of automorphisms of X as $\mathrm{Aut}_C(X) = \{ f \in C(X,X) \,;\, f \text{ iso} \}$.

1.13 Def: Let $F, G: C \to D$ be two functors. A *natural transformation* $\eta: F \Rightarrow G$ associates to each $X \in C$ a morphism $\eta_X \in D(F(X), G(X))$ such that for any $f: X \to Y$ in C, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

commutes.

1.14 Example: (a) The identity $\mathrm{Id}_F: F \Rightarrow F$
(b) Consider the "abelianisation" functor $Gp \xrightarrow{F} Gp$, $G \mapsto G^{ab} = G/[G,G]$
Then the quotient map $q_G: G \to G/[G,G]$ defines a natural transformation $\mathrm{Id}_{Gp} \Rightarrow F$.

1.15 Definition: Let C, D be categories, with C small.
We define the functor category $\mathrm{Fun}(C, D) = D^C$ by
(i) $\mathrm{Ob}(D^C) = \{\text{functors } C \to D\}$
(ii) $D^C(F, G) = \{\text{natural transf. } F \Rightarrow G\}$
(iii) Obvious composition and identities.

1.16 Example (Presheaves): Let C be a small category.
We call $\mathrm{Fun}(C^{op}, \mathrm{Sets})$ the category of presheaves on C;
it is often denoted $\hat{C}$.
Thus an object of $\hat{C}$ is a functor $F: C^{op} \to \mathrm{Sets}$. (presheaf)
For $X \xrightarrow{f} Y$ in C, let $f^* = F(f): F(Y) \to F(X)$.
Note that for any $X \in C$, we obtain a presheaf

$$h_X : C^{op} \to \mathrm{Sets}, \quad h_X(Y) = C(Y, X)$$

(h_X is called the cofunctor represented by X)

Moreover, $h: C \to \hat{C}$, $X \mapsto h_X$ is a functor!

1.17 Yoneda Lemma: For any $F \in \hat{C}$ and any $X \in C$, there is a natural isomorphism

$$\hat{C}(h_X, F) \to F(X)$$
$$(\eta: h_X \Rightarrow F) \mapsto \eta_X(1_X)$$

Proof: obvious!

1.18 Corollary: The functor $h: C \to \hat{C}$ is fully faithfull.

Proof: left as an exercise. //

1.19 Def: Suppose given a pair of functors

$$L : C \to D \quad \text{and} \quad R : D \to C$$

We define two functors $C^{op} \times D \to Sets$ by

$D(L(-), -) : (c, d) \mapsto D(L(c), d)$

$C(-, R(-)) : (c, d) \mapsto C(c, R(d))$

We say that (L, R) is an adjoint pair if there is a natural isomorphism $\tau : D(L(-), -) \to C(-, R(-))$

We also say: L is left adjoint to R, and denote it $L \dashv R$.

1.20 Examples: (a) a standard examples are the "free functors" left adjoint to the forgetful functors:

Let $\mathbb{F}$ be a field, $\mathbb{F}$-Vect the category of vector spaces over $\mathbb{F}$, and let $R : \mathbb{F}\text{-Vect} \to Sets$ be the forget functor.

Let $L : Sets \to \mathbb{F}\text{-Vect}$ be the functor

$$L(X) = \mathbb{F}^{(X)} = \{\alpha : X \to \mathbb{F} \; ; \; \alpha \text{ finitely supported}\}$$

We can view $\mathbb{F}^{(X)}$ as the vector space with basis X.

Then $L(X \xrightarrow{f} Y)$ is the linear extension $F^{(X)} \to F^{(Y)}$ of f.

Then (L, R) is an adjoint pair:

For any set X and any vector space V, let

$$\tau_{(X,V)} : \mathbb{F}\text{-Vect}(F^{(X)}, V) \longrightarrow Sets(X, R(V))$$
$$f \longmapsto f|_X$$

This is an iso (any linear map is uniquely determined by what it does on a basis).

(b) a similar example: the forgetfull functor $R : Gp \to Sets$ and its the "free group on X" functor

$L : Sets \to Gp, \quad X \mapsto \langle X \rangle$

1.21 Def Let $L : C \rightleftarrows D : R$ with $L \dashv R$. There are obvious natural transf. $Id_C \Rightarrow R \circ L$ and $L \circ R \Rightarrow Id_D$ (unit, resp counit of $L \dashv R$)

Recall the definition of a limit: For I small, let us denote $\mathrm{Fun}(I, C)$ by C^I.

1.22 Definition: If $F \in C^I$, a limit of F is a pair (x, η_x) where $x \in C$, $\eta_x : \Delta_x \Rightarrow F$, such that

$$C((-), x) \Rightarrow C^I(\Delta_{(-)}, F)$$
$$f \longmapsto \eta_x \circ \Delta_f$$

is a natural isomorphism of functors $C^{op} \to \mathrm{Sets}$.

Dually, a colimit of F is a pair (y, ν_y) where $y \in C$, $\nu_y : F \Rightarrow \Delta_y$, such that

$$C(y, (-)) \Rightarrow C^I(F, \Delta_{(-)})$$
$$f \longmapsto \Delta_f \circ \nu_y$$

is a natural iso.

1.23 Remark if it exists, the limit (colimit) of $F \in C^I$ is unique up to canonical isomorphism.
We denote it by $\lim F$ or $\lim_I F$ ($\mathrm{colim}\, F$ or $\mathrm{colim}_I F$). The natural tr. η, ν are omitted from the not.

1.24 Definition A category C is called (co)complete if for any small category I, a right adjoint $\lim$ (resp. a left adjoint colim) of Δ exists and is given.

