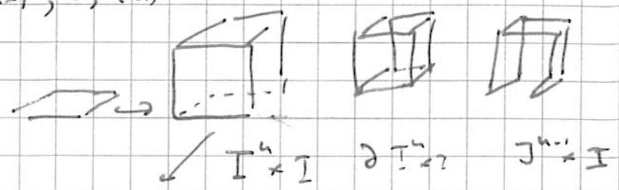
(b) $\ker j_* \subset \operatorname{Im} i_*$. Let $[f] \in \ker j_*$. Thus $f \overset{H}{\simeq} c$ for $H : (I^n, \partial I^n, J^{n-1}) \to (X, A, *)$

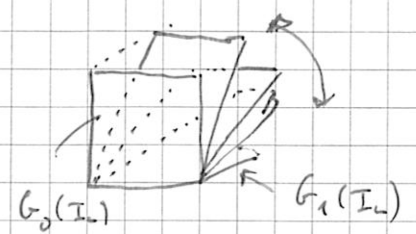
Take $g : (I^n, \partial I^n) \to (A, *)$ given by

$$g(t_1, \dots, t_n) = H(t_1, \dots, t_{n-1}, 0, t_n)$$

Then $ig \overset{H\circ G}{\simeq} f : (I^n, \partial I^n) \to (X, *)$



where $G : I^n \times I \to I^{n+1}$ "claps the front face down":



$$G(t_1, \dots, t_n, s) = \begin{cases} (t_1, \dots, t_n, 2st_n) & 0 \le s \le \frac{1}{2} \\ (t_1, \dots, t_{n-1}, (2-2s)t_n, t_n) & \frac{1}{2} \le s \le 1 \end{cases}$$

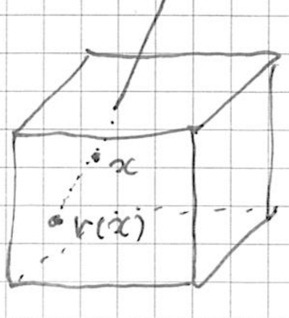
The other cases are easier and left as ~~exercices~~ #

We can wonder about the dependence of $\pi_n(X, A, *)$ on the choice of the base-point $* \in A \subset X$.

<u>3.12 Def</u>: Let $X \in Top$, C a category. A <u>local system</u> of objects in C is a functor $\pi X \to C$.

We can promote $\pi_n(X, -)$ to a local system of groups. We consider the subspace $K_n \subset \partial I^{n+1} \subset I^{n+1}$ formed by "$\partial I^{n+1} \setminus$ top face": $K_n = (\partial I^n \times I) \cup (I^n \times \{0\})$.

Then $K_n \hookrightarrow I^{n+1}$ admits a restraction $r : I^{n+1} \to K_n$; Choose one (for example a stereographic projection from a point P above the center of the top face).



In particular, any map $f : K_n \to X$ can be extended to a map $\bar{f} : I^{n+1} \to X$ (for example $\bar{f} = f \circ r$).

3.13 Proposition: For $u : I \to X$ a path from $a = u(0)$ to $b = u(1)$ in X, and for $f : (I^n, \partial I^n) \to (X, a)$, let $F_u : K_n \to X$ be defined by
$$f_u(\underbrace{x, t}_{\in I^n \times I}) = \begin{cases} f(x) & \text{if } t = 0 \\ u(t) & \text{if } x \in \partial I^n. \end{cases}$$
Choose an extension $\overline{F}_u : I^{n+1} \to X$, and let
$$\bar{f}_u : (I^n, \partial I^n) \to (X, b), \quad \bar{f}_u(x) = \overline{F}_u(x, 1).$$
Then, for $n \geq 1$, the rule
$$([u] : a \to b) \mapsto \left(\begin{array}{c} \pi_n(X, a) \to \pi_n(X, b) \\ [f] \mapsto [\bar{f}_u] \end{array} \right)$$
is well-defined and defines a local system of groups on X.

#

3.14 Remark: for $n = 1$ we recover the local system $\Pi X \to$ Groups, $a \mapsto \mathrm{Aut}_{\Pi X}(a)$ discussed in chapter 2, and have for $\pi_n(X, a)$ the same conclusions.

Note that we obtain a group action of $\pi_1(X, a)$ on $\pi_n(X, a)$, $\forall n \geq 1$.

For $\pi_n(X, A, *)$, we can adapt the construction to get a local system on ΠA.

One can show: the long exact sequence of the pair (X, A, a) admits a compatible action of $\pi_1(A, a)$.

3.15 Def Let $n \in \mathbb{N}$. (a) $X \in \mathrm{Top}$ is called <u>n-connected</u> if $\pi_k(X, x) = 0 \quad \forall\ 0 \leq k \leq n, \ \forall\ x \in X$.

(b) $(X, A) \in \mathrm{Top}$ is called <u>n-connected</u> if

- each path-con. component of X meets A, if $n = 0$
- It is 0-connected, and for all $1 \leq k \leq n$, for all $a \in A$ we have $\pi_k(X, A, a) = 0$, if $n \geq 1$

(c) A map $f : X \to Y$ in Top is called <u>n-connected</u> if $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an iso for all $0 \le k \le n-1$, and is surjective for $k = n$, $\forall x \in X$.

(d) A map $f : (X, A) \to (Y, B)$ in Top^2 is called <u>n-connected</u> $(n \geq 1)$ if for all $a \in A$, $f_* : \pi_k(X, A, a) \to \pi_k(Y, B, f(a))$ is an iso for all $1 \le k \le n-1$ and is surjective for $k = n$, $\forall a \in A$.

<u>3.16 Theorem</u> (Blackers-Massey: "homotopy excision")
Let $X \in$ Top, $U_0, U_1 \subset X$ open, $U_{01} = U_0 \cap U_1$.
Suppose given $m, n \in \mathbb{N}$ such that

(a) $U_{01} \neq \emptyset$

(b) (U_0, U_{01}) is m-connected,

(c) (U_1, U_{01}) is n-connected.

Then the inclusion $(U_0, U_{01}) \hookrightarrow (X, U_1)$ is $(m+n)$-connected.

proof: See tom Dieck. #

<u>3.17 Definition</u>: For $(X, *) \in Top_*$ define the (reduced) suspension of X as $(\Sigma X, *) = (X \times I/\sim, [*])$ where $\sim$ is the equiv. rel. generated by $(x, 0) \sim (y, 0)$, $(x, 1) \sim (y, 1)$ and $(x_0, t) \sim (x_0, s)$ for all $x, y \in X$ and s, t.

