2. Topological Manifolds

2.1 Def: Let $n \in \mathbb{N}$. An <u>n-dimensional topological manifold</u> M is a Hausdorff space which is second countable and locally homeomorphic to $\mathbb{R}^n$. We say also: M is an n-manifold.
Here, <u>second countable</u> means that the topology of M admits a countable basis, i.e. a basis of the form $\{U_i\}_{i \in \mathbb{N}}$.
Here, M <u>locally homeomorphic to $\mathbb{R}^n$</u> means: for any $x \in M$, there exists an open nbhd U of x, an open subset V of $\mathbb{R}^n$, and a homeo $U \xrightarrow{h} V$.
<u>Exercise</u>: We can assume $V = \mathbb{R}^n$.

2.2 Examples: (1) $\mathbb{R}^n$ is an n-dim'l topological manifold: we know it is Hausdorff; it is also second-countable: take the basis of the topology consisting of open subsets of the form $\prod_{i=1}^{n}]a_i, b_i[$ with $a_i, b_i \in \mathbb{Q}$ and $a_i < b_i$.
(2) If M is an n-manifold, and $U \subset M$ is an open subset, then U is also an n-manifold.

2.3 Theorem (Brouwer's Invariance of domain): If $U \subset \mathbb{R}^n$ is an open subset, and if $f: U \to \mathbb{R}^n$ is continuous and injective, then $f(U)$ is an open subset of $\mathbb{R}^n$ and $U \xrightarrow{f} f(U)$ is a homeomorphism. $\left(\Rightarrow \left[(\mathbb{R}^m \cong \mathbb{R}^n) \Rightarrow (m = n) \right] \right)$

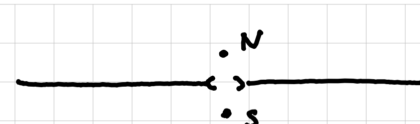
2.4 Def: If M is an n-manifold, we call n the <u>dimension of M</u>; it is unique if $M \neq \emptyset$. A 1-manifold is called a <u>curve</u>, and a 2-manifold is called a <u>surface</u>.

2.5 Examples (1) For any $n \in \mathbb{N}$, we have that $S^n = \{x \in \mathbb{R}^{n+1} \,;\, \|x\|_2 = 1\}$ is a (compact) n-manifold.
For $n = 0$: clear (in fact, any countable discrete space is a 0-manifold). For $n \geq 1$ we have seen in our description of S^n as push-out of $\mathbb{R}^n \xleftarrow{i} \mathbb{R}^n \setminus \{0\} \xrightarrow{h} \mathbb{R}^n$ that it is locally homeo to $\mathbb{R}^n$; as a subspace of $\mathbb{R}^{n+1}$, it is automatically Hausdorff and second-countable.

(2) Note that in the push-out description of S^n above, i can be taken as the inclusion, but not both i and h!
For example, consider the push-out

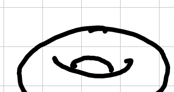
$$\begin{tikzcd} \mathbb{R}\setminus\{0\} \arrow[r, hook, "inc"] \arrow[d, hook, "inc"'] & \mathbb{R} \arrow[d] \\ \mathbb{R} \arrow[r] & M \end{tikzcd}$$

Then M is locally homeo to $\mathbb{R}$, second countable, but not Hausdorff! compare with Ex. 1.35, prop 1.35.

For $n = 1$, M "looks like" 

where the nbhds of N are of the form $\{N\} \cup U \setminus \{0\}$ where U is a nbhd of 0 in $\mathbb{R}$, and similarly for the nbhds of S. (see ex. 4.1).

2.6 Proposition: If $k \in \mathbb{N}$, $M_1, \dots, M_k$ are manifolds of dim. $n_1, \dots, n_k$, then $M_1 \times \dots \times M_k$, with the product topology, is a manifold of dimension $n_1 + \dots + n_k$.
proof: exercise. ∎

2.7 Example $T^n = (S^1)^{\times n}$, the n-torus, is an example of an n-manifold. In exercise 1.6, we have seen how T^2 is homeomorphic to a subspace in $\mathbb{R}^3$:  T^2
the surface of a "tire" or "donut".

2.8 Definition : X, Y be spaces, and $f : X \to Y$ a map. We say that f is an embedding of X into Y if the restriction $f : X \to f(X)$, where $f(X) \subset Y$ has the subspace topology, is a homeomorphism.

2.9 Theorem : Let M be a compact n-manifold. Then there exist $N \in \mathbb{N}$ and an embedding $M \xhookrightarrow{f} \mathbb{R}^N$.
We sketch the proof. It is a consequence of the Urysohn lemma:

2.10 Urysohn lemma : let X be a normal space, and A, B be closed, disjoint subspaces. Then there exists a continuous map $f : X \to [0,1]$ with $A \subset f^{-1}(\{0\})$ and $B \subset f^{-1}(\{1\})$. ☒

2.11 Definition Let X be a space and $\{U\}_{i=1}^n$ be an open cover of X. A partition of unity of X (relative $U_1, \dots, U_n$) is a family of functions $\{\varphi_i : X \to [0,1]\}_{i=1}^n$ such that $\operatorname{supp}(\varphi_i) \subset U_i$ and $\sum_{i=1}^n \varphi_i = 1$ (the constant function with value 1). Recall : $\operatorname{supp}(\varphi_i) = \overline{\{x \in X \,;\, \varphi_i(x) \neq 0\}}$ (closure).

