trivially imply some other properties. Every continuous real-valued function on a hyperconnected space is constant, so such spaces are necessarily pseudocompact. On the other hand, no nontrivial ultraconnected space can have more than one closed point, so none are T_1, even though they must all be T_4, trivially.

Quasicomponents and components are equal if (but not only if; see Example 26) a space has a basis consisting of connected sets; we call such a space **locally connected**. Equivalently, X is locally connected if the components of open subsets of X are open in X. Local connectedness clearly does not imply connectedness, but neither does connectedness imply local connectedness (Example 116). However, every hyperconnected space is clearly locally connected, since in such spaces every open set is connected. Figure 8 summarizes the relevant counterexamples.

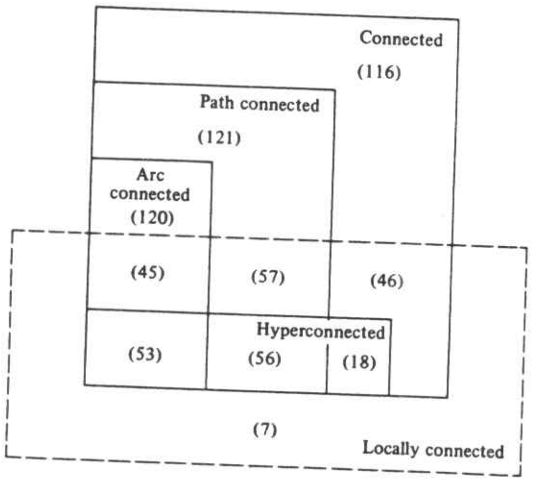


Figure 8.

Path components are equal to quasicomponents if a space has a basis consisting of path connected sets; such a space is called **locally path connected**. Equivalently, X is locally path connected if the path components of open subsets of X are open in X. Analogously, arc components are equal to quasicomponents if a space has a basis of arc connected sets; such a space is said to be **locally arc connected**. As above, locally arc connected implies locally path connected, which implies locally connected, but neither converse holds (Examples 4 and 18). Furthermore, locally path connected is independent of path connected and locally arc connected is independent of arc connected (Examples 118 and 32.5).

Functions and Products

Any set S which is the union of connected sets A_α and a connected set B where $B \cap A_\alpha \neq \emptyset$ for each α must be connected since a separation of S would necessarily separate B. Since any finite product $\prod_{i=1}^{n} X_i$ of connected sets X_i can be written as the union of spaces homeomorphic to $\prod_{i=1}^{n-1} X_i$ and X_n, a simple induction argument shows that any finite product of connected spaces is connected. In fact, a straightforward argument by transfinite induction can be used to show that any product $\prod_{\alpha\in A} X_\alpha$ of connected spaces X_α is connected. If the index set A is well ordered and if $x = \langle x_\alpha\rangle \in X = \prod X_\alpha$ is some fixed point, let $S_\alpha = \{\langle y_\beta\rangle \in X \mid y_\beta = x_\beta \text{ for all } \beta \geq \alpha\}$. Then S_α is connected whenever $S_{\alpha-1}$ is since S_α is homeomorphic to $S_{\alpha-1} \times X_\alpha$. If α is a limit ordinal, $S_\alpha = \overline{\bigcup_{\beta<\alpha} S_\beta}$, so if each S_β is connected for $\beta < \alpha$, S_α must be also, since the collection $\{S_\gamma\}$ is nested. Thus $X = \overline{\bigcup_{\alpha\in A} S_\alpha}$ is connected. Indeed we have proved more since the proof uses only the facts that in the product topology the subsets $X'_\alpha \subset \prod X_\alpha$ where $X'_\alpha = \{\langle y_\beta\rangle \in X \mid y_\beta = x_\beta,\ \beta \neq \alpha\}$ are homeomorphic to the X_α's and that $X = \overline{\bigcup_{\alpha\in A} S_\alpha}$. Thus this proof applies to the Cartesian product of the X_α with any topology in which the sets X'_α are copies of the corresponding X_α, and $X = \overline{\bigcup_{\alpha\in A} S_\alpha}$.

If X is connected and f is a continuous function on X, then $f(X)$ must be connected, for if A and B separate $f(X)$, $f^{-1}(A)$ and $f^{-1}(B)$ separate X. Though the continuous image of a locally connected space need not be locally connected, it is true that local connectedness is preserved under continuous maps f from a compact space X onto a Hausdorff space Y. For suppose E is a component of an open subset U of Y. Then each component of $f^{-1}(E)$ is a component of $f^{-1}(U)$ since if G is a component of $f^{-1}(U)$, then $f(G)$ is connected and thus either contained in E or disjoint from it. But if X is locally connected, the components of the open set $f^{-1}(U)$ are open, so $f^{-1}(E)$ must be open. Its complement is closed, thus compact, so $f(X - f^{-1}(E)) = Y - E$ is compact, hence closed (since Y is Hausdorff). Thus E is open, and therefore Y must be locally connected.