For any $n \geq 0$, the suspension homomorphism $\Sigma_* : \pi_n(X, *) \to \pi_{n+1}(\Sigma X, *)$ is defined as follows: for $f : (I^n, \partial I^n) \to (X, *)$, $f \times id_I : (I^{n+1}, \partial I^n \times I) \to (X \times I, \{*\} \times I)$ induces on quotient spaces $\Sigma f : (I^{n+1}, \partial I^{n+1}) \to (\Sigma X, *)$; we put $\Sigma_*([f]) = [\Sigma f]$.

<u>Exercise</u> (1) Check well define and a group homomorphism if $n \geq 1$. (2) $\Sigma S^n \cong S^{n+1}$

3.18 Corollary: (a) For $n \geq 1$, S^n is $(n-1)$-connected.
(b) $\Sigma_*: \pi_k(S^n,*) \to \pi_{k+1}(S^{n+1},*)$ is an isomorphism if $k \leq 2n-2$ and is surjective if $k = 2n-1$
(c) We have $\Sigma_*: \pi_1(S^1,*) \twoheadrightarrow \pi_2(S^2,*)$ and $\Sigma_*: \pi_n(S^n,*) \xrightarrow{\cong} \pi_{n+1}(S^{n+1},*)$ if $n \geq 2$. In particular, $\pi_n(S^n,*)$ is cyclic for $n \geq 2$.

proof: (a) Induction on n: true for $n=1$ since S^1 path-connected; assume $n \geq 1$ and S^n $(n-1)$-connected. Let S, N be the south and north pole of S^{n+1}.
Let $U_0 = S^{n+1} \setminus \{S\}$, $U_1 = S^{n+1} \setminus \{N\}$, $U_{01} = U_0 \cap U_1$
Consider the long exact sequences of

$$\begin{array}{ll}(U_0, U_{01}, *): & 0 \to \pi_{k+1}(U_0, U_{01}) \xrightarrow[\cong]{\partial} \pi_k(U_{01}) \to 0 \quad (k \geq 1) \\ & \qquad\qquad \downarrow i_* \\ (S^{n+1}, U_1, *): & \\ & 0 \to \pi_{k+1}(S^{n+1}) \xrightarrow[\cong]{j} \pi_{k+1}(S^{n+1}, U_1) \to 0\end{array}$$

The zeros correspond to $U_0 \simeq *$, $U_1 \simeq *$, and by induction hypothesis we know that $\pi_k(U_{0,1}) = 0$ $\forall$ $1 \leq k \leq n-1$, since $U_{0,1} \cong S^n \times \mathring{I} \simeq S^n$.
In particular, the pairs (U_0, U_{01}) (and by symmetry (U_1, U_{01})) are n-connected. By 3.16 we deduce that i_* is an iso if $k+1 < 2n$ and surjective if $k+1 = 2n$. This proves (a), as well as (b) with
Exercise: Check that Σ corresponds here to $j_*^{-1} \circ i_* \circ \partial_*^{-1}$.
(c) is just a special case of (b).

3.19 Remark (a) If X is nice enough, the Freudenthal isomorphism theorem states that if X is nice enough, and n-connected, then $\Sigma_*: \pi_k(X,*) \to \pi_{k+1}(X,*)$ is an iso if $k \leq 2n$ and surjective if $k = 2n+1$

(b) We will see that $\pi_1(S^1) \xrightarrow{\Sigma} \pi_2(S^2)$ is an iso, thus $\pi_n(S^n) \cong \mathbb{Z}$ $\forall n \geq 1$.

(c) From 3.18 (b) we deduce that if n is fixed then $\pi_{n+k}(S^k)$ does not depend on k (up to a Σ_* isomorphism) if $k \geq n+2$.

In other words, the sequence

$$(*) \quad \pi_n(S^0) \xrightarrow{\Sigma_*} \pi_{n+1}(S^1) \xrightarrow{\Sigma_*} \cdots \to \pi_{n+k}(S^k) \to \cdots$$

stabilizes if $k \geq n+2$, so that its colimit is isomorphic to $\pi_{2n+2}(S^{n+2})$.

We call this colimit the stable homotopy group of rank n (of S^0) : $\pi_n^s = \operatorname{colim}_k (*)$.

A central goal of stable htpy theory is to understand the general structure. $\pi_0^s \cong \mathbb{Z}$ and π_n^s finite $\forall n \geq 1$.

n	0	1	2	3	4	5	6	7	8	9	10	...
π_n^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	

<u>3.20 Definition</u> A map $p: E \to B$ has the homotopy lifting property (HLP) with respect to a space X if for any commutative square of the form below (full arrows), there exists a "lift" H of h.

$$\begin{tikzcd} X \arrow[r, "H_0"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\exists H"] & B \end{tikzcd}$$

Here $i_0 : X \to X \times I$, $x \mapsto (x,0)$.

We say that p is a <u>Hurewicz fibration</u> if it has the HLP with respect to any $X \in \mathrm{Top}$. We say p is a <u>Serre fibration</u> if it has the HLP with respect to I^n for all n $(I = [0,1] \subset \mathbb{R})$.

<u>3.21 Example</u>: Suppose that $E = B \times F$ and $p: E \to B$ is the projection on the first factor. Then p is a Hurewicz fibration: Take $H(x,t) = (h(x,t), q H_0(x))$, $q = \mathrm{pr}_2 : E \to F$.

3.22 Def: Suppose that $p:(E,e_0)\to(B,b_0) \in Top_*$ is a fibration (when forgetting the base point). Then $(F,*) = (p^{-1}(b_0), e_0) \overset{i}{\hookrightarrow} (E,e_0)$ is called the $\underline{\text{fibre}}$ of E (over b_0).

3.23 Proposition: Assume $p:(E,e_0)\to(B,b_0)$ is a Serre fibration with fibre $i:(F,e_0)\to(E,e_0)$. Then $p_*: \pi_n(E,F,e_0)\to\pi_n(B,b_0)$ is an isomorphism for for all $n\geq 1$.

proof: more or less obvious: just have to "twist" $i_0: I^{n-1}\to I^n$ by a homeomorphism.

(a) p_* is surjective. Let $f:(I^n,\partial I^n)\to(B,b_0)$. Consider

$$\begin{tikzcd} J_{n-1} \arrow[r, "c_{e_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \arrow[r, "f"'] \arrow[ur, dashed, "\exists \bar f"] & B \end{tikzcd}$$

Then $\bar f: (I^n,\partial I^n, J_{n-1})\to(E,F,e_0)$ is a lift. Thus $p_*[\bar f]=[f]$.

