I Constructions of topological spaces

1.1. Notation: If X, Y are (topological) spaces, we will denote by $\mathrm{Set}(X,Y)$ the <u>set</u> of all maps of (underlying) sets, and $\mathrm{Top}(X,Y) = \{ f \in \mathrm{Set}(X,Y) ;\ f \text{ continuous} \}$.

1.2 Examples: (a) For X a space, $\mathrm{Top}(X,X)$ contains at least 1 element: $id_X : X \to X$.

(2) For any space Z, $\mathrm{Top}(\emptyset, Z)$ and $\mathrm{Top}(Z, \{*\})$, where $\{*\}$ is a 1-point space, contain exactly 1 element

1.3 Remark: In example (2), $\emptyset$ and $\{*\}$ are essentially caraterized by the given condition (up to homeo for $\{*\}$).

Goal of this chapter: define constructions of a new topological space T, out of previously constructed spaces (or maps) such that certain properties of $\mathrm{Top}(Z,T)$ or $\mathrm{Top}(T,Z)$ (as <u>sets</u>), when Z is any space, caracterize T (up to homeo).

$\boxed{\text{The Subspace topology}}$

1.4 Proposition Let X be a space, and $A \subset X$ be a <u>subset</u> (of the underlying set of X). Denote $i : A \to X$ the inclusion. Then there <u>exists</u> on A a <u>unique</u> topology $\mathcal{T}_A$ such that i is continuous, and for any space Z, the following map is bijective:

$$\phi : \mathrm{Top}(Z, \underset{\text{with } \mathcal{T}_A}{A}) \longrightarrow \{ f \in \mathrm{Set}(Z,A) ;\ i \circ f \in \mathrm{Top}(Z,X) \}$$
$$f \mapsto f$$

Picture:

$$\begin{tikzcd} & X \\ Z \arrow[ur, "i \circ f"] \arrow[r, "f"'] & A \arrow[u, "i"'] \end{tikzcd}$$

proof: let τ_X be the topology of X. Existence:
let $\tau_A = \{ V \in \mathcal{P}(A) \,;\, \exists\, U \in \tau_X,\ V = U \cap A \}$.
Then it is easy to check that τ_A is a topology for which $i : A \to X$ is continuous. Now let Z be any space, and $f : Z \to A$ a map of sets

(1) f continuous $\Rightarrow$ $i \circ f$ continuous (known: composition of continuous maps is continuous).

(2) $i \circ f$ continuous $\Rightarrow$ f continuous: let $V \subset A$ open. Then $\exists\, U \subset X$ open with $U \cap A = V$. We have $f^{-1}(V) = (i \circ f)^{-1}(U)$ open in Z since $i \circ f$ is continuous. Thus f is continuous.

Unicity: let τ'_A be another topology on A satisfying

that ϕ is bijective.

$$\begin{tikzcd} & X \\ (A,\tau_A) \arrow[ur, "i"] \arrow[r, "id"'] & (A,\tau'_A) \arrow[u, "i'"'] \end{tikzcd}$$

Since i is continuous, so is $id_A\ (A,\tau_A) \to (A,\tau'_A)$.
Since i' is continuous, so is $id_A : (A,\tau'_A) \to (A,\tau_A)$.

Thus $id_A : (A,\tau_A) \to (A,\tau_A')$ is a homeomorphism, which means $\tau_A = \tau_A'$. ∎

1.5 Definition: If X is a space, $A \subset X$, then the top. τ_A given by Prop 1.4 is called the *subspace topology* on A (relative to the inclusion $i : A \hookrightarrow X$).
We also say: $A \overset{i}{\hookrightarrow} X$ is a *subspace* of X.

1.6 Examples If X is a space, $A \subset X$ a subspace, an open subset $V \subset A$ is also a subset of X, but need not be open in X ! ⚠

For example: $A = \{0\} \subset \mathbb{R}$ (usual topology).
Then of course A is open in A (eg $A = A \cap \underbrace{\mathbb{R}}_{\text{open in } \mathbb{R}}$)
but not in $\mathbb{R}$.

If $A \subset X$ open, the notions agree !

1.7 Lemma. Let X be a space, $A \subset X$ a subspace, and $B \subset A$. Then subspace topologies of B relative to A and relative to X agree ! (Exercise).

1.8 Definition Let $n \in \mathbb{N}$ and $\mathbb{R}^{n+1}$ endowed with the Euclidean metric. We call the subspace

$$S^n = \{ x \in \mathbb{R}^{n+1} ; \ \|x\|_2 = 1 \} \subset \mathbb{R}^{n+1}.$$

the n-th dimensional sphere.

1.9 Remark If X is a space, B a set, and $i : B \to X$ is an injective map, by the same procedure as in Prop 1.4 produces a topology on B, such that

$$\phi : \operatorname{Top}(Z, B) \xrightarrow{\text{bij.}} \{ f \in \operatorname{Set}(Z, B) ; \ f \circ i \in \operatorname{Top}(Z, X) \}.$$

It is the coarsest topology making i continuous.

We can also characterize it as follows:

$$\begin{tikzcd} Z \arrow[rr, "\forall f,"', "\operatorname{Im}(f) \subset A"] \arrow[dr, dashed, "\exists! g"'] & & X \\ & A \arrow[ur, "i"'] & \end{tikzcd}$$

For any $f \in \operatorname{Top}(Z, X)$ such that $\operatorname{Im}(f) \subset A$, $\exists!\ g \in \operatorname{Top}(Z, A)$ with $i \circ g = f$.