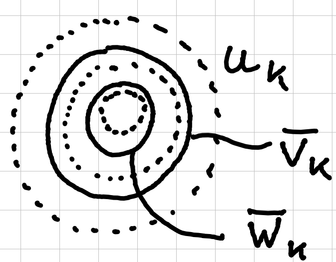
2.12 Theorem : If X is normal, and $\{U_i\}_{i=1}^n$ is a finite open cover of X, then there exists a partition of unity of X relative $\{U_i\}_{i=1}^n$.

proof: One can construct by induction an open cover $\{V_i\}_{i=1}^n$ of X with $\overline{V_i} \subset U_i$ $\forall\, i = 1, \dots, n$ (since X is normal): induction on i.

- $A = (X \setminus (U_2 \cup \dots \cup U_n))$ and $B = (X \setminus U_1)$ are closed and disjoint ; $\exists$ V_1, W_1 open, $A \subset V_1$, $B \subset W_1$ and $V_1 \cap W_1 = \emptyset$. Then $A \subset V_1 \subset \overline{V_1} \subset U_1$, and $\{V_1, U_2, \dots, U_n\}$ covers X.

- Suppose $1 \le k < n$ given, and $V_1, \dots, V_k$ defined so that $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ is an open cover of X with $\overline{V_i} \subset U_i$ for $1 \le i \le k$. Repeat with
$A = X \setminus (V_1 \cup \dots V_k \cup U_{k+2} \cup \dots \cup U_n)$ and
$B = X \setminus U_{k+1}$

Now take such $(V_1, \dots, V_n)$; repeat to get $(W_1, \dots, W_n)$



Urysohn: $\exists\, \psi_i : X \to [0,1]$ with
$\psi_i(\overline{W_i}) = \{1\}$ and
$\psi_i(X \setminus V_i) = \{0\}$.
Then $\mathrm{Supp}(\psi_i) \subset \overline{V_i} \subset U_i$.

Since $\{W_i\}_{i=1}^n$ covers X, $\forall x$, $\psi(x) := \sum_{i=1}^n \psi_i(x) > 0$.
Take $\varphi_i = \frac{\psi_i}{\psi}$. ▨

<u>Proof of Thm 2.9</u> By compactness, we can choose $m \in \mathbb{N}$ and an open cover $\{U_i\}_{i=1}^m$ of M together with homeos $h_i : U_i \to \mathbb{R}^n$. We know M is normal, so $\exists\, \{\varphi_i\}_{i=1}^n$ partition of unity relative $\{U_i\}_{i=1}^m$. We define

$$g_i : X \to \mathbb{R}^n, \quad g_i(x) = \begin{cases} \varphi_i(x) \cdot h_i(x), & x \in U_i \\ 0, & x \in X \setminus \mathrm{supp}(\varphi_i). \end{cases}$$

Take $g : X \to (\mathbb{R} \times \mathbb{R}^n)^m$, $x \mapsto ((\varphi_1(x), g_1(x)), \dots, (\varphi_m(x), g_m(x)))$

Then g is continuous and injective: assume $g(x) = g(y)$: then $\forall i$, $\varphi_i(x) = \varphi_i(y)$ and $g_i(x) = g_i(y)$.

Then $\exists\, j$, $\varphi_j(x) > 0$; $\Rightarrow x, y \in U_j$. Then $x = y$ since $g_j : U_j \to \mathbb{R}^n$ injective. Take $N = m(n+1)$.

Since X compact, and $\mathbb{R}^N$ Hausdorff, g is also closed, thus an embedding (ex. 4.2) ▨

We now introduce n-manifolds with boundary: they are locally diffeomorphic with $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} =: \mathbb{H}^n$.

<u>2.13 Definition</u> An n-manifold with boundary is a Hausdorff, second countable space M that is locally homeo to $\mathbb{H}^n$.

<u>2.14 Lemma + Def</u>: Let M be an n-manifold with boundary, and $x \in M$. Then one and only one of the following assertion holds:

(a) x has an open nbhd U homeomorphic to $\mathbb{R}^n$.
we then say that x is an <u>interior point of M</u>

(b) x has no open nbhd homeomorphic to $\mathbb{R}^n$.
we then say that x is a <u>boundary point of M</u>.

We denote $\mathrm{Int}(M) = \{x \in M \,;\, x \text{ interior point}\}$ (the interior of M), and $\partial M = M \setminus \mathrm{Int}(M)$

(the boundary of M).

<u>proof</u>: this is obvious by invariance of domain: no nbhood of 0 in $\mathbb{H}^n$ can be homeomorphic to a nbhd of 0 in $\mathbb{R}^n$.

<u>2.15 Remark</u>: we see that the notion of a manifold with boundary generalizes that of a manifold (with $\partial M = \emptyset$)

<u>2.16 Proposition</u>: Let M be an n-manifold with boundary.

(a) $\mathrm{Int}(M)$ is an open subset of M and an n-manifold without boundary.

(b) ∂M is a closed subset of M and an $(n-1)$-manifold without boundary.

<u>proof</u>: obvious (see lecture).

