3. The fundamental group

3.1 Def.: a *category* C consists of:

(1) a *class* of objects, denoted $Ob(C)$ (or just C).

(2) for any pair $X, Y \in Ob(C)$, a *set* $C(X,Y)$, called the *set of morphisms* from X to Y; $f \in C(X,Y)$ is usually denoted $X \xrightarrow{f} Y$.

(3) for any tripple $X, Y, Z \in Ob(C)$, a "*composition law*"
$C(Y,Z) \times C(X,Y) \xrightarrow{\circ} C(X,Z)$, $(g,f) \mapsto g \circ f$ (or gf)

such that:

(a) The composition is associative

(b) For any $X \in Ob(C)$, there exists an element $1_X \in C(X,X)$ neutral for the composition (left or right), called the *identity* of X.

3.2 Remark: If $Ob(C)$ is a set, we say that C is a *small* category.

3.3 Examples: (a) $\underline{Set}$, the category with:

- $Ob(Set)$: the class of sets.
- $Set(X,Y) =: Map(X,Y)$: the set of maps $X \to Y$
- $1_X : X \to X$, $x \mapsto x$; usual composition.

(b) Top, the category of topological spaces:
$Ob(Top)$: the class of topological spaces.
$Top(X,Y) = \{ f \in Set(X,Y) ;\ f \text{ continuous} \}$

(c) Gp, the category of groups and group homomorphisms.

(d) Ring, the category of unital, associative rings and ring homomorphisms.

(e) R a ring; R-Mod (Mod-R), the category of left (right) R-modules.

Notice that none of the examples above are small.

The following are examples of small categories:

(f) If G is a group, define a category $\underline{G}$ with a unique object $*$: $Ob(\underline{G}) = \{*\}$ and morphisms $\underline{G}(*,*) := G$ (underlying set of G), together with:

- $1_* = e$ (neutral element of G)
- Composition $\underline{G}(*,*) \times \underline{G}(*,*) \to \underline{G}(*,*)$ is the product of G: $g \circ f = gf$.

(g) $\mathcal{F}in$, the category with objects

$Ob(\mathcal{F}in) = \{ X \in \mathcal{P}(\mathbb{N}) ; \exists n \in \mathbb{N}, X = \underline{n} \}$

where $\underline{n} = \{ a \in \mathbb{N} ; 1 \leq a \leq n \}$

$\mathcal{F}in(\underline{m}, \underline{n}) = Set(\underline{m}, \underline{n})$, and usual composition.

(h) We can also take smaller sets of morphisms:

$\mathcal{I}$: $ob(\mathcal{I}) = Ob(\mathcal{F}in)$

$\mathcal{I}(\underline{m}, \underline{n}) = \{ \alpha \in \mathcal{F}in(\underline{m}, \underline{n}) ; \alpha \text{ injection} \}$

Σ : $Ob(\Sigma) = Ob(\mathcal{F}in)$

$\Sigma(\underline{m}, \underline{n}) = \{ \alpha \in \mathcal{F}in(\underline{m}, \underline{n}) ; \alpha \text{ bijection} \}$

<u>3.4 Definition</u>: let C be a category. A morphism $f \in C(X,Y)$ is called an <u>isomorphism</u> if $\exists g \in C(Y,X)$, $f \circ g = id_Y$ & $g \circ f = id_X$ (then unique! Noted f^{-1})

A <u>groupoid</u> is a small category C where all morphisms are isomorphisms.

<u>3.5 Examples</u>: In 3.3, $\underline{G}$ and Σ are groupoids.

<u>3.6 Definition</u>: let C be a category. We define the

opposite category C^{op} by : $Ob(C^{op}) = Ob(C)$, and for any $X, Y \in Ob(C^{op})$,

$$C^{op}(X,Y) := C(Y,X), \text{ together with}$$

$$C^{op}(Y,Z) \times C^{op}(X,Y) \longrightarrow C(X,Z)$$

$$(f^{op}, g^{op}) \longmapsto f^{op} \circ g^{op} := (\underbrace{g \circ f}_{\text{in } C})^{op}$$

"Revert all arrows".

3.7 Definition Let C, D be categories. A *functor* $F: C \to D$ is

(a) A relation assigning to each $X \in Ob(C)$ a unique $F(X) \in Ob(D)$.

(b) For all $X, Y \in Ob(C)$, $F = F_{X,Y}: C(X,Y) \to D(F(X), F(Y))$ such that $F(1_X) = 1_{F(X)}$, for all $X \in Ob(C)$, and $F(g) \circ F(f) = F(g \circ f)$ for all composable pair (g, f) of morphisms in C.

A *cofunctor* or a *contravariant functor* $C \to D$. is a functor $C^{op} \to D$ (Spell it out!).

3.8 Examples: (a) If C, D are categories, and $Z \in Ob(D)$, define the *constant functor* $c_Z : C \to D$ by $c_Z(X) = Z$ for all $X \in Ob(C)$, and $c_Z(f) = 1_Z$ for all morphisms of C.

(b) We have the *forgetfull functors*

F: Top $\to$ Sets , F: Gp $\to$ Sets , F: R-Mod $\to$ Sets :

$F(X)$ = underlying set of X, $F(f)$ = underlying map of sets.

(c) If C is a category and $X \in Ob(C)$, we have a cofunctor $F^X : C \to$ Sets, given by

$F^X(Y) = C(Y, X)$ for all $Y \in Ob(C)$, and for $f: Y \to Z$ in C, $F^X(f): C(Z,X) \to C(Y,X)$, $g \mapsto g \circ f$.

It is called the co-functor represented by X.

Similarly, we have a functor $F_X : \mathcal{C} \to \mathrm{Sets}$ called the functor co-represented by X : $F_X(Y) = \mathcal{C}(X,Y)$, and $F_X(f) : \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$, $\alpha \mapsto f\circ\alpha$, for any $f : Y \to Z$ in $\mathcal{C}$.

(d) If $\mathcal{C}$ is a category, we have the identity functor

$Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, $Id_{\mathcal{C}}(X) = X$ and $Id_{\mathcal{C}}(f) = f$.

<u>3.9 Definition</u> : Let $\mathcal{C}$ be a category. A <u>sub-category</u> of $\mathcal{C}$ is a category $\mathcal{D}$ with

(i) $Ob(\mathcal{D}) \subset Ob(\mathcal{C})$

(ii) $\forall X, Y \in Ob(\mathcal{D})$, $\mathcal{D}(X,Y) \subset \mathcal{C}(X,Y)$

(iii) The inclusion $\mathcal{D} \to \mathcal{C}$ is a functor.

We say that $\mathcal{D}$ is a <u>full</u> subcategory of $\mathcal{C}$ if for any $X, Y \in Ob(\mathcal{D})$, $\mathcal{D}(X,Y) = \mathcal{C}(X,Y)$.

<u>3.10 Examples</u> : (a) $\mathcal{F}in$ is a full subcategory of $\mathcal{S}et$; Σ is a (non full) subcategory of $\mathcal{I}$, which is a (non-full) subcategory of $\mathcal{F}in$.

(b) $\mathcal{G}p$ is not a subcategory of $\mathcal{S}et$!

Top is not a subcategory of Set !

(c) $\mathcal{A}b$, the full subcategory of $\mathcal{G}p$, consisting of <u>abelian</u> groups.

(d) If $\mathcal{D}$ is a subcategory of $\mathcal{C}$, we have a functor $i : \mathcal{D} \to \mathcal{C}$, $i(X) = X$, $i(f) = f$, by definition of a subcategory.

3.11 Definition : If $C \xrightarrow{F} D$ and $D \xrightarrow{G} \mathcal{E}$ are functors, we can define the composition functor $G \circ F : C \to \mathcal{E}$ by $(G \circ F)(X) = G(F(X))$ for any $X \in Ob(C)$, and $(G \circ F)(f) = G(F(f))$ for any $f : X \to Y$ in C.

