3. The fundamental group

3.1 Def. : a <u>category</u> C consists of :

(1) a <u>class</u> of objects, denoted $Ob(C)$ (or just C).

(2) for any pair $X, Y \in Ob(C)$, a <u>set</u> $C(X,Y)$, called the <u>set of morphisms</u> from X to Y ; $f \in C(X,Y)$ is usually denoted $X \xrightarrow{f} Y$.

(3) for any tripple $X, Y, Z \in Ob(C)$, a "<u>composition law</u>"

$$C(Y,Z) \times C(X,Y) \xrightarrow{\circ} C(X,Z), (g,f) \mapsto g \circ f \text{ (or } gf)$$

such that :

(a) The composition is associative

(b) For any $X \in Ob(C)$, there exists an element $1_X \in C(X,X)$ neutral for the composition (left or right), called the <u>identity</u> of X.

3.2 Remark : If $Ob(C)$ is a set, we say that C is a <u>small</u> category.

3.3 Examples : (a) <u>Set</u>, the category with :

- $Ob(Set)$: the class of sets.
- $Set(X,Y) =: Map(X,Y)$: the set of maps $X \to Y$
- $1_X : X \to X,\ x \mapsto x$; usual composition.

(b) Top, the category of topological spaces :

$Ob(Top)$: the class of topological spaces.

$Top(X,Y) = \{ f \in Set(X,Y) ;\ f \text{ continuous} \}$

(c) Gp, the category of groups and group homomorphisms.

(d) Ring, the category of unital, associative rings and ring homomorphisms.

(e) R a ring ; R-Mod (Mod-R), the category of left (right) R-modules.

Notice that none of the examples above are small.

The following are examples of small categories:

(f) If G is a group, define a category $\underline{G}$ with a unique object $*$: $Ob(\underline{G}) = \{*\}$ and morphisms $\underline{G}(*,*) := G$ (underlying set of G), together with:

- $1_* = e$ (neutral element of G)
- Composition $\underline{G}(*,*) \times \underline{G}(*,*) \to \underline{G}(*,*)$ is the product of G: $g \circ f = gf$.

(g) $\mathcal{F}in$, the category with objects

$Ob(\mathcal{F}in) = \{ X \in \mathcal{P}(\mathbb{N}) \, ; \, \exists n, X = [n] \}$

where $\underline{n} = \{ a \in \mathbb{N} \, ; \, 1 \leq a \leq n \}$

$\mathcal{F}in(\underline{m}, \underline{n}) = Set(\underline{m}, \underline{n})$, and usual composition.

(h) We can also take smaller sets of morphisms:

$\mathcal{I}$: $ob(\mathcal{I}) = Ob(\mathcal{F}in)$

$\mathcal{I}(\underline{m}, \underline{n}) = \{ \alpha \in \mathcal{F}in(\underline{m}, \underline{n}) \, ; \, \alpha \text{ injective} \}$

Σ : $Ob(\Sigma) = Ob(\mathcal{F}in)$

$\Sigma(\underline{m}, \underline{n}) = \{ \alpha \in \mathcal{F}in(\underline{m}, \underline{n}) \, ; \, \alpha \text{ bijective} \}$

<u>3.4 Definition</u>: Let C be a category. A morphism $f \in C(X,Y)$ is called an <u>isomorphism</u> if $\exists g \in C(Y,X)$, $f \circ g = id_Y$ & $g \circ f = id_X$ (then unique! Noted f^{-1})

A <u>groupoïd</u> is a small category C where all morphisms are isomorphisms.

<u>3.5 Examples</u>: In 3.3, $\underline{G}$ and Σ are groupoïds.

<u>3.6 Definition</u>: Let C be a category. We define the

opposite category C^{op} by: $Ob(C^{op}) = Ob(C)$, and for any $X, Y \in Ob(C^{op})$,

$$C^{op}(X,Y) := C(Y,X), \text{ together with}$$

$$C^{op}(Y,Z) \times C^{op}(X,Y) \longrightarrow C(X,Z)$$

$$(f^{op}, g^{op}) \longmapsto f^{op} \circ g^{op} := (g \underset{\text{in } C.}{\circ} f)^{op}$$

"Revert all arrows".

3.7 Definition Let C, D be categories. A functor $F: C \to D$ is

(a) A relation assigning to each $X \in Ob(C)$ a unique $F(X) \in Ob(D)$.

(b) For all $X, Y \in Ob(C)$, $F = F_{X,Y}: C(X,Y) \to D(F(X), F(Y))$ such that $F(1_X) = 1_{F(X)}$, for all $X \in Ob(C)$, and $F(g) \circ F(f) = F(g \circ f)$ for all composable pair (g, f) of morphisms in C.

