Generalized vector multiplicative cascades.

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Abstract

We define the extension of the so-called "martingales in the branching random walk" in $\mathbb{R}$ or $\mathbb{C}$ to some Banach algebras B of infinite dimension and give conditions for their convergence, almost surely and in L^p norm. This abstract approach gives conditions for the simultaneous convergence of uncountable families of such martingales constructed simultaneously in $\mathbb{C}$, the idea being to consider such a family as a function valued martingale in a Banach algebra of functions. The approach is an alternative to those of Biggins (1989, 1992) and Barral (2000), and it applies to a class of families on which the previous approach did not.

We also show a result of continuity on these multiplicative processes.

Our results extend to a varying environment version of the usual construction : instead of attaching i.i.d. copies of a given random vector to the nodes of the tree $\bigcup_{n\geq 0} \mathbb{N}_+^n$, the distribution of the vector depends on the node in the multiplicative cascade. In this context, when $B = \mathbb{R}$ and in the non-negative case, we generalize the measure on the boundary of the tree usually related to the construction; then we evaluate the dimension of this non statistically self-similar measure.

In the self-similar case, our convergence results make it possible to simultaneously define uncountable families of such measures, and then to estimate their dimension simultaneously.

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1 Introduction.

The processes we generalize in this paper are real or complex valued multiplicative cascades, called "martingales in the branching random walk" (MBRW), that have been studied by many authors during the last twenty five years (Mandelbrot (1974 a,b), Kingman (1975), Kahane and Peyrière (1976), Peyrière (1977), Biggins (1977, 1979, 1989, 1992), Durrett and Liggett (1983), Kahane (1987, 1991), Guivarc'h (1990), Collet and Koukiou (1992), Waymire and Williams (1995), Molchan (1996), Liu and Rouault (1996), Liu (1997, 1998, 1999, 2000), Barral (1999, 2000 a,b)). One of our purposes is to study the convergence of function valued MBRW to answer some questions left open in the study of the simultaneous convergence of uncountable families of $\mathbb{C}$-valued MBRW defined simultaneously.

Let us describe these multiplicative cascades in the context in which they were mostly investigated, namely the non-negative case. We follow the Ulam-Harris construction of such a branching process:

Let $(\Omega, \mathcal{A}, \mathbb{P})$ stand for the probability space on which the random variables in this paper are defined. Define $\mathbb{N}_+ = \mathbb{N}\backslash\{0\}$, $T = \bigcup_{n\geq 0}\mathbb{N}_+^n$, the set of finite words on $\mathbb{N}_+$, equipped with the concatenation operation (ϵ stands for the empty word and $\mathbb{N}_+^0 = \{\epsilon\}$). Let $A = (A_i)_{i\geq 1}$ be a sequence of non negative random variables such that

$$N_A = \sum_{i\geq 1} \mathbf{1}_{\{A_i \neq 0\}} < \infty \tag{1}$$

almost surely,

$$\sum_{i\geq 1} A_i \text{ is integrable} \tag{2}$$

and

$$\mathbb{E}\left(\sum_{i\geq 1} A_i\right) = 1. \tag{3}$$

Without loss of generality in Sections 1 to 4, except in the "i.i.d." case, we assume that the A_i are ordered with the non-zero ones first: $A_1, \ldots, A_{N_A}$ are the non zero components of A if $N_A > 0$.

Fix $(A(a))_{a\in T}$ a sequence of independent copies of A. Denote by $(M_n)_{n\geq 1}$ the size of the nth generation of the Galton Watson branching process with offspring distribution the probability distribution of N_A, and generated with the $N_{A(a)}$, $a \in T$, that is

$$M_n = \sum_{a_1\ldots a_n \in \mathbb{N}_+^n} \prod_{j=1}^{n} \mathbf{1}_{\{1\leq a_j \leq N_A(a_1\ldots a_{j-1})\}}.$$

Denote by $\tilde{T}$ the Galton-Watson subtree of T associated with M_n: $\tilde{T} = \{a_1 \ldots a_n \cdots \in T;\ \forall\, i \geq 1,\ 1 \leq a_i \leq N(a_1 \ldots a_{i-1})\}$.

Then for every $n \geq 1$ multiply the A_i values together down each line of descent in $\tilde{T} \cap \mathbb{N}_+^n$ and sum the results over the nth generation to obtain

$$Y_n = Y_{A,n} = \sum_{a_1\ldots a_n \in \tilde{T}\cap\mathbb{N}_+^n} A_{a_1}(\epsilon) A_{a_2}(a_1) \ldots A_{a_n}(a_1 \ldots a_{n-1}). \tag{4}$$

It is easily verified that $(Y_n)_{n\geq 1}$ is a non-negative martingale with respect to the filtration $(F_n = \sigma(A_i(a); i \geq 1,\, a \in \cup_{j=0}^{n-1}\mathbb{N}_+^k)_{n\geq 1}$, with mean 1. So it converges almost surely to a random variable with mean ≤ 1. To find a necessary and sufficient condition for $(Y_n)_{n\geq 1}$ to converge to a non trivial limit in L^1 or in L^p $(p > 1)$ has been one of the main problems studied in the papers mentioned above. Final answers are given in Lyons (1996) and Liu (1997, 2000).

The martingale $(Y_n)_{n\geq 1}$ has an immediate extension to $\mathbb{C}$ if one allows the A_i to belong to $\mathbb{C}$ and replaces (2) by

$$\mathbb{E}(\sum_{i\geq 1} |A_i|) < \infty.$$

Biggins (1989 or 1992) gives a sufficient condition for the L^p convergence $(1 < p \leq 2)$, which is the natural extension of the condition in the non-negative case. To do this he uses the complex version of a result of von Bahr and Esseen (1965).

Then, it is natural to try to extend the construction of Y_n to more general Banach spaces over $\mathbb{R}$ that are equipped with a multiplication operation having the property to respect the multiplication in $\mathbb{R}$, and equipped with a submultiplicative norm. Such spaces are Banach algebras with unity (see Larsen (1973) for details).

Thus, let B be a separable Banach algebra over $\mathbb{R}$ with unity denoted by I_B (for all $U \in B$, $UI_B = I_BU$) and equipped with a submultiplicative norm denoted by $\mathcal{N}$: for every $(U, V) \in B^2$,

$$\mathcal{N}(UV) \leq \mathcal{N}(U)\mathcal{N}(V). \tag{5}$$

For $x \in \mathbb{R}_+$ and $U \in B$, $\mathcal{N}(U)^x$ or $\mathcal{N}^x(U)$ will denote $(\mathcal{N}(U))^x$.

Assume now that the A_i are random variables taking values in the separable Banach space B (equipped with the Borel σ-algebra relative to $\mathcal{N}$). Replace (2) by

$$\mathbb{E}\left(\sum_{i\geq 1} \mathcal{N}(A_i)\right) < \infty. \tag{6}$$

This implies that the random variable $\sum_{i\geq 1} A_i$ is Bochner integrable with respect to the probability $\mathbb{P}$ (cf. Diestel and Uhl (1977) for this extension of the Lebesgue integral theory to separable Banach spaces valued functions). Then replace (3) by

$$\mathbb{E}\left(\sum_{i\geq 1} A_i\right) = I_B. \tag{7}$$

Then (4) defines a F_n-martingale with mean I_B.

Define

$$\mathcal{E}_B = \{B^{\mathbb{N}_+}\text{–valued random variable } A;\ (6) \text{ and } (7) \text{ hold}\}. \tag{8}$$

Our initial purpose is to show that the conditions for the L^p convergence obtained in Biggins (1989, 1992) in the case $B = \mathbb{C}$ can be naturally extended to B provided B satisfies the two following conditions: we need an extension to B (Lemma 1) of the essential von Bahr and Esseen result used by Biggins, so B will be supposed to be of type p for some $1 < p \leq 2$ (cf. Ledoux and Talagrand (1991), Chapter 9). Moreover from now B is supposed to satisfy the Radon-Nikodym property (cf. Diestel and Uhl (1977)) in order to extend martingales convergence theorems to B. Concrete examples of spaces of type p are finite dimensional vector spaces over $\mathbb{R}$ and closed subspaces of L^p. The Radon-Nikodym property is satisfied by the closed subspaces of reflexive Banach spaces, in particular by finite dimensional vector spaces and closed subspaces of L^p ($1 < p < \infty$). Two examples of the kind of B then allowed are the space of square complex matrices of order $k \geq 2$ and certain Sobolev spaces (described in Section 2.2).

