Therefore, by the compactness of $[0,1]$, there exist a finite sequence of numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ and open neighborhoods $U(i)$ of b in B such that $\xi \mid (U(i) \times [t_{i-1},t_i])$ is trivial for $1 \leqq i \leqq n$. Let $U = \bigcap_{1 \leqq i \leqq n} U(i)$. Then the bundle $\xi \mid (U \times [0,1])$ is trivial by an application of Lemma (4.1) $n - 1$ times. Therefore, there is an open covering $\{U_i\}$, $i \in I$, of B such that $\xi \mid (U_i \times I)$ is trivial.

The next theorem is the first important step in the development of the homotopy properties of vector bundles.

4.3 Theorem. Let $r: B \times I \to B \times I$ be defined by $r(b,t) = (b,1)$ for $(b,t) \in B \times I$, and let $\xi^k = (E, p, B \times I)$ be a vector bundle over $B \times I$, where B is a paracompact space. There is a map $u: E \to E$ such that $(u,r): \xi \to \xi$ is a morphism of vector bundles and u is an isomorphism on each fibre.

Proof. Let $\{U_i\}$, $i \in I$, be a locally finite open covering of B such that $\xi \mid (U_i \times I)$ is trivial. This covering exists by (4.2) and the paracompactness of B. Let $\{\eta_i\}$, $i \in I$, be an envelope of unity subordinate to the open covering $\{U_i\}$, $i \in I$, that is, the support of η_i is a subset of U_i and $1 = \max_{i \in I} \eta_i(b)$ for each $b \in B$. Let $h_i: U_i \times I \times F^k \to p^{-1}(U_i \times I)$ be a $(U_i \times I)$-isomorphism of vector bundles.

We define a morphism $(u_i,r_i): \xi \to \xi$ by the relations $r_i(b,t) = (b,\max(\eta_i(b),t))$, u_i is the identity outside $p^{-1}(U_i \times I)$, and $u_i(h_i(b,t,x)) = h_i(b,max(\eta_i(b),t),x)$ for each $(b,t,x) \in U_i \times I \times F^k$. We well order the set I. For each $b \in B$, there is an open neighborhood $U(b)$ of b such that $U_i \cap U(b)$ is nonempty for $i \in I(b)$, where $I(b)$ is a finite subset of I. On $U(b) \times I$, we define $r = r_{i(n)} \cdots r_{i(1)}$, and on $p^{-1}(U(b) \times I)$, we define $u = u_{i(n)} \cdots u_{i(1)}$, where $I(b) = \{i(1), \ldots, i(n)\}$ and $i(1) < i(2) < \cdots < i(n)$. Since r_i on $U(b) \times I$ and u_i on $p^{-1}(U(b) \times I)$ are identities for $i \notin I(b)$, the maps r and u are infinite compositions of maps where all but a finite number of terms are identities near a point. Since each u_i is an isomorphism on each fibre, the composition u is an isomorphism on each fibre.

4.4 Corollary. With the notations of Theorem (4.3), $\xi \cong r^*(\xi \mid (B \times 1))$ over $B \times I$.

Proof. This result is a direct application of Theorem (3.2) to Theorem (4.3).

Let $\xi = (E,p,B)$ be a vector bundle, and let Y be a space. We use the notation $\xi \times Y$ for the vector bundle $(E \times Y, p \times 1_Y, B \times Y)$. The fibre over $(b,y) \in B \times Y$ is $p^{-1}(b) \times y$, which has a natural vector space structure that it derives from $p^{-1}(b)$. If $h: U \times F^k \to p^{-1}(U)$ is a U-isomorphism, the $h \times 1_Y: U \times Y \times F^k \to p^{-1}(U) \times Y = (p \times 1_Y)^{-1}(U \times Y)$ is a $(U \times Y)$-isomorphism. Consequently, $\xi \times Y$ is a vector bundle, and this leads to the following version of (4.3).

4.5 Corollary. With the notations of Theorem (4.3),

$$\xi \cong (\xi \mid (B \times 1)) \times I$$

are vector bundles over $B \times I$.

Proof. For this, it suffices to observe that $r^*(\xi \mid (B \times 1)) = (\xi \mid B \times 1) \times I$. In both cases the total space of the bundles is the subspace of $(b,t,x) \in B \times I \times E(\xi \mid (B \times 1))$ such that $(b,1) = p(x)$, and the projection is the map $(b,t,x) \mapsto (b,t)$.