A cofunctor or a contravariant functor $C \to D$. is a functor $C^{op} \to D$ (Spell it out!).

3.8 Examples: (a) If C, D are categories, and $Z \in Ob(D)$, define the constant functor $c_Z: C \to D$ by $c_Z(X) = Z$ for all $X \in Ob(C)$, and $c_Z(f) = 1_Z$ for all morphisms of C.

(b) We have the forgetfull functors

$F: Top \to Sets$, $F: Gp \to Sets$, $F: R\text{-Mod} \to Sets$:

$F(X)$ = underlying set of X, $F(f)$ = underlying map of sets.

(c) If C is a category and $X \in Ob(C)$, we have a cofunctor $F^X: C \to Sets$, given by

$F^X(Y) = C(Y,X)$ for all $Y \in Ob(C)$, and for $f: Y \to Z$ in C, $F^X(f): C(Z,X) \to C(Y,Z)$, $g \mapsto g \circ f$.

It is called the <u>co-functor represented</u> by X.

Similarly, we have a functor $F_X : C \to \text{Sets}$ called the functor co-represented by X : $F_X(Y) = C(X,Y)$, and $F_X(f) : C(X,Y) \to C(X,Z)$, $\alpha \mapsto f \circ \alpha$. for any $f : Y \to Z$ in C.

(d) If C is a category, we have the identity functor
$$Id_C : C \to C, \quad Id_C(X) = X \text{ and } Id_C(f) = f.$$

<u>3.9 Definition</u> : let C be a category. A <u>sub-category</u> of C is a category D with

(i) $Ob(D) \subset Ob(C)$

(ii) $\forall X, Y \in Ob(D)$, $D(X,Y) \subset C(X,Y)$

(iii) The inclusion $D \to C$ is a functor.

We say that D is a <u>full</u> subcategory of C if for any $X, Y \in Ob(D)$, $D(X,Y) = C(X,Y)$.

<u>3.10 Examples</u> : (a) $\mathcal{F}in$ is a full subcategory of Set. Σ is a (non full) subcategory of $\mathcal{I}$, which is a (non-full) subcategory of $\mathcal{F}in$.

(b) $\mathcal{G}p$ is not a subcategory of Set !
Top is not a subcategory of Set !

(c) $\mathcal{A}b$, the full subcategory of $\mathcal{G}p$, consisting of <u>abelian</u> groups.

(d) If D is a subcategory of C, we have a functor $i : D \to C$, $i(X) = X$, $i(f) = f$, by definition of a subcategory.

3.11 Definition: If $C \xrightarrow{F} D$ and $D \xrightarrow{G} \mathcal{E}$ are functors, we can define the composition functor $G \circ F : C \to \mathcal{E}$ by $(G \circ F)(X) = G(F(X))$ for any $X \in Ob(C)$, and $(G \circ F)(f) = G(F(f))$ for any $f : X \to Y$ in C.

3.12 Remark: the composition of functors is associative, and identity functors are neutral for the composition.

3.13 Exercise: we have defined the notion of a product, coproduct, pull-back, push-out of spaces by solely using the notion of a category. Can do the same for any category (but they do not nec. exist!).

⊳ We will introduce more categorical notions when needed! We now introduce the homotopy category of spaces

As above (and as in chapter 1), we denote by Top the category of topological spaces and continuous maps.

3.14 Def For $f, g \in Top(X, Y)$, a homotopy from f to g is a map $H : X \times I \to Y$, $I = [0,1] \subset \mathbb{R}$, such that

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes; here $i_t : X \to X \times I$, $x \mapsto (x, t)$

We denote this by $f \overset{H}{\simeq} g$. We say that f and g are homotopic (denoted $f \simeq g$) if $\exists H$, $f \overset{H}{\simeq} g$.
We denote $H_t := H \circ i_t : X \to Y$.

3.15 Lemma: The relation $\simeq$ on $Top(X, Y)$ is an equivalence rel. This relation is compatible with composition in the sense that

$$\begin{array}{ccc} Top(Y, Z) \times Top(X, Y) & \xrightarrow{\circ} & Top(X, Z) \\ \Downarrow \qquad\qquad \Downarrow & & \\ g, g' \qquad\qquad f, f' & & \end{array}$$

$f \simeq f'$ and $g \simeq g'$ $\Rightarrow$ $g \circ f \simeq g' \circ f'$.

proof: let $f, g, h \in Top(X,Y)$.

By $X\times I \xrightarrow{H} Y$, $H(x,t) = f(x)$ $\forall (x,t)$, we see $f \overset{H}{\simeq} f$.