This abstract point of view is motivated by the following problem: consider an open subset Λ of $\mathbb{R}^k$ or $\mathbb{C}^k$, $\lambda \mapsto A(\lambda)$ a stochastic process taking values in $\mathbb{C}^{\mathbb{N}_+}$, and $(\lambda \mapsto A(\lambda)(a))_{a\in T}$ a sequence of independent copies of $\lambda \mapsto A(\lambda)$. Then assume that for every $\lambda \in \Lambda$, $A(\lambda) \in \mathcal{E}_{\mathbb{C}}$ (see (8)) so that $(Y_{A(\lambda),n})_{n\geq 1}$ is the martingale defined by (4). A natural question is to determine whether $Y_{A(\lambda),n}$ converges for every λ almost surely, and if so, whether the convergence holds almost surely simultaneously for all λ. Then, if the random function $\lambda \mapsto A(\lambda)$ possesses some regularity, a companion question is to determine whether or not the limit process $\lambda \mapsto \lim_{n\to\infty} Y_{A(\lambda),n}$ possesses some related regularity (by the regularity of $\lambda \mapsto A(\lambda)$ we mean the regularity of its components, the $\lambda \mapsto A_i(\lambda)$).

Satisfactory answers to these questions in particular cases are given in Joffe, Le Cam and Neveu (1973) when $\Lambda \subset \mathbb{C}$ and $\lambda \mapsto A(\lambda)$ is analytic; in Biggins (1989) when $\Lambda \subset \mathbb{R}$ and $\lambda \mapsto A(\lambda)$ is non-negative and is differentiable, and also when $\Lambda \subset \mathbb{C}$ and $\lambda \mapsto A(\lambda)$ is analytic; in Biggins (1992) when $\Lambda \subset \mathbb{C}^k$ and $\lambda \mapsto A(\lambda)$ is analytic; in Barral (2000) when $\Lambda \subset \mathbb{R}$ and $\lambda \mapsto A(\lambda)$ is non-negative and continuously differentiable. Roughly speaking these results conclude that there is simultaneous and even uniform convergence, and that the limit possesses the same (or almost the same) regularity as $\lambda \mapsto A(\lambda)$.

But the approach used in these papers does not work if $\Lambda \subset \mathbb{R}^k$ ($k \geq 2$) and $\lambda \mapsto A(\lambda)$ is not analytic. It then appears that using Sobolev spaces is a good way forward in solving the problems of convergence and regularity (Theorem 3). Moreover our approach makes it possible to obtain in another way some results in the papers mentioned above (see Section 2.2 and Section 4).

The next purpose of the paper (Section 5) is to generalize the usual construction by changing the distribution of $A(a)$ at each node a of T. This is the varying environment counterpart of the usual construction. Indeed, the offspring distribution of $N_A(a)$ depends upon the node a, thus the multiplications defining the martingale is made down the line of descent of a varying environment branching process (see Liu (1996) or D'Souza and Hambly (1997)). Results obtained in the Galton-Watson case possess immediate extensions to this context. Then, in the non-negative one dimensional case, we define the varying environment counterpart of the measure on the boundary of T, naturally associated with $(Y_{A,n})_{n\geq 1}$. Results on L^p convergence yield conditions for this measure to be non-degenerate, and we give bounds for its dimension. This generalizes results by Peyrière (1977) and Liu and Rouault

(1996). Then, our result in Section 2.2 allows us to define and describe (via their dimension), simultaneously, uncountable families of such measures in the self-similar case (i.e. in the Galton-Watson case).

The results on the continuity of the generalized process (Theorem 7) and the dimension of the generalized limit measure (Theorem 8) are interesting because they give information on the difference between the usual statistically self-similar construction and a perturbation thereof.

We end this section by giving three areas in which the martingales we deal with are used to construct some models. They are used to construct a model for turbulence in Mandelbrot (1974 a) where the distribution of the energy in a turbulent fluid is described by certain of the random measures mentioned above. Certain of these martingales depending on the inverse temperature are used to model the free energy of spin glasses and directed polymers via a partition function (see Koukiou (1997) and references therein). Also in mathematical finance the associated measures are used to produce a model of price variations with multifractal trading time (see Mandelbrot, Calvet and Fisher (1997)). Results like Theorem 3 give information on the dependence of these models with respect to some parameters.

The paper is organized as follows. Section 2, and 3, are devoted to the proofs of L^p convergence ($p > 1$) of $(Y_n)_{n\geq 1}$ in B, with the application to the study of simultaneous convergence of uncountable families of such martingales taking values in $\mathbb{C}$, using a Banach algebra of functions for B. Section 4 deals with the continuity of the multiplicative process which defines the limit martingale. Section 5 extends the results of previous sections to the generalized process obtained when one retains the independence between the $A(a)$ but one allows the distribution of $A(a)$ to depend upon the node a of T in the process of multiplications and addition yielding the martingale $(Y_n)_{n\geq 1}$. In the non-negative case the associated measure, μ, on the boundary of T is defined, and we give almost surely μ-almost everywhere the $\liminf$ and the $\limsup$ of the logarithmic density of μ, and so a lower bound and a upper bound for the dimension of the measure μ. The paper ends with the same estimates for uncountable families of measures in the self-similar case.

2 L^p convergence of $(Y_n)_{n\geq 1}$ and application.

2.1 L^p convergence.

For $x \in \mathbb{R}_+$, $n \in \mathbb{N}_+$ and $(A(a) = (A_i(a))_{i\geq 1})_{a\in T}$ a sequence of random elements of $B^{\mathbb{N}_+}$ define

$$S_{A,n}(x) = \mathbb{E}\left[\sum_{a_1\dots a_n\in\mathbb{N}_+^n} \mathcal{N}\left(A_{a_1}(\epsilon)A_{a_2}(a_1)\dots A_{a_n}(a_1\dots a_{n-1})\right)^x\right],\ S_{A,0}(x) = 1,$$

$$\psi_A(x) = \mathbb{E}[\mathcal{N}(\sum_{i\geq 1} A_i(\epsilon))^x],$$

and

$$\bar{\psi}_{A,n}(x) = \mathbb{E}\left[\left[\sum_{a_1\ldots a_n\in\mathbb{N}_+^n} \mathcal{N}\left(A_{a_1}(\epsilon)A_{a_2}(a_1)\ldots A_{a_n}(a_1\ldots a_{n-1})\right)\right]^x\right].$$

In this subsection, we write $S_n(x)$, $\psi(x)$ and $\bar{\psi}_n(x)$ for $S_{A,n}(x)$, $\psi_A(x)$ and $\bar{\psi}_{A,n}(x)$ respectively and we define

$$\bar{I}_B = (\mathbf{1}_{\{i=1\}} I_B)_{i\geq 1}.$$

Theorem 1 *Fix $p > 1$. Suppose one of the three following assertions holds.*

i) $p \in]1,2]$, B is of type p, $\sum_{n\geq 1}(S_n(p))^{1/p} < \infty$ (satisfied as soon as $S_1(p) < 1$) and $\psi(p) < \infty$;

ii) $p \in \mathbb{N}$, $p \geq 3$, $(Y_n)_{n\geq 1}$ is bounded in L^{p-1}, $S_n(p) < 1$ and $\bar{\psi}_n(p) < \infty$ for some $n \geq 1$;

iii) $\bar{p}$ denotes the integer such that $\bar{p} < p \leq \bar{p}+1$;

a) $p > 2$, $(Y_n)_{n\geq 1}$ is bounded in $L^{\bar{p}}$, $\operatorname{ess\,sup} N_A < \infty$, $S_n(p) < 1$ for some $n \geq 1$;

b) $p > 2$, $(Y_n)_{n\geq 1}$ is bounded in $L^{\bar{p}}$, $S_n(p) < 1$ and $S_n(ph) < \infty$ for some $n \in \mathbb{N}_+$ and $h > 1$, and $\mathbb{E}(N_A^{(\bar{p}+2)\frac{h}{h-1}+\varepsilon}) < \infty$ for some $\varepsilon > 0$.