1.25 Rem. If C is complete, and I is small, we thus have a given functor $\lim : C^I \to C$. In more details, if $\mu : F \Rightarrow G$ is a nat. transfo, then the morphism $\mu : \lim F \to \lim G$ in C is defined by

$$C(\lim F, \lim G) \xrightarrow{\cong} C^I(\Delta_{\lim F}, G)$$
$$\mu \longmapsto \mu \circ \eta_{\lim F}$$

We have the dual property for colim.

1.26 Examples (a) I discrete (only id-morphisms), and if $F \in C^I$, we use the terminology

$$\lim_I F =: \prod_{i \in I} F(i) \quad \text{(product)}$$

$$\operatorname{colim}_I F =: \coprod_{i \in I} F(i) \quad \text{(coproduct)}$$

(b) If $I = \{ 0 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 1 \}$, F given by $F(0) \overset{a}{\underset{b}{\rightrightarrows}} F(1)$ then $\lim F$ is called the <u>equalizer</u>:

$$eq(a,b) = \lim F \xrightarrow{\eta} F(0) \rightrightarrows F(1)$$

and $\operatorname{colim} F$ is called the <u>co-equalizer</u>:

$$F(0) \rightrightarrows F(1) \to \operatorname{colim} F = coeq(a,b)$$

(c) If $I = \{1 \to 0 \leftarrow 2\}$, and $F \in C^I$, A limit X of F is called a <u>pull-back</u>.
We then say that the commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F(2) \arrow[d] \\ F(1) \arrow[r] & F(0) \end{tikzcd}$$

is <u>cartesian</u> (or is a pull-back square). We add $\lrcorner$ in the square if it is cartesian

(d) Dually, if $J = I^{op} = \{1 \leftarrow 0 \to 2\}$ and $F \in C^J$, a colimit Y of is called a <u>push-out</u>.
We then say that the commutative square

$$\begin{tikzcd} F(0) \arrow[r] \arrow[d] & F(2) \arrow[d] \\ F(1) \arrow[r] & Y \end{tikzcd}$$

is cocartesian (or is a push-out square).

(e) In <u>Sets</u>, a pull-back of $A \xrightarrow{f} C \xleftarrow{g} B$ is given by $X = \{(a,b) \in A \times B \mid f(a) = g(b)\}$ with $X \xrightarrow{\eta_1} A$, $X \xrightarrow{\eta_2} B$ the restriction of the projections:

$$\begin{tikzcd} & X \arrow[dl, bend right, "\eta_1"'] \arrow[d, dashed, "\eta_0"] \arrow[dr, bend left, "\eta_2"] & \\ A \arrow[r, "f"'] & C & B \arrow[l, "g"] \end{tikzcd}$$

$\eta_0 = f\eta_1 = g\eta_2$ (no need to give it) we write

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

(f) In Sets, colim $(A \xleftarrow{f} C \xrightarrow{g} B)$ (the push-out) is given by $Y = A \amalg_C B := (A \amalg C)/\sim$, where $\sim$ is the equivalence rel. generated by $a \sim b$ if $\exists\, c \in C$, $f(c) = a$ and $g(c) = b$. The structure maps ν are obvious.

1.27 Def: A functor $F: C \to D$ is called an equivalence of categories if there exists a functor $G: D \to C$ and natural isomorphisms $id_C \overset{\cong}{\Rightarrow} G \circ F$, $id_D \overset{\cong}{\Rightarrow} F \circ G$.

1.28 Example: Let $\mathcal{F}in$ be the category of finite sets. Let $[n] = \{0, 1, \dots n\} \subset \mathbb{N}$, and let $Sk(\mathcal{F}in)$ be the full subcategory of $\mathcal{F}in$ with objects $\{[n] ; n \in \mathbb{N}\} \cup \{\emptyset\}$. The the inclusion $Sk(\mathcal{F}in) \to \mathcal{F}in$ is an equivalence of categories. We call $Sk(\mathcal{F}in)$ a "small skeleton of $\mathcal{F}in$" (Skeleton: one object per isomorphism class).

1.29 Remark: an equivalence of categories is fully faithfull. Conversely note that if $F: C \to D$ is a fully faithfull functor, and if $X, Y \in Ob(C)$, then

$$F(X) \cong F(Y) \iff X \cong Y$$

There is an equivalence of categories from C to its image $F(C)$ (the full subcat with objects $\{F(X) ; X \in Ob(C)\}$).

II The homotopy category of spaces

Recall that Top is the category of (topological) spaces and continuous maps. Let $I = [0,1] \subset \mathbb{R}$.

2.1 Def: Let $X, Y \in \mathrm{Top}$, and $f, g \in \mathrm{Top}(X, Y)$. A homotopy from f to g is a map $H : X \times I \to Y$ such that

$$\begin{tikzcd} X \arrow[r, hook, "i_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, hook', "i_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes,

where for $k = 0, 1$, $i_k : X \to X \times I$, $x \mapsto (x, k)$.
We say that f and g are homotopic if there exists a homotopy from f to g; we denote it by $f \simeq g$, or $f \overset{H}{\simeq} g$ if we want to specify the homotopy H.

2.2 Lemma: The relation $\simeq$ is an equivalence relation on $\mathrm{Top}(X, Y)$, compatible with composition in the following sense: for $f, f' \in \mathrm{Top}(X, Y)$ with $f \simeq f'$, and $g, g' \in \mathrm{Top}(Y, Z)$ with $g \simeq g'$, we have
$$g \circ f \simeq g' \circ f'.$$

proof: $\simeq$ is an equivalence relation:
(a) for $f \in \mathrm{Top}(X, Y)$, $f \overset{H}{\simeq} f$ for $H : X \times I \to Y$, $(x, t) \mapsto f(x)$
(b) for $f, g \in \mathrm{Top}(X, Y)$ with $f \overset{H}{\simeq} g$, we have $g \overset{K}{\simeq} f$ for
$$K : X \times I \to Y, \quad K(x, t) := H(x, 1-t)$$
(c) for $f, g, h \in \mathrm{Top}(X, Y)$ with $f \overset{F}{\simeq} g$ and $g \overset{G}{\simeq} h$, we have $f \overset{F*G}{\simeq} h$, for $F * G : X \times I \to Y$ given by
$$F * G\,(x, t) = \begin{cases} F(x, 2t) & \text{if } 0 \le t \le 1/2 \\ G(x, 2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$
For the compatibility with composition, we use