Disconnectedness

A space is **totally pathwise disconnected** if the only continuous maps from the unit interval into X are constant, or, equivalently, if its path components are single points. A space with single point components is said

Seebach, Steen: Counterexamples in Topology.

Das ganze ist doch ein bisschen nicht-triviale (zumindest hab ich's immer noch nicht raus...
wer noch meint ne korrekte Lösung zu haben, bitte nochmal zu mir ins Büro kommen oder zumindest den Zettel mitbringen nächste Übung.)

$f: S^4 \longrightarrow W$

„Ich" zeig nen bisschen weniger: (als in der Übung behauptet)

Hab ich doch... siehe 2 Zettel.

ja / bla

ist $f(S^4) \cap A \neq \emptyset$

so gibt es eine Umgebung $U \subseteq W$ von A, so dass A, B, C disjunkt

$U \cap B \cap f(S^4) = \emptyset$.

<u>Bew</u> Nach dem beigelegten Lemma ist $f(S^4)$ als Raum lokal zusammenhängend. (das ist das, was ich in der Übung nicht hinbekommen habe!)

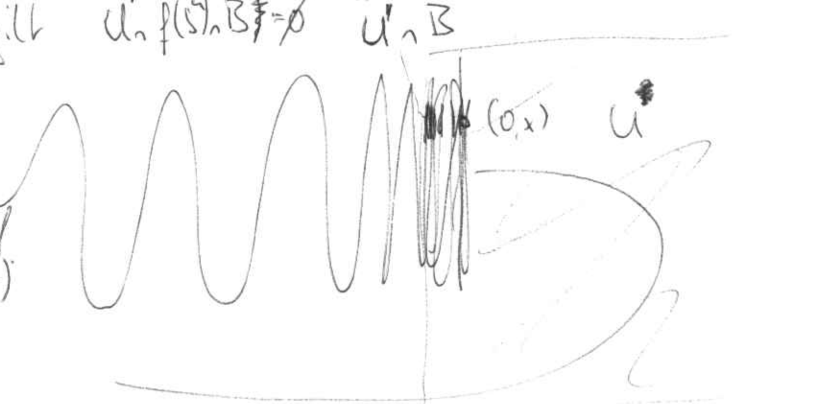
Nehmen wir uns also einen Punkt $(0,x)$ aus $A \cap f(S^4)$ her. Dann hat er in $f(S^4)$ eine zusammenhängende Umgebung V die in $U_{\frac{1}{2}}(0,x)$ enthalten ist. Mit dem „üblichen" Argument folgt $V \cap B = \emptyset$.

Ist jetzt $U' \subseteq W$ eine Umgebung von $(0,x)$ mit $U' \cap f(S^4) = V$, so gilt $U' \cap f(S^4) \cap B = \emptyset$

also ist $f(S^4) \cap B$ komplett „links" von U', weil weg zusammenhängend

Man kann also $U = \{(x,y) \in \mathbb{R}^2 \mid \exists (z,z) \in U' \; z < x\}$ setzen (oder leer)

Das ist ne Umgebung von A und $U \cap B = U'$, also folgt die Behauptung. $\square$



Der Rest geht für f mit $f(S^n) \cap A \neq \emptyset$ wie in der Übung besprochen:

$f(S^n)$ ist nach dem obigen in einem ~~zu einem Intervall homöomorphen~~ ~~homotop~~ zusammenziehbaren Teilraum von W enthalten. nämlich $W \setminus (U \cap B)$

$\Rightarrow$ Also ist f nullhomotop.

Mir fällt grad auf, dass aus dem oben doch das folgt, was ich in der Übung gesagt habe.

Wenn $f(S^n) \cap A = \emptyset$ ist ja alles trivial, weil dann ja direkt $f(S^n)$ von A positiven Abstand hat

($f(S^n)$ kompakt, A abgeschlossen)

wir also wieder so ein U finden!

Also geht das immer und wir

sind fertig!

(1) $f : S^n \to WK$ ist nicht surjektiv: ansonsten wäre $WK = f(S^n)$ lokalzusammenhängend.

(2) $B \not\subset f(S^n)$: ansonsten, da $f(S^n)$ abgeschlossen (kompakt in Hausdorff) gilt $\overline{B} = B \cup A \subset f(S^n)$, und da $f(S^n)$ wegzusammenhängend ist folgt C ist auch $f(S^n)$, also f surjektiv ↯.

(3) Sei $b \in B \setminus f(S^n)$, dann hat $WK \setminus \{b\}$ zwei Wegzusammenhangskomponenten, sagen wir B' und $A \cup C'$. Dann gilt $f(S^n) \subset B' \simeq *$ oder $f(S^n) \subset A \cup C' \simeq *$.