Then the martingale $(Y_n)_{n\geq 1}$ converges, almost surely, and in L^p norm to a random variable $Y = Y_A$ with mean I_B. Moreover

$$Y = \sum_{i\geq 1} A_i(\epsilon)Y(i). \tag{E}$$

where $(Y(i))_{i\geq 1}$ is a sequence of independent copies of Y, which are also independent of $A(\epsilon)$.

Remark 1 1) The conditions involved in *i)* are the natural extension of those of Liu (2000) or Biggins (1989, 1992) which need $S_1(p) < \infty$ and $\psi(p) < \infty$ when $\mathcal{N}$ is the usual norm on $\mathbb{R}$ or $\mathbb{C}$.

2) *ii)* and *iii)* are the extensions of results in Liu (1997), and they really apply, via *i)*, if B is of type 2. It is not possible to apply the approach used in Liu (2000) to completely solve the problem of L^p convergence when $B = \mathbb{R}$ and the A_i's are non-negative, because it is based on the equality $\mathcal{N}(Y) = \sum_{i\geq 1}\mathcal{N}(A_i(\epsilon))\mathcal{N}(Y(i))$ which holds only under strong hypotheses in the general case.

3) If we assume that B is a space of square matrices and that their exists a non-random invertible matrix M such that almost surely the MA_iM^{-1} are diagonal, then the problem is reduced to the study of a finite number of martingales of this kind, but in dimension 1. So in the case of non-negative diagonals, necessary and sufficient conditions for the L^p convergence ($p \geq 1$) are easily deduced from Lyons (1996) and Liu (2000).

4) Given A, if one considers equation (E) as an equality between distribution laws (then one speaks of a "fixed point of a generalized smoothing transformation"), as in Biggins (1977), Durrett and Liggett (1983), Guivarc'h (1990) and Liu (1998), it is natural to ask whether Y is the unique non trivial solution of (E). This is false in general since when

$B = \mathbb{R}$ and $A = (\frac{1}{2}\mathbf{1}_{\{1\le i\le 2\}})_{i\ge 1}$, the Cauchy law and the law δ_1 are solutions. This raises the question of whether restrictions on A can insure the uniqueness of the solution.

5) As we shall see in the proof of Theorem 1, the condition $\mathcal{C}_1$: $\mathbb{E}(N_A^{(\bar{p}+2)\frac{h}{h-1}+\varepsilon}) < \infty$ for some $\varepsilon > 0$ in $iii)b)$ could be replaced by the weaker condition $\mathcal{C}_2$: $\sum_{\ell\ge 1} \ell^{\bar{p}+1}\mathbb{P}(M_n = \ell)^{\frac{h-1}{h}} < \infty$. To see this apply the result in Liu (2000) on positive moments of MBRW to the martingale $M_n/(\mathbb{E}(N_A))^n$. This shows that if $\mathbb{E}(N_A^\beta) < \infty$ for some $\beta > 1$ then $\mathbb{E}(M_n^\beta) < \infty$. So if $\mathcal{C}_1$ holds the Chebitshev inequality yields $\mathcal{C}_2$. Unfortunately $\mathcal{C}_2$ is not easy to verify without using the moments of the offspring distribution.

We now give similar but more precise results in the so called "i.i.d. case". We assume that $A = (A_1, A_2, \ldots, A_{N_A}, 0, 0, \ldots)$ possesses a Bellman-Harris like structure where the A_i are identically distributed and independent conditionally on N_A. More precisely, there exists a Bochner integrable random variable $W \in B$ with mean I_B, a positive measurable function g and a random integer $N \in \mathbb{N}_+$ such that

$$\mathbb{E}\left(\frac{N}{g(N)}\right) = 1,$$

and

$$A = (A_i)_{i\ge 1} = \left(\mathbf{1}_{\{1\le i\le N\}}\frac{W_i}{g(N)}\right)_{i\ge 1}$$

where the W_i's are identically distributed and independent of one another and of N.

Thus for every $a \in T$, $A(a) = (\mathbf{1}_{\{1\le i\le N(a)\}}\frac{W_i(a)}{g(N(a))})_{i\ge 1}$.

We shall need the following proposition.

Proposition 1 *Fix $h \ge 1$ a real number and assume that $\mathbb{E}(\mathcal{N}(W)^h) < \infty$.*

1) The sequence $\frac{1}{n}\log\mathbb{E}(\mathcal{N}(W_1W_2\ldots W_n)^h)$ converges to a limit $\rho(h) \ge 0$.

2) The mapping $h' \mapsto \rho(h')$ is convex non decreasing on $[1, h]$.

3) If $\rho(h) + \log\mathbb{E}(N/g^h(N)) < 0$ then $\sum_{n\ge 1}(S_n(h))^{1/h} < \infty$.

Proof. 1) By the assumption on the W_i's the sequence $s_n = \log\mathbb{E}(\mathcal{N}(W_1W_2\ldots W_n)^h)$ is subadditive and so s_n/n converges to $\rho(h) \in \mathbb{R}\cup\{-\infty\}$. Moreover $\mathbb{E}(\mathcal{N}(W_1W_2\ldots W_n)^h) \ge \mathcal{N}(\mathbb{E}(W_1W_2\ldots W_n))^h = \mathcal{N}(I_B)^h$. So $\rho(h) \ge 0$.

2) Follows from the log-convexity of $h \mapsto \mathbb{E}(\mathcal{N}(W_1W_2\ldots W_n)^h)$ for every $n \ge 1$.

3) In this context, we have

$$\begin{aligned} S_n(h) &= \mathbb{E}(\sum_{a_1\ldots a_n\in\mathbb{N}_+^n}\prod_{j=1}^n \frac{\mathbf{1}_{\{1\le a_j\le N(a_1\ldots a_{j-1})\}}}{g(N(a_1\ldots a_{j-1}))^h})\mathbb{E}(\mathcal{N}(W_1W_2\ldots W_n)^h)) \\ &= \left(\mathbb{E}(N/g^h(N))\right)^n\mathbb{E}(\mathcal{N}(W_1W_2\ldots W_n)^h) \end{aligned}$$

from which the conclusion follows.

We can now state our result in this situation.

Theorem 2 (The i.i.d. case) *Fix $p > 1$. Assume one of the following assertions:*

i) $p \in]1, 2]$, $\rho(p) + \log \mathbb{E}(N/g^p(N)) < 0$, $\mathbb{E}(N^p/g^p(N)) < \infty$ and B is of type p;

ii) $p \in \mathbb{N}$, $p \geq 3$, $\max_{h\in\{2,p\}} \rho(h) + \log \mathbb{E}(N/g^h(N)) < 0$, $\mathbb{E}(N^p/g^p(N)) < \infty$ and B is of type 2;

iii) $p > 2$, $\max_{h\in\{2,p\}} \rho(h) + \log \mathbb{E}(N/g^h(N)) < 0$, $\mathbb{E}(N^{\bar{p}+1}/g^p(N)) < \infty$ where $\bar{p}$ denotes the integer such that $\bar{p} < p \leq \bar{p} + 1$, and B is of type 2.

Then the conclusion of Theorem 1 holds.

Remark 2 $i)$ and $ii)$ are direct consequences of Theorem 1. Part $iii)$ needs some additional argument; it implies $ii)$ and generalizes the corresponding result of Peyrière (1977) where $B = \mathbb{R}$, the W_i's are non negative, $g(N) = N$ and $\rho(p) = \log \mathbb{E}(W^p)$.

