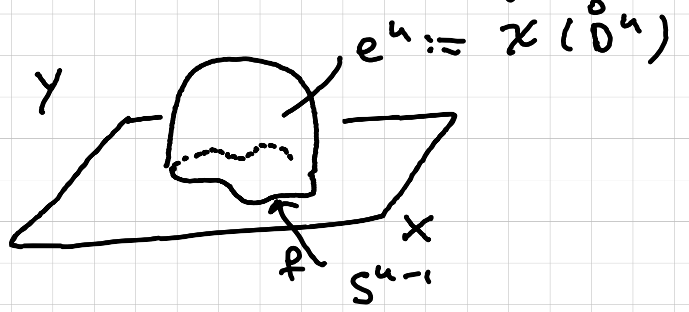
4. The fundamental group of closed surfaces

In this chapter we finish the proof of the Classification theorem 2.29, by computing $\pi_1(S,*)$ for any closed (i.e. compact without boundary) surface S.

<u>4.1 Definition</u>: Let X be a space, $n \geq 1$, and $f: S^{n-1} \to X$ a map. We say that Y is <u>obtained from X by attaching an n-cell</u> e^n if Y can be given as a push-out of the following form, for $\varphi: S^{n-1} \to X$:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ D^n \arrow[r, "\chi"] & Y \end{tikzcd}$$



<u>4.2 Definition</u> A finite cell complex is a space obtained by successively attaching finitely many cells to a finite discrete space.

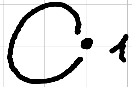
<u>4.3 Examples</u> The closed surfaces of Theorem 2.29 are finite cell complexes. Indeed, they are all realized as quotients of a regular polygon with $2n$ edges, $P \xrightarrow{q} S$, where all vertices of P map to the same point, and where edges are identified pairwise (See Th. 2.29 and Exercises 5.2, 5.3). If ∂P denotes the boundary of P (formed by the $2n$ edges), we see that $q(\partial P)$ is homeomorphic to $\bigvee_{i=1}^{n}(S^1,1)$: indeed, each pair of edges that are

identified give a copy of S^1: $\bar q$ is a homeo (continuous bijective from compact to Hausdorff).

$$\begin{tikzcd} \partial P \arrow[rr, "q|\partial P"] \arrow[dr, "\pi"'] & & S \\ & \bigvee_{i=1}^{n} S^1 \arrow[ur, "\bar q"'] & \end{tikzcd}$$

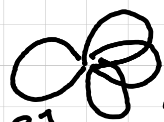
By induction, we see easily that $\bigvee_{i=1}^{n} S^1$ is obtained from a point 1 by attaching n 1-cells:

$$\begin{tikzcd} S^0 \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \{1\} \arrow[d] \\ D^1 \arrow[r] & S^1 \end{tikzcd}$$



e^1

$$\begin{tikzcd} S^0 \arrow[r, "\varphi"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \bigvee_{i=1}^{n-1} S^1 \arrow[d, "n"] \\ D^1 \arrow[r] & \bigvee_{i=1}^{n} S^1 \end{tikzcd}$$



e^1 $\bigvee_{i=1}^{n-1} S^1$

$\varphi(1) = \varphi(-1) = $ base point.

Now consider the following push-out square:

$$\begin{tikzcd} \partial P \arrow[r, "\pi"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\ulcorner"] & \bigvee_{i=1}^{n} S^1 \arrow[d] \arrow[dd, bend left, "\bar q"] \\ P \arrow[r] \arrow[dr, bend right, "q"'] & X \arrow[d, dashed, "f"] \\ & S \end{tikzcd}$$

Since the outer square commutes, we have an induced map $f : X \to S$.

Obviously f is continuous and bijective, and X is compact while S is Hausdorff; so f is a homeo. Since the pair $\partial P \hookrightarrow P$ is homeomorphic to $S^1 \hookrightarrow D^2$, we have shown that S can be given as a push-out:

$$\begin{tikzcd} S^1 \arrow[r, "\varphi"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\ulcorner"] & \bigvee_{i=1}^{n} S^1 \arrow[d] \\ D^2 \arrow[r] & S \end{tikzcd}$$

Thus it is a finite cell complex, obtained by attaching n 1-cells to a point 1, obtaining $\bigvee_{i=1}^{n} S^1$, and one 2-cell to $\bigvee_{i=1}^{n} S^1$, obtaining S.

<u>4.4 Proposition</u> Let X be a path-connected space, $n \geq 1$, $f : (S^{n-1}, *) \to (X, x_0)$ a pointed map, and Y obtained by attaching an n-cell to X; take $y_0 = i(x_0)$.

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "i"] \\ D^n \arrow[r, "\chi"] & Y \end{tikzcd}$$

(1) If $n \geq 3$, $i : X \to Y$ induces an isomorphism
$i_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

(2) If $n = 2$, f provides an element $[f] \in \pi_1(X, x_0)$, and i induces an isomorphism

$$i_* : \underbrace{\pi_1(X, x_0) / [f]} \to \pi_1(Y, y_0)$$

quotient $\pi_1(X, x_0)/N$, N = normal subgroup generated by $[f]$.