the transitivity of $\simeq$ as follows: $X \underset{f'}{\overset{f}{\rightrightarrows}} Y \underset{g'}{\overset{g}{\rightrightarrows}} Z$

with $f \overset{F}{\simeq} f'$ and $g \overset{G}{\simeq} g'$, then we have

$$g \circ f \overset{K}{\simeq} g' \circ f \overset{L}{\simeq} g' \circ f' \quad \text{for} \quad \begin{cases} K : X \times I \xrightarrow{f \times Id} Y \times I \xrightarrow{G} Z \\ L : X \times I \xrightarrow[F]{} Y \xrightarrow{g'} Z . \end{cases}$$

#

<u>2.3 Definition</u>: We define the <u>homotopy category of Top</u>, hoTop, with $Ob(\text{hoTop}) = Ob(\text{Top})$, and

$$\text{hoTop}(X, Y) = \text{Top}(X, Y)/_{\simeq}$$

$\text{hoTop}(X, Y)$ is often denoted by $[X, Y]$, and the class of $f \in \text{Top}(X, Y)$ by $[f] \in [X, Y]$ (often we write f instead of $[f]$).

The composition is defined by $[g] \circ [f] = [g \circ f]$ (well defined by 2.1).

The axioms are clear, with $1_X = [1_X]$ in hoTop.

We have an obvious functor

$$ho : \text{Top} \longrightarrow \text{hoTop}$$

<u>2.4 Definition</u> A map $f \in \text{Top}(X, Y)$ is called a <u>homotopy equivalence</u> if $[f] \in \text{hoTop}(X, Y)$ is an isomorphism.

We say that $X, Y \in \text{Top}$ are <u>homotopy equivalent</u>, denoted $X \simeq Y$, if they are isomorphic in hoTop.

We say that X is <u>contractible</u> if $X \simeq \{*\}$ (one point space).

<u>2.5 Example</u>: Let $X \in \text{Top}$, $n \geq 0$, and let $f, g \in \text{Top}(X, \mathbb{R}^n)$. Then $f \overset{H}{\simeq} g$ by $H : X \times I \to \mathbb{R}^n$, $(x, t) \mapsto (1-t) f(x) + t g(x)$

Thus $[X, \mathbb{R}^n]$ has just one element, represented for example by the constant map $c_0 : X \to \mathbb{R}^n$ with image 0.

In particular, $\{0\} \hookrightarrow \mathbb{R}^n$ is a homotopy equivalence, and $\mathbb{R}^n$ is contractible!

We are interested in Homotopy-invariants for Top: functors: $Top \xrightarrow{F} C$ that factor through $HTop$:
This is equivalent to $\forall X, Y \in Top$, $\forall f, g \in Top(X, Y)$,

$$f \simeq g \Rightarrow F(f) = F(g).$$

2.6 Rem $\mathbb{R}^n \simeq *$! $[S^k, S^\ell]_*$ very hard !!!

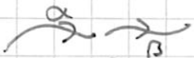
First invariant: πX, the fundamental groupoid.

2.7 Def: A groupoid is a small category in which all morphisms are isomorphisms.
The groupoids and functors of groupoids form a category. $\mathcal{GP}$.

2.8 Def: $X \in Top$, $a, b \in X$, Path α
Form $\Omega(X, a, b)$

$\alpha \sim \beta$: rel $\{0,1\}$. $\rightsquigarrow$ $[\alpha]$

Let us denote $\pi X(a, b)$ the quotient $\Omega(X, a, b)/\sim$.



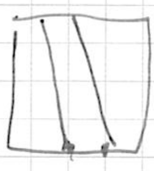
2.9 Def: If α, β path, $\alpha(1) = \beta(0)$ $\rightsquigarrow$ $\alpha \cdot \beta$. concat.
$\rightleftarrows$ α^{-1}: inverse dir. $\quad$ c_a: const. in a.

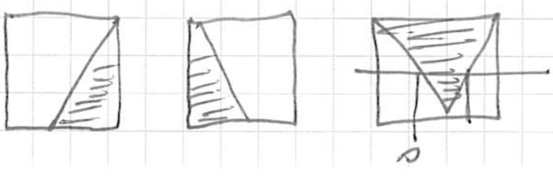
2.10 Lemma: We have the following propert.:
(a) The concatenation is compatible $\sim$
(b) The con. is associative up to hpy.
(c) If $\alpha \in \Omega(X, a, b)$, then $\alpha \cdot c_b \sim \alpha$, $c_a \alpha \sim \alpha$
(d) If " $\alpha\alpha^{-1} \sim c_a$, $\alpha^{-1}\alpha \sim c_b$

proof: (1) $\alpha \overset{F}{\sim} \alpha'$, $\beta \overset{G}{\sim} \beta'$, then $\alpha\beta \overset{H}{\sim} \alpha'\beta'$ by

$$H(s,t) = \begin{cases} F(s, 2t) & 0 \le t \le 1/2 \\ G(s, 2t-1) & 1/2 \le t \le 1, \end{cases}$$

(2)





2.12 Lemma + Def Let $X \in Top$. We define

$$Ob(\Pi X) := X \quad (\text{the underlying set of points})$$

and, for $a, b \in X$,

$$\Pi X(a,b) = \Omega(X,a,b)/_{\sim} \quad \text{as above.}$$

We define $id_a = [c_a] \in \Pi X(a,b)$

We define $\Pi X(b,c) \times \Pi X(a,b) \xrightarrow{\circ} \Pi X(a,c)$

$$([\beta], [\alpha]) \longmapsto [\beta\alpha]$$

Then, with this structure, ΠX is a groupoid, called the fundamental groupoid of X.

proof: by 2.12: (a) $\Rightarrow$ $\circ$ is well defined, (b) $\Rightarrow$ $\circ$ assoc., (c) = id_a is an identity, (d) $\Rightarrow$ all morph. are iso's: $[\alpha]^{-1} = [\alpha^{-1}]$ #

2.13 Definition: If G is a groupoid, let

$$\pi_0(G) = Ob(G)/_{\sim}, \text{ where}$$

$\sim$ is the equivalence relation on G defined by

$$a \sim b \iff G(a,b) \neq \emptyset.$$

We call $\pi_0(G)$ the <u>set con. components</u> of G.