2.2 Application to uniform convergence of complex valued martingales in the branching random walk.

Fix $k \geq 1$ and let Λ be an open subset of $\mathbb{R}^k$. Consider a stochastic process $\lambda \in \Lambda \mapsto A(\lambda)$ such that every $A(\lambda)$ belongs to $\mathcal{E}_{\mathbb{C}}$ (see (8)). Such families appear naturally (see Biggins (1989, 1992), Barral (1999, 2000 b) and references therein). Let us give a simple example in the i.i.d case described in Section 2.1: if W is almost surely positive and satisfies the assumption $i)$ of Theorem 2 for some $p > 1$, and if f is a continuous function from a neighbourhood U of 0 in $\mathbb{R}^k$ to $\mathbb{C}$ such that $f(0) = 1$, taking an open set $\Lambda \subset U$ small enough yields the family

$$\lambda \mapsto A(\lambda) = \left(\mathbf{1}_{\{1\leq i\leq N\}} \frac{W_i^{f(\lambda)}/g^{f(\lambda)}(N)}{\mathbb{E}(W^{f(\lambda)})\mathbb{E}(N/g^{f(\lambda)}(N))}\right)_{i\geq 1}.$$

As we said in Section 1, it is natural to seek conditions under which the associated martingales $(Y_{A(\lambda),n})_{n\geq 1}$ converge almost surely simultaneously and then if the $\lambda \mapsto A_i(\lambda)$ possess some regularity almost surely , to know whether the limit martingale $\lambda \mapsto Y_A(\lambda)$ possesses a related regularity.

Our aim is essentially to deal with the case where $k \geq 2$ and the $\lambda \mapsto A_i(\lambda)$ are not analytic when k is an even integer. This is the case where the approach developed until now does not apply. See Section 1 for details.

The key idea is to imbed the $\lambda \mapsto A_i(\lambda)$ in a convenient Sobolev space. Moreover, this approach also yields Biggins (1992) results in certain cases when the $\lambda \mapsto A_i(\lambda)$ are analytic, and it gives an alternative to the Biggins (1989) and Barral (2000) approach when $k = 1$ (see Section 4).

To state our result some preliminaries are needed.

Assume that Λ is bounded and possesses the cone property (see Adams (1975)). This holds, for example, if Λ is an open ball. Fix $p \in]1, 2]$ and an integer $m \geq 1$ such that

$mp > k$. These conditions are needed to get a Sobolev space with the property to be a Banach algebra:

Let $\mathcal{W}^{m,p}(\Lambda)$ be the Sobolev space of complex valued functions m times weakly differentiable on Λ and with weak partial derivatives of order $\leq m$ belonging to $L^p(\Lambda, \ell)$, where ℓ denotes the restriction of Lebesgue measure to Λ. Then for $f \in \mathcal{W}^{m,p}(\Lambda)$, with standard notations

$$\|f\|_{\mathcal{W}^{m,p}(\Lambda)} = \left\{ \sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Lambda,\ell)} \right\}^{1/p}.$$

By Theorem 5.23 of Adams (1975), the hypothesis made on the geometry of Λ and the inequality $mp > k$ are sufficient for $\mathcal{W}^{m,p}(\Lambda)$ to be a Banach algebra with unity Id_Λ. In particular there exists a constant $C > 0$ depending upon Λ, p and m only such that

$$\forall\ f, g \in \mathcal{W}^{m,p}(\Lambda),\ \|fg\|_{\mathcal{W}^{m,p}(\Lambda)} \leq C\|f\|_{\mathcal{W}^{m,p}(\Lambda)}\|g\|_{\mathcal{W}^{m,p}(\Lambda)}. \tag{9}$$

This property yields the two following norms on $\mathcal{W}^{m,p}(\Lambda)$, both equivalent to $\|\ \|_{\mathcal{W}^{m,p}(\Lambda)}$ and satisfying (5):

$$\mathcal{N}_1 :\ f \in \mathcal{W}^{m,p}(\Lambda) \mapsto C\,\|f\|_{\mathcal{W}^{m,p}(\Lambda)}$$

and

$$\mathcal{N}_2 :\ f \in \mathcal{W}^{m,p}(\Lambda) \mapsto \sup_{g \in \mathcal{W}^{m,p}(\Lambda)\setminus\{0\}} \|fg\|_{\mathcal{W}^{m,p}(\Lambda)}/\|g\|_{\mathcal{W}^{m,p}(\Lambda)}.$$

We compare these norms in Remark 3.
Here $\mathcal{W}^{m,p}(\Lambda)$ is a closed subset of $(L^p(\Lambda, \ell))^2$, which is separable and reflexive. So by Cor. III.2.13 and Thm III.3.2. of Diestel and Uhl (1977) it satisfies the Radon-Nikodym property. Moreover, by Ledoux and Talagrand (1991, p. 247), since it is a closed subset of $(L^p(\Lambda, \ell))^2$, $\mathcal{W}^{m,p}(\Lambda)$ is of type p. So it is possible to apply Theorem 1 and 2 with $B = \mathcal{W}^{m,p}(\Lambda)$ and $\mathcal{N} = \mathcal{N}_1$ or $\mathcal{N} = \mathcal{N}_2$. This leads to the following result.

Theorem 3 *Assume that: i) for every $i \in \mathbb{N}_+$, $A_i : \lambda \in \Lambda \mapsto A_i(\lambda)$ is a random variable taking values in $\mathcal{W}^{m,p}(\Lambda)$ that is almost surely continuous;*

ii) $A = (A_i)_{i\geq 1}$ belongs to $\mathcal{E}_{\mathcal{W}^{m,p}(\Lambda)}$;

iii) A satisfies one of the assumptions of Theorem 1 or 2 with $B = \mathcal{W}^{m,p}(\Lambda)$ and $\mathcal{N} \in \{\mathcal{N}_1, \mathcal{N}_2\}$.

Then $Y_{A,n}$ converges in $\mathcal{W}^{m,p}(\Lambda)$ almost surely and in L^p norm. Consequently the martingales $(Y_{A(\lambda),n})_{n\geq 1}$, $\lambda \in \Lambda$, converge almost surely simultaneously, and uniformly on the compact subsets of Λ; almost surely the limit function $\lambda \mapsto Y_{A(\lambda)}$ is continuous and belongs to $\mathcal{W}^{m,p}(\Lambda)$. Moreover if k is an even integer, $\mathbb{R}^k$ is identified with $\mathbb{C}^{k/2}$, and the $\lambda \mapsto A_i(\lambda)$ are almost surely analytic then $\lambda \mapsto Y_{A(\lambda)}$ is almost surely analytic.

Remark 3 1) The main reason for introducing the second norm $\mathcal{N}_2$ is the following. To apply Theorem 1, we need to check conditions like $S_1(p) < 1$. If $\mathcal{N} = \mathcal{N}_1$ and the mapping

$\lambda \mapsto A(\lambda)$ is almost surely equal to a constant $\bar{A} \in \mathcal{E}_{\mathbb{C}}$, it is easily seen that the condition $S_1(p) < 1$ does not coincide with the usual condition $S_{\bar{A},1}(p) < 1$ for the convergence in L^p of $Y_{\bar{A},n}$. Indeed, in this case

$$S_1(p) = C\,\ell(\Lambda)\mathbb{E}\left(\sum_{i\geq 1} |\bar{A}_i|^p\right) < 1.$$

So, as $C\,\ell(\Lambda) > 1$, which can be seen from (9), this yields a more restrictive sufficient condition than $S_{\bar{A},1}(p) = \mathbb{E}\left(\sum_{i\geq 1} |\bar{A}_i|^p\right) < 1$.

On the other hand, when $\mathcal{N} = \mathcal{N}_2$ one checks that $S_1(p) = S_{\bar{A},1}(p)$. Moreover, even if $\lambda \mapsto A(\lambda)$ is not constant, the condition $S_1(p) < 1$ implies that for all $\lambda_0 \in \Lambda$, the usual condition for L^p convergence of $Y_{A(\lambda_0),n}$ holds, and $S_{A(\lambda_0),1}(p) < 1$.