(b) p_* is injective: Suppose given $f_0, f_1:(I^n,\partial I^n,J^{n-1})\to(E,F,e_0)$ with $p_*[f_0]=p_*[f_1]$. Thus we have a homotopy

$h:(I^n,\partial I^n)\times I\to(B,b_0)$ with $h_i = p f_i$, $i=0,1$.

Consider

$$\begin{tikzcd} K \arrow[r, "H_0"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ I^{n+1} \arrow[r, "h"'] \arrow[ur, dashed, "\exists H"] & B \end{tikzcd}$$

Where $K=(I^n\times 0)\cup(I^n\times 1)\cup(J^{n-1}\times I) \subset \partial I^{n+1} \subset I^n$, $i: K \to I^n$.

Define H_0 to be f_i on $I^n\times\{i\}$ and c_{e_0} on $J^{n-1}\times I$. Then $f_0 \overset{H}{\simeq} f_1 : (I^n,\partial I^n,J^{n-1})\to(E,F,e_0)$. Thus $[f_0]=[f_1]$ in $\pi_n(E,F,e_0)$. #

3.24 Proposition Let $p: E \to B$ be a Serre fibration, $e_0 \in E$ and $(F, e_0) \xrightarrow{i} (E, e_0)$ the fibre of p at e_0. Then there is a long homotopy exact sequence

$$\begin{tikzcd} \arrow[r, "\partial_{n+1}"] & \pi_n(F,e_0) \arrow[r, "i_*"] & \pi_n(E,e_0) \arrow[r, "p_*"] & \pi_n(B, p(e_0)) \arrow[dll, rounded corners, "\partial_n"'] \\ & \pi_{n-1}(F,e_0) \arrow[r] & \cdots \quad \pi_0(E,e_0) \arrow[r] & \pi_0(B, p(e_0)) \end{tikzcd}$$

Here ∂_n is defined as

$$\pi_n(B, p(e_0)) \xrightarrow{p_*^{-1}} \pi_n(E, F, e_0) \xrightarrow{\partial_n'} \pi_{n-1}(F, e_0)$$

where ∂_n' is the connecting hom for the long exact sequence of the pair (E, F, e_0).

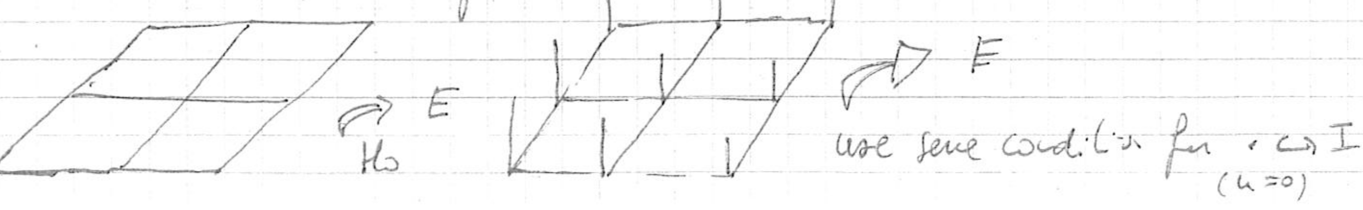
proof: combine 3.10 and 3.23. #

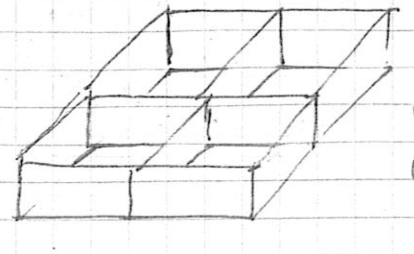
The property of a Serre Fibration is a local one:

3.24bis Proposition Let $p: E \to B$ in Top, and assume that there exist an open covering $\{U_j\}_{j \in J}$ of B such that $p_j = p|_{p^{-1}(U_j)} : p^{-1}(U_j) \to U_j$ is a Serre fibration for all $j \in J$. Then $p: E \to B$ is a Serre Fibration.

proof: Suppose given $n \geq 0$ and
$$\begin{tikzcd} I^n \arrow[r, "H_0"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ I^{n+1} = I^n \times I \arrow[r, "h"] & B \end{tikzcd}$$
commutative. Subdivide I^{n+1} in N^{n+1} small cubes $\{W_k\}_{1 \leq k \leq N^{n+1}}$ such that for all k, $h(W_k) \subset U_j$ for some $j \in J$ (Lebesgue number lemma). We can extend H_0 to $H: I^n \times I \to E$ compatibly working inductively on these cubes W_k.

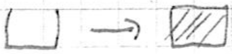
We illustrate this for $n = 2$ and $N = 2$.

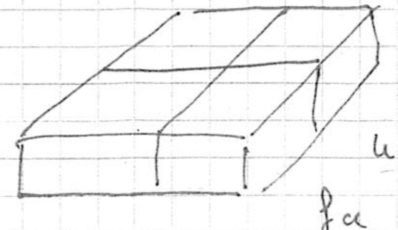




$\rightarrow$ E

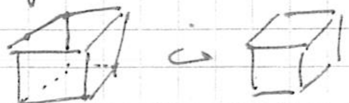
use Serre cond. for $n=1$





$\rightarrow$ E

use Serre cond for $n=2$



Then continue with next floor until reached top.

$\#$

3.25 Example: fibre bundles are Serre fibrations: they are locally a projection, so have locally the property of a Serre Fibration. More precisely:

$p: E \to B$ is a fibre-bundle with fibre $F \in \mathrm{Top}$ if there exists an open cover $\{U_j\}_{j \in J}$ of B and homeomorphisms $h_j : p^{-1}(U_j) \to U_j \times F$ such that

$$\begin{tikzcd} p^{-1}(U_j) \arrow[r, "h_j"] \arrow[dr, "p"'] & U_j \times F \arrow[d, "pr_1"] \\ & U_j \end{tikzcd}$$

commutes. By Example 3.21, a projection is a Serre fibration, so fibre bundles are Serre fibrations.

"Homogeneous spaces" offer very nice examples of fiber bundles.

3.26 Proposition Let G be a Hausdorff topological group and H a closed subgroup of G. Let $p: G \to G/H = \{gH \,;\, g \in G\}$ be the quotient map onto the H-orbits, with G/H endowed with the quotient topology. Suppose that p has a local section at $[e]_H$: $\exists$ U an open nb. of $[e]_H$ in G/H, $s: U \to G$ with $ps = id_U$. Let $K \subset H$ be a closed subgroup. Then the quotient map $q: G/K \to G/H$ is a fibre bundle with fibre $H/K \subset G/K$.

proof: first, notice that p has a local section at $[g]_H$ for any $g \in H$, defined by $g s g^{-1} : gU \to G$

for a chosen local section $s : U \to G$ at $[e]_H$.
For $s: U \to G$ a local section of q at $[e]_H$,
define $U \times H/K \to q^{-1}(U)$

$$h_U : ([g]_H, [h]_K) \mapsto [gh]_K .$$

Then h_U is continuous, with inverse

$$k_U : q^{-1}(U) \to U \times H/K$$
$$[g]_K \mapsto ([g]_H, \; s([g]_H)^{-1} \cdot [g]_K)$$

Moreover, $q \circ h_U([g]_H, [h]_K) = q([gh]_K) = [g]_H$
$= pr_1([g]_H, [h]_K)$. Thus $q|_{q^{-1}(U)} : q^{-1}(U) \to U$
is homeo to a product, and by "symmetry",
q is a fibre bundle.