3.12 Remark : the composition of functors is associative, and identity functors are neutral for the composition.

3.13 Exercise : we have defined the notion of a product, coproduct, pull-back, push-out of spaces by solely using the notion of a category. Can do the same for any category (but they do not nec. exist!).

☞ We will introduce more categorical notions when needed! We now introduce the homotopy category of spaces

As above (and as in chapter 1), we denote by Top the category of topological spaces and continuous maps.

3.14 Def For $f, g \in Top(X, Y)$, a homotopy from f to g is a map $H : X \times I \to Y$, $I = [0,1] \subset \mathbb{R}$, such that

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes ; here $i_t : X \to X \times I$, $x \mapsto (x, t)$

We denote this by $f \overset{H}{\simeq} g$. We say that f and g are homotopic (denoted $f \simeq g$) if $\exists H$, $f \overset{H}{\simeq} g$.

We denote $H_t := H \circ i_t : X \to Y$.

3.15 Lemma : The relation $\simeq$ on $Top(X, Y)$ is an equivalence rel. This relation is compatible with composition in the sense that

$$Top(Y,Z) \times Top(X,Y) \xrightarrow{\circ} Top(X,Z)$$
$$g, g' \qquad f, f' \qquad g \circ f, g' \circ f'$$

$f \simeq f'$ and $g \simeq g'$ $\Rightarrow$ $g \circ f \simeq g' \circ f'$.

proof: let $f, g, h \in Top(X,Y)$.
By $X \times I \xrightarrow{H} Y$, $H(x,t) = f(x)\ \forall (x,t)$, we see $f \overset{H}{\simeq} f$.
If $f \overset{K}{\simeq} g$, then $g \overset{L}{\simeq} f$ by $L: X \times I \to Y$, $L(x,t) = K(x, 1-t)$
If $f \overset{K}{\simeq} g \overset{L}{\simeq} h$, then $f \overset{K*L}{\simeq} g$ where $K*L : X \times I \to Y$ is given by
$$K*L(x,t) = \begin{cases} K(x, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ L(x, 2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Thus $\simeq$ is an equivalence relation on $Top(X,Y)$.
For the compatibility with composition, given $f \overset{F}{\simeq} f'$ and $g \overset{G}{\simeq} g'$, we use transitivity $g \circ f \overset{K}{\simeq} g' \circ f \overset{L}{\simeq} g' \circ f'$ for
$K: X \times I \xrightarrow{f \times id} Y \times I \xrightarrow{G} Z$, $L: X \times I \xrightarrow{F} Y \xrightarrow{g'} Z$. ▨

<u>**3.16 Def**</u> We define $ho(Top)$, the <u>homotopy category of spaces</u> by $Ob(ho(Top)) = Ob(Top)$, $hoTop(X,X] := [X,Y] := Top(X,Y)/_{\simeq}$.
The class of $f \in Top(X,Y)$ in $[X,Y]$ is denoted $[f]$, and

composition is defined by $[f] \circ [g] := [f \circ g]$ (well defined by 3.15). The identity of $X \in hoTop$ is $1_X = [id_X]$

<u>Remark</u>: We have an obvious functor $Top \to hoTop$.

3.<u>17 Def</u>: A map $f: X \to Y$ is called a homotopy equivalence if $[f] \in hoTop(X,Y)$ is an isomorphism.
We say that spaces X, Y are <u>homotopy equivalent</u>, (denoted $X \simeq Y$), if they are isomorphic in $hoTop$. We say that X is <u>contractible</u> if $X \simeq \{*\}$ (one-point space).

<u>3.18 Examples</u> (a) Given X a space, any $f, g : X \to \mathbb{R}^n$ $(n \in \mathbb{N})$ are homotopic: $f \overset{H}{\simeq} g$ with $H: X \times I \to \mathbb{R}^n$, $H(x,t) = (1-t)f(x) + t g(x)$.
Thus $[X, \mathbb{R}^n]$ has a unique element!
In particular, $\{0\} \hookrightarrow \mathbb{R}^n$ is a homotopy equivalence, thus $\mathbb{R}^n$ is contractible.

(b) Suppose $f : X \to S^n$ is non surjective. Then f is homotopic to a constant map.

Proof: Assume $\mathrm{Im}(f) \subset S^n \setminus \{N\}$. North, south pole

We have a homeomorphism $\mathbb{R}^n \xrightarrow{h} S^n \setminus \{N\}$, $0 \mapsto S$

Therefore f factorises as $X \xrightarrow{g} \mathbb{R}^n \xrightarrow{h} S^n \setminus \{N\}$.

If $c_0 : X \to \mathbb{R}^n$ is the constant map to $0 \in \mathbb{R}^n$, we know by (a) that $\exists H : X \times I \to \mathbb{R}^n$, $g \overset{H}{\simeq} c_0$.

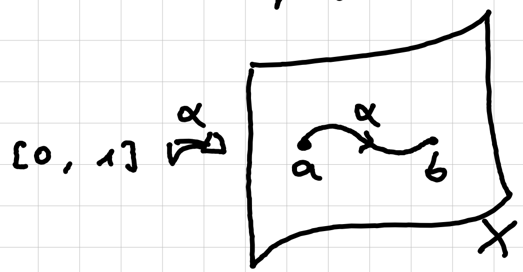
Thus by Lemma 3.15, $i h H : X \times I \to S^n \setminus \{N\} \overset{i}{\hookrightarrow} S^n$ is a homotopy between f and c_S.

(c) The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence (exercise).

(d) $[S^n, S^m]$ is very hard to study!

<u>3.19 Definition</u>: Let X be a space, and $a, b \in X$.

A <u>path</u> from a to b in X is a continuous map $\alpha : I \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$.



Denote by $\Omega(X, a, b)$ the set of all paths from a to b in X.

We say that $\alpha, \beta \in \Omega(X, a, b)$ are <u>homotopic relative to $\partial I = \{0, 1\}$</u> if $\exists H : I \times I \to X$, s.t.

(i) $H_0(s) := H(s, 0) = \alpha(s)$ and $H_1(s) := H(s, 1) = \beta(s)$ for all $s \in I$;

(ii) $H_t(0) = H(0, t) = a$ and $H_t(1) = H(1, t) = b$ for all $t \in I$.

We denote it by $\alpha \sim \beta$ or $\alpha \overset{H}{\sim} \beta$.

<u>3.20 Lemma</u> $\sim$ is an equivalence relation on $\Omega(X, a, b)$.

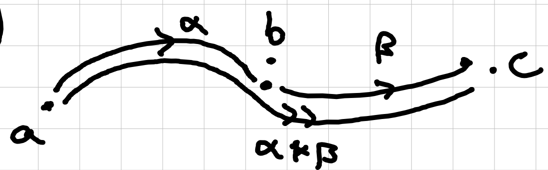
proof: it is the same as the proof of 3.15. $\boxtimes$

<u>3.21 Def</u>: let $a, b, c \in X$. We define a map

$$\Omega(X, a, b) \times \Omega(X, b, c) \to \Omega(X, a, c)$$
$$(\alpha, \beta) \mapsto \alpha * \beta$$

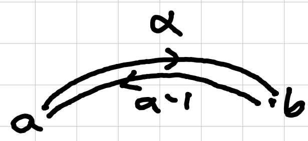
where $(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$

(called <u>the concatenation of paths</u>)



We define a map

$$\Omega(X, a, b) \to \Omega(X, b, a)$$
$$\alpha \mapsto \alpha^{-1}$$



by $\alpha^{-1}(t) = \alpha(1-t) \quad \forall t \in I$

We define $c_a \in \Omega(X, a, a)$ by $c_a(t) = a \quad \forall t \in I$

<u>3.22 Lemma</u>:

(a) The concatenation is compatible with $\sim$:
If $\alpha, \alpha' \in \Omega(X, a, b)$, $\beta, \beta' \in \Omega(X, b, c)$, and if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha * \beta \sim \alpha' * \beta'$.