4.6 Corollary. With the notations of Theorem (4.3), there exists, after restriction, an isomorphism $(u,r): \xi \mid (B \times 0) \to \xi \mid (B \times 1)$.

Proof. This is a direct application of Theorem (2.5) to the situation described in (4.3) where $r = 1$ on $B \times 0 = B \times 1 = B$.

Finally, we have the following important application of (4.6) in the framework of homotopy theory.

4.7 Theorem. Let $f,g: B \to B'$ be two homotopic maps, where B is a paracompact space, and let ξ be a vector bundle over B'. Then $f^*(\xi)$ and $g^*(\xi)$ are B-isomorphic.

Proof. Let $h: B \times I \to B'$ be a map with $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Then $f^*(\xi) \cong h^*(\xi) \mid (B \times 0)$ over B, and $g^*(\xi) \cong h^*(\xi) \mid (B \times 1)$ over B. By (4.6), $h^*(\xi) \mid (B \times 0)$ and $h^*(\xi) \mid (B \times 1)$ are B-isomorphic, and, therefore, $f^*(\xi)$ and $g^*(\xi)$ are B-isomorphic.

4.8 Corollary. Every vector bundle over a contractible paracompact space B is trivial.

Proof. Let $f: B \to B$ be the identity, and let $g: B \to B$ be a constant map. For each vector bundle ξ over B, $f^*(\xi)$ is B-isomorphic to ξ, and $g^*(\xi)$ is B-isomorphic to the product bundle $(B \times F^k, p, B)$. Since f and g are homotopic, ξ is isomorphic to the product bundle $(B \times F^k, p, B)$, by (4.7).

Theorem (4.7) is the first of the three main theorems on the homotopy classification of vector bundles.

5. CONSTRUCTION OF GAUSS MAPS

5.1 Definition. A *Gauss map* of a vector bundle ξ^k in F^m ($k \leqq m \leqq +\infty$) is a map $g: E(\xi^k) \to F^m$ such that g is a linear monomorphism when restricted to any fibre of ξ.

Recall that $E(\gamma_k^m)$ is the subspace of $(V,x) \in G_k(F^m) \times F^m$ with $x \in V$. Then the projection $q: E(\gamma_k^m) \to F^m$, given by the relation $q(V,x) = x$, is a Gauss map. In the next proposition, we see that every Gauss map can be constructed from this map and vector bundle morphisms.

5.2 Proposition. If $(u,f): \xi^k \to \gamma_k^m$ is a vector bundle morphism that is an isomorphism when restricted to any fibre of ξ^k, then $qu: E(\xi^k) \to$

F^m is a Gauss map. Conversely, if $g\colon E(\xi^k) \to F^m$ is a Gauss map, there exists a vector bundle morphism $(u,f)\colon \xi^k \to \gamma_k^m$ such that $qu = g$.

Proof. The first statement is clear. For the second, let $f(b) = g(p^{-1}(b)) \in G_k(F^m)$, and let $u(x) = (f(p(x)),g(x)) \in E(\gamma_k^m)$ for $x \in E(\xi^k)$. We see that f is continuous by looking at a local coordinate of ξ, and from this u is also continuous.

5.3 Corollary. There exists a Gauss map $g\colon E(\xi) \to F^m$ $(k \leqq m \leqq +\infty)$ if and only if ξ is $B(\xi)$-isomorphic with $f^*(\gamma_k^m)$ for some map $f\colon B(\xi) \to G_k(F^m)$.

Proof. This follows from Proposition (5.2) and Theorem (3.2).

In Theorem (5.5), we construct a Gauss map for each vector bundle over a paracompact space. First, we need a preliminary result concerning the open sets over which a vector bundle is trivial.

5.4 Proposition. Let ξ be a vector bundle over a paracompact space B such that $\xi \mid U_i$, $i \in I$, is trivial, where $\{U_i\}$, $i \in I$, is an open covering. Then there exists a countable open covering $\{W_j\}$, $1 \leqq j$, of B such that $\xi \mid W_j$ is trivial. Moreover, if each $b \in B$ is a member of at most n sets U_i, there exists a finite open covering $\{W_j\}$, $1 \leqq j \leqq n$, of B such that $\xi \mid W_j$ is trivial.