3.27 Example (Stiefel manifolds)

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, $1 \le k \le n$ in $\mathbb{N}$, and let $\mathbb{F}^n$ be endowed with an euclidean (hermitian) pairing $\langle , \rangle$. Let
$$V_{n,k} = V^{\mathbb{F}}_{n,k} = \{ (v_1, \dots, v_k) \in \mathbb{F}^n \; ; \; \langle v_i, v_j \rangle = \delta_{ij} \}$$
be endowed with the subspace topology of $V_{k,n} \subset \mathbb{F}^{nk}$.
(the space of k-frames in $\mathbb{F}^n$, a Stiefel manifold).
Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{F}^n$.
Take $x_0 = (e_{n-k+1}, \dots, e_n) \in V_{n,k}$ as base point.
Consider $G(n) = O(n)$ if $\mathbb{F} = \mathbb{R}$ or $U(n)$ if $\mathbb{F} = \mathbb{C}$.
We have a continuous, transitive action

$$G(n) \times V_{n,k} \to V_{n,k}, \; A \cdot (v_1, \dots, v_k) = (Av_1, \dots, Av_k)$$

The stabilizer of x_0 is $G(n)_{x_0} = G(n-k) \hookrightarrow G(n)$,
$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

and we obtain a homeo (compact $\to$ Hausdorff).

$$h_{n,k} : G(n)/G(n-k) \longrightarrow V_{n,k}$$

We get a commutative diagram for $1 \le k \le l \le n$

$$\begin{array}{ccc} V_{n,l} & \xrightarrow{p_{n,l}} & V_{n,k} \\ \uparrow & & \uparrow \\ G(n)/G(n-l) & \xrightarrow{q} & G(n)/G(n-k) \end{array}$$

with $p(v_1, \dots, v_l) = (v_{l-k+1}, \dots, v_l)$, and p, q quotient maps.

These are fibre bundles: it is easy to define a local section of $p_{n,n}$ at x_0: take

$U = \{ (v_1, \dots, v_k) \in V_{n,k} \ ; \ (e_1, \dots, e_{n-k}, v_1, \dots, v_k)$ is a basis of $\mathbb{F}^n \}$

Check that U is an open nbhd of x_0; let

$S : U \longrightarrow V_{n,l} \cong G(n)/\ldots, (v_1, \dots, v_k) \mapsto (f_{n-l+1}, \dots, f_{n-k}; v_1, \dots, v_k)$

where $(f_1, \dots f_{n-k}, v_1, \dots, v_k)$ is obtained by applying Gram-Schmidt to $(e_1, \dots, e_{n-k}, v_1, \dots, v_k)$ backwards.

Thus we have fibre bundles with inclusion of fibres

$$\begin{array}{ccccc} V_{n-k,l-k} & \xrightarrow{i} & V_{n,n-l} & \xrightarrow{p} & V_{n,n-k} \\ \cong \uparrow h & & \cong \uparrow h & & \cong \uparrow h \\ G(n-k)/G(n-l) & \xrightarrow{j} & G(n)/G(n-l) & \xrightarrow{q} & G(n)/G(n-k) \end{array}$$

One can show: $V_{n,k}$ is a smooth manifold.

<u>3.28 Example</u> (The Grassmann manifolds) In the situation of 3.27, let

$$G_{n,k} = \{ V \subset \mathbb{F}^n \ ; \ V \text{ is a subvect. space of dim } k \}.$$

We then have a map

$$V_{n,k} \xrightarrow{\pi} G_{n,k} \ , \ (v_1, \dots, v_k) \mapsto \underbrace{\langle v_1, \dots, v_k \rangle}_{\text{subspace gen. by } \dots}$$

and endow $G_{n,k}$ with the quotient topology.

Take $\langle x_0 \rangle = \pi(x_0) = \langle e_{n-k+1}, \dots, e_n \rangle$ as base-point.

We have a transitive action of $G(n)$ on $G_{n,k}$,

and $G(n)_{\langle x_0 \rangle} = G(n-k) \times G(k) \hookrightarrow G(n)$

$$(A, B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Thus the induced map ϕ is a homeomorphism, and we have a com. diagram

$$\begin{tikzcd} G(n)/G(n-k) \arrow[r, "h", "\cong"'] \arrow[d] & V_{n,k} \arrow[d, "\pi"] \\ G(n)/G(n-k)\times G(k) \arrow[r, "\phi", "\cong"'] & G_{n,k} \end{tikzcd}$$

We can define a local section of π at y_0: take
$V = \{ W \in G_{n,k} \mid W \cap \langle e_1, \dots, e_{n-k} \rangle = 0 \}$.
Then V is open and $y_0 \in V \subset G_{n,k}$.
Define $t : V \to V_{n,k}$, $W \mapsto (f^W_{n-k+1}, \dots, f^W_n)$
where $(f^W_{n-k+1}, \dots, f^W_n)$ is obtained by applying Gram-Schmidt to the orthogonal projections of $e_{n-k+1}, \dots, e_n$ onto W. We have $t(V) \subset U$ (as in 3.27), and composing with $s : U \to G(n)$ we obtain a local section $st : V \to G(n)$ of $G(n) \to V_{n,k} \to G_{n,k}$ at y_0.
Thus, we have a fibre bundle π with inclusion of fiber i

$$\begin{tikzcd} V_{k,k} \arrow[r, hook, "i"] \arrow[d, phantom, "\cong" rotate=90] & V_{n,k} \arrow[r, "\pi"] \arrow[d, phantom, "\cong" rotate=90] & G_{n,k} \arrow[d, phantom, "\cong" rotate=90] \\ G(k) \arrow[r, hook] & G(n)/G(n-k) \arrow[r] & G(n)/G(n-k)\times G(k) \end{tikzcd}$$

<u>3.29 Remark</u>: the same construction works for $\mathbb{F} = \mathbb{H}$, the quaternion algebra structure (non-commutative field) on $\mathbb{R}^4$, leading to fibre bundles $V^{\mathbb{H}}_{n-k,\, l-k} \hookrightarrow V^{\mathbb{H}}_{n,l} \to V^{\mathbb{H}}_{n,k}$ and $V^{\mathbb{H}}_{k,k} \hookrightarrow V^{\mathbb{H}}_{n,k} \to G^{\mathbb{H}}_{n,k}$.