The quotient topology

1.10 Recollections : An equivalence relation on a set X is a relation $R \subset X \times X$ (with $x \sim y \Leftrightarrow (x, y) \in R$)

that is reflexive, symmetric and transitive.
We denote by X/R (or $X/\sim$ if R is understood) the <u>set</u> of equivalence classes: For $x \in X$, let
$[x] = \{ y \in X \,;\, x \sim y \}$; let
$X/R = \{ A \in \mathcal{P}(X) \,;\, \exists x \in X,\ A = [x] \}$.
The map of sets $\pi : X \to X/R$, $x \mapsto [x]$ is obviously surjective, and is called the (canonical) quotient map (of sets)

<u>1.11 Proposition</u>. Let X be a space, R an equivalence relation on X, and $\pi : X \to X/R$ the quotient map. Then there <u>exist</u> on X/R a <u>unique</u> topology such that the map $\pi : X \to X/R$ is continuous, and for any space Z, the following map of sets is bijective:

$$\Psi : Top(X/R, Z) \to \{ f \in Set(X/R, Z) \,;\, f \circ \pi \in Top(X, Z) \}$$
$$f \mapsto f$$

Picture:

$$\begin{tikzcd} X \arrow[r, "f \circ \pi"] \arrow[d, "\pi"'] & Z \\ X/R \arrow[ur, "f"'] & \end{tikzcd}$$

(See also Lemma 1.15 and Remark 1.16)

<u>proof</u> Let τ_X be the topology of X. We define
$\tau_{X/R} = \{ U \in \mathcal{P}(X/R) \,;\, \pi^{-1}(U) \text{ open in } X \}$.
It is easy to check that $\tau_{X/R}$ is a topology on X/R, for which π is continuous.
The rest of the proof is left as an exercise: follow the same steps as the proof of Prop. 1.4. ∎

<u>1.12 Definition</u>: Let X be a space and R an equivalence relation. The topology provided by 1.11 on X/R

is called the quotient topology on X/R (relative to X), and the continuous map $\pi : X \to X/R$ is called the quotient (map).

1.13 Example Consider the topological vector space $\mathbb{R}^n$, and V a sub-vector space. Recall that the quotient vector space $\mathbb{R}^n/V$ is the quotient by the equivalence relation
$x \sim y \iff x - y \in V$.

We can endow $\mathbb{R}^n/V$ with the quotient topology.
Let $U \subset \mathbb{R}^n$ be a supplementary subspace of V (so that $U \oplus V = \mathbb{R}^n$).
We can endow U with the subspace topology relative U.
Then the composition $U \xrightarrow{i} \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n/V$ is a homéomorphism! We know that $\pi \circ i$ is continuous and bijective. It remains to show that it is open.
We have the projection map $p : \mathbb{R}^n \to U$ along V, which is continuous (linear!).
Now, if $O \subset U$ is open, we have

$$\pi^{-1}(\pi \circ i(O)) = \{x \in \mathbb{R}^n ; \exists\, a \in O, x \sim a\}$$
$$= p^{-1}(O) \text{ open. which implies } \pi \circ i(O) \text{ open.}$$

$\square$

1.14 Definition Let X be a space and A a subspace.
Consider the equivalence relation R_A on X given by

$$x \sim y \iff \begin{cases} x = y, \text{ or} \\ x \in A \text{ and } y \in A. \end{cases}$$

(Thus $R_A = A \times A \cup \mathrm{Diag} \subset X \times X$).
We denote the quotient $\pi : X \to X/R_A =: X/A$

and call it the <u>quotient of X by the subspace A</u>.

We can reformulate slightly the (universal) property of the quotient topology:

<u>1.15 Lemma</u>: Let X be a space, R an equivalence relation on X, and $A \subset X$ a subspace, $i: A \hookrightarrow X$ inclusion.

(a) For any space Z, and any continuous map $f: X \to Z$ such that for any $x, y \in X$, $xRy \Rightarrow f(x) = f(y)$, there exists a unique map $\bar{f}: X/R \to Z$ with $f = \bar{f} \circ \pi$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & X/R \arrow[ur, dashed, "\bar{f}", "\exists!"'] & \end{tikzcd}$$

(b) For any space Z and any continuous map $f: X \to Z$ such that $f \circ i$ is constant, there exists a unique map

$\bar{f}: X/A \to Z$ such that $f = \bar{f} \circ \pi$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & X/A \arrow[ur, dashed, "\exists!", "\bar{f}"'] & \end{tikzcd}$$

proof: obvious.

<u>1.16 Remark</u> Let X be a space, and Y a set, and $\pi: X \to Y$ a surjective map. Then, using ψ as in 1.11 (replace $\pi: X \to X/R$ by $\pi X \longrightarrow Y$), endow Y with a topology, which we can call the "quotient topology relative π".

<u>Exercise</u>: In the remark above, we can define on X the relation $x \overset{R}{\sim} y \iff \pi(x) = \pi(y)$.

$$\begin{tikzcd} & X \arrow[dl, "\pi_R"'] \arrow[dr, "\pi"] & \\ X/R \arrow[rr, dashed, "\exists!\, \bar{\pi}"'] & & Y \end{tikzcd}$$

$\exists!\ \bar{\pi}: X/R \to Y$ with $\bar{\pi} \circ \pi_R = \pi$, and $\bar{\pi}$ is a homeomorphism!