For $a \in Ob(G)$, the group of aut. of a, is sometimes denoted $\pi_1(G,a) =: Aut_G(a)$.

2.14 Definition If $X \in Top$, define the set of path con. components of X by

$$\pi_0(X) := \pi_0(\Pi X).$$

For $X \in Top$ and $a \in X$, define the fundamental group of X at a by

$$\pi_1(X,a) = \pi_1(\Pi X, a) = Aut_{\Pi X}(a).$$

2.15 Remarks (Functorial properties)

(1) If $f : X \to Y$ is a map in Top, we get an induced morphism of groupoids $f_* : \Pi X \to \Pi Y$, defined as follows:

- on objects, $f_*(a) := f(a)$.
- On morphisms: $\Pi X(a,b) \longrightarrow \Pi Y(f(a), f(b))$

$$[\alpha] \longmapsto f_*[\alpha] = [f \circ \alpha]$$

This is well defined ($\alpha \sim \beta \Rightarrow f\circ\alpha \sim f\circ\beta$), and is a functor since $f(\alpha\beta) = f(\alpha) f(\beta)$, (check!) etc...

(2) If $f \overset{H}{\simeq} g : X \to Y$, then H defines a natural isomorphism $f_* \overset{\eta}{\Rightarrow} g_* : \Pi X \to \Pi Y$ as follows: for $a \in X$, let $\eta_a = [H_a]$ $\quad H_a : I \to Y \cdot t \mapsto H_a(t) = H(a,t)$

is a path from $f(a)$ to $g(a)$.

Indeed, η_a is an iso (we are in a groupoid!), and if $[\alpha] \in \Pi X(a,b)$, the diagram

$$\begin{tikzcd} f(a) \arrow[r, "{[f\circ\alpha]}"] \arrow[d, "\eta_a"'] & f(b) \arrow[d, "\eta_b"] \\ g(a) \arrow[r, "{[g\circ\alpha]}"] & g(b) \end{tikzcd}$$

commutes! We must show only that we have: a homotopy of paths

$$H_b(f\circ\alpha) \overset{K}{\simeq} (g\circ\alpha) H_a$$

Define K as a composition

$$I \times I \xrightarrow{\Phi} I \times I \xrightarrow{\alpha \times id} X \times I \xrightarrow{H} Y$$

where 

(3) We deduce that a map $f : X \to Y$ induces a map of sets $f_* : \Pi_0 X \to \Pi_0 Y$, and that if $f \overset{H}{\simeq} g$, then $f_* = g_* : \Pi_0 X \to \Pi_0 Y$: indeed, if $a \in \Pi_0 X$, $f(a) \simeq g(a)$ via the path H_a.

$\Rightarrow$ We get a functor $\Pi_0 : \mathrm{Top} \to \mathrm{Sets}$

$$\begin{tikzcd} \mathrm{Top} \arrow[r] \arrow[dr] & \mathrm{Sets} \\ & \mathrm{HTop} \arrow[u] \end{tikzcd}$$

(4) If $f : (X, x_0) \to (Y, y_0)$ is a pointed map, we have an induced group homomorphism

$$f_* \ \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

If $f \overset{H}{\simeq} g : (X, x_0) \to (Y, y_0)$ is a pointed homotopy, then $\eta_{x_0} = id_{y_0}$, thus, if $[\alpha] \in \pi X(x_0, x_0)$, $f_*[\alpha] = g_*[\alpha]$, i.e. $f_* = g_*$.

We obtain a homotopy invariant functor

$$\begin{tikzcd} \underline{Top}_* \arrow[rr, "\pi_1"] \arrow[dr] & & \underline{Grp} \\ & H\underline{Top}_* \arrow[ur] & \end{tikzcd}$$

(5) If we have $f, g : (X, x_0) \to (Y, y_0)$, and an <u>unpointed</u> Homotopy H, then by (3) we have a com. diagram for all $[\alpha] \in \pi X(x_0, x_0)$:

$$\begin{tikzcd} y_0 \arrow[r, "f_*[\alpha]"] \arrow[d, "h_{x_0}"'] & y_0 \arrow[d, "h_{x_0}"] \\ y_0 \arrow[r, "g_*[\alpha]"] & y_0 \end{tikzcd}$$

If $\tau_H : \pi_1(Y, y_0) \circlearrowleft$

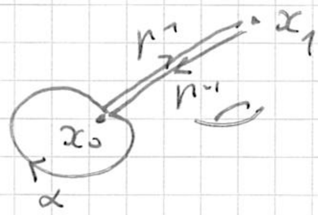
$[\beta] \mapsto [h_{x_0}][\beta][h_{x_0}^{-1}]$

then $g_* = \tau_H \circ f_* \quad : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

<u>2.16 Remark</u> (Dependence on Basepoint):

Note that if $\gamma : I \to X$ is a path from x_0 to x_1, we get a <u>non-canonical</u> isomorphism

$$\pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$
$$[\alpha] \longmapsto [\gamma \alpha \gamma^{-1}]$$
$$= c_\gamma([\alpha])$$



As a corollary:

<u>2.17 Proposition</u> If $X \xrightarrow{f} Y$ is a homotopy equivalence in Top, and if $x_0 \in X$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ <u>iso</u>.