2.17 Examples (a) $\mathbb{H}^n$ is an n-manifold with boundary, and $\partial \mathbb{H}^n = \{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} = \mathbb{H}^n$.
Obviously $\partial \mathbb{H}^n$ is an $(n-1)$-manifold.

(b) If M is an m-manifold with boundary and if N is an n-manifold without boundary, then $M \times N$ is an $(m+n)$-manifold with boundary, and $\partial(M \times M) = (\partial M) \times N$
This is obvious since $\mathbb{H}^{m+n} = \mathbb{H}^m \times \mathbb{R}^n$ and $\partial(\mathbb{H}^{m+n}) = \partial(\mathbb{H}^m) \times \mathbb{R}^n$.

(c) $D^n = \{ x \in \mathbb{R}^n ; \|x\| \leq 1 \}$ is an n-manifold with boundary, and $\partial D^n = S^{n-1}$. Exercise!

2.18 Proposition: Let M and N be two n-manifolds with boundary, and suppose given a homeomorphism $h : \partial M \to \partial N$. Consider the push-out (in spaces)

$$\begin{tikzcd} \partial M \arrow[r, "j"] \arrow[d, "inc"'] \arrow[dr, phantom, "\ulcorner"] & N \arrow[d, "g"] \\ M \arrow[r, "f"'] & P \end{tikzcd}$$

where $j : \partial M \xrightarrow{h} \partial N \xrightarrow{inc} N$.
Then P is an n-manifold, and $\partial P = \emptyset$.
Moreover, f and g are closed embeddings.

proof: we have the quotient map $q : (M \amalg N) \to P$.
Let $Q = q(\partial M) = q(\partial N) \subset P$; Q is closed in P.
Obviously the restriction $q : \mathrm{Int}(M) \amalg \mathrm{Int}(N) \to P \setminus Q$ is a homeomorphism; therefore $P \setminus Q$ is locally homeo to $\mathbb{R}^n$. Now take $x \in Q$; $q^{-1}(x) = \{y, z\}$ with $y \in \partial M$ and $z = h(y) \in \partial N$.
We can choose open nbhds of U of y in M, V of z in N, and homeos $\varphi : U \to \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$

$\psi : V \to \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$

such that $h : U \cap \partial M \to V \cap \partial N$ homeo

$$\begin{tikzcd}
U\cap\partial M \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d] \arrow[ddr, bend left, "\tilde\Psi"] & \\
U \arrow[r] \arrow[drr, bend right, "\varphi"'] & W \arrow[dr, dashed] & \\
& & \mathbb{R}^n
\end{tikzcd}$$

where

$\tilde\Psi = \chi \circ \Psi$

$\chi : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} \to \mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$

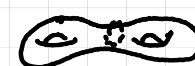
$(t,x) \mapsto (-t, \alpha)$

$\alpha : \{0\} \times \mathbb{R}^{n-1} \xrightarrow{\varphi^{-1}} U \cap \partial M \xrightarrow{h} V \cap \partial N \xrightarrow{\Psi} \{0\} \times \mathbb{R}^{n-1}$ homeo.

Then the outer diagram commutes, and induces $\beta : W \to \mathbb{R}^n$.
Obviously W is an open nbhd of $x \in Q$ in P, and β is a homeo. Thus P locally homeo to $\mathbb{R}^n$.

- Hausdorff: let $x, y \in P$. consider the cases
  - (a) $x, y \in P \setminus Q$ (obvious can separate)
  - (b) $x \in Q$, $y \in P \setminus Q$
  - (c) $x, y \in Q$.

- Second countable: see exercises ☒

A nice way of constructing new manifolds out of known-ones is the notion of connected sum: for example 

2.19 Def: M an n-manifold. A coordinate chart of M (at a point $x \in M$) is a pair (U, φ) where U is an open nbhd of x in M, $\varphi : U \to V \subset \mathbb{R}^n$ a homeo onto some open subset of $\mathbb{R}^n$ ($\varphi : U \to V \subset \mathbb{H}^n$ if $x \in \partial M$)

2.20 Definition: let M and N be two n-manifolds. Suppose given coordinate charts (U, φ) and (V, ψ) of M and N, with $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$.
Let $M' = M \setminus \varphi^{-1}(D^n)$, $N' = N \setminus \psi^{-1}(D^n)$.
(Then M' and N' are manifolds with boundary
$\partial M' = \varphi^{-1}(S^{n-1}) \cong S^{n-1}$ and $\partial N = \psi^{-1}(S^{n-1}) \cong S^{n-1}$)
Let $h : \partial M' \to \partial N'$ be the homeomorphism

given by restricting $\psi^{-1} \circ \varphi$.
The connected sum of M and N (along φ and ψ), denoted $M \# N$, is the push-out

$$\begin{tikzcd} \partial M' \arrow[r] \arrow[d, hook] & N' \arrow[d] \\ M' \arrow[r] & M \# N \end{tikzcd}$$

as in proposition 2.18.

<u>2.21 Remark</u>: We deduce from 2.18 that $M \# N$ is an n manifold, and $\partial(M \# N) = \partial M \amalg \partial N$. It contains M', N' as closed embedded subspaces.
One can show that up to homeo there are at most 2 manifolds that can be constructed as $M \# N$ with given M and N: the choice of U, V doesn't matter; it matters if $\varphi \psi^{-1} : \mathbb{R}^n \circlearrowleft$ reverses the orientation or not.