1.17 Example

Let $X = [0,1]$ (subspace topology of $[0,1] \subset \mathbb{R}$)

Let $A = \{0,1\}$.

Consider the map $X \xrightarrow{f} \mathbb{R}^2$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$

Then $\mathrm{Im}(f) \subset S^1 \subset \mathbb{R}^2$; so by prop 1.4,

$\exists!\ g : X \to S^1$.

Then $g(A) = (1,0) \in S^1$, so by Lemma 1.15,

$\exists!\ \bar{g} : X/A \to S^1$ with $\bar{g} \circ \pi = g$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "\pi"'] & S^1 \\ & X/A \arrow[u, dashed, "\exists! \bar{g}"'] \end{tikzcd}$$

Claim: $\bar{g}$ is a homéomorphism.

Obviously (?) $\bar{g}$ is continuous and bijective. Compact to Hausdorff (but not yet seen).

The product topology

Recall that the cartesian product of sets $X \times Y$, together with the "canonical" projections $X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$.

A map (of sets) $Z \xrightarrow{f} X \times Y$ corresponds precisely to a pair of maps (of sets) $Z \xrightarrow{f_X} X$, $Z \xrightarrow{f_Y} Y$, such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, bend right, "f_X"'] \arrow[d, "f"] \arrow[dr, bend left, "f_Y"] & \\ X & X \times Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

commutes.

In other words, $f(z) = (f_X(z), f_Y(z))\ \forall z \in Z$.

The same holds for the product of topological spaces and continuous maps.

1.18 Proposition Let X and Y be topological space. Then there exists on $X \times Y$ a unique topology

$\tau_{X\times Y}$, such that p_X and p_Y are continuous, and such that for any space Z, there is a bijection of sets

$$P : Top(Z, X\times Y) \longrightarrow Top(Z,X) \times Top(Z,Y)$$
$$f \longmapsto (p_X \circ f, \; p_Y \circ f)$$

proof: Existence: We define the topology $\tau_{X\times Y}$ by saying that it admits $\mathcal{B} = \{ U\times V \subseteq \mathcal{P}(X\times Y) ; U \in \tau_X, V \in \tau_Y \}$ as a basis(*), where τ_X, τ_Y are the topologies of X and Y, respectively. Obviously this makes p_X and p_Y continuous, so that P is well defined.
Since P is the restriction of a bijection, it is obviously injective. It is also surjective: For $f_X \in Top(Z,X)$ and $f_Y \in Top(Z,Y)$, let $f : Z \to X\times Y$
$\qquad z \mapsto (f_X(z), f_Y(z))$.

((*) check 2 conditions)

Assume $U\times V$ is a basis element of $\tau_{X\times Y}$.
Then $f^{-1}(U\times V) = f_X^{-1}(U) \cap f_Y^{-1}(V)$ open in Z.

■

1.19 Example The Euclidean Topology on $\mathbb{R}^2$ is equal to the product topology.
Indeed, we know that all norms are equivalent, obviously the basis elements (open balls) for $\| \ \|_\infty$ are products of basis elements for $\mathbb{R}$ (open intervals)
Conclude with the following exercise:
If B_X, B_Y are basis for topologies for X and Y, then $\mathcal{B}' = \{ (U\times V) \in \mathcal{P}(X\times Y) ; U \in B_X, V \in B_Y \}$ is also a basis for the product topology.

We now generalize to arbitrary products.

1.20 Proposition: let I be a set and $\{X_i\}_{i\in I}$ a family of topological spaces indexed by I. Then there exists on the product $X = \prod_{i\in I} X_i$ a unique topology such that:

(i) $\forall i \in I$, the projection $p_i : \prod_{i\in I} X_i \to X_i$ is continuous

(ii) The map

$$\mathrm{Top}(Z, \prod_{i\in I} X_i) \xrightarrow{\chi} \prod_{i\in I} \mathrm{Top}(Z, X_i)$$
$$f \mapsto \{p_i \circ f\}_{i\in I}$$

is a bijection for any space Z.

__proof__ (sketch): Take $S = \{A \in \mathcal{P}(\prod_{i\in I} X_i) ; \exists J \subset I$ finite, such that $A = \prod_{i\in I} A_i$, where $A_i \subset X_i$ and $i \notin J \Rightarrow A_i = X_i\}$. This is a subbasis of a topology on X, and if τ_X is the corresponding topology on X, then (i) holds! The proof that χ is a bijection is the same as in 1.18. We now prove the uniqueness.

Suppose τ' is another topology on X such that (i) and (ii) hold; let $X' = (X, \tau')$. Then $id : X' \to X$ is continuous since $\forall i \in I$, $p_i \circ id : X' \to X \to X_i$ is continuous (i'). Similarly, $id : X \to X'$ is continuous. So $\tau = \tau'$. ▨.

1.21 Remark: ⚠ The product topology is quite coarse; for example, an (infinite) product of discrete spaces need not be discrete. Famous example: $\{0,1\}$ discrete space: Then $\prod_{\mathbb{N}} \{0,1\}$ is not discrete. In fact, it is homeomorphic to the Cantor set (as a subsp. of $[0,1]$). See exercises.

We now see the "dual" notion: coproduct. Treated in the Exercises. We treat directly the coproduct indexed over an arbitrary set.