<u>proof</u>: Choose $g : Y \to X$, and $gf \overset{H}{\simeq} id_X$, $fg \overset{K}{\simeq} id_Y$

$$\begin{tikzcd} & \pi_1(Y, f(x_0)) \arrow[dr, "g_*"] & \\ \pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[rr, "g_* f_*"] \arrow[dr, "id = id_*"', "\cong"] & & \pi_1(X, gf(x_0)) \arrow[dl, "\cong", "\tau_H"'] \\ & \pi_1(X, x_0) & \end{tikzcd}$$

Then the diag. commutes $\Rightarrow f_*$ inj, g_* surj. Repeat for K. #

2.18 Theorem (Seifert - Van Kampen, Groupoid version)

Let $X \in \underline{\mathrm{Top}}$, U_0, U_1 ~~open~~ subsets with $X = \mathring{U}_0 \cup \mathring{U}_1$.

Then

$$\begin{tikzcd} \Pi(U_0 \cap U_1) \arrow[r, "h_0"] \arrow[d, "h_1"'] & \Pi U_0 \arrow[d, "i_0"] \arrow[ddr, bend left, "j_0"] & \\ \Pi U_1 \arrow[r, "i_1"] \arrow[drr, bend right, "j_1"'] & \Pi X \arrow[dr, dashed, "\exists ! \, j"] & \\ & & C \end{tikzcd}$$

$(*)$ is a push-out in $\underline{GP}$.

proof:

Assume $C \in \underline{GP}$,

and $j_0 : \Pi U_0 \to C$, $j_1 : \Pi U_1 \to C$ with $j_0 h_0 = j_1 h_1$.

We must show unicity and existence of $j : \Pi X \to C$

with $j i_0 = j_0$ and $j i_1 = j_1$.

We need

→ Lemma (Lebesgue number) Let M be a compact metric space and $\mathcal{U} = \{U_i\}_{i \in \Theta}$ an open cover of M.
Then $\exists\, \varepsilon > 0$ with the following property:
There is a function $M \xrightarrow{\sigma} \Theta$ such that $\forall x \in M$,
$B(x, \varepsilon) \subset U_{\sigma(x)}$. ε is a Lebesgue number for $\mathcal{U}$

On objects, $(*)$ is just the diagram of sets

$$\begin{tikzcd} U_0 \cap U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_1 \arrow[r] & X \end{tikzcd}$$

which is a push-out on sets!

$\Rightarrow$ Existence and unicity of j on objects is clear.

Unicity on morphisms: let $a, b \in X$, $[\alpha] \in \Pi X(a, b)$.

$\exists\, N \in \mathbb{N}$ such that $\frac{1}{N}$ is a Lebesgue number for $\mathcal{U} = \{\alpha^{-1}(\mathring{U}_0), \alpha^{-1}(\mathring{U}_1)\}$

$[0,1] = I = I_1 \cup I_2 \cup \ldots \cup I_N$, $I_k = [\frac{k-1}{N}, \frac{k}{N}]$,

$$I_k \cap I_{k+1} = \{\tfrac{k}{N}\}.$$

If $\alpha_k = \alpha|_{I_k}$ reparametrized to have length 1,

we have $\alpha = \alpha_N \alpha_{N-1} \cdots \alpha_1 \alpha_0$.

We have a function $\{1, \ldots, N\} \xrightarrow{\sigma} \{0, 1\}$ with α_k path in $U_{\sigma(k)}$,

thus $[\alpha_k] = i_{\sigma(k)}[\alpha_k]$, $j_{\sigma(k)}[\alpha_k] = j(\alpha_k)$, $j([\alpha]) = j_{\sigma(N)}[\alpha_N] \cdots j_{\sigma(1)}[\alpha_1]$.
(☆)

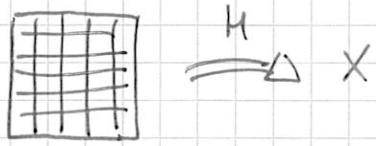
Existence on morphisms: We need to show that we can use formula (✡) to define $j[\alpha]$: this means:

(a) It does not depend on the subdivision of I and the function σ used: assume given two subdivisions
$0 = t_0 < t_1 < \dots < t_N = 1$ and $0 = s_0 < s_1 < \dots < s_M = 1$,
and functions $\sigma: \{1, \dots, N\} \to \{0,1\}$ $\tau: \{1, \dots, M\} \to \{0,1\}$ with

$$\alpha([t_{k-1}, t_k]) \subset U_{\sigma(k)}, \quad \alpha([s_{\ell-1}, s_\ell]) \subset U_{\tau(\ell)}.$$

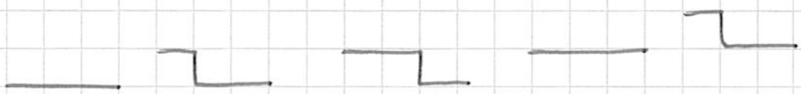
Refine to a common subdivison $0 = r_0 < r_1 < \dots < r_L = 1$.
Then $\alpha = \alpha_L \cdots \alpha_1$ with $\alpha_k = \alpha|_{[r_{k-1}, r_k]}$ reparam.
If the σ, τ differ doesn't matter since agree on $U_0 \cap U_1$.

(b) It does not depend on the homotopy class of α:
$\alpha \overset{H}{\sim} \beta \in \Omega(X, a, b)$.



can subdivide $I \times I$ in sq. of size $\frac{1}{N} \times \frac{1}{N}$ so that sub. to $\{H^{-1}(U_0), H^{-1}(U_1)\}$.

We can order the paths in this manner



equivalent.