We will construct many examples of compact surfaces by $\#$ from "building blocks": T^2 and $\mathbb{RP}^2$, which we now introduce.
<u>2.22 Definition</u> Let $n \geq 1$. We consider on $\mathbb{R}^{n+1} \setminus \{0\}$ the equiv. relation given by $x \sim y \iff \exists \lambda \in \mathbb{R}^*$, $x = \lambda y$. Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\} /_\sim =: \mathbb{RP}^n$ be the quotient space; it is called the <u>n-th (real) projective space</u>.
<u>2.23 Remark</u>: The equivalence class of $x \in \mathbb{R}^{n+1} \setminus \{0\}$ is given by $[x] = \langle x \rangle \cap (\mathbb{R}^{n+1} \setminus \{0\})$, where $\langle x \rangle$ is the subspace generated by x: $\mathbb{RP}^n$ is the "space of lines" in $\mathbb{R}^{n+1}$.
<u>2.24 Lemma</u>: the inclusion $i : S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ induces a homeo on quotients: $\bar{i} : S^n / C_2 \xrightarrow{\cong} \mathbb{RP}^n$, where

$C_2 = \langle t \rangle / t^2 = 1 \cong \mathbb{Z}/2$ is the cyclic group of order 2, acting on S^n by $t \cdot x = -x$.

proof: we have map:

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "q"] \arrow[rr, bend right, "f"] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "\pi"] & \mathbb{R}P^n \\ S^n / C_2 \arrow[urr, dashed, "\exists ! \bar{f}"'] & & \end{tikzcd}$$

For any $x \in S^n$, we have $f(x) = f(tx) = [i(x)]$.

So by the universal property, $\exists ! \bar{f}$ with $\bar{f} \circ q = f$.

An inverse of $\bar{f}$ is given by $\bar{g}$,

$$\begin{tikzcd} \mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "h"] \arrow[d, "\pi"] \arrow[rr, bend right, "g"] & S^n \arrow[r, "q"] & S^n / C_2 \\ \mathbb{R}P^n \arrow[urr, dashed, "\exists ! \bar{g}"'] & & \end{tikzcd}$$

$h(x) = \frac{x}{\|x\|}$

which exists since $\forall \lambda \in \mathbb{R}^*$

$$h(\lambda x) = \begin{cases} x & \text{if } \lambda > 0 \\ t \cdot x & \text{if } \lambda < 0 . \end{cases}$$

We leave the details (check $\bar{f} \circ \bar{g} = id_{\mathbb{R}P^n}$ and $\bar{g} \circ \bar{f} = id_{S^n/C_2}$). $\square$

<u>2.25 Proposition</u>: $\mathbb{R}P^n$ is compact n-manifold, connected and without boundary.

proof: By 1.46, the map $q : S^n \to \mathbb{R}P^n$ is open. We deduce that since S^n is Hausdorff and second countable, so is $\mathbb{R}P^n$. We now define coordinate charts:

For $i = 1, \dots, n+1$, let $[x]$ be the class of $x \in \mathbb{R}^{n+1} \setminus \{0\}$, and

$$U_i = \{ [x] \in \mathbb{R}P^n \; ; \; x_i \neq 0 \} \subset \mathbb{R}P^n .$$

Then $$\pi^{-1}(U_i) = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{i-1 \text{ factors}} \times (\mathbb{R} \setminus \{0\}) \times \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n+1-i \text{ factors}}$$

is open in $\mathbb{R}^{n+1} \setminus \{0\}$, so U_i is open in $\mathbb{R}P^n$.

Then $\varphi_i : U_i \to \mathbb{R}^n$, $[x] \mapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots \frac{x_{n+1}}{x_i} \right)$ is well defined and continuous; it admits

$\phi : \mathbb{R}^n \to U_i$, $x \mapsto (x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_{n+1})$

as continuous inverse (exercise!), thus is a homeomorphism. Obviously we have $\mathbb{RP}^n = \bigcup_{i=1}^{n+1} U_i$ which proves that $\mathbb{RP}^n$ is locally Euclidean. ▨

<u>2.26 Examples</u>: note that as opposed to our construction of S^n or the embedded Torus T^2 in $\mathbb{R}^3$ (ex. 1.6), $\mathbb{RP}^n$ is not constructed as a subspace of $\mathbb{R}^N$.

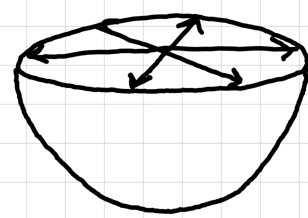
To "visualize it", notice that $\mathbb{RP}^n$ can be obtained from D^n by identifying antipodal points of its boundary: View D^n as the south hemisphere of S^n, then the composite

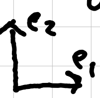
$$\begin{tikzcd} D^n \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & S^n \arrow[r, "q"] & \mathbb{RP}^n \\ & D^n/\sim \arrow[ur, "\bar{q}", "\cong"'] & \end{tikzcd}$$

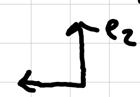
qi factors over a homeomorphism $D^n/\sim \ \to \mathbb{RP}^n$ (compact to Hausdorff).