Proof. By paracompactness, let $\{\eta_i\}$, $i \in I$, be a partition of unity with $V_i = \eta_i^{-1}(0,1] \subset U_i$. For each $b \in B$, let $S(b)$ be the finite set of $i \in I$ with $\eta_i(b) > 0$. For each finite subset $S \subset I$, let $W(S)$ be the open subset of all $b \in B$ such that $\eta_i(b) > \eta_j(b)$ for each $i \in S$ and $j \notin S$.

If S and S' are two distinct subsets of I each with m elements, then $W(S) \cap W(S')$ is empty. In effect, there exist $i \in S$ with $i \notin S'$ and $j \in S'$ with $j \notin S$. For $b \in W(S)$ we have $\eta_i(b) > \eta_j(b)$, and for $b \in W(S')$ we have $\eta_j(b) > \eta_i(b)$. Therefore, $W(S) \cap W(S')$ is empty.

Let W_m be the union of all $W(S(b))$ such that $S(b)$ has m elements. Since $i \in S(b)$ yields the relation $W(S(b)) \subset V_i$, the bundle $\xi \mid W(S(b))$ is trivial, and since W_m is a disjoint union, $\xi \mid W_m$ is trivial. Finally, under the last hypothesis, W_j is empty for $n < j$.

5.5 Theorem. For each vector bundle ξ^k over a paracompact space B there is a Gauss map $g\colon E(\xi) \to F^\infty$. Moreover, if B has an open covering of sets $\{U_i\}$, $1 \leqq i \leqq n$, such that $\xi \mid U_i$ is trivial, ξ has a Gauss map $g\colon E(\xi) \to F^{kn}$.

Proof. Let $\{U_i\}$ be the countable or finite open covering of B such that $\xi \mid U_i$ is trivial, let $h_i\colon U_i \times F^k \to \xi \mid U_i$ be U_i-isomorphisms, and let $\{\eta_i\}$ be a partition of unity with closure of $\eta_i^{-1}((0,1]) \subset U_i$. We define $g\colon E(\xi) \to \sum_i F^k$ as $g = \sum_i g_i$, where $g_i \mid E(\xi \mid U_i)$ is $(\eta_i p)(p_2 h_i^{-1})$ and $p_2\colon U \times F^k \to F^k$ is the projection on the second factor. Outside $E(\xi \mid U_i)$, the map g_i is zero.

Since each $g_i\colon E(\xi) \to F^k$ is a monomorphism on the fibres of $E(\xi)$ over b with $\eta_i(b) > 0$, and since the images of g_i are in complementary subspaces, the map g is a Gauss map. In general, $\sum_i F^k$ is F^∞, but if there are only n sets U_i, then $\sum_i F^k$ is F^{kn}.

Theorem (5.5) with Corollary (5.6) is the second main homotopy classification theorem for vector bundles.

5.6 Corollary. Every vector bundle ξ^k over a paracompact space B is B-isomorphic to $f^*(\gamma_k)$ for some $f\colon B \to G_k(F^\infty)$.

The following concept was suggested by Theorem (5.5).

5.7 Definition. A vector bundle ξ is of finite type over B provided there exists a finite open covering $U_1, \ldots, U_n$ of B such that $\xi \mid U_i$ is trivial, $1 \leqq i \leqq n$.

In the next theorem we derive other formulations of the notion of finite type. By 1(2.6) and (4.8) every vector bundle over a finite-dimensional CW-complex is of the finite type.

5.8 Proposition. For a vector bundle ξ over a space B, the following are equivalent.

(1) The bundle ξ is of the finite type.

(2) There exists a map $f\colon B \to G_k(F^m)$ for some m such that $f^*(\gamma_k^m)$ and ξ are B-isomorphic.

(3) There exists a vector bundle η over B such that $\xi \oplus \eta$ is trivial.

Proof. By the construction in (5.5), statement (1) implies (2). Since $\gamma_k^m \oplus {}^*\gamma_k^m$ is trivial over $G_k(F^m)$, then $f^*(\gamma_k^m) \oplus f^*({}^*\gamma_k^m)$ and θ^m are B-isomorphic. Let η be $f^*({}^*\gamma_k^m)$. Since $f^*(\gamma_k^m \oplus {}^*\gamma_k^m)$ is trivial, the bundle $\xi \oplus \eta$ is trivial. Finally, the composition $E(\xi) \to E(\xi \oplus \eta) \to B \times F^m \to F^m$ is a Gauss map.