(c) It defines a functor: $a, b, c \in X$,
$\alpha \in \Pi X(b, c)$, $\beta \in \Pi X(a, b)$
Then $j(\alpha\beta) = j(\alpha) j(\beta)$: can subdivide $\alpha\beta$ in
$\underbrace{\gamma_N \cdots \gamma_M}_{\alpha} \underbrace{\gamma_{M-1} \cdots \gamma_1}_{\beta}$ with $\sigma: \{1, \dots, N\} \to \{0,1\}$ as above,

Then $j(\alpha\beta) \overset{\text{def}}{=} \underbrace{j_{\sigma(N)}(\gamma_N) \cdots j_{\sigma(M)}(\gamma_M)}_{\text{def } j(\alpha)} \underbrace{j_{\sigma(M-1)}(\gamma_{M-1}) \cdots j_{\sigma(1)}(\gamma_1)}_{\text{def } j(\beta)}$

$j(\mathrm{id}_a)$ clear: take $a \in U_0$ or U_1, etc.

#.

2.21 Theorem (Seifert-van Kampen, group version)
Let $X \in Top$, $U_0, U_1 \subset X$ open with $X = U_0 \cup U_1$, such that $U_{01} := U_0 \cap U_1$, U_0, U_1 are path connected. Take $x_0 \in U_{01}$. Then the square

$$\begin{array}{ccc} \pi_1(U_{01}, x_0) & \longrightarrow & \pi_1(U_0, x_0) \\ \downarrow & & \downarrow \\ \pi_1(U_1, x_0) & \longrightarrow & \pi_1(X, x_0) \end{array}$$

where the morphisms are induced by the inclusions, is a pushout in Groups.

This the special case $A = \{x_0\}$ of the following proposition.

2.22 Proposition Let $X \in Top$, $U_0, U_1 \subset X$ open with $X = U_0 \cup U_1$ and let $U_{01} = U_0 \cap U_1$. Let $A \subset X$ be a subset such that A meets every path-connected component of U_{01}, of U_0 and U_1 (thus of X also). Denote by $\Pi(X, A)$ the full subcat. of ΠX with ob $\Pi(X, A) = A$. Then the square

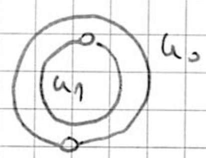
$$\begin{array}{ccc} \Pi(U_{01}, A) & \longrightarrow & \Pi(U_0, A) \\ \downarrow & & \downarrow \\ \Pi(U_1, A) & \longrightarrow & \Pi(X, A) \end{array}$$

is a push-out in GP.

proof:
Exercise: In any category, a retract of push-out square is a push-out square ($(a \xrightarrow{i} b)$ is a retract of b if $\exists\ r: b \to a$ with $ri = id_a$).
It is easy to see that the square in proposition 2.22 is a retract of the square in 2.18.
The retraction $r_x : \Pi(X) \to \Pi(X, A)$ is defined by
(1) Chosing a path $\omega_x : x \to a_x$ $\forall x \in X$, with $a_x \in A$ (if possible, take ω_x in U_0, U_1, U_{01}).
(2) Define $r_x : \Pi(X) \to \Pi(X, A)$ on objects by $x \mapsto a_x$, on $x \xrightarrow{u} y$ by $r(u) : a_x \xrightarrow{\omega_x^{-1} * u\, \omega_y} a_y$. #

2.23 Theorem: Let $\omega : I \to S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the path given by $t \mapsto e^{2\pi i t}$.
Then $\pi_1(S^1, 1)$ is the free group generated by $[\omega]$:
$$\pi_1(S^1, 1) \cong \mathbb{Z}, \text{ generated by } [\omega] \quad (\mathbb{Z}\{[\omega]\})$$

proof: Let $A = \{1, -1\} \subset S^1$, $U_0 = S^1 \setminus \{-i\}$, $U_1 = S^1 \setminus \{i\}$. By 2.22 we have a push-out diagram



$$\begin{tikzcd}
\Pi(U_{01}, A) \arrow[r] \arrow[d] & \Pi(U_0, A) \arrow[d] \\
\Pi(U_1, A) \arrow[r] & \Pi(S^1, A).
\end{tikzcd}$$

We see that $\Pi(U_{01}, A) = \begin{tikzcd} -1 \arrow[loop below, "id"] & 1 \arrow[loop below, "id"] \end{tikzcd}$

$\Pi(U_0, A) = \begin{tikzcd} -1 \arrow[loop below, "id"] \arrow[r, bend right, "{[u]^{-1}}"'] & 1 \arrow[loop below, "id"] \arrow[l, bend right, "{[u]}"'] \end{tikzcd}$ $\qquad v : I \to U_0$, $t \mapsto e^{\pi i t}$

$\Pi(U_1, A) = \begin{tikzcd} -1 \arrow[loop below, "id"] \arrow[r, bend right, "{[v]}"'] & 1 \arrow[loop below, "id"] \arrow[l, bend right, "{[v]^{-1}}"'] \end{tikzcd}$ $\qquad u : I \to U_1$, $t \mapsto e^{\pi i (t+1)}$

Thus the push-out is the groupoid generated by

$$\Pi(S^1, A) = \begin{tikzcd} -1 \arrow[loop left, "id"] \arrow[r, bend right, "{[v]}"'] & 1 \arrow[loop right, "id"] \arrow[l, bend right, "{[u]}"'] \end{tikzcd}$$

$$\pi_1(S^1, 1) = \operatorname{Aut}_1(\Pi(S^1, A)) = \{ ([u][v])^n ; n \in \mathbb{Z} \}$$
$$= \mathbb{Z}\{[\omega]\}.$$

Equivalence of groupoids!