It is based on the coproduct of sets:
If I is a set and $\{A_i\}_{i\in I}$ a collection of sets indexed by I. The coproduct is a set A, denoted $\coprod_{i\in I} A_i$, together with maps $\{j_i : A_i \to A\}$, such that for any set B,

$$\mathrm{Set}\left(\coprod_{i\in I} A_i, B\right) \longrightarrow \prod_{i\in I} \mathrm{Set}(A_i, B)$$
$$f \longmapsto \{f\circ j_i\}_{i\in I}$$

is a bijection.
It exists and is unique up to canonical bijections compatible with the j_i's. A model for it

$$\coprod_{i\in I} A_i = \{(k,x) \in I \times (\bigcup_{i\in I} A_i) \,;\, x \in A_k\}$$

with $j_i : A_i \longrightarrow \coprod_{i\in I} A_i \subset I \times (\bigcup_I A_i)$
$$x \longmapsto (i, x).$$

Note that each of the j_i is injective, also, if $i \neq k$, then $j_i(A_i) \cap j_k(A_k) = \emptyset$ (even if $A_i \cap A_k \neq \emptyset$).
However, if $\forall\, i,j \in I$, $A_i \cap A_j = \emptyset$, then we have a canonical iso $\coprod_{i\in I} A_i \to \bigcup_{i\in I} A_i$ ← disjoint union.
Thus $\coprod$ is sometimes called the <u>disjoint union</u>. This can be confusing.
Example: take $I = \mathbb{R}$, and $\forall\, i \in I$, take $A_i = \mathbb{R}$. Then we have a bijection

$$\coprod_{i\in I} A_i \longrightarrow \mathbb{R}\times\mathbb{R} \qquad (\neq \bigcup_I \mathbb{R} \text{ since not disjoint})$$
$$(k,x) \longmapsto (k,x).$$

<u>1.22 Proposition</u>: Let I be a set, $\{X_i\}_{i\in I}$ a family of topological spaces index by I. Then there

exists on $\coprod_{i\in I} X_i$ a unique topology such that:

(i) $\forall i \in I$, $j_i : X_i \to \coprod_{i\in I} X_i$ is continuous;

(ii) The map $Top(\coprod_{i\in I} X_i, Z) \to \prod_{i\in I} Top(X_i, Z)$ is bijective for any space Z. $f \mapsto \{f \circ j_i\}_{i\in I}$

proof: exercise

<u>1.23 Lemma</u>: In the above situation, $\forall i \in I$, the map $j_i : X_i \to \coprod_{i\in I} X_i$ is open; In particular, it is a homeomorphism onto its image: we can identify X_i with a <u>subspace</u> of $\coprod_{i\in I} X_i$. As such, X_i is both an open and closed subspace.

<u>proof</u>: exercise.

<u>1.24 Lemma</u>: let X be a topological space, let I be a set, and let $\{A_i\}_{i\in I}$ be a family of disjoint subspaces of X. Then the following conditions are equivalent:

(1) $\bigcup_{i\in I} A_i$, with the subspace topology relative X, and the inclusion $j_i : A_i \hookrightarrow \bigcup_{i\in I} A_i$, is the coproduct $\coprod_{i\in I} A_i$.

(2) $\forall k \in I$, $\exists\, U_k \subset X$ open, $(\bigcup_{i\in I} A_i) \cap U_k = A_k$

<u>proof</u>: Since A_i open in $\coprod_{i\in I} A_i$, (1) ⇒ (2) by def of the subspace topology.

(2) ⇒ (1) $j_i : A_i \to \bigcup_{i\in I} A_i$ is continuous: both have the subspace topology, and if $U \subset X$, then $j_i^{-1}(U) = U \cap A_i$.

This implies that we have a map, $\forall$ space Z

$$Top(\bigcup_I A_i, Z) \to \prod_I Top(A_i, Z),$$ injective (restr. of an iso).

It is also surjective: given $\{f_i : A_i \to Z\}_{i \in I}$ collection of continuous maps, $f : \bigcup_I A_i \to Z$ is defined since the A_i are disjoint, and if $V \subset Z$ is open, then for each $i \in I$, $\exists\, W_i$ open in X, $f_i^{-1}(V) = A_i \cap W_i$ (open in A_i). The condition on $\exists\, U_i$ implies that $f_i^{-1}(V) = (\bigcup_{i \in I} A_i) \cap (W_i \cap U_i)$ is open in $\bigcup_{i \in I} A_i$; Thus $f^{-1}(V) = \bigcup_{i \in I} f_i^{-1}(V)$ is open in $\bigcup_{i \in I} A_i$. ☒

For example, as set, $\mathbb{R} = \underbrace{]-\infty, 0[}_{A_0 =} \amalg \underbrace{[0, \infty[}_{A_1 =}$, but not as coproduct of spaces, of course!
We have continuous maps $f_0 : A_0 \to \mathbb{R}$, $t \mapsto 0$ and $f_1 : A_1 \to \mathbb{R}$, $t \mapsto 1$, but the resulting map $f : \underset{A_0 \amalg A_1}{\overset{\shortparallel}{\mathbb{R}}} \to \mathbb{R}$ is continuous bijection that is not open!

<u>1.25 Example</u>: For any space X and any set I, for the constant family $\{X_i\}_{i \in I}$ with $X = X_i\ \forall i$, we have a homeo $\coprod_I X \longrightarrow I \times X$ (product top for I discret).