To see this, consider $g \neq 0$ an element of $C_0^\infty(\mathbb{R}^k)$. Then fix $\lambda_0 \in \Lambda$ and for $\varepsilon > 0$ define $g_{\lambda_0,\varepsilon}(.) = g(\frac{\lambda_0 - .}{\varepsilon})$. It follows from the definitions of $g_{\lambda_0,\varepsilon}$ and $\mathcal{N}_2$ that for $i \geq 1$, almost surely $|A_i(\lambda_0)|^p = \lim_{\varepsilon\to 0} \frac{\|g_{\lambda_0,\varepsilon}(.)A_i(.)\|^p_{\mathcal{W}^{m,p}(\Lambda)}}{\|g_{\lambda_0,\varepsilon}\|^p_{\mathcal{W}^{m,p}(\Lambda)}}$. So by the Fatou Lemma

$$\mathbb{E}[|A_i(\lambda_0)|^p] \leq \liminf_{\varepsilon\to 0} \mathbb{E}\left(\frac{\|g_{\lambda_0,\varepsilon}(.)A_i(.)\|^p_{\mathcal{W}^{m,p}(\Lambda)}}{\|g_{\lambda_0,\varepsilon}\|^p_{\mathcal{W}^{m,p}(\Lambda)}}\right) \leq \mathbb{E}[\mathcal{N}_2^p(A_i(.))].$$

In particular,

$$\sup_{\lambda\in\Lambda} S_{A(\lambda),1}(p) \leq S_1(p), \text{ so } \sup_{\lambda\in\Lambda} S_{A(\lambda),1}(p) < 1. \tag{10}$$

We end this comparison by noting that under the assumption $i)$ of Theorem 1 with $S_1(p) < 1$ and $\mathcal{N} = \mathcal{N}_1$, at least if the $A_i(\lambda)$ are non-negative, since Id_Λ is the mean of the limit function $\lambda \mapsto Y_{A(\lambda)}$, for every $\lambda \in \Lambda$, we have $S_{A(\lambda),1}(p) < 1$ by Thm 5.1 of Liu (1997). But the definition of $\mathcal{N}_1$ implies that $C\int_\Lambda S_{A(\lambda),1}(p)\ell(d\lambda) < 1$. So the set of λ's such that $S_{A(\lambda),1}(p) < C^{-1}\left(\ell(\Lambda)\right)^{-1}$ is of positive ℓ-measure.

2) We have to produce sufficient conditions on the $\lambda \mapsto A_i(\lambda)$ for the condition $S_1(p) < 1$ to be satisfied under $\mathcal{N}_1$ or $\mathcal{N}_2$. A condition like $\psi(p) < \infty$ is easily satisfied.

Example 1. Once $\lambda_0 \in \Lambda$ is fixed, natural conditions are the following (i), (ii) and (iii) or (i'), (ii) and (iii'):

(i) $\mathcal{N} = \mathcal{N}_1$ and $S_{A(\lambda_0),1}(p) < C^{-1}\left(\ell(\Lambda)\right)^{-1}$;
(i') $\mathcal{N} = \mathcal{N}_2$ and $S_{A(\lambda_0),1}(p) < 1$;
For $i \geq 1$ define

$$\varepsilon_i : \lambda \mapsto \varepsilon_i(\lambda) = A_i(\lambda) - A_i(\lambda_0)$$

and define $\varepsilon = (\varepsilon_i)_{i\geq 1}$. In particular
(ii) $\varepsilon(\lambda_0) = 0$;

(iii) $\mathcal{N} = \mathcal{N}_1$ and $\operatorname{ess\,sup} \sup_{1\leq i\leq N_{A(\lambda)}, \lambda\in\Lambda} \frac{|\varepsilon_i(\lambda)|}{|A_i(\lambda_0)|}$ and $S_{\varepsilon,1}(p)$ are small enough so that $S_{A,1}(p) < 1$ holds;

(iii') $\mathcal{N} = \mathcal{N}_2$ and ess sup $\sup_{1 \le i \le N_{A(\lambda)}, \lambda \in \Lambda} \frac{|\varepsilon_i(\lambda)|}{|A_i(\lambda_0)|}$ and ess sup $\sum_{i \ge 1} \sum_{0 < |\alpha| \le m} \sup_{\lambda \in \Lambda} |D^\alpha \varepsilon_i(\lambda)|$ are small enough so that $S_{A,1}(p) < 1$ holds.

In other words one can generate an example to which Theorem 3 applies by considering for every $\lambda \in \Lambda$, $A(\lambda) = A(\lambda_0) + \varepsilon(\lambda)$ where $A(\lambda_0) \in \mathcal{E}_{\mathbb{C}}$ and ε is a random variable with mean 0 taking values in $\mathcal{W}^{m,p}(\Lambda)^{\mathbb{N}_+}$, such that (i), (ii), (iii) or (i'), (ii), (iii') are satisfied.

Here is a type of family on which these conditions can be verified. Assume that Λ is a neighbourhood of $\lambda_0 \in \mathbb{R}^k$. Fix $\hat{A} = (\hat{A}_i)_{i \ge 1} \in \mathcal{E}_{\mathbb{R}}$ such that the $\hat{A}_i$ are non-negative and for every $i \ge 1$, fix $f_i \in \mathcal{W}^{m,p}(\Lambda)$ with $f_i(\lambda_0) = 1$ and positive values. Then define for every $i \ge 1$, $\lambda \in \Lambda \mapsto A_i(\lambda) = \hat{A}_i^{f_i(\lambda)} / \mathbb{E}(\sum_{j \ge 1} \hat{A}_j^{f_j(\lambda)})$. The derivatives of the A_i can be explicited in terms of the $\hat{A}_i$ and the derivatives of the f_i, and simple conditions yield $\varepsilon = (\lambda \mapsto [\hat{A}_i^{f_i(\lambda)} / \mathbb{E}(\sum_{j \ge 1} \hat{A}_j^{f_j(\lambda)})] - \hat{A}_i)_{i \ge 1}$ and $A(\lambda_0) = \hat{A}$ satisfying the previous conditions.

Example 2. Fix a random variable f taking values in $\mathcal{W}^{m,p}(\Lambda)$ with mean Id_Λ, and independently $(\hat{A}_i)_{i \ge 1} \in \mathcal{E}_{\mathbb{C}}$ with $S_{\hat{A},1}(p) < \infty$ and $\psi_{\hat{A}}(p) < \infty$. Assume that $\mathbb{E}[\mathcal{N}_j^p(f)] S_{\hat{A},1}(p) < 1$ ($j = 1$ or $j = 2$) and define for every $i \ge 1$, $\lambda \in \Lambda \mapsto A_i(\lambda) = f(\lambda) \hat{A}_i$.

3) The result in the analytic case is of the same nature as Theorem 2 and Corollary 3 of Biggins (1992). Our conditions are more restrictive because we deal with $\mathcal{W}^{m,p}(\Lambda)$, which is largest than the space of bounded analytic functions, and we do not work finely with the Cauchy formula as Biggins (1992). When $\mathcal{N} = \mathcal{N}_2$, we see via (10) that the main assumption of Theorem 2 of Biggins (1992) is satisfied if $S_1(p) < 1$. Moreover under $\mathcal{N}_2$ the same argument as the one leading to (10) shows that the condition $\psi_A(p) < \infty$ implies the second assumption of Theorem 2 of Biggins (1992), namely $\sup_{\lambda \in \Lambda} \psi_{A(\lambda)}(p) < \infty$.

Proof of Theorem 3. It is an immediate consequence of the convergence of $Y_{A,n}$ in B using Theorem 1 and the Sobolev imbedding $\mathcal{W}^{m,p}(\Lambda) \to C^0(\bar{\Lambda})$ (see Lemma 5.15 of Adams (1975)) that the elements of $\mathcal{W}^{m,p}(\Lambda)$ are essentially bounded and there exists a constant C_1 depending only upon Λ, k, p, and m such that for every $f \in \mathcal{W}^{m,p}(\Lambda)$

$$\text{ess sup}\, |f| \le C_1 \, \|f\|_{\mathcal{W}^{m,p}(\Lambda)}.$$

The result in the analytic case is a consequence of the uniform convergence.

3 Proofs of the results of Section 2.1.

We shall need an extension to B of the fundamental Lemma 1 used in Biggins (1992) for the case $B = \mathbb{C}$. It is given by the following immediate consequence of Proposition 9.11 of Ledoux and Talagrand (1991).