#

2.24 Proposition: The group structure of S^1 induces a group structure on $[S^1, S^1] = \text{hoTop}(S^1, S^1)$, and $[S^1, S^1] = \mathbb{Z}\{\mathrm{Id}_{S^1}\}$.

proof: First, notice that for any $u \in \Omega(X, x_0, x_0)$ we have a unique $f_u : S^1 \to X$ with $f(1) = x_0$:

$$\begin{tikzcd} I \arrow[rr, "u"] \arrow[dr, "w"] \arrow[d, "q"'] & & X \\ I/\partial I \arrow[r, "\bar{w}", "\cong"'] & S^1 \arrow[ur, "\exists! f_u"'] & \end{tikzcd}$$

since w is, up to an isomorphism, the quotient map by $0 \sim 1$, and $u(0) = x_0 = u(1)$.

Similarly, if $u \sim v$, then $f_u \simeq f_v$ (pointed homotopy).

Thus we deduce that we have a bijection

$$\pi_1(S^1, 1) \xrightarrow{\cong} [(S^1,1),(X,x_0)]_* \qquad [u] \longmapsto [f_u]$$

Now take $X = S^1$.

Then, the map $[(S^1,1),(S^1,1)]_* \xrightarrow{F} [S^1, S^1]$ (forget the base point) is bijective:

- Surjective: clear, since for any $z \in S^1$, $z = e^{2\pi i t}$ and for $S^1 \xrightarrow{m_z} S^1$, $x \longmapsto z \cdot x$, we have $m_z \overset{H}{\simeq} id_{S^1}$ via

$$H : S^1 \times I \longrightarrow S^1, \quad H(-, s) = m_{e^{2\pi i t(1-s)}}$$

So any map in $Top(S^1, S^1)$ is homotopic to a map in the image of $F : Top_*((S^1,1),(S^1,1))$.

- Injective: If $H : S^1 \times I \to S^1$ is an unpointed homotopy between pointed maps,

$$\tilde{H} : S^1 \times I \to S^1, \quad (z,t) \longmapsto H(z,t) \cdot H(1,t)^{-1}$$

is a pointed homotopy. Group structure: exercise !!! #

2.25 Remark (a) We know that $[*, X] \cong \pi_0 X$, the set of path connected component of X.

Proposition 2.24 gives a first (most basic but non-trivial) example that the homotopy category, for connected spaces, contains interesting information!

III INTRO TO HIGHER HOMOTOPY GROUPS

For X a space, we have defined ΠX, $\pi_0 X$ and $\pi_1(X, x_0)$. The purpose of this chapter is to define the higher htpy groups, $\pi_n(X, x_0)$ $n \geq 1$, and to sketch: $\pi_n(S^n, *) \cong \mathbb{Z}$, as well as the beautiful result (Hopf) that $\pi_3(S^2, *) \cong \mathbb{Z}$.

3.1 Def hoTop^2.

3.2 Def: Recall $I = [0,1] \subset \mathbb{R}$. For $n \in \mathbb{N}$, let $I^n \subset \mathbb{R}^n$ be defined as the n-cube

$$I^n = \begin{cases} \{0\} & n = 0 \\ \underbrace{I \times \dots \times I}_{n} \end{cases} \quad \text{and let} \quad \partial I^n = \begin{cases} \emptyset & n = 0 \\ \{(t_1, \dots, t_n) \in I_n \mid \exists i, t_i = 0\} \end{cases}$$

For $(X, *) \in \mathrm{Top}_*$, let

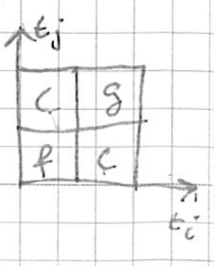
$$\pi_n(X, *) = [(I^n, \partial I^n), (X, *)] \quad (\text{in } \mathrm{hoTop}^2)$$

For $n \geq 1$ and $1 \leq k \leq i$ define a product $+_i$ on $\pi_n(X, x_0)$ as follows: $[f] +_i [g] = [f +_i g]$ where

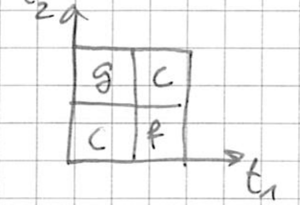
$$f +_i g : I^n \to X, \quad (f +_i g)(t_1, \dots, t_n) = \begin{cases} f(t_1, \dots, t_{i-1}, 2t_i, t_{i+1}, \dots t_n) & \text{if } 0 \leq t_i \leq 1/2 \\ g(\dots\dots, t_{i-1}, 2t_i - 1, t_{i+1}, \dots) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

3.3 Proposition For $n \geq 1$ and $1 \leq i \leq n$, $+_i$ defines a group structure on $\pi_n(X, x_0)$, with the constant map $[c]$ as neutral element. Moreover, for $n \geq 2$ and $1 \leq i, j \leq n$, we have $+_i = +_j$ on $\pi_n(X, x_0)$, and the group $\pi_n(X, x_0)$ is <u>abelian</u>.

<u>proof</u>: that $+_i$ defines a group structure is analogous to the $\pi_1(X, x)$ case. If $n \geq 2$ and $1 \leq i, j \leq 2$, notice that $f +_i g \simeq (f +_j c) +_i (c +_j g) \overset{(1)}{=} (f +_i c) +_j (c +_i g) \simeq f +_j g$, thus $+_i = +_j$.