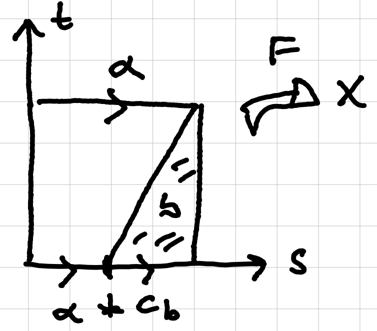
(b) Given $\alpha \in \Omega(X, a, b)$, $\beta \in \Omega(X, b, c)$, $\gamma \in \Omega(X, c, d)$, we have $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$ in $\Omega(X, a, d)$.

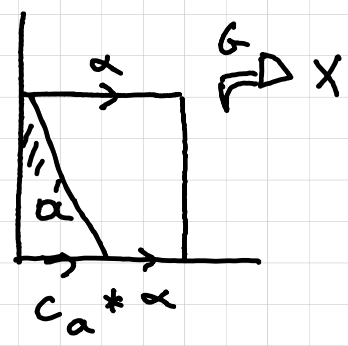
(c) Given $\alpha \in \Omega(X, a, b)$, $c_a * \alpha \sim \alpha \sim \alpha * c_b$.

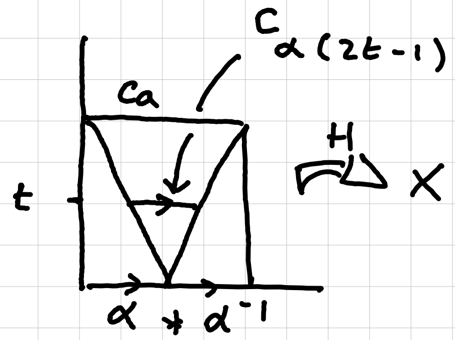
(d) Given $\alpha \in \Omega(X, a, b)$, $\alpha * \alpha^{-1} \sim c_a$ and $\alpha^{-1} * \alpha \sim c_b$.

proof: (a) Take $\alpha \overset{G}{\sim} \alpha'$ and $\beta \overset{H}{\sim} \beta'$.
Define $G * H : I \times I \to X$, $G * H(s, t) = \begin{cases} G(s, 2t) & t \le \frac{1}{2} \\ H(s, 2t-1) & t \ge \frac{1}{2} \end{cases}$
Then $\alpha * \beta \overset{G*H}{\sim} \alpha' * \beta'$.

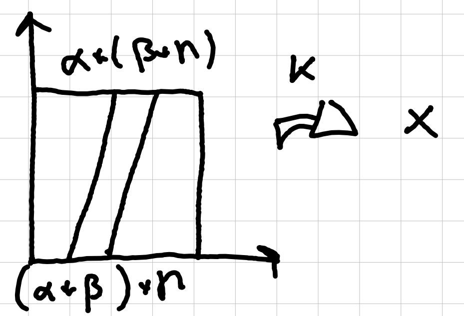
(b) - (d) We just indicate by a picture how the homotopies can be defined, and leave it as an an exercise to write down the precise definition of the homotopies with a formula:







$$\alpha * c_b \overset{F}{\sim} \alpha \qquad c_a * \alpha \overset{G}{\sim} \alpha \qquad \alpha * \alpha^{-1} \sim c_a$$



$$(\alpha * \beta) * \gamma \overset{K}{\sim} \alpha * (\beta * \gamma)$$ ▨

<u>3.23 Lemma + Def</u>: Let X be a space. Define a category πX by $Ob(\pi X) = X$ (underlying set of X), and, for any $a, b \in X$, $\pi X(a,b) = \Omega(X,a,b)/\sim$, for $\sim$ defined in 3.19.

We define $1_a = [c_a] \in \pi X(a,a)$, and

⚠ Beware the reversed order

$$\pi X(b,c) \times \pi X(a,b) \xrightarrow{\circ} \pi X(a,c)$$
$$([\beta],[\alpha]) \longmapsto [\beta]\circ[\alpha] := [\alpha * \beta]$$

This is well defined, and πX is a groupoid called <u>the fundamental groupoid of X</u>.

<u>proof</u>: By 3.22(a), the composition is well defined: If $\beta \sim \beta'$, $\alpha \sim \alpha'$, then $[\alpha * \beta] = [\alpha' * \beta']$.
By 3.22(b), $\circ$ is associative, and by 3.22.(c), 1_a is neutral for composition on the left / right.

Finally, by 3.22 (d), any morphism $[\alpha] \in \Pi X(a,b)$ is an isomorphism, since $[\alpha] \circ [\alpha^{-1}] = [\alpha^{-1} * \alpha] = [c_b] = 1_b$ and $[\alpha^{-1}] \circ [\alpha] = [\alpha * \alpha^{-1}] = [c_a] = 1_a$, so that $[\alpha^{-1}] = [\alpha]^{-1}$. ▨

3.24 Definition If $\mathcal{G}$ is a groupoid, let $\pi_0(\mathcal{G}) := Ob(\mathcal{G})/\approx$, where $\approx$ is the equivalence relation(*) on $Ob(\mathcal{G})$ defined by $a \approx b \iff \mathcal{G}(a,b) \neq \emptyset$.

We call $\pi_0 \mathcal{G}$ the set of connected components of $\mathcal{G}$.

For $a \in Ob(\mathcal{G})$, the automorphism group of a, $Aut_{\mathcal{G}}(a) := \{\alpha \in \mathcal{G}(a,a) ; \alpha \text{ iso}\} = \mathcal{G}(a,a)$ is also denoted $\pi_1(\mathcal{G}, a)$.

$\mathcal{G}$ a groupoid!

((*) This is an equiv. relation since $\mathcal{G}$ is a groupoid)

3.25 Def: If X is a space, $\pi_0(X) := \pi_0(\Pi X)$ is called the set of path-connected components of X.

For $a \in X$, $\pi_1(X,a) := \pi_1(\Pi X, a) = Aut_{\Pi X}(a)$ is called the fundamental group of the pointed space (X,a).

3.26 Examples: (a) We have $Ob(\Pi \mathbb{R}^n) = \mathbb{R}^n$ (as sets), and for any $a,b \in \mathbb{R}^n$, $\Pi\mathbb{R}^n(a,a) = \{f_{a,b}\}$ is a singleton: $f_{a,b} = [\alpha]$ where $\alpha : I \to \mathbb{R}^n$, $\alpha(t) = (1-t)a + tb$. Indeed, $\alpha(0) = a$ and $\alpha(1) = b$, so $[\alpha] \in \Pi\mathbb{R}^n(a,b)$, and for any $\beta \in \Omega(\mathbb{R}^n, a, b)$, $\alpha \sim \beta$ (same argument as in 3.18. (a)).

In summary: between any two objects a,b of $\Pi\mathbb{R}^n$, there is a unique morphism $a \to b$ (in fact an isomorphism).

Thus $\pi_0(\mathbb{R}^n) = \{[0]\}$ (the $\approx$ class of $0 \in \mathbb{R}^n$) and

$\pi_1(X, a) = \{1_a\}$, the trivial group.

(b) Let X be a space with the discrete topology (any point forms an open subset). Then $\mathrm{Ob}(\pi X) = X$ (as a set), and for all $a, b \in X$,

$$\pi X(a,b) = \begin{cases} \emptyset & a \neq b \\ \{1_a\}, & a = b \end{cases}$$

We deduce $\pi_0 X = X$ and $\pi_1(X, a) = \{1_a\}$ (the trivial group).