Lemma 1 *Fix $p \in]1, 2]$. Assume that B is of type p. Then there exists a constant C_p such that for any sequences $(U_i)_{i \ge 0}$ and $(V_i)_{i \ge 0}$ of Radon r. v. in B such that $\sigma(U_i;\, i \ge 0)$ and*

$\sigma(V_i;\, i \geq 0)$ are independent, $\sum_{i\geq 0} \mathbf{1}_{\{U_iV_i\neq 0\}} < \infty$ almost surely, and the V_i's are mutually independent, integrable and with mean 0,

$$\mathbb{E}(\mathcal{N}(\sum_{i\geq 0} U_iV_i)^p) \leq C_p \sum_{i\geq 0} \mathbb{E}(\mathcal{N}(U_i)^p)\mathbb{E}(\mathcal{N}(V_i)^p).$$

Proof of Theorem 1. *i)* For $n \geq 1$ and $a_1 \dots a_n \in \mathbb{N}_+^n$, define

$$U_{a_1\dots a_n} = A_{a_1}(\epsilon)A_{a_2}(a_1)\dots A_{a_n}(a_1\dots a_{n-1})$$

and

$$V_{a_1\dots a_n} = [\sum_{i\geq 1} A_i(a_1\dots a_n)] - I_B.$$

The sequences $(U_{a_1\dots a_n})$ and $(V_{a_1\dots a_n})$, $n \geq 1$, $a_1 \dots a_n \in \mathbb{N}_+^n$, satisfy the hypotheses of Lemma 1 and

$$Y_{n+1} - Y_n = \sum_{a_1\dots a_n \in \mathbb{N}_+^n} U_{a_1\dots a_n} V_{a_1\dots a_n}.$$

So by Lemma 1 and by taking into account the fact that the $V_{a_1\dots a_n}$'s are identically distributed

$$\mathbb{E}(\mathcal{N}(Y_{n+1} - Y_n)^p) \leq C_p S_n(p)\psi_{A-\bar{I}_B}(p).$$

Since $\psi_{A-\bar{I}_B}(p) \leq 2^{p-1}(\psi(p) + \mathcal{N}(I_B)^p)$, we have $\sum_{n\geq 1}[\mathbb{E}(\mathcal{N}(Y_{n+1} - Y_n)^p)]^{1/p} < \infty$ by the hypotheses of the Theorem and so $(Y_n)_{n\geq 1}$ is bounded in L^p norm. The conclusion comes then from Corollary V.2.4 and Theorem V.2.8 of Diestel and Uhl (1977).

In particular we obtain

$$(\mathbb{E}(\mathcal{N}(Y - I_B)^p))^{1/p} \leq C_p^{1/p}\psi_{A-\bar{I}_B}^{1/p}(p) \sum_{n\geq 0} S_n^{1/p}(p). \tag{11}$$

ii), *iii)a)b)*: as in Liu (1997), the proof is based on the approach of Kahane and Peyrière (1976).

Fix n as in the statement. It is easily seen by definition (see also the proof of Theorem 4) that for every $m \geq n+1$, Y_m can be written

$$Y_m = \sum_{a_1\dots a_n \in \tilde{T}\cap\mathbb{N}_+^n} A_{a_1}(\epsilon)A_{a_2}(a_1)\dots A_{a_n}(a_1\dots a_{n-1})Y_{m-n}(a_1\dots a_n)$$

where the $Y_{m-n}(a_1 \dots a_n)$'s are independent copies of Y_{m-n} and are independent of the σ-algebra F_n. Thus

$$\mathcal{N}(Y_m) \leq \sum_{a_1\dots a_n \in \tilde{T}\cap\mathbb{N}_+^n} \mathcal{N}(A_{a_1}(\epsilon)A_{a_2}(a_1)\dots A_{a_n}(a_1\dots a_{n-1})) .\mathcal{N}(Y_{m-n}(a_1\dots a_n)). \tag{12}$$

Let $\bar{p}$ stand for the integer such that $\bar{p} < p \leq \bar{p}+1$. We shall use the fact that by the subadditivity of the function $x \mapsto x^{p/\bar{p}+1}$ on $[0,\infty[$, for all integer $M \geq 1$ and non negative real numbers $x_1,\ldots,x_M$,

$$\left(\sum_{i=1}^{M} x_i\right)^p \leq \sum_{i=1}^{M} x_i^p + \sum \alpha_{j_1\ldots j_M} (x_1^{j_1}\ldots x_M^{j_M})^{p/\bar{p}+1} \tag{13}$$

where in the last sum the j_i's are $\leq \bar{p}$, $j_1+\cdots+j_M = \bar{p}+1$, $j_i \geq 0$ and $\sum \alpha_{j_1\ldots j_M} = M^{\bar{p}+1} - M$.

In order to simplify the mathematical expressions for every $i \geq 1$ define $\hat{A}_i$ and $Y_{m-n,i}$ the random variables equal to 0 if $i > M_n$ and equal to $A_{a_1}(\epsilon)A_{a_2}(a_1)\ldots A_{a_n}(a_1\ldots a_{n-1})$ and $Y_{m-n}(a_1\ldots a_n)$ respectively if $a_1\ldots a_n$ is the i^{th} element of $\tilde{T}\cap\mathbb{N}_+$ in lexicographical order.

By using (12) and (13), the independencies between random variables, the equidistribution of the $Y_{m-n,i}$'s given M_n, and the fact that given $j_1,\ldots,j_{M_n}$ as in (13) one has $\frac{j_i p}{\bar{p}+1} \leq \bar{p}$ and so $\mathbb{E}(\mathcal{N}(Y_{m-n})^{j_i p/\bar{p}+1}) \leq (\mathbb{E}(\mathcal{N}(Y_{m-n})^{\bar{p}}))^{j_i p/\bar{p}(\bar{p}+1)}$ for every $1 \leq i \leq M_n$, we obtain

$$\begin{aligned}\mathbb{E}(\mathcal{N}(Y_m)^p) \;&\leq\; \mathbb{E}(\sum_{i=1}^{M_n} \mathcal{N}(\hat{A}_i)^p)\mathbb{E}(\mathcal{N}(Y_{m-n})^p)\\ &\quad+\mathbb{E}\left(\sum \alpha_{j_1\ldots j_{M_n}} (\mathcal{N}(\hat{A}_1)^{j_1}\ldots\mathcal{N}(\hat{A}_{M_n})^{j_{M_n}})^{p/\bar{p}+1}\right)\, \mathbb{E}(\mathcal{N}(Y_{m-n})^{\bar{p}})^{p/\bar{p}}.\end{aligned}$$

As $(\mathcal{N}(Y_m)^p)_{m\geq 1}$ is a submartingale and $\mathbb{E}(\sum_{i=1}^{M_n}\mathcal{N}(\hat{A}_i)^p) = S_n(p)$, this yields

$$\mathbb{E}(\mathcal{N}(Y_m)^p)(1-S_n(p)) \leq \mathbb{E}(\sum \alpha_{j_1\ldots j_{M_n}} (\mathcal{N}(\hat{A}_1)^{j_1}\ldots\mathcal{N}(\hat{A}_{M_n})^{j_{M_n}})^{p/\bar{p}+1})(\mathbb{E}(\mathcal{N}(Y_{m-n})^{\bar{p}}))^{p/\bar{p}}.$$

Now if p is an integer, that is $p = \bar{p}+1$,

$$\mathbb{E}(\sum \alpha_{j_1\ldots j_{M_n}} \mathcal{N}(\hat{A}_1)^{j_1}\ldots\mathcal{N}(\hat{A}_{M_n})^{j_{M_n}}) \leq \mathbb{E}\left[\left(\sum_{i=1}^{M_n}\mathcal{N}(\hat{A}_i)\right)^p\right] = \bar{\psi}_n(p).$$

If p is not an integer, define

$$\begin{aligned}\Gamma &= \mathbb{E}(\sum \alpha_{j_1\ldots j_{M_n}} (\mathcal{N}(\hat{A}_1)^{j_1}\ldots\mathcal{N}(\hat{A}_{M_n})^{j_{M_n}})^{p/\bar{p}+1})\\ &= \sum_{M\geq 1}\mathbb{E}\left(\mathbf{1}_{\{M_n=M\}}\sum \alpha_{j_1\ldots j_\ell}(\mathcal{N}(\hat{A}_1)^{j_1}\ldots\mathcal{N}(\hat{A}_M)^{j_M})^{p/\bar{p}+1}\right)\end{aligned}$$