We recall:

<u>1.26 Definition</u>: a space X is called connected if it is not the disjoint union of two open, non-empty subspaces.
The <u>connected component</u> of $x \in X$ is the equivalence class of x for the relation $x \sim y \iff \exists$ a connected subspace A of X with $x \in A$ and $y \in A$.

We treat examples in the exercises.

We now study constructions that combine the above construction.

1.27 Definition: Consider continuous maps $f_0 : X_0 \to Y$ and $f_1 : X_1 \to Y$. A *pull-back* of f_0, f_1 is a space X together with cont. maps $q_0 : X \to X_0$, $q_1 : X \to X_1$, such that $f_0 q_0 = f_1 q_1$ and the map

$$\begin{aligned} Top(Z, X) &\longrightarrow \{ (g_0, g_1) \in Top(Z, X_0) \times Top(Z, X_1) ; \\ g &\longmapsto (q_0 g, q_1 g) \qquad\qquad f_0 g_0 = f_1 g_1 \} \end{aligned}$$

$$\begin{tikzcd} Z \arrow[dr, dotted, "\exists ! g"] \arrow[drr, bend left, "g_0"] \arrow[ddr, bend right, "g_1"'] & & \\ & X \arrow[r, "q_0"'] \arrow[d, "q_1"] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d, "f_0"] \\ & X_1 \arrow[r, "f_1"'] & Y \end{tikzcd}$$

1.28 Proposition: The pull-back exists and is unique up to canonical homeomorphisms. A model is given

as $X = \{ (x_0, x_1) \in X_0 \times X_1 ; f_0(x_0) = f_1(x_1) \}$ (with the subspace topology of the product topology) and $q_j : X \subset X_0 \times X_1 \xrightarrow{p_j} X_j$ for $j = 0, 1$.

proof: obviously q_0, q_1 are continuous, and $f_0 q_0 = f_1 q_1$. Given Z and $g_j : Z \to X_j$ with $f_0 g_0 = f_1 \circ g_1$, $\exists !\ g' : Z \to X_0 \times X_1$ with $g_j = p_j g$; its image lies in X, so this factorizes uniquely as $g' : Z \xrightarrow{g} X \subset X_0 \times X_1$. This proves that (q_0, q_1) is a pull-back of (f_0, f_1), which completes the existence.

The proof of uniqueness up to homeo is a (by now) standard exercise.

1.28 Examples: (1) The pull-back of $X \longrightarrow \{*\} \longleftarrow Y$ is $(X \times Y, p_X, p_Y)$.

(c) The pull back of $Y \xrightarrow{f} X \xleftarrow{id} X$ is $Y \xleftarrow{id} Y \xrightarrow{f} X$.
See the exercises for more examples.
We now consider the dual notion.

<u>1.29 Definition</u> : Let $A \xrightarrow{f_0} Y_0$ and $A \xrightarrow{f_1} Y_1$ be continuous maps. A push-out of (f_0, f_1) is a space Y together with continuous maps $i_0 : Y_0 \to Y$, $i_1 : Y_1 \to Y$, such that $i_0 f_0 = i_1 f_1$ and for any space Z,

$$\begin{aligned} Top(Y,Z) &\to \{(g_0,g_1) \in Top(Y_0,Z) \times Top(Y_1,Z) ;\ g_0 f_0 = g_1 f_1\} \\ g &\mapsto (g i_0, g i_1) \end{aligned}$$

is a bijection.

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "f_1"'] & Y_0 \arrow[d, "i_0"] \arrow[ddr, bend left, "g_0"] & \\ Y_1 \arrow[r, "i_1"] \arrow[drr, bend right, "g_1"'] & Y \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dotted, "\exists ! g"] & \\ & & Z \end{tikzcd}$$

<u>1.30 Proposition</u> The pushout of (f_0, f_1) exists and is unique up to canonical homeomorphisms. A model is given by

$$\begin{tikzcd} Y_0 \arrow[r, hook] \arrow[dr, "i_0"'] & Y_0 \amalg Y_1 \arrow[d, "\pi"] & Y_1 \arrow[l, hook'] \arrow[dl, "i_1"] \\ & Y = (Y_0 \amalg Y_1)/\sim & \end{tikzcd}$$

where $Y_0 \hookrightarrow Y_0 \amalg Y_1 \hookleftarrow Y_1$ are the canonical maps into the coproduct, and π is the quotient map in spaces, for the equivalence relation generated by

$y_0 \sim y_1$ if $\exists\, a \in A$ with $f_0(a) = y_0$ and $f_1(a) = y_1$.

<u>proof</u> : left as an exercise.

<u>1.31 Examples</u> : (a) The push-out of $X \leftarrow \emptyset \to Y$ is $X \hookrightarrow X \amalg Y \hookleftarrow Y$.

(b) The pushout of $X \xleftarrow{f} A \xrightarrow{id} A$ is
$X \xrightarrow{id} X \xleftarrow{f} A$.

<u>1.32 Remark</u> an obvious example that we might wonder about is the following one: If Y is a space, $Y_1, Y_2 \subset Y$ subspaces with $Y_1 \cup Y_2 = Y$, and $A = Y_1 \cap Y_2$, is Y the push-out of $Y_0 \leftarrow A \rightarrow Y_1$?
By the universal property, we have a continuous map $g : P \to Y$ from the pushout. It is obviously bijective, but need not be open!