(c) We will see that $\pi_1(S^1, a) \cong \mathbb{Z}$ $\forall a \in S^1$, which is intuitive but not obvious to prove!

<u>3.27 Remark (Functorial properties)</u>: If $f : X \to Y$ is a continuous map, we can define a functor

$f_* : \pi X \to \pi Y$ defined as follows:

(i) On objects, f_* is the underlying map of sets of f:

$$\begin{aligned} \mathrm{Ob}\, X = X &\xrightarrow{f} Y = \mathrm{Ob}\, Y \\ a &\mapsto f(a) \end{aligned}$$ (Here X, Y are viewed as sets).

(ii) On morphisms,

$$\begin{aligned} \pi X(a,b) &\longrightarrow \pi Y(f(a), f(b)) \\ [\alpha] &\longmapsto f_*([\alpha]) = [f \circ \alpha] \end{aligned}$$

(note that this is well-defined since $\alpha \sim \alpha' \Rightarrow f\circ\alpha \sim f\circ\alpha'$).

It is easy to check that this defines a functor:

$f_*(1_a) = 1_{f(a)}$ and $f_*([\alpha]\circ[\beta]) = f_*([\alpha]) \circ f_*([\beta])$

(left as an exercise!).

<u>3.28 Definition</u>: Let $F, G : C \to D$ be two functors. A natural transformation $\eta : F \Rightarrow G$ from F to G is a collection of morphisms of D $\eta_a : F(a) \to G(a)$ indexed by $a \in \mathrm{Ob}(C)$, such that for any morphism $\alpha : a \to b$ in C, the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{F(\alpha)} & F(b) \\ \downarrow{\scriptstyle \eta_a} & & \downarrow{\scriptstyle \eta_b} \\ G(a) & \xrightarrow{G(\alpha)} & G(b) \end{array}$$

commutes.

3.29 Example: let $f, g : X \to Y$ be continuous maps, and $H : X \times I \to Y$ with $f \overset{H}{\simeq} g$.

Then H provides a natural transformation $\eta : f_* \Rightarrow g_*$ of functors $\Pi X \to \Pi Y$, defined as follows:

for any $a \in X = Ob(\Pi X)$, let $\eta_a := [H_a]$, where $H_a : I \to Y$, $t \mapsto H(a,t)$ is a path in Y from $f_*(a)$ to $g_*(a)$. We must show that if $[\alpha] \in \Pi X(a,b)$, then the following diagram commutes:

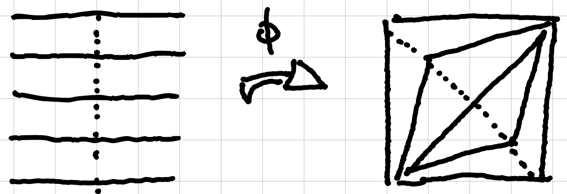
$$\begin{tikzcd} f(a) \arrow[r, "{[f\circ\alpha]}"] \arrow[d, "\eta_a"'] & f(b) \arrow[d, "\eta_b"] \\ g(a) \arrow[r, "{[g\circ\alpha]}"'] & g(b) \end{tikzcd}$$

In more details, we must show that there exists a homotopy K:

$(f\circ\alpha) * H_b \overset{K}{\simeq} H_a * (g\circ\alpha)$ (which implies $[H_b]\circ[f\circ\alpha] = [g\circ\alpha]\circ[H_a]$).

We define K as a composition:

$I \times I \xrightarrow{\phi} I \times I \xrightarrow{\alpha \times id} X \times I \xrightarrow{H} Y$ where ϕ is a piecewise affine map described by the following picture:



Exercise: write out a formula for ϕ

3.30 Example: let $f, g : X \to \mathbb{R}^n$ be two continuous map. We know that $f \simeq g$; as homotopy, we can take $H : X \times I \to \mathbb{R}^n$, $H(x,t) = (1-t)f(x) + t\,g(x)$.

For example, if $X = \mathbb{R}^n$, $f = id_{\mathbb{R}^n}$ and g is the constant map with value $0 \in \mathbb{R}^n$, then the natural transformation given by H is $\eta_x : x \xrightarrow{[H_x]} 0$ (the path $t \mapsto (1-t)\cdot x$)

Notice in this example of $Y = \mathbb{R}^n$, the natural transfo is unique ($\forall\, x, y \in \mathbb{R}^n$, up to homotopy there is only one path from x to y).

3.31 Remark: If $F, G, H : C \to D$ are functors, and if $\varepsilon : F \Rightarrow G$, $\eta : G \Rightarrow H$ are natural transformations, then we have an obvious composition of natural transformations: $\eta \circ \varepsilon : F \Rightarrow H$, defined by
$(\eta \circ \varepsilon)_a = \eta_a \circ \varepsilon_a$ for all $a \in Ob(C)$.
This composition is associative, and the identity natural transformation $id_F : F \Rightarrow F$, $(id_F)_a = id_{F(a)}$ $\forall a \in Ob(C)$ is neutral.

3.32 Proposition:
(1) $\pi_0 : Top \to Sets$ is a functor, and for $f \simeq g : X \to Y$, we have $f_* = g_* : \pi_0 X \to \pi_0 Y$.
In particular, π_0 factorizes as $Top \longrightarrow hoTop \xrightarrow{\pi_0} Set$.

(2) $\pi_1 : Top_* \to Gp$ is a functor from the category of pointed spaces to the category of groups.
If $f \overset{H}{\simeq} g : (X, x_0) \to (Y, y_0)$ is a pointed homotopy (i.e. $H(x_0, t) = y_0 \ \forall t \in I$), then

$$f_* = g_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

In particular, π_1 factorizes as $Top_* \to hoTop_* \xrightarrow{\pi_1} Gp$.
(3) Let $f, g : (X, x_0) \to (Y, y_0)$ be a pointed maps, and $H : X \times I \to Y$ be a (not necessarily pointed) homotopy between f and g. Denote by $h_{x_0} = [H(x_0, -)]$ the element of $\pi_1(Y, y_0)$ corresponding to the loop $H(x_0, -) : I \to Y$, $t \mapsto H(x_0, t)$. Let $\tau_H : \pi_1(Y, y_0) \circlearrowleft$ be the conjugation by h_{x_0}: $\tau_H([\alpha]) = h_{x_0} [\alpha] \cdot h_{x_0}^{-1}$.
Then $g_* = \tau_H \circ f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

proof: obvious. Let us provide some details:

(1) $f: X \to Y$ induces a functor of groupoids $f_*: \Pi X \to \Pi Y$. Obviously, if $a \approx b$ in $\mathrm{Ob}\,\Pi X = X$, then $f(a) \approx f(b)$ since if α is a path in X, $f\circ\alpha$ is a path in Y. So f induces $f_*: \pi_0 X \to \pi_0 Y$.
$$[a] \mapsto [f(a)]$$

If $f \overset{H}{\simeq} g: X \to Y$, and $a \in \mathrm{Ob}(X)$, then Ha is a path in Y from $f(a)$ to $g(a)$, therefore $f(a) \approx g(a)$ and $f_*([a]) = [f(a)] = [g(a)] = g_*([a])$, so $f_* = g_*$.