If $\operatorname{ess\,sup} N_A < \infty$ we have $\operatorname{ess\,sup} M_n \leq M_0 < \infty$ for some $M_0 \in \mathbb{N}_+$ and

$$\Gamma \leq \sum_{M=1}^{M_0}\sum \alpha_{j_1\ldots j_M}\mathbb{E}\left((\mathcal{N}(\hat{A}_1)^{j_1}\ldots\mathcal{N}(\hat{A}_M)^{j_M}))^{p/\bar{p}+1}\right).$$

If ess sup $N_A = \infty$, we use an Hölder inequality and obtain for any $h > 1$

$$\Gamma \le \sum_{M\ge 1} \sum \alpha_{j_1\dots j_M} \mathbb{P}(M_n = M)^{h-1/h} \mathbb{E}\left([\mathcal{N}(\hat{A}_1)^{j_1} \dots \mathcal{N}(\hat{A}_M)^{j_M}]^{ph/\bar{p}+1} \right)^{1/h}.$$

By the generalized Hölder inequality, for every integer $M \ge 1$ and $h \ge 1$,

$$\begin{aligned} \mathbb{E}\left([\mathcal{N}(\hat{A}_1)^{j_1} \dots \mathcal{N}(\hat{A}_M)^{j_M}]^{ph/\bar{p}+1} \right) &\le \prod_{i=1}^{M} (\mathbb{E}(\mathcal{N}(\hat{A}_i)^{ph})^{j_i/\bar{p}+1} \\ &\le \prod_{i=1}^{M} (\mathbb{E}(\sum_{i=1}^{M} \mathcal{N}(\hat{A}_i)^{ph})^{j_i/\bar{p}+1} \le S_n(ph). \end{aligned}$$

By using the fact that $\sum \alpha_{j_1\dots j_M} \le M^{\bar{p}+1}$ we conclude that if ess sup $M_n \le M_0$ then we have $\Gamma \le M_0^{\bar{p}+2} S_n(p)$ and elsewhere $\Gamma \le \sum_{M\ge 1} M^{\bar{p}+1} \mathbb{P}(M_n = M)^{h-1/h} S_n^{1/h}(ph)$ for any $h > 1$.

Thus, in all the cases $ii)$, $iii)a)$ and $iii)b)$ (see remark 1.5)), we have

$$\mathbb{E}(\mathcal{N}(Y_m)^p)(1 - S_n(p)) \le T(p) \sup_{q\ge 1} (\mathbb{E}(\mathcal{N}(Y_q)^{\bar{p}}))^{p/\bar{p}}.$$

with $1 - S_n(p) > 0$ and $T(p) \sup_{q\ge 1} (\mathbb{E}(\mathcal{N}(Y_q)^{\bar{p}}))^{p/\bar{p}} < \infty$. So $(Y_m)_{m\ge 1}$ is bounded in L^p norm and the conclusion comes again from Corollary V.2.4 and Theorem V.2.8 of Diestel and Uhl (1977).

Proof of Theorem 2. Since the mapping $h \mapsto \rho(h) + \mathbb{E}(N/g^h(N))$ is convex, $i)$ and $ii)$ (by induction on p) are direct consequences of Theorem $1i)ii)$ and Proposition 1, if we show that $\bar{\psi}_n(p) < \infty$ for every $n \ge 1$ ($\psi(p) \le \bar{\psi}_1(p)$). Write

$$Z_n = \sum_{a_1\dots a_n \in \mathbb{N}_+^n} \prod_{j=1}^{n} \frac{\mathbf{1}_{\{1\le a_j \le N(a_1\dots a_{j-1})\}} \mathcal{N}(W_{a_n}(a_1 \dots a_{j-1}))}{g(N(a_1 \dots a_{j-1}))};$$

$\bar{\psi}_n(p) \le \mathbb{E}(Z_n^p)$ and $Z_n = \sum_{i=1}^{N(\epsilon)} \frac{\mathcal{N}(W_i(\epsilon))}{g(N(\epsilon))} Z_{n-1}(i)$ where the $Z_{n-1}(i)$'s are independent copies of Z_n and are also independent of $N(\epsilon)$ and the $W_i(\epsilon)$'s. So the independencies and a convexity inequality yield

$$\begin{aligned} \mathbb{E}(Z_n^p) &\le \mathbb{E}\left[(N(\epsilon))^{p-1} \sum_{i=1}^{N(\epsilon)} \frac{\mathcal{N}(W_i(\epsilon))^p}{g(N(\epsilon))^p} Z_{n-1}^p(i) \right] \\ &\le \mathbb{E}\left(\frac{N^p}{g^p(N)} \right) \mathbb{E}(\mathcal{N}(W)^p) E(Z_{n-1}^p) \le \mathbb{E}\left(\frac{N^p}{g^p(N)} \right)^n \mathbb{E}(\mathcal{N}(W)^p)^n. \end{aligned}$$

$iii)$ With the notations of the proof of Theorem 1, for every $1 \le i \le M_n$, the offspring distribution being the one of N, $\hat{A}_i$ is of the form

$$\frac{W_{a_1}(\epsilon) W_{a_2}(a_1) \dots W_{a_n}(a_1 \dots a_{n-1})}{g(N(\epsilon)) g(N(a_1)) \dots g(N(a_1 \dots a_{n-1}))}.$$

Then if we denote the product $g(N(\epsilon))g(N(a_1))\dots g(N(a_1\dots a_{n-1}))$ by $g_{\hat{A}_i}$, and the norm $\mathcal{N}(W_{a_1}(\epsilon)W_{a_2}(a_1)\dots W_{a_n}(a_1\dots a_{n-1}))$ by $w_{\hat{A}_i}$, by using the independencies between r. v.'s and the generalized Hölder inequality to get $\mathbb{E}((w^{j_1}_{\hat{A}_1}\dots w^{j_M}_{\hat{A}_M})^{p/\bar{p}+1})\leq \mathbb{E}(\mathcal{N}(W_1\dots W_n)^p)$, we obtain

$$\begin{aligned}
&\mathbb{E}(\sum \alpha_{j_1\dots j_{M_n}}(\mathcal{N}(\hat{A}_1)^{j_1}\dots \mathcal{N}(\hat{A}_{M_n})^{j_{M_n}})^{p/\bar{p}+1})\\
\leq\ & \sum_{M\geq 1}\sum \alpha_{j_1\dots j_M}\mathbb{E}(\mathbf{1}_{\{M_n=M\}}(g^{-j_1}_{\hat{A}_1}\dots g^{-j_M}_{\hat{A}_M})^{p/\bar{p}+1})\mathbb{E}((w^{j_1}_{\hat{A}_1}\dots w^{j_M}_{\hat{A}_M})^{p/\bar{p}+1})\\
\leq\ & \sum_{M\geq 1}\mathbb{E}(\mathbf{1}_{\{M_n=M\}}\sum \alpha_{j_1\dots j_M}(g^{-j_1}_{\hat{A}_1}\dots g^{-j_M}_{\hat{A}_M})^{p/\bar{p}+1})\mathbb{E}(\mathcal{N}(W_1\dots W_n)^p)\\
=\ & \mathbb{E}(\left[\sum_{a_1\dots a_n\in\mathbb{N}_+^n}\prod_{j=1}^n \frac{\mathbf{1}_{\{1\leq a_j\leq N(a_1\dots a_{j-1})\}}}{g^{p/\bar{p}+1}(N(a_1\dots a_{j-1}))}\right]^{\bar{p}+1})\mathbb{E}(\mathcal{N}(W_1\dots W_n)^p).
\end{aligned}$$

Define $L_n=\displaystyle\sum_{a_1\dots a_n\in\mathbb{N}_+^n}\prod_{j=1}^n \frac{\mathbf{1}_{\{1\leq a_j\leq N(a_1\dots a_{j-1})\}}}{g^{p/\bar{p}+1}(N(a_1\dots a_{j-1}))}$. An argument similar to the one giving the bound for $\mathbb{E}(Z_n^p)$ yields $\mathbb{E}(L_n^{\bar{p}+1})\leq \mathbb{E}(N^{\bar{p}+1}/g^p(N))^n$. Then the conclusion (by induction on $\bar{p}$) is obtained as for *ii*).