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_0 \arrow[d] \arrow[dl] \\ Y_1 \arrow[r] \arrow[u, phantom] & Y \end{tikzcd} \qquad \begin{tikzcd} Y_0 \arrow[r] & P & Y_1 \arrow[l] \\ & P \arrow[r, dotted] & Y \end{tikzcd}$$

Example: If $Y = \mathbb{R}$, $Y_0 =]-\infty, 0]$ and $Y_1 =]0, \infty[$, then $\mathbb{R} = Y_0 \cup Y_1$, and $Y_0 \cap Y_1 = \emptyset$, so the pushout of $(Y_0 \hookleftarrow Y_0 \cap Y_1 \rightarrow Y_1)$ is $Y_0 \amalg Y_1$, not homeo to $\mathbb{R}$ by Lemma 1.24. !

<u>1.33 Proposition</u>: If Y is a space, $Y_0, Y_1 \subset Y$ subspaces such that $Y = Y_0 \cup Y_1$. Suppose

$$\begin{tikzcd} Y_0 \cap Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_0 \arrow[d, "i_0"] \\ Y_1 \arrow[r, "i_1"'] & P \end{tikzcd}$$

is the push-out. Then the canonical map $P \xrightarrow{h} Y$ is a homeo provided:
(a) Y_0, Y_1 are both open in Y, or
(b) Y_0, Y_1 are both closed in Y.

<u>proof</u>. We know that h is continuous and bijective, it suffices to show it is open (or closed). Assume (a).
Denote

$$\begin{tikzcd} Y_0 \arrow[r, hook] \arrow[dr, bend right, "i_0"'] & Y_0 \amalg Y_1 \arrow[d, "\pi"] & Y_1 \arrow[l, hook'] \arrow[dl, bend left, "i_1"] \\ & P & \end{tikzcd}$$

Suppose $U \subset P$ open.
Then by definition of the quotient topology, $\pi^{-1}(U)$ open in $Y_0 \amalg Y_1$, which means $\pi^{-1}(U) \cap Y_j = i_j^{-1}(Y_j)$ open in Y_j, for $j = 0, 1$. Since Y_j is open

in Y for $j = 0, 1$, $\pi^{-1}(U) \cap Y_j$ is open in Y.
Thus $h(U) = \left(\pi^{-1}(U) \cap Y_0\right) \cup \left(\pi^{-1}(U) \cap Y_1\right)$ is open in Y.

The proof for (b) is similar and left as exo (essentially replace every occurence of "open" by "closed"). ☒

<u>1.34 Examples</u>: We will treat in the exercises the following examples, which follow from 1.33: let $n \geq 1$, and define $D^n := \{ x \in \mathbb{R}^n ;\ \|x\|_2 \leq 1 \}$ (n-disk)

Consider the following push-out,

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & D^n \arrow[d] \\ D^n \arrow[r] & P \end{tikzcd}$$

where $i : S^{n-1} \hookrightarrow D$. Then P is homeomorphic to S^n (See 1.35).

Same example with open disks.

<u>1.35 Proposition</u> The product, co-product, pull-back and push-out are invariant (up to homeo) under replacement of the "universal problem" by a homeomorphic one.

More precisely for push-outs:

Suppose we have a commutative diagram

$$\begin{tikzcd} Y_0 \arrow[d, "h_0"] & A \arrow[l, "f_0"'] \arrow[r, "f_1"] \arrow[d, "h_{01}"] & Y_1 \arrow[d, "h_1"] \\ Y_0' & A' \arrow[l, "f_0'"'] \arrow[r, "f_1'"] & Y_1' \end{tikzcd}$$

where h_0, h_{01} and h_1 are homeomorphisms.

If we denote the pushouts:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_0 \arrow[d] \\ Y_1 \arrow[r] & P \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_0' \arrow[d] \\ Y_1' \arrow[r] & P' \end{tikzcd}$$

then there is a (canonical) homeomorphism $h : P \to P'$!

<u>Proof</u>: We sketch the proof for the push-out. The case of the pull-back is similar, and the case of (co)product easier.

We have a commutative diagram : indeed

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "f_1"'] & Y_0 \arrow[d, "i_0"] \arrow[ddr, bend left, "h_0"] & \\ Y_1 \arrow[r, "i_1"] \arrow[ddr, bend right, "h_1"'] & P \arrow[dr, dotted, "h"] & Y_0' \arrow[d, "i_0'"] \\ & & P' \\ & Y_1' \arrow[ur, "i_1'"'] & \end{tikzcd}$$

$i_0' h_0 f_0 = i_0' f_0' h_{01} = i_1' f_1' h_{01}$

$= i_1' h_1 f_1$. Thus $\exists !\, h$ making the diagram commute.

Now we can repeat the same argument with

$$\begin{tikzcd} Y_0' \arrow[d, "h_0^{-1}"'] & A_0' \arrow[l] \arrow[r] \arrow[d, "h_{01}^{-1}"] & Y_1' \arrow[d, "h_1^{-1}"] \\ Y_0 & A_0 \arrow[l] \arrow[r] & Y_1 \end{tikzcd}$$

which also commutes; $\Rightarrow$ we obtain $g : P' \to P$.

Now gh and id are both a solution to the problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y_0 \arrow[d, "i_0"] \arrow[dr, bend left, "i_0"] & \\ Y_0 \arrow[r, "i_1"] \arrow[rr, bend right, "i_1"'] & P \arrow[r, dotted] & P \end{tikzcd}$$

so they must be equal by uniqueness. Thus h is open and is a homeo. (or : $id_{P'}$ and hg have to be equal).