(2) left as an exercise

(3) By 3.29 H provides a natural transformation $\eta: f \Rightarrow g$; In particular, for any $[\alpha] \in \Pi X(x_0, x_0)$, we have a commutative diagram of the form

$$\left(\begin{array}{ccc} f_*(x_0) & \xrightarrow{f([\alpha])} & f_*(x_0) \\ \eta_{x_0}\downarrow & & \downarrow \eta_{x_0} \\ g_*(x_0) & \xrightarrow{g_*([\alpha])} & g_*(x_0) \end{array}\right) = \left(\begin{array}{ccc} y_0 & \xrightarrow{f_*([\alpha])} & y_0 \\ \downarrow h_{x_0} & & \downarrow h_{x_0} \\ y_0 & \xrightarrow{g_*([\alpha])} & y_0 \end{array}\right)$$

Thus $g_*([\alpha]) \cdot h_{x_0} = h_{x_0} \cdot f_*([\alpha])$ implying $g_* = \tau_H \circ f_*$. ◪

__Note__ (Dependence of π_1 on the basepoint): If x_0, x_1 lie in the same path-connected component of X (i.e. $[x_0] = [x_1]$ in $\pi_0 X$), then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are (non-canonically) isomorphic: If $\gamma: I \to X$ is a path from x_0 to x_1 in X, then
$$\tau_\gamma: \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1),\ [\alpha] \mapsto [\gamma * \alpha \gamma^{-1}]$$
is an iso.

As a corollary, we obtain:

3.33 Proposition: If $f: X \to Y$ in Top is a homotopy equivalence, and if $x_0 \in X$, then

$$f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

is an isomorphism.

proof: Choose $g: Y \to X$ with homotopies $g \circ f \overset{H}{\simeq} id_X$, $f \circ g \overset{K}{\simeq} id_Y$. Then the following diagram commutes:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[rr, "g_* \circ f_*"] \arrow[dr, "id = id_{X*}"'] & & \pi_1(X, g \circ f(x_0)) \arrow[dl, "\tau_H", "\cong"'] \\ & \pi_1(X, x_0) & \end{tikzcd}$$

by 3.32 (3).

Thus $g_* \circ f_* = \tau_H^{-1}$ is an iso, so f_* injective (and g_* surjective). Repeat with $f \circ g \overset{K}{\simeq} id_Y$ to deduce that f_* is surjective. ▨

3.34 Theorem (Seifert - Van Kampen, groupoid version)

Let $X \in$ Top, U_0, U_1 subsets of X with $\mathring{U}_0 \cup \mathring{U}_1 = X$ (where $\mathring{U}$ denotes the interior of $U \subset X$).
Let $U_{01} = U_0 \cap U_1$, and $U_{01} \xrightarrow{h_i} U_i \xrightarrow{i_i} X$, $i = 0, 1$, be the inclusion. Then

$$\begin{tikzcd} \Pi U_{01} \arrow[r, "h_0"] \arrow[d, "h_1"'] & \Pi U_0 \arrow[d, "i_0"] \\ \Pi U_1 \arrow[r, "i_1"] & \Pi X \end{tikzcd}$$

is a push-out in the category of groupoids and functors. (denoted GP)

We will need the following result of general topology, which we recall:

Lemma (Lebesgue Number) Let M be a compact metric space and $\mathcal{U} = \{U_i\}_{i \in \Theta}$ an open cover of M. Then $\exists\, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$, and a function $\sigma: M \to \Theta$, with the following property:

$\forall x \in M,\ B(x,\varepsilon) \subset \mathcal{U}_{\sigma(x)}$,

where $B(x,\varepsilon) = \{ y \in M ;\ d(x,y) < \varepsilon \}$.

Here ε is called a <u>Lebesgue Number</u> for $\mathcal{U}$.

<u>proof of 3.34</u>: Suppose given a commutative diagram in GP:

$$\begin{tikzcd}
\pi \mathcal{U}_{01} \arrow[r, "h_{0*}"] \arrow[d, "h_{1*}"'] & \pi \mathcal{U}_0 \arrow[d, "i_{0*}"] \arrow[ddr, bend left, "j_0"] & \\
\pi \mathcal{U}_1 \arrow[r, "i_{1*}"] \arrow[drr, bend right, "j_1"'] & \pi X \arrow[dr, dashed, "j"] & \\
 & & G
\end{tikzcd} \qquad (\star)$$

We must show: $\exists!$ functor $j : \pi X \to G$ making the resulting diagram commute.

(1) On objects: Obviously
$$\begin{tikzcd}
\mathcal{U}_{01} \arrow[r] \arrow[d] & \mathcal{U}_0 \arrow[d] \\
\mathcal{U}_1 \arrow[r] & X
\end{tikzcd}$$
is a pushout in Sets. Thus $\exists!$ map of sets $Ob\,\pi X \xrightarrow{j} Ob(G)$ making the diagram commute on Objects. The "harder" part of the proof is to:

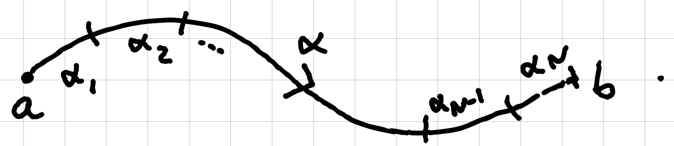
(1) define $\pi X(a,b) \longrightarrow G(j(a), j(b))\ \forall a,b \in X$

(2) show this makes j a functor

(3) show such a j is unique.

We start with (3), this will also indicate how to proceed for (1). So assume $\exists\ j : \pi X \to G$ a functor making $(\star)$ commute. Suppose $a,b \in Ob(\pi X) = X$, and $[\alpha] \in \pi X(a,b)$. Thus $\alpha : I \to X$, $\alpha(0) = a$, $\alpha(1) = b$.
$\exists N \in \mathbb{N}^*$ s.t. $\varepsilon = \frac{1}{N}$ is a Lebesgue number for $\mathcal{U} = \{ \mathring{\mathcal{U}}_0, \mathring{\mathcal{U}}_1 \}$. Then $I = I_1 \cup I_2 \cup \dots \cup I_N$ where $I_k = [\frac{k-1}{N}, \frac{k}{N}]$ and $I_k \cap I_{k+1} = \{\frac{k}{N}\}$.
Let $\alpha_k : I \longrightarrow X$, $\alpha_k(t) = \alpha\left(\frac{k-1+t}{N}\right)$ for $k = 1, \dots N$. Then $\alpha = \alpha_N * \alpha_{N-1} * \dots * \alpha_2 * \alpha_1$ the concatenation is chosen so that $\alpha(I_k) = \alpha_k(I)$

so that we have $\alpha =$ (and not $\alpha \sim \alpha_1 * \dots * \alpha_N$). Picture:



Since $\frac{1}{N}$ is a Lebesgue Number for $\{\mathring{U}_0, \mathring{U}_1\}$,

we have $\sigma : \{1, \dots, N\} \to \{0,1\}$ s.t. $\alpha_k(I) \subset U_{\sigma(k)}$ $\forall\, 1 \leq k \leq N$.

If $\beta_k : I \longrightarrow U_{\sigma(k)}$, $t \mapsto \alpha_k(t)$, then we have

$\alpha_k = i_{\sigma(k)} \circ \beta_k$. In particular, we have

$$j([\alpha_k]) = j([i_{\sigma(k)} \circ \beta_k]) = j \circ i_{\sigma(k)*}([\beta_k])$$

$= j_{\sigma(k)}([\beta_k])$. This shows that ($\star$)

$$j([\alpha]) = j([\alpha_1 * \dots * \alpha_N]) = j([\alpha_N] \circ \dots \circ [\alpha_1]) \overset{j \text{ is a functor}}{=}$$

$$j([\alpha_N]) \circ \dots \circ j([\alpha_1]) = j_{\sigma(N)}([\alpha_N]) \circ \dots \circ j_{\sigma(1)}([\alpha_1])$$

The last term does not depend on j; this shows that j, if it exists as a functor, is also <u>unique</u> on morphisms.

(1) Definition of $\Pi X(a,b) \longrightarrow G(j(a), j(b))$:

given $[\alpha] \in \Pi X(a,b)$, we use the equalities ($\star$) above to define $j([\alpha]) \in G(j(a), j(b))$. We show that this is well defined, i.e. it does not depend on the choices we made:

(i) For $\alpha \in [\alpha]$, we chose N and $\{1, \dots, N\} \xrightarrow{\sigma} \{0,1\}$.