<u>3.30 Definition</u> Taking $k=1$, we obtain the Hopf fiber bundle

$$\left.\begin{array}{l} S^0 \to S^n \to \mathbb{R}P^n \\ S^1 \to S^{2n+1} \to \mathbb{C}P^n \\ S^3 \to S^{4n+3} \to \mathbb{H}P^n \end{array}\right\} \text{ as } V^{\mathbb{F}}_{1,1} \to V^{\mathbb{F}}_{n+1,1} \to G^{\mathbb{F}}_{n+1,1} \text{ for } \mathbb{F} = \left\{\begin{array}{l} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H} \end{array}\right.$$

and, for $n=1$, using homeos $\mathbb{R}P^1 \cong S^1$, $\mathbb{C}P^1 \cong S^2$, $\mathbb{H}P^1 = S^4$
$S^0 \to S^1 \xrightarrow{2} S^1$, $S^1 \to S^3 \xrightarrow{\eta} S^2$, $S^3 \to S^7 \xrightarrow{\nu} S^4$

3.31 Remark The Octonion $\mathbb{O}$, given by an algebra structure on $\mathbb{R}^8$, are not commutative nor associative.
We cannot define $\mathbb{O}P^n$ for $n \geq 2$ (lack of associativity) but one can show that $\mathbb{O}P^1$ is well defined, leading to a "Hopf fibre bundle" $S^7 \to S^{15} \xrightarrow{\sigma} S^8$.

3.32 Lemma $\pi_n(S^1, 1) = 0$ for $n \geq 2$.
proof $\mathbb{Z} \subset \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ is a Serre fibration and $\mathbb{R} \simeq *$.

3.33 Theorem: For all $n \geq 1$, we have an iso $\mathbb{Z} \to \pi_n(S^n, *)$
$1 \mapsto [id_{S^n}]$.
proof Use the Serre Fibration $S^1 \to S^3 \to S^2$ together with $\pi_2(S^3) = 0$, $\pi_1(S^1) = 0$, to conclude by the long exact sequence that $\partial : \pi_2(S^2) \to \pi_1(S^1)$ is an iso. Use 3.18 (c) to conclude #

3.34 Theorem: The map $\eta : S^3 \to S^2$ induces an isomorphism $\pi_n(S^3, *) \to \pi_n(S^2, *)$ for all $n \geq 3$.
In particular, $\mathbb{Z} \to \pi_3(S^2)$ is an isomorphism.
$1 \mapsto [\eta]$
proof: Long exact seq. of the fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$.

3.35 Remark $\pi_4(S^3)$ is stable; we have a surjection $\pi_3(S^2) \xrightarrow{\Sigma_*} \pi_4(S^3) \cong \mathbb{Z}/2$.

3.36 Remark As a corollary of 3.33 we have:
$S^n \hookrightarrow D^{n+1}$ has no retract
(if $S^n \hookrightarrow D^{n+1} \to S^n$ is id_{S^n}, apply π_n: $\mathbb{Z} \to 0 \to \mathbb{Z}$ with composite id) ↯.
This has consequences like Brouwers fixed point Theorem.

IV SIMPLICIAL SETS AND SINGULAR HOMOLOGY

<u>4.1 Definition</u>: Let $\mathbb{\Delta}$ be the category with objects the ordered sets $[n] = \{ i \in \mathbb{N} \,;\, 0 \leq i \leq n \}$ for $n \in \mathbb{N}$, and morphisms the order preserving maps.
(We can view $\mathbb{\Delta}$ as the full subcategory of Cat (the cat. of small categories), on the objects $[n]$ viewed as categories.)

<u>4.2 Definition</u>: A <u>simplicial set</u> is a functor $\mathbb{\Delta}^{op} \to \mathrm{Set}$, thus is a presheaf of sets on $\mathbb{\Delta}$.
We denote $\widehat{\mathbb{\Delta}}$ or SSets the category of simplicial sets (with morphisms the natural transformations).

<u>4.3 Example</u>: a most basic example is the representable presheaf $\Delta_n = h_{[n]} = \mathbb{\Delta}(-,[n])$, called the <u>standard n-simplex</u>

We can give a decomposition of the morphisms in $\mathbb{\Delta}$ leading to a more explicit description of a simplicial set $X \in \widehat{\mathbb{\Delta}}$.

<u>4.4 Definition</u>: For $n \geq 1$ and $0 \leq i \leq n$, let
$\delta^i = \delta^i_n : [n-1] \to [n]$ be the unique injective morphism in $\mathbb{\Delta}([n-1],[n])$ not taking i as value
For $n \geq 0$ and $0 \leq i \leq n$, let
$\sigma^i = \sigma^i_n : [n+1] \longrightarrow [n]$ be the unique surjective morphism in $\mathbb{\Delta}([n+1],[n])$ "take twice i as value".
The δ^i are called the <u>cofaces</u> and the σ_i the codegeneracies

<u>4.5 Example</u>

$$\begin{tikzcd}[4] \arrow[d, "\sigma_3^2"'] & 0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[dl] & 4 \arrow[dl] \\ [3] & 0 & 1 & 2 & 3 & \end{tikzcd}
\qquad
\begin{tikzcd}[4] & 0 & 1 & 2 & 3 & 4 \\ [3] \arrow[u, "\delta_4^1"] & 0 \arrow[u] & 1 \arrow[ur] & 2 \arrow[ur] & 3 \arrow[ur] & \end{tikzcd}$$

4.6 Lemma The morphisms defined above satisfy the relations

$$\delta^j_{n+1}\,\delta^i_n = \delta^i_{n+1}\,\delta^{j-1}_n \quad \text{if} \quad i<j$$

$$\sigma^j_n\,\sigma^i_{n+1} = \sigma^i_n\,\sigma^{j+1}_{n+1} \quad \text{if} \quad i\le j$$

$$\sigma^j_{n-1}\,\delta^i_n = \begin{cases} \delta^i_{n-1}\,\sigma^{j-1}_{n-2} & \text{if} \quad i<j \\ 1_{[n-1]} & \text{if} \quad i=j,j+1 \\ \delta^{i-1}_{n-1}\,\sigma^j_{n-2} & \text{if} \quad i>j \end{cases}$$

(whenever the morphism exist)

These relations are called the cosimplicial identities #

4.7 Lemma: Any morphism $\alpha \in \Delta([m],[n])$ admits a unique decomposition $\alpha = \underbrace{\delta^{i_p}_{n-1} \cdots \delta^{i_1}_{m-q+1}}_{\alpha_2}\,\underbrace{\sigma^{j_1}_{m-q} \cdots \sigma^{j_q}_{m-1}}_{\alpha_1}$

where $i_1<\dots<i_p$ and $j_1<\dots<j_q$.

proof: Take $[n] \setminus \mathrm{Im}(\alpha) =: \{i_1<\dots<i_p\} \subset [n]$

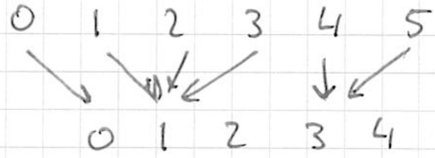
and $\{j\in[m] \,;\, \alpha(j)=\alpha(j+1)\} = \{j_1<\dots<j_q\} \subset [m]$.