(a) $\mathbb{RP}^1$ can be viewed as $D^1/\sim \ = [-1,1]/\{-1,1\}$

$\cong S^1$.

(2) $\mathbb{RP}^2$ can be viewed as  D^2

Observation: if you move a frame 

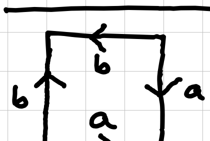
over the surface by crossing $\bar{q}(S^1)$, it will return as 

($\rightsquigarrow$ surface is non-orientable)

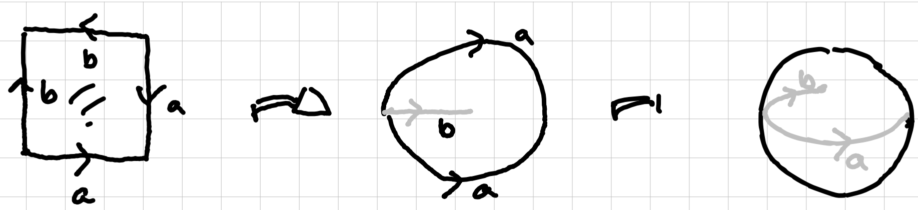
We can now construct many new examples of surfaces: $T^2 \# T^2$, $T^2 \# T^2 \# T^2$, $\mathbb{RP}^2 \# T^2$, $\mathbb{RP}^2 \# \mathbb{RP}^2$, etc...

To better study these, we describe them as quotients of $2n$-polygons by identifying edges pairwise.

<u>2.27 Examples</u> (a) S^2: is obtained as the quotient

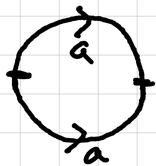


Here edges with same label are "glued" to each other isometrically, so that the arrows match



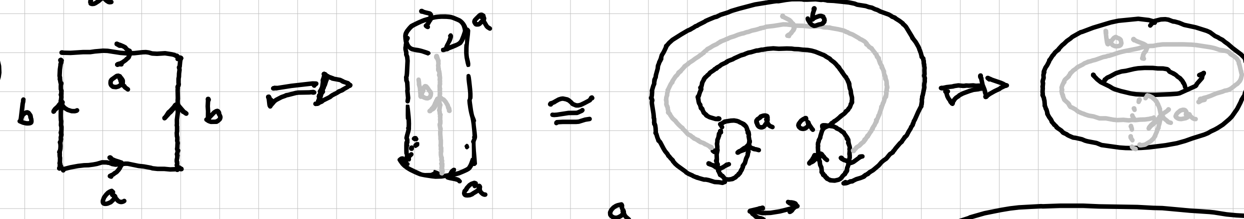
$= S^2$

Note that if we are ready to work with a disk, we

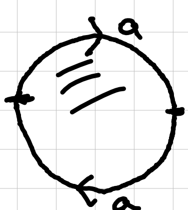


need only two "edges".

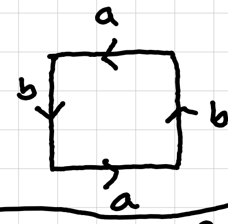
(b)



(c)



$\cong \mathbb{RP}^2 \cong$

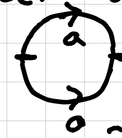


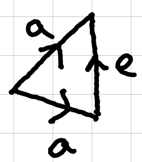
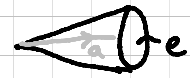
"Klein bottle"

(d)

K^2

2.28 Example How do we interpret connected sums?

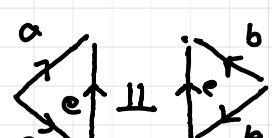
(a) Recall S^2 as the quotient 

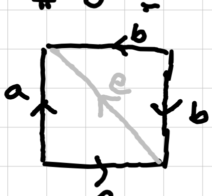
Note that  $\cong$  $\cong S^2 \setminus D$

where D is a small disk on S^2:

Since the edge e is alone, it is the boundary of the quotient obtained by gluing $\vec{a}$ with $\vec{a}$.

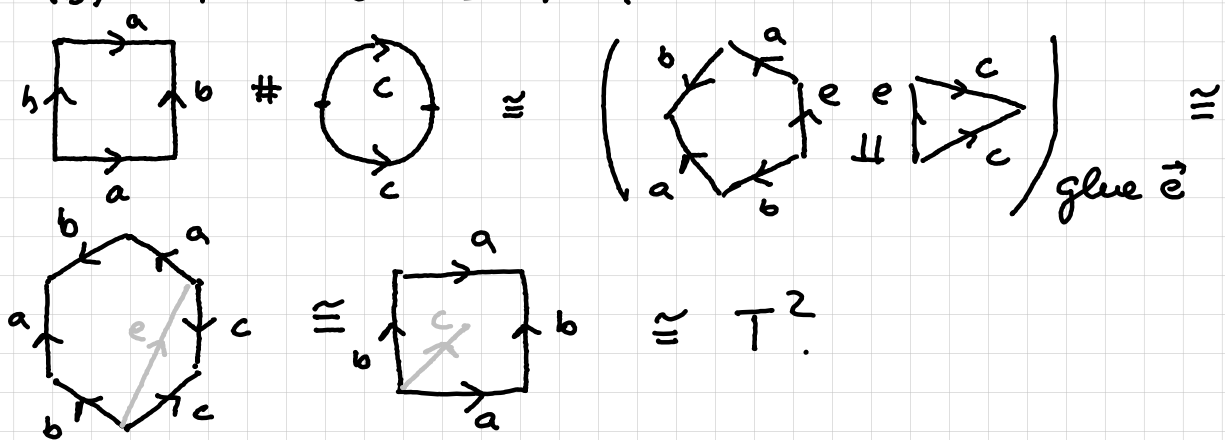
$$S^2 \# S^2 = (S^2 \setminus D) \amalg_h (S^2 \setminus D) \cong \left(\text{[figure]} \right) \Big/ \text{glue along } \vec{e}$$