6. HOMOTOPIES OF GAUSS MAPS

Let F^{ev} denote the subspace of $x \in F^\infty$ with $x_{2i+1} = 0$, and F^{odd} with $x_{2i} = 0$ for $i \geqq 0$. For these subspaces, $F^\infty = F^{\text{ev}} \oplus F^{\text{odd}}$. Two homotopies $g^e\colon F^n \times I \to F^{2n}$ and $g^o\colon F^n \times I \to F^{2n}$ are defined by the following formulas:

$$g_t^e(x_0,x_1,x_2,\ldots) = (1-t)(x_0,x_1,x_2,\ldots) + t(x_0,0,x_10,x_2,\ldots)$$

$$g_t^o(x_0,x_1,x_2,\ldots) = (1-t)(x_0,x_1,x_2,\ldots) + t(0,x_0,0,x_10,x_2,\ldots)$$

The properties of these homotopies are contained in the following proposition. In the above formulas and in the next proposition, we have $1 \leqq n \leqq +\infty$.

6.1 Proposition. With the above notations, these homotopies have the following properties:

(1) The maps g_0^e and g_0^o each equal the inclusion $F^n \to F^{2n}$.
(2) For $t = 1$, $g_1^e(F^n) = F^{2n} \cap F^{ev}$ and $g_1^o(F^n) = F^{2n} \cap F^{odd}$.
(3) There are vector bundle morphisms $(u^e,f^e): \gamma_k^n \to \gamma_k^{2n}$ and $(u^o,f^o): \gamma_k^n \to \gamma_k^{2n}$ such that $qu^e = g_1^e$, $qy^o = g_1^o$.
(4) f^e and f^o are homotopic to the inclusion $G_k(F^n) \to G_k(F^{2n})$.

Proof. Statements (1) and (2) follow immediately from the formulas for g_t^e and g_t^o. For (3), we use (5.2). Finally, the homotopies g_t^e and g_t^o define homotopies of f^e and f^o with 1.

The next theorem describes to what extent Gauss maps are unique in terms of homotopy properties of their associated bundle morphisms. We use the above notations.

6.2 Theorem. Let $f, f_1: B \to G_k(F^n)$ be two maps such that $f^*(\gamma_k^n)$ and $f_1^*(\gamma_k^n)$ are B-isomorphic and let $j: G_k(F^n) \to G_k(F^{2n})$ be the natural inclusion. Then the maps jf and jf_1 are homotopic for $1 \leqq n \leqq +\infty$.

Proof. By hypothesis, there is a vector bundle ξ over B and two morphisms $(u,f): \xi \to \gamma_k^n$ and $(u_1,f_1): \xi \to \gamma_k^n$ which are isomorphisms when restricted to the fibres of ξ. Let $g = qu: E(\xi) \to F^n$ and $g_1 = qu_1: E(\xi) \to F^n$ be the associated Gauss maps. Composing with the above maps, we have morphisms $(u^e u, f^e f): \xi \to \gamma_k^{2n}$ with a Gauss map $g_1^e g: E(\xi) \to F^{ev} \cap F^{2n}$ and $(u^o u, f^o f): \xi \to \gamma_k^{2n}$ with a Gauss map $g_1^o g_1: E(\xi) \to F^{odd} \cap F^{2n}$. We define a Gauss map $h: E(\xi) \times I \to F^{2n}$ by the relation $h_t(x) = (1-t)g_1^e g(x) + t g_1^o g_1(x)$. For a fibre $p^{-1}(b) \subset E(\xi)$, the linear maps $g_1^e g: p^{-1}(b) \to F^{ev}$ and $g_1^o g_1: p^{-1}(b) \to F^{odd}$ are monomorphisms, and since $F^{ev} \cap F^{odd} = 0$, the map $h_t: p^{-1}(b) \to F^{2n}$ is a linear monomorphism. Therefore, there is a Gauss map $h: E(\xi) \times I \to F^{2n}$ which determines a bundle morphism $(w,k): \xi \to \gamma_k^{2n}$. The map $k: B \times I \to G_k(F^{2n})$ is a homotopy from $f^e f$ to $f^o f_1$. Since f and $f^e f$ are homotopic and $f^o f_1$ and f_1 are homotopic, f and f_1 are homotopic. This proves the theorem.