Check that

$$\alpha : [m] \overset{\alpha_1}{\twoheadrightarrow} [m-q] = [n-p] \overset{\alpha_2}{\hookrightarrow} [n]$$

#

4.8 Example

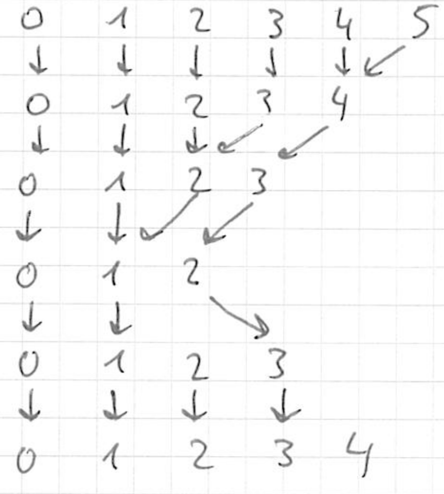


$$\begin{array}{c}[5]\\ \downarrow\alpha \\ [3]\end{array}$$

Then $(i_1,i_2) = (2,4)$

$(j_1,j_2,j_3) = (1,2,4)$

$\Rightarrow \alpha = \delta^4_4\,\delta^2_3\,\sigma^1_2\,\sigma^2_3\,\sigma^4_4$



4.9 Proposition: The category Δ is freely generated by the morphisms of type δ_n^i and σ_n^i, modulo the cosimplicial identities.

4.10 Definition: If $X \in \hat{\Delta}$, $X(\delta_n^i) : X([n]) \to X([n-1])$ is called the i-th face map of X. We usually write $X_n := X[n]$ and $d_i = d_i^n = X(\delta_n^i) : X_n \to X_{n-1}$. Then $s_i = s_i^n = X(\sigma_n^i) : X_{n+1} \to X_n$ is called the i-th degeneracy of X.

4.11 Remark: 4.9 says that $X \in \hat{\Delta}$ precisely corresponds to the data

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} \xrightarrow{s_0} X_1 \; \substack{\leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow} \; X_2 \; \substack{\leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow} \; X_3 \; \cdots$$

where $X_i \in$ Set, and where d_i, s_i are given and satisfy the simplicial identities, dual to the cosimplicial identities:

$$d_i d_j = d_{j-1} d_i \quad i < j$$

$$s_i s_j = s_{j+1} s_i \quad i \leq j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ id, & i = j, j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

A small category $C \in CAT$, can be viewed as a simplicial set via the "nerve functor" N.

4.12 Definition: If $C \in CAT$, we define the nerve of C, $NC \in \hat{\Delta}$, by

$$NC : \Delta^{op} \to Sets, \quad [n] \mapsto CAT([n], C)$$

where $[n] \in \Delta$ is viewed as an element of CAT.
(by its structure of an ordered set).

<u>4.13 Remark</u> let's describe NC more explicitly:
A functor $[n] = \{0 \to 1 \to 2 \to \cdots \to n\} \xrightarrow{x} C$ is just a sequence of n composable morphisms
$$x_0 \xrightarrow{\alpha_0} x_1 \xrightarrow{\alpha_1} x_2 \to \cdots \xrightarrow{\alpha_{n-1}} x_n \quad \text{in cat.}$$
Thus
$$NC_0 = Ob(C)$$
$$NC_1 = Mor(C)$$
$$NC_2 = \{x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2\} = Mor \underset{Ob}{\times} Mor$$
etc ...

We see that

$$0 \to 1 \cdots \to i \to i+1 \to \cdots \to n+1 \qquad\qquad x_0 \to \cdots \to x_i \xrightarrow{id} x_i \to \cdots \to x_n$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow s_i$$
$$0 \to 1 \cdots \to i \to \cdots \to n \qquad\qquad x_0 \xrightarrow{\alpha_1} \cdots\cdots \xrightarrow{\alpha_{n-1}} x_n$$

σ_n^i sends $i \to i+1$ to id_i

and that

$$0 \to 1 \to \cdots \to i-1 \to i \to i+1 \to \cdots \to n$$
$$\uparrow$$
$$0 \to \cdots \to i-1 \to i \to \cdots \to n-1$$

and we deduce

δ_n^i sends $(i-1) \to i$ to the composition $(i-1) \to i \to (i+1)$

$$d_i(x_0 \xrightarrow{\alpha_0} \cdots \to x_n) = \begin{cases} (x_1 \xrightarrow{\alpha_1} \cdots \to x_n) & \text{if } i = 0 \\ (x_0 \to \cdots \to x_{i-1} \xrightarrow{\alpha_i \alpha_{i-1}} x_{i+1} \to \cdots x_n) & 0 < i < n \\ (x_0 \to \cdots \to x_{n-1}) & \text{if } i = n. \end{cases}$$

<u>4.14 Remarks</u> (a) We can show (exercise) that N is fully faithful.
(b) N is right adjoined to the "truncation" $\hat{\Delta} \xrightarrow{\tau_1} CAT$ where $Ob(\tau_1(X)) = X_0$, the morphism are generated by X_1, source $= d_1$, target $= d_0$, identity $= s_0$,

with relation of the form $h = g \circ f$ for every $x \in X_2$ of the form

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "x"] & & \cdot \end{tikzcd}$$

(meaning $f = d_2 x$, $g = d_0 x$, $h = d_1 x$)

Check the details (eg: when does $f \circ id = f$ come from?).

(c) $\widehat{\Delta}$ is a complete and cocomplete category: the limits and colimits are formed level-wise (true for any category of presheaves of sets on a small category).
In particular, as a right adjoint, the nerve N preserves products.