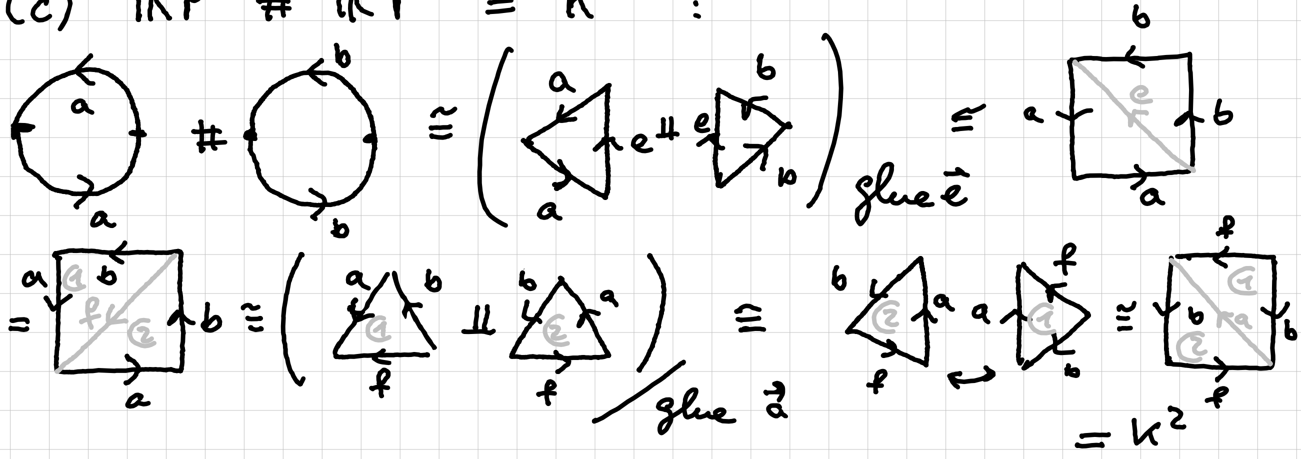
$=$  $\cong S^2.$

This shows: $S^2 \# S^2 \cong S^2$

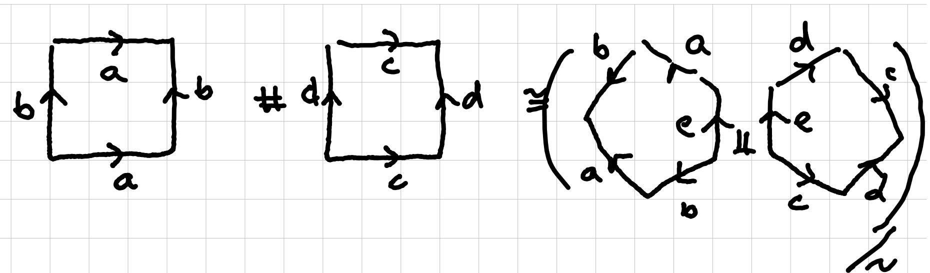
(b) $T^2 \# S^2 \cong T^2$:

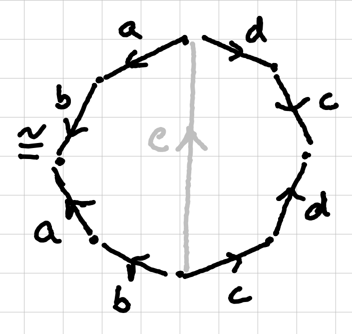


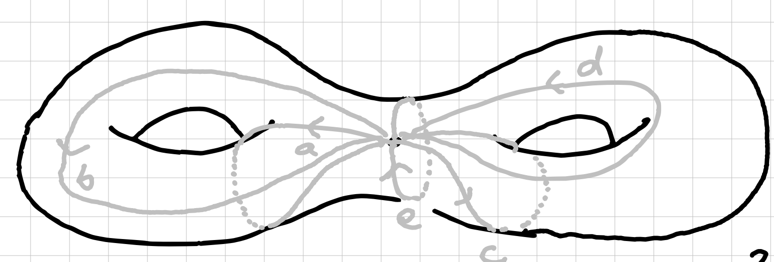
(c) $\mathbb{R}P^2 \# \mathbb{R}P^2 \cong K^2$:



(d) $T^2 \# T^2$:







$T^2 \# T^2$

<u>Lemma</u> (Exercise 5.2): Let P be a regular polygon in $\mathbb{R}^2$ (with its interior) having $2n$ edges, $n \geq 2$.
Label each edge with a label choosen in
$$\{ a_1^{\varepsilon}, a_1'^{\varepsilon}, a_2^{\varepsilon}, a_2'^{\varepsilon}, \dots, a_n^{\varepsilon}, a_n'^{\varepsilon} \ ; \ \varepsilon \in \{\pm 1\} \}$$
such that each $a_1, a_1', \dots, a_n, a_n'$ appears exactly once.