Theorem (6.2) is the third of the three main homotopy classification theorems.

7. FUNCTORIAL DESCRIPTION OF THE HOMOTOPY CLASSIFICATION OF VECTOR BUNDLES

Let **P** denote the category paracompact spaces and homotopy classes of maps. Let **ens** denote, as usual, the category of sets and functions.

Let $\mathrm{Vect}_k(B)$ denote the set of B-isomorphism classes of k-dimensional vector bundles over B. For a k-dimensional bundle ξ, we denote by $\{\xi\}$ the class in $\mathrm{Vect}_k(B)$ determined by ξ. If $[f]: B_1 \to B$ is a homotopy class of maps between paracompact spaces, we define a function $\mathrm{Vect}_k([f]): \mathrm{Vect}_k(B) \to \mathrm{Vect}_k(B_1)$ by the relation $\mathrm{Vect}_k([f])(\{\xi\}) = \{f^*(\xi)\}$. By the remarks at the end of Sec. 3 and Theorem (4.7), $\mathrm{Vect}_k([f])$ is a well-defined function.

7.1 Proposition. The family of functions $\mathrm{Vect}_k: \mathbf{P} \to \mathbf{ens}$ is a cofunctor.

Proof. Since $1^*(\xi)$ and ξ are B-isomorphic, the function $\mathrm{Vect}_k([1])$ is the identity. If $[f]: B_1 \to B$ and $[g]: B_2 \to B_1$ are two homotopy classes of maps, $g^*(f^*(\xi))$ and $(fg)^*(\xi)$ are B_2-isomorphic. Consequently, $\mathrm{Vect}_k([f][g]) = \mathrm{Vect}_k([g])\, \mathrm{Vect}_k([f])$, and Vect_k satisfies the axioms for being a cofunctor.

For each B, we define a function $\phi_B: [B, G_k(F^\infty)] \to \mathrm{Vect}_k(B)$ by the relation $\phi_B([f]) = \{f^*(\gamma_k)\}$. Again by Theorem (4.7), ϕ_B is a well-defined function. The next theorem, together with the definition of Vect_k and ϕ_B, brings together all aspects of the homotopy classification theory of vector bundles.

7.2 Theorem. The family ϕ of functions ϕ_B defines an isomorphism of cofunctors $\phi: [-, G_k(F^\infty)] \to \mathrm{Vect}_k$.

Proof. First, we prove that ϕ is a morphism of cofunctors. For this, let $[f]: B_1 \to B$ be a homotopy class of maps. Then the following diagram is commutative.

$$\begin{array}{ccc} [B, G_k(F^\infty)] & \xrightarrow{\phi_B} & \mathrm{Vect}_k(B) \\ \Big\downarrow{\scriptstyle [[f],\, G_k(F^\infty)]} & & \Big\downarrow{\scriptstyle \mathrm{Vect}_k([f])} \\ [B_1, G_k(F^\infty)] & \xrightarrow{\phi_{B_1}} & \mathrm{Vect}_k(B_1) \end{array}$$

In effect, if $[g] \in [B, G_k(F^\infty)]$, we have

$$\mathrm{Vect}_k([f])\phi_B([g]) = \mathrm{Vect}_k([f])\{g^*(\gamma_k)\} = \{f^*g^*(\gamma_k)\}$$

and $\phi_{B_1}[[f], G_k(F^\infty)][g] = \phi_{B_1}([g][f]) = \{(gf)^*(\gamma_k)\}$.

Finally, ϕ is an isomorphism because each ϕ_B is a bijection. The function ϕ_B is surjective by (5.5) and (5.6), and it is injective by (6.2). This proves the theorem.

7.3 The isomorphism $\phi: [-, G_k(F^\infty)] \to \mathrm{Vect}_k$ is called a corepresentation of the cofunctor Vect_k. The preceding four sections have been dedicated to proving that the cofunctor Vect_k is corepresentable. In this way the problem of classifying vector bundles, i.e., of computing $\mathrm{Vect}_k(B)$, has been reduced to the calculation of sets of homotopy classes of maps, i.e., the sets $[B, G_k(F^\infty)]$.