(3) If $n = 1$, $\pi_1(Y, y_0) \cong \pi_1(X, x_0) * \mathbb{Z}$.

proof: We apply Seifert-Van Kampen, as follows:
U_0 is the push-out:

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "j"] \\ D^n \setminus \{0\} \arrow[r] & U_0 \end{tikzcd}$$

and U_1 is $\chi(\{x \in D^n ; \|x\| < \frac{1}{2}\})$.
Then obviously U_0, U_1, $U_{01} = U_0 \cap U_1$ are open subspaces of Y; moreover, we have:

(*)
- $\chi : \underbrace{\{x \in D^n ; 0 < \|x\| < \frac{1}{2}\}}_{\cong S^{n-1} \times \mathbb{R}} \to U_{01}$ is a homeo
- $\chi : \underbrace{\{x \in D^n ; \|x\| < \frac{1}{2}\}}_{\cong \mathbb{R}^n} \to U_1$ is a homeo
- $j : X \to U_0$ is a homotopy equivalence

(As homotopy inverse, it admits the obvious map $k : U_0 \to X$, given by $k' : D^n \setminus \{0\} \to X$, $k'(ty) = f(y)$ $\forall\, t \cdot y \in D^n \setminus \{0\}$ with $t \in (0, 1]$ and $y \in S^{n-1}$, and $k'' = id_X : X \to X$. We have $k \circ j = k'' = id_X$, and

There is an obvious homotopy $j \circ k \simeq \mathrm{id}_{U_0}$ (Exercise!)).

(ii) Assume $n \geq 2$, so that U_{01}, U_0, U_1 are path connected. Choose a base point $z \in U_{01}$.

By Seifert - Van Kampen (group version) we have a push-out diagram, where the 3 first groups and 2 homomorphisms are identified via $(*)$:

$$\begin{tikzcd}
\pi_1(S^{n-1}, z) \arrow[rr, "f_*"] \arrow[dr, "\cong"] \arrow[dd] & & \pi_1(X, z) \arrow[dr, "\cong"] & \\
& \pi_1(U_{01}, z) \arrow[rr] \arrow[dd] \arrow[drr, phantom, "\ulcorner"] & & \pi_1(U_0, z) \arrow[dd] \\
\pi_1(\mathbb{R}^n, *) = 0 \arrow[dr, "\cong"] & & & \\
& \pi_1(U_1, z) \arrow[rr] & & \pi_1(Y, z)
\end{tikzcd}$$

This shows, using $\pi_1(S^{n-1}; z) = 0$ if $n \geq 3$, (Exercise 8.2)

that for $n \geq 3$ we have a push-out:
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_1(X, z) \arrow[d, "i_*"] \\
0 \arrow[r] & \pi_1(Y, z)
\end{tikzcd}$$
(which means that i_* is an iso).

For $n = 2$, we have a push-out:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1 \mapsto {[f]}"'] \arrow[d] & \pi_1(X, z) \arrow[d] \\
0 \arrow[r] & \pi_1(Y, z)
\end{tikzcd}$$
(which means that $\pi_1(X, z)/_{[f]} \longrightarrow \pi_1(Y; z)$ is an iso).

(See exercise 7.5.(b))

Notice that we considered a base point z different from the original base point (we needed $z \in U_{01}$), but the same will hold for x_0 and $i(x_0)$: see the note after Proposition 3.32.

For $n = 1$, U_{01} is not connected, and we need

the groupoid version of Seifert - Van Kampen !
We leave the details as an exercise: it is an obvious adaptation of the proof of Theorem 3.45. ▨

<u>4.5 Notation</u> : If S is a closed surface, we can present it as the quotient $q: P \to S$ of a regular polygon with $2n$ edges, and all vertices identified, and edges glued pairwise (2.29 + Ex. 5.2 + 5.3).
Choose a vertex v_0; running counterclockwise, we obtain a word w of length $2n$, where each letter t_i, $1 \leq i \leq n$, appears twice, possibly with an exponent $\varepsilon = \pm 1$. We call (P, w) a polygonal presentation of S.

<u>4.6 Proposition</u> : Let S be a closed surface, with polygonal presentation (P, w), where w is a word of length $2n$ in the letters $t_1, \dots, t_n$.
Then $\pi_1(S, v_0) \cong \langle t_1, \dots, t_n \mid w \rangle$.
Here $\langle t_1, \dots, t_n \mid w \rangle = F_n / w$, where F_n is the free group on the n generators $t_1, \dots, t_n$, and F_n / w is the quotient of F_n by the normal subgroup generated by $w \in F_n$.

<u>Proof</u>: We have seen in Example 4.3 that S is obtained from $\bigvee_{i=1}^{n} S^1$ by attaching a 2-cell along $S^1 \xrightarrow{f} \bigvee_{i=1}^{n} S^1$, where $f: \partial P \xrightarrow{q} q(\partial P) \cong \bigvee_{i=1}^{n} S^1$.
By Proposition 4.4., $\pi_1(\bigvee_{i=1}^{n} S^1, *) = F_n = \langle t_1, \dots, t_n \rangle$.