Choose homeos $h_x : I \to X^\varepsilon$ such that $h_x(t)$ runs X counter clockwise for each $x \in \{a_i, a_i' \,;\, i = 1, \dots, n\}$.

Let $\sim$ be the equivalence relation on P generated

$$h_{y_i^\varepsilon}(t) = \begin{cases} h_{y_i'}(t) & \text{if } y_i \text{ and } y_i' \text{ have the same power } \varepsilon \\ h_{y_i'}(1-t) & \text{otherwise.} \end{cases}$$

Then $P/\sim$ is a compact, connected surface without boundary.

<u>Notation</u>: choosing one vertex of P and running ∂P counterclockwise, and writing a_i instead of a_i', we obtain a word with $2n$ letters, where each a_i appears twice (with some ε power). Eg (d) above starting at V gives the word $W = aba^{-1}b^{-1}cdc^{-1}d^{-1}$.

<u>2.29 Theorem</u> (Classification of closed surfaces)

Any compact, connected 2-manifold without boundary is homeomorphic to one, and only one, 2-manifold of the following list:

$$S^2, \quad \#_{i=1}^{n} T^2 = \underbrace{T^2 \# \dots \# T^2}_{n \text{ summands}}, \quad \#_{i=1}^{n} \mathbb{R}P^2 = \underbrace{\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2}_{n \text{ summands}}$$

for $n \in \mathbb{N}^*$.

<u>proof</u>: We admit that any compact, connected surface is triangulable: one can "cover it" with triangles that are either:
- disjoint
- meet along exactly one common vertex
- meet along exactly one common edge

Using this (cut the triangles, arrange them on a plane)

it is easy to see that any compact connected surface can be obtained from a Polygon with $2n$ sides as in the lemma above: we say that this is a <u>presentation</u> of the surface: $\langle a_1, \dots, a_n \mid W \rangle$
where W is a word with $2n$ letters, in which each a_i appears exactly twice (with exponent ± 1).
For example we have presentations:

$$S^2 = \langle a, b \mid a a^{-1} b b^{-1} \rangle$$
$$T^2 = \langle a, b \mid a b a^{-1} b^{-1} \rangle$$
$$\mathbb{R}P^2 = \langle a, b \mid a b a b \rangle$$
$$K = \langle a, b \mid a b a b^{-1} \rangle$$
$$T^2 \# T^2 = \langle a, b, c, d \mid a b a^{-1} b^{-1} c d c^{-1} d^{-1} \rangle.$$

We say that two presentations are equivalent ($\simeq$) if they give homeomorphic surfaces.
We now suppose given a surface S with a presentation $\langle a_1, \dots, a_n \mid W \rangle$, and we show this presentation is equivalent to a presentation of a surface from the list.

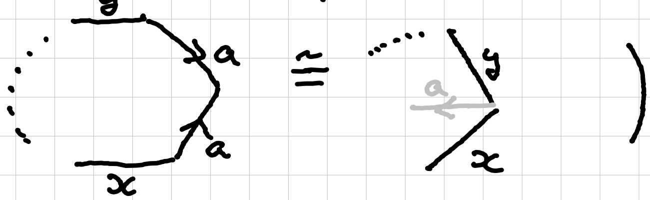
<u>Remark</u>: We have seen that if S, T are surfaces with presentations $\langle a_1, \dots, a_n \mid W \rangle$ and $\langle b_1, \dots, b_m \mid X \rangle$, then $\langle a_1, \dots, a_n, b_1, \dots, b_m \mid WX \rangle$ is a presentation of $S \# T$.

We define operations that replace a presentation by an equivalent one:
(R1) Exchange $a \leftrightarrow a^{-1}$ or rename a letter with one not used: obvious.

(R2) Rotation of a word: cyclic permutation of the letters (just start at another vertex of P).

(R3) Suppress aa^{-1} from the word.

(We accept $\langle a \mid aa^{-1} \rangle = \langle 1 \rangle$ as presentation of S^2, and $\langle a \mid aa \rangle$ as presentation of $\mathbb{RP}^2$; this is the operation already described:

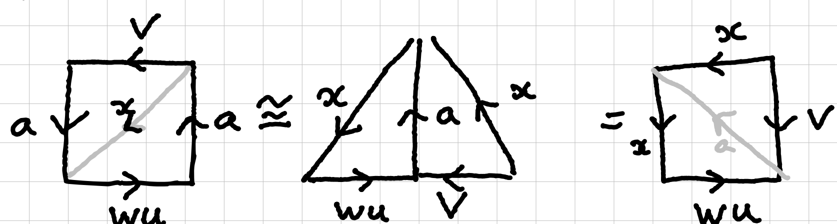


)

(R4) Denote portions of words by capital letters. Then $UaVaW \approx WUV^{-1}aa$