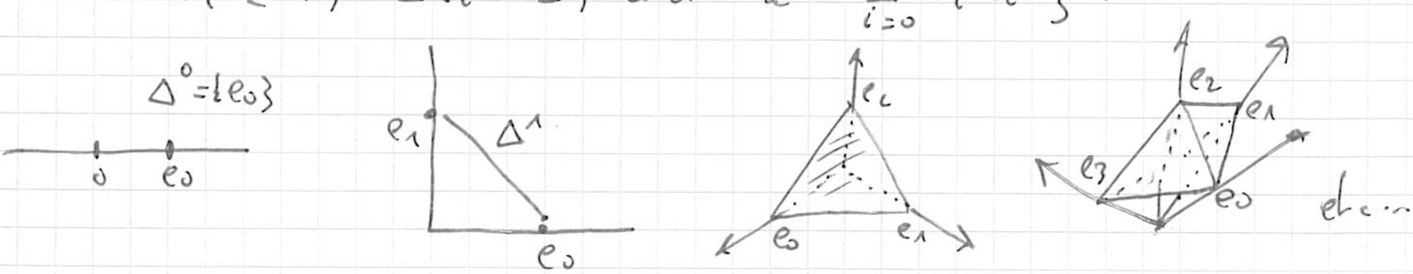
<u>4.15 Definition</u>: Let C be any category. A <u>simplicial object in C</u> is a functor $\Delta^{op} \xrightarrow{X} C$. A <u>cosimplicial object in C</u> is a functor $\Delta \xrightarrow{Y} C$.

<u>4.16 Example</u> We define a cosimplicial space

$$\Delta^{\bullet} : \Delta \to Top, \quad [n] \mapsto \Delta^n \quad \text{as follows.}$$

In $\mathbb{R}^{(\mathbb{N})} = \bigoplus_{\mathbb{N}} \mathbb{R}$ consider the basis $\{e_0, e_1, e_2 \dots\}$ defined by $e_i : \mathbb{N} \to \mathbb{R}$, $e_i(j) = \delta_{i,j}$.
Let Δ^n be the convex hull of $\{e_0, \dots, e_n\}$:

$$\Delta^n = \{ x \in \mathbb{R}^{n+1} \subset \mathbb{R}^{(\mathbb{N})} \mid \exists (t_0, \dots, t_n) \in \mathbb{R}, \; 0 \leq t_i \leq 1, \; \Sigma t_i = 1, \text{ and } x = \sum_{i=0}^{n} t_i e_i \}.$$

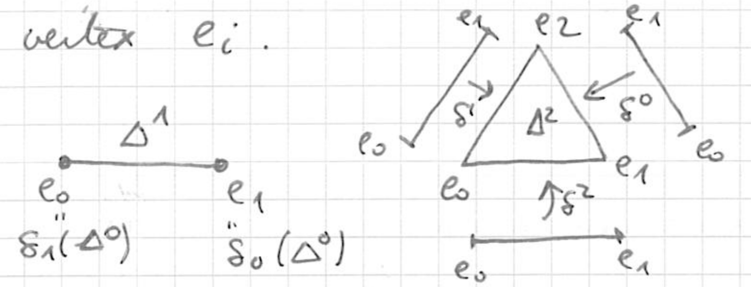


and for $[m] \xrightarrow{\alpha} [n]$ in Δ, $\Delta^m \xrightarrow{\alpha_*} \Delta^n$ is the affine extension of the map $\{e_0, \dots, e_m\} \to \{e_0, \dots, e_n\}$:

$$e_i \longmapsto e_{\alpha(i)}$$

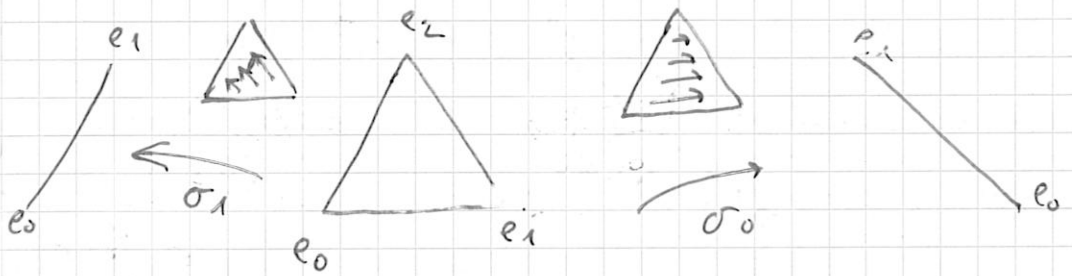
$$\alpha_* : \Delta^m \to \Delta^n, \quad \alpha_*\left(\sum_{i=1}^{m} t_i e_i\right) = \sum_{j=0}^{n} \left(\sum_{i \in \alpha^{-1}(j)} t_i\right) e_j$$

This is obviously a functor. It also explains, why $\delta^i : [n-1] \to [n]$ is called a (co)face: $\delta^i_* : \Delta^{n-1} \hookrightarrow \Delta^n$ is the face of Δ^n opposite to the vertex e_i.



etc.

Similarly, σ^i collapse Δ^{n+1} onto the face opposite to e_i



<u>4.17 Definition</u>: For $X \in Top$, we define $Sing\, X \in \hat{\Delta}$ by $Sing_\bullet X = Top(\Delta^\bullet, X)$.
This obviously defines a functor $Sing : Top \longrightarrow \hat{\Delta}$.

<u>4.18 Remark</u>: More explicitly,

$$Sing_n X = Top(\Delta^n, X)$$

An element $\tau : \Delta^n \to X$ of $Sing_n X$ is called a <u>singular simplex in X</u>.
Then $d_i(\tau) = \tau \circ \delta^i_* : \Delta^{n-1} \xrightarrow{\delta^i_*} \Delta^n \xrightarrow{\tau} X$
is the i-th face of τ, and

$$s_i(\tau) = \tau \circ \sigma^i_* \text{ is its ith degeneracy.}$$

The functor Sing. admits a <u>left adjoint</u>.