Suppose we chose N' and $\{1, \dots, N'\} \xrightarrow{\sigma'} \{0,1\}$,

giving $\alpha = \alpha'_1 * \dots * \alpha'_{N'} = \alpha_1 * \dots * \alpha_N$.

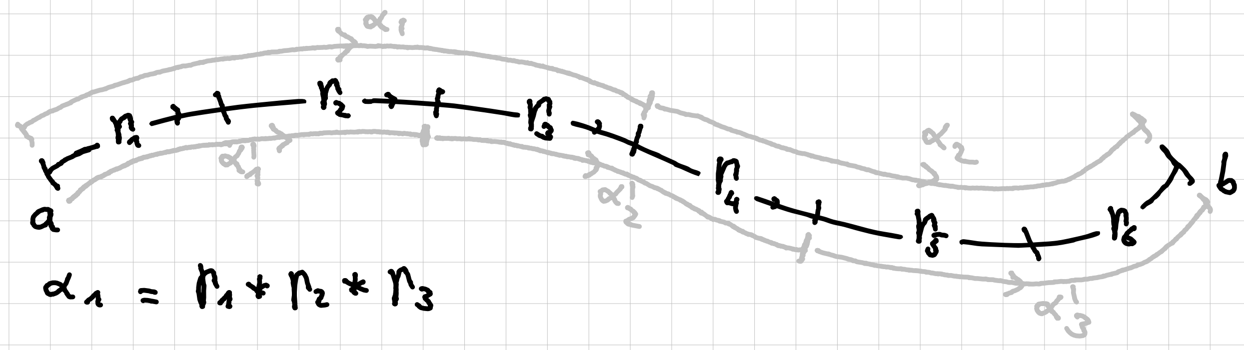
Now we can subdivide further to a common subdivision:

We can choose $\tau : \{1, \dots, N \cdot N'\} \longrightarrow \{0,1\}$ and

$\alpha = \gamma_1 * \dots * \gamma_{NN'}$ s.t. $\alpha'_k = \gamma_{(k-1)N+1} * \dots * \gamma_{kN}$

and $\alpha_k = \gamma_{(k-1)N'+1} * \dots * \gamma_{kN'}$.

Picture for $N = 2$ and $N' = 3$:



$$\alpha_1 = \gamma_1 * \gamma_2 * \gamma_3$$

$$\alpha_2 = \gamma_4 * \gamma_5 * \gamma_6$$

$$\alpha'_1 = \gamma_1 * \gamma_2 \qquad \alpha'_2 = \gamma_3 * \gamma_4 \;, \quad \alpha'_3 = \gamma_5 * \gamma_6$$

$$\alpha_1 * \alpha_2 = \gamma_1 * \dots * \gamma_6 = \alpha'_1 * \alpha'_2 * \alpha'_3 .$$

Notice that for $1 \leq \ell \leq NN'$, $\tau(\ell) \in \{0,1\}$ might be different then the choice of σ or σ' for that portion of α: but this doesn't matter! Indeed, if 2 choices are possible, this means that $\mathrm{Im}(\gamma_\ell) \subset U_{01}$, so by commutativity of ($\star$), we know $j_0([\gamma_\ell]) = j_0 h_0 [\gamma_\ell] = j_1 h_1 [\gamma_\ell] =$

$j_1([\gamma_\ell])$. This shows that $j([\alpha])$ as defined in ($\star$) does not depend on the choice of N and σ:

$$j_{\sigma(N)}([\alpha_N]) \circ \dots \circ j_{\sigma(1)}([\alpha_1]) =$$

$$\left(j_{\sigma(N)}([\gamma_{NN'}]) \circ \dots \circ j_{\sigma(N)}([\gamma_{N(N'-1)+1}]) \right) \circ \dots$$

$$\dots \circ \left(j_{\sigma(1)}([\gamma_{N'}]) \circ \dots \circ j_{\sigma(1)}([\gamma_1]) \right) =$$

$$= j_{\tau(NN')}([\gamma_{NN'}]) \circ \dots \circ j_{\tau(1)}([\gamma_1])) =$$

$$= \left(j_{\sigma'(N')}([\gamma_{NN'}]) \circ \dots j_{\sigma'(N')}([\gamma_{N'(N-1)+1}]) \right) \circ \dots$$

$$\dots \circ \left(j_{\sigma'(1)}([\gamma_N]) \circ \dots \circ j_{\sigma'(1)}([\gamma_1]) \right) =$$

$$j_{\sigma'(N')}([\alpha'_{N'}]) \circ \dots \circ j_{\sigma'(N')}([\alpha'_1]) .$$

(We only used compatibility of j_0 and j_1 with composition).

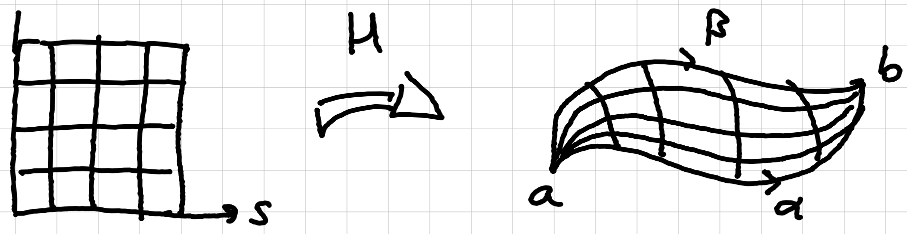
(ii) We must show $j([\alpha])$ does not depend on the choice of the representant α of the homotopy class $[\alpha]$.

Suppose $\alpha, \beta \in \Omega(X, a, b)$, $\alpha \overset{H}{\sim} \beta$, $H : I \times I \to X$.

$\exists\, N \in \mathbb{N}^*$, so that we can subdivide $I \times I$ in sub-squares of side-length $\frac{1}{N}$, and

$$\sigma : \{1, \dots, N\} \times \{1, \dots, N\} \to \{0, 1\}$$

with $H\left(\left[\frac{k-1}{N}, \frac{k}{N}\right] \times \left[\frac{\ell-1}{N}, \frac{\ell}{N}\right]\right) \subset \mathring{U}_{\sigma(k,\ell)}$.



Inductively, we can interpolate from α to β one small square at a time (since each small square is in U_0 or U_1, and $j_0([\alpha_k])$ or $j_1([\alpha_k])$ are well defined and depend only on the homotopy class.

This show that (✡) does not depend of the choice of α in $[\alpha]$.

(3) j is a functor:

(i) $\forall\, a \in Ob(\Pi X)$, $j(1_a) = 1_{j(a)}$: obvious, since if $a \in U_i$, $j(1_a) = j_i(1_a) = 1_{j(a)}$, since j_i is a functor.