Obviously, $f_*: \pi_1(S^1,*) \to \pi_1(\bigvee_{i=1}^n S^1,*)$
$$\begin{array}{ccc} \| & & \| \\ \mathbb{Z} \ni 1 & \mapsto & w \in F_n \end{array}$$

Again by 4.4. (2), we see that
$$\pi_1(S,*) = F_n/_{w} = \langle t_1,\dots,t_n \mid w\rangle. \qquad \square$$

We can now finish the proof of 2.29:

<u>4.7 Proposition:</u> We have the following isomorphisms for any $n \in \mathbb{N}$, $n \geq 1$:

- $\pi_1(S^2,*) = 0$;
- $\pi_1(\#_{i=1}^n \mathbb{R}P^2,*) \cong \langle t_1,\dots,t_n \mid \prod_{i=1}^n t_i^2\rangle$;
- $\pi_1(\#_{i=1}^n T^2,*) \cong \langle s_1,\dots,s_n,t_1,\dots,t_n \mid \prod_{i=1}^n [s_i,t_i]\rangle$, where $[s_i,t_i] = s_i t_i s_i^{-1} t_i^{-1}$.

None of these groups are isomorphic; in particular, each pair of surfaces from the list
$$S^2, \quad \#_{i=1}^n \mathbb{R}P^2, \quad \#_{i=1}^n T^2$$
is a non-homeomorphic pair (finishing the proof of 2.29).

<u>Proof</u>: If $X \xrightarrow{f} Y$ is a homeomorphism, then for any $x \in X$, $\pi_1(X,x) \to \pi_1(Y,f(x))$ is an isomorphism (a homeo is a homotopy equivalence; apply Prop. 3.33).
None of these groups are isomorphic: indeed, their abelianization are not isomorphic:
$$\pi_1(\#_{i=1}^n \mathbb{R}P^2;*) = \langle t_1,\dots,t_n \mid t_1^2\cdot\dots\cdot t_n^2\rangle_{ab} =$$
$$\cong \bigoplus_{i=1}^n \mathbb{Z}\Big/_{(2,\dots,2)} \underset{\text{Ex. 8.1}}{\cong} \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2$$
$$\pi_1(\#_{i=1}^n T^2;*) = \langle s_1,\dots,s_n,t_1,\dots,t_n \mid \prod [s_i,t_i]\rangle_{ab}$$

$\cong \bigoplus_{i=1}^{2n} \mathbb{Z}$. This follows from the next proposition. $\blacksquare$

<u>4.8 Proposition</u>: If $G \cong \langle t_1, \dots, t_n , w \rangle$, then $G_{ab} \cong \bigoplus_{i=1}^{n} \mathbb{Z} / f(w)$, where

$$f: F_n \longrightarrow \bigoplus_{i=1}^{n} \mathbb{Z} \text{ is given by}$$
$$t_i \mapsto e_i := (0, \dots, 0, 1, 0, \dots, 0)$$

↑ i-th coordinate.

<u>Proof</u>: G_{ab} and $\bigoplus_{i=1}^{n} \mathbb{Z}/f(w)$ satisfy the same universal property: the one of the Abelianization.

First, $(F_n)_{ab} = \bigoplus_{i=1}^{n} \mathbb{Z}$:

We have the map $f: F_n \to \bigoplus_{i=1}^{n} \mathbb{Z}$ given above.

Suppose given a hom. $g: F_n \to A$, A Abelian; g is determined by $a_1, \dots, a_n \in A$ and $g(t_i) = a_i$.

Since $\oplus$ is the coproduct in Abelian groups, we deduce that $\exists! \, \bar{g} : \bigoplus_{i=1}^{n} \mathbb{Z} \to A$ with $e_i \mapsto a_i$

Obviously $g = \bar{g} \circ f$. Thus f is the abelianization.

The general case follows:

If $g: \langle t_1, \dots, t_n \mid w \rangle \longrightarrow A$ is given, it is again determined by $a_1, \dots, a_n \in A$ with $g(t_i) = a_i$; we must have $g(w) = 0$, forcing $\bar{g}(f(w)) = 0$.

$\exists! \, \bar{g} : \bigoplus_{i=1}^{n} \mathbb{Z}/f(w) \longrightarrow A$ with $e_i \mapsto a_i$. $\blacksquare$

<u>4.9 Proposition</u> $\pi_1(SO(3); id) \cong \mathbb{Z}/2$, with generator the path $[\omega]$, $\omega: I \to SO(3)$

$$t \mapsto \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & \\ 0 & 0 & 1 \end{pmatrix} \in$$

proof: use the homeomorphism $\mathbb{R}P^3 \xrightarrow{f} SO(3)$ constructed in Exercise 5.6, and Exercise 8.4.
Then, show that $[w]$ corresponds to a generator of π_1. $\boxtimes$

This is reflected in the following trick:
Standing on the floor without moving your feet, hold an object in your hand. You can rotate it by $2 \times 360°$ by holding it firmly, and recovering your initial position after $720°$ are reached.
This requires some elasticity, but shows you can rotate the object indefinitely without breaking your arm!