<u>4.19 Definition</u>: For $X \in \hat{\Delta}$, define its <u>geometric realization</u> $|X| \in Top$ by

$$|X| = \mathrm{Coequalizer}\left(\coprod_{\alpha \in Mor\, \Delta} X_{t(\alpha)} \times \Delta^{s(\alpha)} \underset{\coprod (\alpha^* \times id)}{\overset{\coprod (id \times \alpha_*)}{\rightrightarrows}} \coprod_{n \in \mathbb{N}} X_n \times \Delta^n \right)$$

Here, for $\alpha : [m] \to [n]$, X_n, X_m are equipped with the discrete topology, and $X_n \times \Delta^n \xleftarrow{id \times \alpha_*} X_n \times \Delta^m \xrightarrow{\alpha^* \times id} X_m \times \Delta^m$

<u>4.20 Example</u> We have a homeo $\varphi : |\Delta_n| \to \Delta^n$.
Indeed, recall that $\Delta_{n,m} = \mathbb{\Delta}([m],[n])$
It is obvious that the map $\coprod_m \mathbb{\Delta}([m],[n]) \times \Delta^m \to \Delta^n$, $(\alpha, x) \mapsto \alpha_*(x)$
equalizes the diagram in 4.19,
thus inducing a map $\varphi : |\Delta_n| \to \Delta^n$.
An inverse of φ is given by $\Delta^n = \{id_{[n]}\} \times \Delta^n \hookrightarrow \coprod_m \Delta_{n,m} \times \Delta^m$
$\to |\Delta_n|$.

<u>4.21 Proposition</u>: we have an adjoint pair $(|\ |, \mathrm{Sing})$

<u>proof</u>: by def we have

$$\mathrm{Top}(|\Delta_n|, Y) \cong \mathrm{Top}(\Delta^n, Y) = \mathrm{Sing}_n(Y) = \hat{\mathbb{\Delta}}(\Delta_n, \mathrm{Sing}(Y))$$

by the above calculation $|\Delta_n| \cong \Delta^n$ and Yoneda.

For the general case, use that $X \in \hat{\mathbb{\Delta}}$ is a colimit of Δ_n's.
Indeed, let $\mathbb{\Delta} \downarrow X$ be the category with
- objects: pairs (Δ^n, x) with $x : \mathbb{\Delta}^n \to X$ in $\hat{\mathbb{\Delta}}$
- morphism $(\Delta^n, x) \xrightarrow{\alpha} (\Delta^m, y)$
  is a morphism $\alpha : \Delta^n \to \Delta^m$ such that $x = y\alpha$

$$\begin{tikzcd} \Delta^n \arrow[dr, "x"] \arrow[dd, "\alpha"'] & \\ & X \\ \Delta^m \arrow[ur, "y"'] & \end{tikzcd}$$

We have a functor
$\varphi_X : \mathbb{\Delta} \downarrow X \to \hat{\mathbb{\Delta}}$
$(\Delta^n, x) \mapsto \Delta^n$
$\alpha \mapsto \alpha$

We have a natural transfo
$\eta : \varphi_X \to \Delta X$ of functors $\mathbb{\Delta} \downarrow X \to \hat{\mathbb{\Delta}}$ given by
$\eta_{(\Delta^n, x)} = x : \Delta^n \to X$.

It is tautological that the induced maps

$$\operatorname*{Colim}_{\Delta \downarrow X} \varphi_X \xrightarrow{\eta} \operatorname*{Colim}_{\Delta \downarrow X} \Delta X = X$$

is an isomorphism!

Similarly, composing φ_X and ΔX with the realization

$$\hat{\Delta} \xrightarrow{|\,|} Top,$$

we get functors $|\varphi_X| : \Delta \downarrow X \to Top$

$$\Delta |X| = |\Delta X| : \Delta \downarrow X \to Top$$

and essentially by definition of $|X|$ we have a homeo

$$\operatorname*{Colim}_{\Delta \downarrow X} |\varphi_X| \xrightarrow{\cong} \operatorname*{Colim}_{\Delta \downarrow X} \Delta|X| = |X|.$$

To conclude, for any $Y \in Top$, we have natural iso's

$$Top(|X|, Y) = Top(\operatorname*{Colim}_{\Delta \downarrow X} |\varphi_X|, Y) = \lim_{\Delta \downarrow X} Top(|\Delta_n|, Y) \cong$$

$$\lim_{\Delta \downarrow X} \hat{\Delta}(\Delta^n, Sing(Y)) \cong \hat{\Delta}(\operatorname*{Colim}_{\Delta \downarrow X} \varphi_X, Sing(Y))$$

$$= \hat{\Delta}(X, Sing(Y)) \quad \#$$

<u>4.22 Definition</u>: Let R be a commutative, unital ring.

A <u>chain complex</u> of R-modules is a sequence

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

of R-module maps indexed by $n \in \mathbb{Z}$, such that

$d^2 = d_n \circ d_{n+1} = 0 \quad \forall\, n \in \mathbb{Z}$.

If $\mathbb{Z}$ is viewed as a category (ordered set), then C is a functor

$C : \mathbb{Z}^{op} \to R\text{-mod}, \quad (n < n+1) \mapsto (C_{n+1} \xrightarrow{d} C_n)$

with the condition $d^2 = 0$.

A chain map $C \xrightarrow{f} D$ (or morphism of chain complexes)

is then just a natural transformation: f is given by $(f_n : C_n \to D_n)_n$

with $f_{n-1} \circ d^C = d^D \circ f_n$:

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "d^C"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_n \arrow[r, "d^D"] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Chain complexes of R-modules and chain maps obviously form a category, that we will denote $Ch(R)$.
We denote $Ch(R)_{\geq 0}$ the full subcategory on chain complexes C with $C_n = 0$ for all $n < 0$.

4.23 Definition: If $\mathbb{Z}$ is viewed as a discrete category, a graded R-module is a functor $\mathbb{Z} \xrightarrow{M_*} R\text{-mod}$, $n \mapsto M_n$
and a morphism of graded R-modules is a natural transf. $M_* \xrightarrow{f_*} N_*$. $\rightsquigarrow$ Category $Gr(R\text{-Mod})$

4.24 Definition: If $C \in Ch(R)$, we define its homology $H_*(C) \in Gr(R\text{-Mod})$ as follows:
For $n \in \mathbb{Z}$, Let $Z_n = Z_n(C) = \ker(C_n \xrightarrow{d} C_{n-1})$
$B_n = B_n(C) = \mathrm{Im}(C_{n+1} \xrightarrow{d} C_n)$
By $d^2 = 0$ we have $B_n \subset Z_n$, and we define
$$H_n(C) = Z_n / B_n,$$ the nth homology group of C.
A $x \in Z_n$ is called an $(n\text{-})$ cycle, $x \in B_n$ an $(n\text{-})$ boundary.
A chain map $f: C \to D$ restricts to $Z_n(C) \to Z_n(D)$ and $B_n(C) \to B_n(D)$ by definition ($f \circ d = d \circ f$), and we get an induced map
$$f_*: H_*(C) \to H_*(D).$$
This obviously defines a functor $Ch(R) \xrightarrow{H_*} Gr(R\text{-mod})$, and similarly $Ch(R)_{\geq 0} \xrightarrow{H_*} Gr_{\geq 0}(R\text{-Mod})$.