(ii) $\forall\, [\alpha] \in \Pi X(a,b)$ and $[\beta] \in \Pi X(b,c)$,
$j([\beta] \circ [\alpha]) = j([\beta]) \circ j([\alpha])$:

Obvious by definition: $j([\beta] \circ [\alpha]) = j([\alpha * \beta])$, and if we have choices $(M, \sigma), (N, \tau)$ for α, β as in (✡), we find $(M+N, \zeta)$, $\zeta(k) = \begin{cases} \sigma(k) & k \le M \\ \tau(k-M) & k > M \end{cases}$
and then take

$$\alpha * \beta = \underbrace{\gamma_1 * \dots * \gamma_m}_{\alpha} * \underbrace{\gamma_{m+1} * \dots * \gamma_{m+n}}_{\beta} . \qquad \blacksquare$$

<u>3.35 Theorem</u> (Seifert-Van Kampen, group version).
Let $X \in \mathrm{Top}$, $U_0, U_1 \subset X$ open, such that U_0, U_1 and $U_{01} = U_0 \cap U_1$ are path connected. Then, for any $x_0 \in U_{01}$, the following square is a push-out of groups:

$$\begin{tikzcd} \pi_1(U_{01}, x_0) \arrow[r, "h_0"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\ulcorner"] & \pi_1(U_0, x_0) \arrow[d, "i_0"] \\ \pi_1(U_1, x_0) \arrow[r, "i_1"'] & \pi_1(X, x_0) \end{tikzcd}$$

This is a special case of the following proposition, where $A = \{x_0\}$:

<u>3.36 Proposition:</u> Let $X \in \mathrm{Top}$, U_0, U_1 open in X with $X = U_0 \cup U_1$, and let $U_{01} = U_0 \cap U_1$. Suppose $A \subset X$ is a subset that meets each path connected component of U_{01}, U_0 and U_1 (and thus also of X).
Denote by $\Pi(X, A)$ the full-subcategory of ΠX with $\mathrm{Ob}(\Pi X(X, A)) = A$.
Then the square:
is a push-out in GD

$$\begin{tikzcd} \Pi(U_{01}, A) \arrow[r] \arrow[d] & \Pi(U_0, A) \arrow[d] \\ \Pi(U_1, A) \arrow[r] & \Pi(X, A) \end{tikzcd} .$$

<u>proof</u>: It is easy to see that this square is a retract of the square given in 3.34, and therefore is also a push-out (Exercise: a retract of a push-out is a push-out).
We have the inclusion $\Pi(X, A) \xhookrightarrow{i} \Pi X$ as a full subcategory. We have to define a functor $r : \Pi X \to \Pi(X, A)$ with $r \circ i = \mathrm{Id}_{\Pi(X,A)}$, the identity functor.
We proceed as follows:

For any $x \in X$, we choose a path ω_x as follows:

(i) If $x \in A$, ω_x is the constant path at x.

(ii) If $x \in U_{01}$, ω_x is a path in U_{01} from x to some chosen $a_x \in U_{01} \cap A$.

(iii) If $x \in U_i \setminus U_{01}$, ω_x is a path in U_i from x to some chosen $a_x \in U_i \cap A$ $(i = 0, 1)$.

Then define $r_X : \pi X \to \pi(X, A)$ as follows:

- On objects, $r_X(x) = \omega_x(1)$
- On morphisms: $\forall\, u \in \pi X(x, y)$,

$$r_X(u) = [\omega_y] \circ u \circ [\omega_x]^{-1} : r_X(x) \longrightarrow r_X(y)$$

Obviously this defines a functor, with $r_X \circ i = id_{\pi(X,A)}$.
Moreover, we have chosen it so that we have a commutative square when restricting to the subcategories πU_{01}, πU_0, πU_1, so that we we have a commutative diagram of functors

$$\begin{tikzcd}
\pi(U_{01},A) \arrow[rrrrr] \arrow[ddddd] \arrow[dr, "i"] & & & & & \pi(U_0,A) \arrow[ddddd] \arrow[dl, "i"'] \\
& \pi U_{01} \arrow[rrr] \arrow[ddd] \arrow[dr, "r_{U_{01}}"] & & & \pi U_0 \arrow[ddd] \arrow[dl, "r_{U_0}"'] & \\
& & \pi(U_{01},A) \arrow[r] \arrow[d] & \pi(U_0,A) \arrow[d] & & \\
& & \pi(U_1,A) \arrow[r] & \pi(X,A) & & \\
& \pi U_1 \arrow[rrr] \arrow[ur, "r_{U_1}"] & & & \pi X \arrow[ul, "r_X"'] & \\
\pi(U_1,A) \arrow[rrrrr] \arrow[ur, "i"] & & & & & \pi(X,A) \arrow[ul, "i"']
\end{tikzcd}$$

with $r_Z \circ i = Id_Z$ for $Z = U_{01}, U_0, U_1, X$.

This is the statement that the outer square is a retract of the middle one.

To be able to use these theorems, we must understand how push-outs of groupoid are formed. We can use the following result:

<u>3.38 Lemma</u> Suppose A_0, A_1 and A_{01} are small categories, and suppose given functors $h_i : A_{01} \to A_i$, $i = 0,1$, such that h_i is injective on objects.
Then the push-out

$$\begin{tikzcd} A_{01} \arrow[r, "h_0"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\ulcorner"] & A_0 \arrow[d, "i_0"] \\ A_1 \arrow[r, "i_1"] & P \end{tikzcd}$$

in small categories and functors exists.

We have $Ob(P) = \text{Push-out}\,(Ob(A_1) \leftarrow Ob(A_{01}) \to Ob(A_1))$ (in Sets), and the morphisms of $P(x,y)$ are equiv. classes of words $a_0 \cdots a_n$ where:

(i) The letters a_i are morphism in A_{01}, or in A_i, $i = 0,1$;
(ii) Source (a_i) = target (a_{i+1}) for $0 \leq i \leq n-1$.
(iii) Source $(a_n) = x$ and target $(a_0) = y$.

(iv) The equivalence relation is generated by the following relations:
- $a_0 \cdots a_i a_{i+1} \cdots a_n \sim a_0 \cdots (a_i \circ a_{i+1}) \cdots a_n$ if a_i and a_{i+1} belong to A_{01}, A_0, or A_1.
- $a_0 \cdots a_{i-1} h_j(a_i) a_{i+1} \cdots a_n \sim a_0 \cdots a_n$.

The composition is given by concatenation of words (when possible); i_0 and i_1 are the obvious functors.
<u>proof</u>: it is easy to see that P is a push-out : given

a diagram of functors of small categories as display, we must show that $\exists!$ functor $j : P \to C$ making the resulting diagram commute.

$$\begin{tikzcd} A_{01} \arrow[r, "h_0"] \arrow[d, "h_1"'] & A_0 \arrow[d, "i_0"] \arrow[ddr, bend left, "j_0"] & \\ A_1 \arrow[r, "i_1"] \arrow[drr, bend right, "j_1"'] & P \arrow[dr, dotted, "\exists! j"] & \\ & & C \end{tikzcd}$$

On objects this is obvious since $Ob(P)$ is the push-out of $(Ob(A_1) \leftarrow Ob(A_{01}) \to Ob(A_1))$.

On morphisms, given $f \in P(x,y)$, we choose a word $a_0 \cdots a_n$ with $[a_0 \cdots a_n] = f$ and each $a_i \in A_0$ or A_1, and a function $\{0, \dots, n\} \xrightarrow{\sigma} \{0,1\}$ with $a_i \in A_{\sigma(i)}$.
Let $j(f) = j_{\sigma(0)}(a_0) \circ \dots \circ j_{\sigma(n)}(a_n)$.
This is well defined (i.e. compatible with the equiv. relation) and obviously a functor; unicity is obvious. $\boxtimes$

<u>3.39 Corollary</u>. (a) Suppose given a diagram of groupoids $(A_0 \xleftarrow{h_0} A_{01} \xrightarrow{h_1} A_1)$ where h_i is injective on objects. Then the push-out exists and is described in 3.38.
(b) The category of groups admits push-out.

<u>Proof</u>: (a) it suffices to notice that in 3.38, if A_{01}, A_0, and A_1 are groupoids, so is P: the inverse of a morphism $[a_0 \dots a_n]$ is $[a_n^{-1} \dots a_0^{-1}]$.
(b) A group is a groupoid with one object. It suffices to notice that given a diagram $A_0 \xleftarrow{h_0} A_{01} \xrightarrow{h_1} A_1$ in groups, obviously h_0 and h_1 are injective on objects (here $Ob(A_{01})$ is a singleton), in fact bijective, so that P is also a groupoid with one object, thus a group. $\boxtimes$

3.40 Definition If G, H are groups, we define the free product $G*H$ as the push-out

$$\begin{tikzcd} \{e\} \arrow[r] \arrow[d] & G \arrow[d] \\ H \arrow[r] & G*H \end{tikzcd}$$

in groups.