We now discuss some topological properties of these constructions: connected, Hausdorff (T2) and compact.

<u>1.36 Theorem</u> : let I be a set and $\{X_i\}_{i\in I}$ be a family of topological spaces, and let $X = \prod X_i$ be their product.

(a) Suppose given for all $i \in I$ a <u>subspace</u> $A_i \subset X_i$. Then, with the product topology, $A = \prod_I A_i$ is a subspace of X. Moreover, $\prod \overline{A_i} = \overline{\prod A_i}$.

(b) If each X_i is Hausdorff, so is X.

(c) If each X_i is connected, so is X, and the converse holds.

<u>proof</u> : For $J \subset I$ finite, and $\{U_i\}_{i\in I}$ a family of open subsets $U_i \subset X_i$ $\forall i$, with $j \notin J \Rightarrow U_j = X_j$, the sets of the form $O(J, \{U_i\}_{i\in I}) = \prod_i U_i$ form a basis of the product topology on X.

The same holds for the product topology on A, with basis open of the form $O(J, \{V\}_{i \in I})$.
But since the A_i are subspaces, for each $i \in I$, we can choose

$$U_i = \begin{cases} X_i & \text{if } i \notin J \\ \text{some open of } X_i \text{ with } U_i \cap A = V_i, & i \in J. \end{cases}$$

Then $O(J, \{U_i\}_{i \in I}) \cap A = O(J, \{V_i\}_{i \in I})$, which shows that the subspace topology of A rel. X is the product topology.

For $\Pi \overline{A_i} \supset \overline{A}$: we have $\Pi \overline{A_i} = \bigcap_i \pi_i^{-1}(\overline{A_i})$ which is closed and contains A, so $\overline{A} \subset \Pi \overline{A_i}$.

Conversely, suppose $x \in \Pi \overline{A_i}$, and let $O = O(J, \{U_i\}_{i \in I})$ be a basis element (for X) that contains x.

Then $\forall i$, $U_i \subset X_i$ is an open that contains $x_i \in \overline{A_i}$.
Thus $\exists\, z_i \in U_i \cap A_i$, and choose such. If $z = \{z_i\}_{i \in I}$, then $z \in O \cap A$. This shows that any nbhd of x meets A, hence $x \in \overline{A}$.

(b) Obvious: if $x \neq y \in X$, $\exists\, i \in I$, $x_i \neq y_i$.
Choose U, V open in X_i separating x_i and y_i.
The sub-basis elements corresponding to U and V, $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$, separate x and y.

(c). Now assume each X_i connected $\neq \emptyset$. Assume $\exists\, U, V$ open in X with $X = U \cup V$. Since each π_i is surjective and open, and X_i connected, $\exists\, z_i \in \pi_i(U) \cap \pi_i(V)$.
But then $z = \{z_i\}_{i \in I} \in U \cap V$. This proves X connected.
If some X_i is empty so X, so connected.

Conversely, assume $\exists\, i,\ X_i = \emptyset$: $X = \emptyset$. Assume $\forall j,\ X_j \neq \emptyset$ and $\exists\, i,\ X_i = U \cup V$ both open, $\neq \emptyset$, disjoint. Then

$\pi_i^{-1}(U) \neq \emptyset \neq \pi_i^{-1}(V)$, and both open, disjoint, and $X = \pi_i^{-1}(U) \cup \pi_i^{-1}(V)$, so X is disconnected. $\blacksquare$

<u>1.37 Remark</u> : By 1.36, if $\{A_i \overset{j_i}{\hookrightarrow} X_i\}_{i\in I}$ is a family of inclusion of subspaces, then $\prod_i j_i : \prod A_i \to \prod_i X_i$ is the inclusion of a subspace.

<u>Note</u>: the product of quotient maps need not be a quotient map (see sheet 3).

<u>1.38 Theorem</u> (Tychonoff): Any product of compact spaces is compact. $\blacksquare$

<u>Remark</u> equivalent to the Axiom of choice or Zorn's lemma.

<u>1.39 Example</u> Take $X = [0,1]$, and $A = [0,1[$.

Then $X/A = \{[0],[1]\}$, but the only open nbhd of $[1]$ is X/A. Thus X/A is not Hausdorff, so quotient aren't Hausdorff in general.

<u>1.40 Theorem</u> Let X be a space, R an equiv. relation on X, and $\pi : X \to X/R$ the quotient.

(a) If X is connected, so is X/R

(b) If X is compact, so is X/R

(c) If X/R is Hausdorff, then R is closed in $X \times X$. If π is open and R is closed, then X/R is Hausdorff.