(V^{-1} is obtained by reversing the order of the letters and changing the exponent: $(bcb^{-1}c)^{-1} = c^{-1}bc^{-1}b^{-1}$)

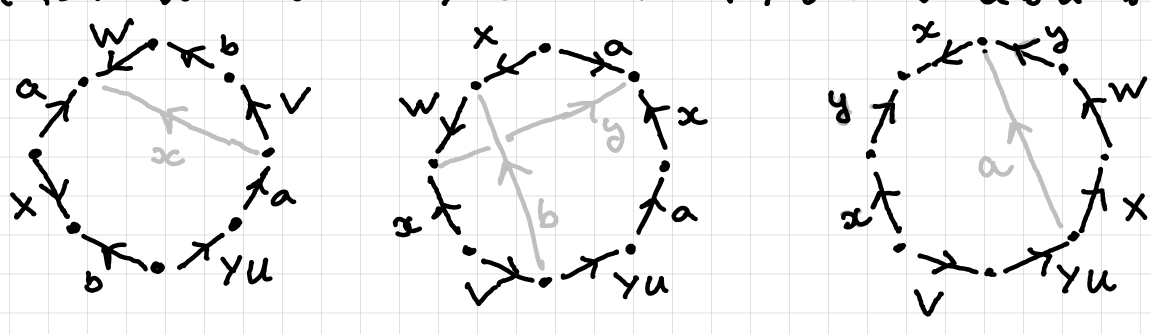
Indeed $UaVaW \approx WUaVa$ (R2), we picture it as:



thus

$WUaVa \approx WUV^{-1}xx \underset{(R1)}{\approx} WUV^{-1}aa$.

(R5) $UaVbWa^{-1}Xb^{-1}Y \approx VYUXW\,aba^{-1}b^{-1}$

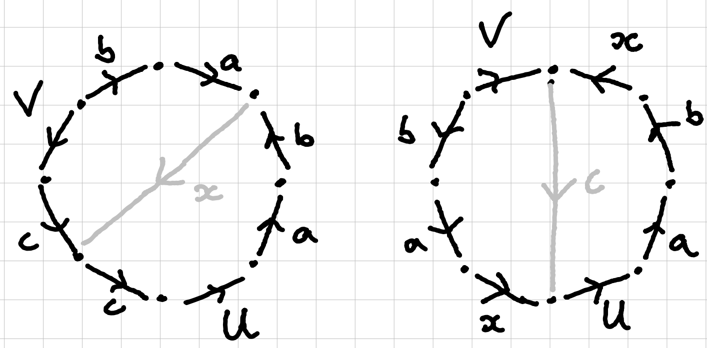


This shows $UaVbWa^{-1}Xb^{-1}Y \approx YUaVbWa^{-1}Xb^{-1}Y$

$\approx VYUXW\,yxy^{-1}x^{-1} \approx VYUXW\,aba^{-1}b^{-1}$.

↑ used also (R2)

(R6) $\quad U a b a^{-1} b^{-1} V c^2 \approx V^{-1} a^2 b^2 U^{-1} c^2$



This shows $U a b a^{-1} b^{-1} V c^2 \approx U a b x V^{-1} b a x$

Now we apply (R4) $\quad E f G f H \approx H E G^{-1} f f$

(we indicate below the word how we use it in the following $\approx$) :

$$\underbrace{U}_{E}\ \underbrace{a}_{f}\ \underbrace{b\, x\, V^{-1} b}_{G}\ \underbrace{a}_{f}\ \underbrace{x}_{H} \overset{(R4)}{\approx} \underbrace{x\, U}_{E}\ \underbrace{b^{-1}}_{f}\ \underbrace{V x^{-1}}_{G}\ \underbrace{b^{-1}}_{f}\ \underbrace{a^2}_{H} \overset{(R4)}{\approx}$$

$$\underbrace{a^2}_{E}\ \underbrace{x}_{f}\ \underbrace{U}_{G}\ \underbrace{x}_{f}\ \underbrace{V^{-1} b^{-2}}_{H} \overset{(R4)}{\approx} V^{-1} b^{-2} a^2 U^{-1} x^2$$

$$\overset{(R1)}{\approx} V^{-1} a^2 b^2 U^{-1} c^2$$

It is now easy to show using (R1), …, (R6) that any presentation is equivalent to one of the following :

$\langle a \mid a a^{-1} \rangle \qquad (S^2)$

$\langle a_1, \dots, a_n, b_1, \dots, b_n \mid (a_1 b_1 a_1^{-1} b_1^{-1}) \dots (a_n b_n a_n^{-1} b_n^{-1}) \rangle \quad (\#_{i=1}^{n} T^2)$

$\langle a_1, \dots, a_n \mid a_1^2 \dots a_n^2 \rangle \qquad (\#_{i=1}^{n} \mathbb{R}P^2)$

Indeed, note that (R1) → (R6) do not increase the number of letters used and (R3) reduces it. In fact,

(R3) allows to suppress $-\# S^2$,
(R4) identifies an $\mathbb{R}P^2$ summand
(R5) identifies an T^2 summand
} can make an induction on number of letters.

(R6) allows to replace $\mathbb{R}P^2 \# T^2$ by $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$

It remains to show that none of the list are homeomorphic: for this we will introduce algebraic topology (π_1). ▨