3.41 Remark: By 3.38, elements of $G*H$ are equivalence classes of words $a_0 \cdots a_n$ with $a_i \in G \cup H$.
Obviously, $G*H$ is the coproduct in the category of groups. Notice that we have an iso $G*H \cong H*G$. If $H \neq \{e\} \neq G$, then $G*H$ is not commutative (even if G and H are).

3.42 Proposition: In the category of groups,

if
$$\begin{tikzcd} H \arrow[r, "h_0"] \arrow[d, "h_1"'] & G_0 \arrow[d, "i_0"] \\ G_1 \arrow[r, "i_1"] & G \end{tikzcd}$$
is a push-out, then we have an isomorphism

$$(G_0 * G_1)/N \to G$$

induced by i_0 and i_1, where N is the normal subgroup of G generated by the subset

$$\{ h_0(h) h_1(h)^{-1} ; h \in H \} \text{ of } G_0 * G_1.$$

Proof: i_0 and i_1 induce $G_0 * G_1 \xrightarrow{i} G$, and obviously $i(h_0(h) h_1(h)^{-1}) = e$, by the definition given in 3.38. Thus i factorises through the quotient: $G_0 * G_1/N \xrightarrow{\bar{i}} G$.

We have a commutative square by definition of N, and thus

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G_0 \arrow[d] \\ G_1 \arrow[r] & G_0 * G_1/N \end{tikzcd}$$

show the existence of an inverse of $\bar{\iota}$. $\boxed{/}$

3.43 Definition: For $n \in \mathbb{N}$, we define the *free group on n generators* inductively as follows:

(i) $F_0 = \{e\}$

(ii) $F_1 = \mathbb{Z} = \langle t \rangle = \{ t^i \, ; \, i \in \mathbb{Z} \}$.

(iii) $F_n = F_1 * F_{n-1} = \langle t_1, \dots, t_n \rangle$.

Its elements are equivalence classes of words $a_0 \cdots a_n$ where $a_i \in \{ t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1} \}$.

3.44 Lemma: Let G be a group, $n \in \mathbb{N}$, and $g_1, \dots, g_n \in G$. $\exists!$ group homomorphism $h : F_n \to G$ with $h(t_i) = g_i$.

proof: For $n = 0, 1$ this is clear; then use an induction, together with the fact that $F_1 * F_{n-1}$ is the coproduct in groups. $\boxed{/}$

3.45 Theorem: Let $\omega : I \to S^1 = \{ z \in \mathbb{C} \, ; \, |z| = 1 \}$ be the path given by $\omega(t) = e^{2\pi i t}$.

Then $\pi_1(S^1, 1)$ is the free abelian group generated by $[\omega]$:

$$\begin{aligned} \mathbb{Z} &\to \pi_1(S^1, 1) \quad \text{is an isomorphism.} \\ 1 &\mapsto [\omega] \end{aligned}$$

proof: Take $A = \{1, -1\} \subset S^1$, $U_0 = S^1 \setminus \{-i\}$, $U_1 = S^1 \setminus \{i\}$, and $U_{01} = U_0 \cap U_1 = S^1 \setminus \{i, -i\}$.

By 3.36 we have a pushout in GP:

$$\begin{tikzcd} \pi(U_{01}, A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \pi(U_0, A) \arrow[d] \\ \pi(U_1, A) \arrow[r] & \pi(S^1, A) \end{tikzcd}$$

We see that we have very explicit descriptions of this push-out:

$\Pi(U_{01}, A)$:
$$\begin{tikzcd} -1 \arrow[loop below, "id"] & 1 \arrow[loop below, "id"] \end{tikzcd}$$
two objects, and only identity morphisms!

This is obvious, since U_{01} is homeomorphic to $I \amalg I$ and I is contractible.

$\Pi(U_0, A)$:
$$\begin{tikzcd} -1 \arrow[r, bend right, "{[u]^{-1}}"'] \arrow[loop below, "id"] & 1 \arrow[l, bend right, "{[u]}"'] \arrow[loop below, "id"] \end{tikzcd}$$
$u : I \to U_0$
$t \mapsto e^{\pi i t}$

This is obvious: since U_0 is homeomorphic to I, there is only one homotopy class of path from 1 to -1.

Similarly:

$\Pi(U_1, A)$:
$$\begin{tikzcd} -1 \arrow[r, bend left, "{[v]}"] & 1 \arrow[l, bend left, "{[v]^{-1}}"] \end{tikzcd}$$
$v : I \to U_1$
$t \mapsto e^{\pi i (t+1)}$.

The pushout $\Pi(S^1, A)$ has 2 objects -1 and 1, and morphisms generated by $[u], [u]^{-1}, [v], [v]^{-1}$

with the only relations $[u][u]^{-1} = id_{-1}$, $[u]^{-1}[u] = id_1$, $[v][v]^{-1} = id_1$, $[v]^{-1}[v] = id_{-1}$.

Therefore
$$\pi_1(S^1, 1) = \mathrm{Aut}_{\Pi(S^1, A)}(1) = \{ ([v][u])^n ; n \in \mathbb{Z} \}.$$

But $[v][u] = [v * u] = [w]$. ▨

3.46 Proposition: The group structure of S^1 induces a group structure on $[S^1, S^1] = \mathrm{hoTop}(S^1, S^1)$, and $[S^1, S^1]$ is the free abelian group generated by id_{S^1}:
$\mathbb{Z} \xrightarrow{\cong} [S^1, S^1]$, $1 \mapsto id_{S^1}$.

proof: the map $w : I \to S^1$ satisfies $w(0) = w(1) = 1 \in S^1 \subset \mathbb{C}$, and is a quotient map. Therefore,
$\forall\, u \in \Omega(S^1, 1, 1)$, $\exists!\, \tilde{u} : (S^1, 1) \to (S^1, 1)$

with $\bar{u} \circ w = u$.

$$\begin{tikzcd} I \arrow[r, "u"] \arrow[dr, "w"'] & S^1 \\ & S^1 \arrow[u, "\bar{u}"'] \end{tikzcd}$$

Similarly, if $u \sim v$, then $\bar{u} \simeq \bar{v}$ (pointed homotopy of maps $(S^1,1) \to (S^1,1)$).

We deduce that

$$\pi_1(S^1,1) \to [(S^1,1),(S^1,1)] \text{ is an isomorphism,}$$
$$[u] \mapsto [\bar{u}]$$

and maps $[w] \mapsto [id_{S^1}]$.

Finally, $[(S^1,1),(S^1,1)] \xrightarrow{F} [S^1,S^1]$ (forget base point) is bijective:

(i) Surjective: If $z = e^{2\pi i t} \in S^1$, then

$m_z : S^1 \to S^1$, $x \mapsto z \cdot x$ (multip. in $S^1 \subset \mathbb{C}$).

is homotopic to the identity: $m_z \overset{H}{\simeq} id_{S^1}$ with

$H : S^1 \times I \to S^1$, $H(-,s) = m_{e^{2\pi i t(1-s)}}$.

we deduce that if $f : S^1 \to S^1$ satisfies $f(1) = z$, then $f \simeq m_{z^{-1}} \cdot f$ and $(m_{z^{-1}} \cdot f)(1) = z^{-1}z = 1$.

(ii) Injective: If $H : S^1 \times I \to S^1$ is an unpointed homotopy of pointed maps, then

$\tilde{H} : S^1 \times I \to S^1$, $(z,t) \mapsto H(z,t) \cdot H(1,t)^{-1}$

is a pointed homotopy of the same maps.

This proves that $\pi_1(S^1,1) \to [S^1,S^1]$ is bijective. Group iso: exercise! $\square$

This examples shows that hoTop contains examples where $[X,Y] \neq 0$!