(d) If X is compact and Hausdorff, then X/R is Hausdorff iff π is closed.

proof (c) Suppose X/R Hausdorff. Let $(x,y) \in X \times X \setminus R$. $\exists\, u, V$ open in X/R such that $\bar{x} \in u$, $\bar{y} \in V$, $u \cap V = \emptyset$. Then $x \in O_1 = \pi^{-1}(u)$, $y \in O_2 = \pi^{-1}(V)$, and $R \cap O_1 \times O_2 = \emptyset$: otherwise $u \cap V \neq \emptyset$. So R is closed. Conversely, if R is closed and π open, and $\bar{x}, \bar{y} \in X/R$ with $\bar{x} \neq \bar{y}$, then $(x,y) \notin R$.
$\exists\, (x,y) \in O_1 \times O_2 \subset X \times X$ open, $O_1 \times O_2 \cap R = \emptyset$.
Then $\pi(O_1)$ and $\pi(O_2)$ separate $\bar{x}, \bar{y}$ and are open (π is).
(d) X/R Hausdorff $\Rightarrow$ π closed is standard!
Conversely, suppose X is compact and Hausdorff, and π closed. We first show:

($\star$) Let $A \subset X$ closed and saturated ($A = \pi^{-1}(\pi(A))$). Suppose given u open with $A \subset u$. Then there exists

there exists V open and saturated with $A \subset V \subset u$.

proof of ($\star$) $X \setminus u$ is closed; thus $\pi(X \setminus u)$ closed in X/R (π closed!) and $\pi^{-1}(\pi(X \setminus u))$ is closed and saturated in X.
Also, $A \cap \pi^{-1}(\pi(X \setminus u)) = \emptyset$ because

$$\pi(A \cap \pi^{-1}(\pi(X \setminus u))) \subset \pi(A) \cap \pi(X \setminus u) = \emptyset$$

the latest: $A \cap (X \setminus u) = \emptyset$ and A saturated.
Thus can take $V = X \setminus \pi^{-1}(\pi(X \setminus u))$. $\square$ ($\star$)
Now suppose $[x], [y] \in X/R$, $[x] \neq [y]$.
Then $\pi^{-1}([x])$ and $\pi^{-1}([y])$ are disjoint and closed (indeed, $[x]$ is closed in X/R since $[x] = \pi(\{x\})$ and $\{x\}$ closed in X by Hausdorff).
X Hausdorff + compact $\Rightarrow$ normal: can separate $\pi^{-1}([x])$ and $\pi^{-1}([y])$ by u and V,

which we can assume to be saturated by (☒)

Then $\pi(U)$ and $\pi(V)$ are open and separate $[x]$, $[y]$ in X/R, which is therefore Hausdorff. ☒ We used:

<u>1.41 Lemma</u>: A compact Hausdorff space X is normal.

proof: let A, B be closed disjoint subsets of X.

For any $a \in A$, can choose U_a and V_a open in X, disjoint, with $a \in U_a$ and $B \subset V_a$: indeed,

Choose $\forall b \in B$, U_b and W_b open disjoint with $a \in U_b$ and $b \in W_b$; B compact (closed in compact) $\Rightarrow \exists\, b_1, \dots, b_n$ and $B \subset W_{b_1} \cup \dots \cup W_{b_n} =: V_a$ open, disjoint from $U_{b_1} \cap \dots \cap U_{b_n} =: U_a \ni a$, also open. Since A is compact, $\exists\, a_1, \dots, a_m \in A$, $A \subset U_A = U_{a_1} \cup \dots \cup U_{a_m}$ open, $B \subset V_{a_1} \cap \dots \cap V_{a_m} =: V_B$ open, $U_A \cap V_B = \emptyset$. ☒

<u>1.42 Corollary</u>: If X is compact and Hausdorff, and A is a closed subspace, then X/A is compact Hausdorff.

<u>proof</u>: $\pi : X \to X/A$ is closed: If $B \subset X$ is closed, then

$$\pi^{-1}\pi(B) = \begin{cases} B & \text{if } B \cap A = \emptyset \\ B \cup A & \text{if } B \cap A \neq \emptyset \end{cases} \quad \text{is closed in } X$$

☒

We now mention quotients by a group action.

<u>1.43 Def</u>: A *topological group* is a space G together with a group structure, such that the multiplication $G \times G \to G$, $(x,y) \mapsto xy$ and the inverse $G \to G$, $x \mapsto x^{-1}$, are continuous.

A continuous action of a topological group G on a space X is a continuous map $G \times X \to X$ that is an action on the underlying sets.

1.44 Examples (a) $\mathbb{Z}$ is a discrete topological group, continuously acting on $\mathbb{R}$ via $\mathbb{Z} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$.

(b) $S^1 \subset \mathbb{C}^\times$ is a top. group continuously acting on $\mathbb{C} = \mathbb{R}^2$: it is the rotation group. Similarly, $S^3 \subset \mathbb{H}^\times$ is acting on $\mathbb{R}^4$ (unit quaternions).

By restriction, (finite) subgroups of S^1 act on $\mathbb{R}^2$, ...

(c) If $H < G$ with G a topological group, so is H, and it acts continuously on G by multiplication.

1.45 Definition: If G is a top group continuously acting on a space X, let R be the equivalence relation on X given by $x \sim y \iff \exists\, g \in G,\ y = g \cdot x$.

The quotient $X \xrightarrow{\pi} X/R$ is denoted $X \xrightarrow{\pi} X/G$: it is the quotient by the group action.

1.46 Proposition: Let G be a topological group continuously acting on a space X. Then the quotient map $\pi : X \to X/G$ is open.

Proof: Let $U \subset X$ open.

Then $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$ where

$g \cdot U = \{ x \in X \ ;\ \exists\, u \in U,\ x = g \cdot u \}$. Since $\tilde{g} : X \to X$, $x \mapsto gx$ is a homeo, $g \cdot U = \tilde{g}(U)$ is open. Thus $\pi^{-1}(\pi(U))$ is open, thus $\pi(U)$ open. $\blacksquare$

1.47 Proposition: $H < G$ topological groups. Then G/H is Hausdorff iff H is closed in G.

proof: exercise.