Moreover $f \underset{1}{+} g \simeq f \underset{2}{+} g \simeq (c +_1 f) +_2 (g +_1 c) =$
$(c \underset{2}{+} g) +_1 (f +_2 c) \simeq g +_1 f$. #



The functoriality is clear:

<u>3.4 Definition</u>: For $n \geq 1$ and $(X, x_0) \in Top_*$, we define the nth homotopy group of (X, x_0) as

$$\left(\pi_n(X, x_0),\ + = +_1,\ [c]\right).$$

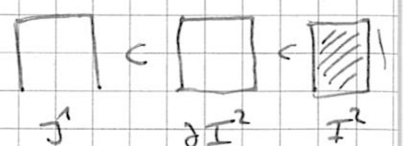
We have induced group homomorphisms, and obtain functors

$$Top_* \xrightarrow{\pi_n} \begin{cases} Grps & \text{if } n = 1 \\ Ab & \text{if } n \geq 2 \end{cases}$$ that are homotopy invariant. ($\pi_0 : Top_* \to Set_*$, pointed sets)

<u>3.5 Remark</u> For $n \geq 2$ the π_n are easy to define, hard to compute (as opposed to homology). A first tool is given by the exact sequence of a pointed pair $(X, A, *) \in Top_*^2$.

<u>3.6 Definition</u>: For $n \geq 1$, let $J^{n-1} \subset \partial I^n \subset I^n$ be defined by

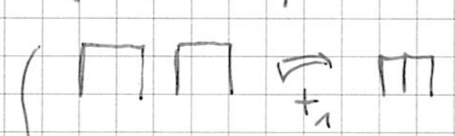
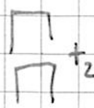
$$J^{n-1} = \begin{cases} \{1\} & \text{if } n = 1 \\ (\partial I^{n-1} \times I) \cup (I^{n-1} \times \{1\}) & n \geq 2 \end{cases}$$

(i.e. the "cube" ∂I^n without its lower face: 

For $(X, A, *) \in Top_*^2$, let

$$\pi_n(X, A; *) = [(I^n, \partial I^n, J^{n-1}), (X, A, *)]$$ in Top^3.

For $n \geq 2$, and $1 \leq i \leq n-1$, and for $[f], [g] \in \pi_n(X, A, *)$, we define $[f] +_i [g] = [f +_i g]$ for the same $+_i$ as above

( ok but not  !!!).

We let $+ = +_1$. As above, we obtain

3.7 Definition: Let $(X, A, *) \in Top^2_*$ and $n \geq 1$. We define the n-th relative homotopy group of $(X, A, *)$ as $\pi_n(X, A, *)$, $+$. We have unique maps, and obtain functors

$$\pi_n : Top^2_* \longrightarrow \begin{cases} Sets_* & \text{if } n = 1 \\ Groups & \text{if } n = 2 \\ Ab & \text{if } n \geq 3 \end{cases}$$

that are homotopy invariants.

3.8 Remark: It is obvious that if $A = \{*\}$, then
$$\pi_n(X, \{*\}, *) = \pi_n(X, *).$$

3.9 Lemma + Def: Let $n \geq 1$, $(X, A, *) \in Top^2_*$.
Let $d_n : I^{n-1} \hookrightarrow I^n$ be the inclusion as "bottom face"
$d_n(t_1, \dots, t_{n-1}) = (t_1, \dots, t_{n-1}, 0)$.
Define
$$\partial_n : \pi_n(X, A, *) \longrightarrow \pi_{n-1}(A, *)$$
$$[f] \longmapsto [f \circ d_n]$$
Then ∂_n is well defined, is a group homomorphism if $n \geq 2$, and defines a natural transformation of functors

$$\partial_n : \pi_n \Longrightarrow \pi_{n-1} \circ r : Top^2_* \longrightarrow \begin{cases} Sets & n = 1 \\ Groups & n = 2 \\ Ab & n \geq 3 \end{cases}$$

where $r : Top^2_* \to Top_*$ is the functor given by $\left((X, A, *) \xrightarrow{f} (Y, B, *)\right) \longmapsto \left((A, *) \xrightarrow{f|_A} (B, *)\right)$

__proof:__ For $f : (I^n, \partial I^n, J^n) \to (X, A, *)$, we have $f \circ d_n : (I^{n-1}, \partial I^{n-1}) \to (A, *)$ by definition, since $(\partial I^n, J^{n-1}) \cap (I^{n-1} \times \{0\}) = (I^{n-1}, \partial I^{n-1})$.
The rest is clear.

3.10 Theorem (Exact sequence for the homotopy groups of a pair)
For $(X, A, *) \in Top^2_*$, let $i : (A,*) \hookrightarrow (X,*)$ in Top_* and $j : (X,*,*) \hookrightarrow (X,A,*)$ be the inclusions.
Then the long sequence

$$\xrightarrow{\partial_{n+1}} \pi_n(A,*) \xrightarrow{i_*} \pi_n(X,*) \xrightarrow{j_*} \pi_n(X,A,*) \xrightarrow{\partial_n} \pi_{n-1}(A,*) \to \cdots \quad \pi_1(X,A,*) \xrightarrow{\partial_1} \pi_0(A,*) \to \pi_0(X,*)$$

is exact.

3.11 Definition Let $A \xrightarrow{f} B \xrightarrow{g} C$ in Groups. We say that this sequence is exact in B if $\mathrm{Ker}(g) = \mathrm{Im}(f)$.
(If C, B and C, or A, B and C are pointed sets, interpret this in the same way by setting
$\mathrm{Ker}(g) = g^{-1}(*)$, $* \in C$ base point.
A long sequence is called exact if exact at any group (pointed set) with incoming and outgoing maps.

Proof of Thm 3.10:
(1) Exactness at $\pi_n(X,*)$, $n \geq 1$:
(a) $\mathrm{Ker}(j_*) \subset \mathrm{Im}(i_*)$ is equivalent to $j_* \circ i_* = 0$.
Take $[f] \in \pi_n(A,*)$. Let $\tilde{f} = j \circ i \circ f$:
Then $\tilde{f} \overset{H}{\simeq} c$ with

$$H : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, *)$$

with $\tilde{f} \circ G$ $\qquad G : I^n \times I \to I^n$,

$$((t_1, \dots, t_n), t) \mapsto (t_1, \dots, t_{n-1}, t + (1-t)t_n$$