If $f \overset{K}{\simeq} g$, then $g \overset{L}{\simeq} f$ by $L: X\times I \to Y$, $L(x,t) = K(x, 1-t)$

If $f \overset{K}{\simeq} g \overset{L}{\simeq} h$, then $f \overset{K*L}{\simeq} g$ where $K*L : X\times I \to Y$ is given by

$$K*L(x,t) = \begin{cases} K(x, 2t) & \text{if } 0\le t\le \frac{1}{2} \\ L(x, 2t-1) & \text{if } \frac{1}{2}\le t\le 1. \end{cases}$$

Thus $\simeq$ is an equivalence relation on $Top(X,Y)$.

For the compatibility with composition, given $f \overset{F}{\simeq} f'$ and $g \overset{G}{\simeq} g'$, we use transitivity $g\circ f \overset{K}{\simeq} g'\circ f \overset{L}{\simeq} g'\circ f'$ for

$K: X\times I \xrightarrow{f\times id} Y\times I \xrightarrow{G} Z$, $L: X\times I \xrightarrow{F} Y \xrightarrow{g'} Z$. ▨

<u>3.16 Def</u> We define $ho(Top)$, the <u>homotopy category of spaces</u> by $Ob(ho(Top)) = Ob(Top)$, $hoTop(X,Y] := [X,Y] := Top(X,Y)/_{\simeq}$.

The class of $f \in Top(X,Y)$ in $[X,Y]$ is denoted $[f]$, and composition is defined by $[f]\circ[g] := [f\circ g]$ (well defined by 3.15). The identity of $X \in hoTop$ is $1_X = [id_X]$

<u>Remark</u>: We have an obvious functor $Top \to hoTop$.

3.<u>17 Def</u>: A map $f: X\to Y$ is called a homotopy equivalence if $[f] \in hoTop(X,Y)$ is an isomorphism.

We say that spaces X, Y are <u>homotopy equivalent</u>, (denoted $X\simeq Y$), if they are isomorphic in $hoTop$. We say that X is <u>contractible</u> if $X \simeq \{*\}$ (one-point space).

<u>3.18 Examples</u> (a) Given X a space, any $f, g: X \to \mathbb{R}^n$ $(n\in\mathbb{N})$ are homotopic: $f \overset{H}{\simeq} g$ with $H: X\times I \to \mathbb{R}^n$, $H(x,t) = (1-t)f(x) + t g(x)$.

Thus $[X, \mathbb{R}^n]$ has a unique element!

In particular, $\{0\} \hookrightarrow \mathbb{R}^n$ is a homotopy equivalence, thus $\mathbb{R}^n$ is contractible.

(b) Suppose $f : X \to S^n$ is non surjective. Then f is homotopic to a constant map.

Proof: Assume $\mathrm{Im}(f) \subset S^n \setminus \{N\}$.

North, south pole

We have a homeomorphism $\mathbb{R}^n \xrightarrow{h} S^n \setminus \{N\}$, $0 \mapsto S$

Therefore f factorises as $X \xrightarrow{g} \mathbb{R}^n \xrightarrow{h} S^n \setminus \{N\}$.

If $c_0 : X \to \mathbb{R}^n$ is the constant map to $0 \in \mathbb{R}^n$, we know by (a) that $\exists\, H : X \times I \to \mathbb{R}^n$, $g \overset{H}{\simeq} c_0$.

Thus by Lemma 3.15, $i h H : X \times I \to S^n \setminus \{N\} \overset{i}{\hookrightarrow} S^n$ is a homotopy between f and c_S.

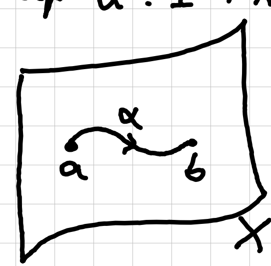
(c) The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence (exercise).

(d) $[S^n, S^m]$ is very hard to study!

<u>3.19 Definition</u>: Let X be a space, and $a, b \in X$. A <u>path</u> from a to b in X is a continuous map $\alpha : I \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$.

Denote by $\Omega(X, a, b)$ the set of all paths from a to b in X.

$[0,1] \xrightarrow{\alpha}$



We say that $\alpha, \beta \in \Omega(X, a, b)$ are <u>homotopic relative to $\partial I = \{0, 1\}$</u> if $\exists\, H : I \times I \to X$, s.t.

(i) $H_0(s) := H(s, 0) = \alpha(s)$ and $H_1(s) = H(s, 1) = \beta(s)$ for all $s \in I$;

(ii) $H_t(0) = H(0, t) = a$ and $H_t(1) = H(1, t) = b$ for all $t \in I$.

We denote it by $\alpha \sim \beta$ or $\alpha \overset{H}{\sim} \beta$.

<u>3.20 Lemma</u> $\sim$ is an equivalence relation on $\Omega(X, a, b)$.