4 Continuity of the process.

In this section, we extend the results obtained in Barral (1999, 2000 a) on the continuity of the mapping "$A\mapsto Y_A$". If a pair $(A,A')\in B^{\mathbb{N}_+}\times B^{\mathbb{N}_+}$ is chosen in such a way that A and A' belong to $\mathcal{E}_B$ and $(A(a),A'(a))_{a\in T}$ is a sequence of independent copies of (A,A'), we obtain two martingales $Y_{A,n}$ and $Y_{A',n}$ which under some conditions converge almost surely and in L^p norm respectively to Y_A and $Y_{A'}$, and we show that the L^p norm of $Y_A-Y_{A'}$ is controlled by that of $A-A'$. What we mean by the L^p norm of $A-A'$ is made explicit in the statement of the theorem.

For, if $p\in]1,2]$ and B is of type p define

$$S(A,A',p)=\max(\sum_{n\geq 0}(S_{A,n}(p))^{1/p},\sum_{n\geq 0}(S_{A',n}(p))^{1/p}),$$

$$\psi(A,A',p)=\max(\psi_{A-\bar{I}_B}(p),\psi_{A'-\bar{I}_B}(p))$$

and then

$$\alpha_{A,A',p}=\max[C_p^{2/p}S(A,A',p)^2\psi^{1/p}(A,A',p),C_p^{1/p}S(A,A',p)].$$

We obtain the following result which when $B=\mathbb{R}$ improves the corresponding results obtained in Barral (1999, 2000 a) for the non-negative case:

Theorem 4 *Fix $p \in]1,2]$; assume that B is of type p and*

$$\psi_A(p) + \psi_{A'}(p) + \sum_{n\geq 0} (S_{A,n}(p))^{1/p} + (S_{A',n}(p))^{1/p} < \infty.$$

Then

$$\begin{aligned} &[\mathbb{E}(\mathcal{N}(Y_A - Y_{A'})^p)]^{1/p} \\ \leq\ & \alpha_{A,A',p}\left(\left[\mathbb{E}\left(\sum_{i\geq 1}\mathcal{N}(A_i - A_i')^p\right)\right]^{1/p} + \left[\mathbb{E}\left(\mathcal{N}\left(\sum_{i\geq 1}[A_i - A_i]'\right)^p\right)\right]^{1/p}\right). \end{aligned}$$

Theorem 5 (Application) *Let I be a compact subinterval of $\mathbb{R}$.*

Let $(A(t))_{t\in I}$ be a stochastic process taking values in $B^{\mathbb{N}_+}$, such that every $A(t) \in \mathcal{E}_B$. Consider a sequence $((A(t)(a))_{t\in I})_{a\in T}$ of independent copies of $(A(t))_{t\in I}$. Fix $p \in]1,2]$ and assume that B is of type p,

$$\sup_{t\in I}\psi_{A(t)}(p) + \sum_{n\geq 1}(S_{A(t),n}(p))^{1/p} < \infty \tag{14}$$

and that there exists $q \in]1,p]$ and $K > 0$ such that for all $t, t' \in I$

$$[\mathbb{E}(\sum_{i\geq 1}\mathcal{N}(A_i(t) - A_i(t'))^p)]^{1/p} + [\mathbb{E}(\mathcal{N}(\sum_{i\geq 1}A_i(t) - A_i(t'))^p)]^{1/p} \leq K|t-t'|^{q/p}. \tag{15}$$

Then

i) all the processes $(A_i(t))_{t\in I}$, $i \geq 1$, possess a continuous modification with Hölder exponent γ for every $\gamma \in]0, \frac{q-1}{p}[$, and the limit process $(Y_{A(t)})_{t\in I}$ also possesses such a version with the same exponent;

ii) if $B = \mathbb{C}$, $q = p$, $\mathcal{N}$ is the usual norm and the $t \mapsto A_i(t)$ are supposed to be differentiable and absolutely continuous, then $t \mapsto Y_{A(t),n}$ converges almost surely uniformly to $t \mapsto Y_{A(t)}$ which is absolutely continuous.

Remark 4 The proof of Theorem 5*i*) is a simple consequence of Theorem 4 and an extension of the Kolmogorov-Tchentov Theorem (Tchentov (1956)). It is possible to state more general versions of Theorem 5*i*) by using results on regularity of processes in Ledoux and Talagrand (1991, Chapter 11).

The approach leading to Theorem 5*ii*) is an alternative to those of Biggins (1989) and Barral (2000 a).

Proof of Theorem 4. By the hypotheses and Theorem 1, $Y_{\tilde{A}}$ exists and has a finite L^p norm for $\tilde{A} \in \{A, A'\}$. Moreover, by construction, for every $n \geq 1$

$$Y_{\tilde{A}} = \sum_{a=a_1\dots a_n\in\mathbb{N}_+^n} \tilde{A}_{a_1}(\epsilon)\tilde{A}_{a_2}(a_1)\dots\tilde{A}_{a_n}(a_1\dots a_{n-1})Y_{\tilde{A}}(a)$$

where

$$Y_{\tilde{A}}(a) = \lim_{l\to\infty} \sum_{a'_1\ldots a'_l\in\mathbb{N}_+^l} \tilde{A}_{a'_1}(a)\tilde{A}_{a'_2}(aa'_1)\ldots\tilde{A}_{a'_l}(aa'_1\ldots a'_{l-1})$$

and the $(Y_A(a), Y_{A'}(a))$'s, $a \in \mathbb{N}_+^n$, are independent copies of $(Y_A, Y_{A'})$. They are also independent of the $(A(b), A'(b))$, $b \in \cup_{l=0}^{n-1}\mathbb{N}_+^l$, and satisfy almost surely

$$Y_{\tilde{A}}(a) = \sum_{i\geq 1} \tilde{A}_i(a) Y_{\tilde{A}}(ai). \tag{E_a}$$

Now by using equations (E_a) one shows by induction that for every integer $m \geq 0$

$$Y_A - Y_{A'} = Q_m + \sum_{l=0}^{m} R_l + R_{l'}$$

with

$$Q_m = \sum_{a\in\mathbb{N}_+^m}\sum_{i\geq 1} A'_{a_1}(\epsilon)A'_{a_2}(a_1)\ldots A'_{a_m}(a_1\ldots a_{m-1})A_i(a)(Y_A(ai) - Y_{A'}(ai)),$$

$$R_{l=} \sum_{a\in\mathbb{N}_+^l}\sum_{i\geq 1} A'_{a_1}(\epsilon)A'_{a_2}(a_1)\ldots A'_{a_l}(a_1\ldots a_{l-1})(A_i(a) - A'_i(a))(Y_A(ai) - I_B)$$

and

$$R'_l = \sum_{a\in\mathbb{N}_+^l} A'_{a_1}(\epsilon)A'_{a_2}(a_1)\ldots A'_{a_l}(a_1\ldots a_{l-1})[\sum_{i\geq 1}(A_i(a) - A'_i(a))].$$

Since $\mathbb{E}(Y_A(ai) - Y_{A'}(ai)) = \mathbb{E}(Y_A(ai) - I_B) = \mathbb{E}[\sum_{i\geq 1}(A_i(a) - A'_i(a))] = 0$, by taking account of the independencies between variables, we can apply Lemma 1 to Q_m, R_l, R'_l and obtain:

$$\mathbb{E}(\mathcal{N}(Q_m)^p) \leq C_p S_{A',m+1}(p)\mathbb{E}(\mathcal{N}(Y_A - Y_{A'})^p),$$

$$\mathbb{E}(\mathcal{N}(R_l)^p) \leq C_p S_{A',l}(p) S_{A-A',1}(p)\mathbb{E}(\mathcal{N}(Y_A - I_B)^p) \tag{16}$$

and

$$\mathbb{E}(\mathcal{N}(R'_l)^p) \leq C_p S_{A',l}(p)\psi_{A-A'}(p). \tag{17}$$

Then by hypotheses $\lim_{m\to\infty}\mathbb{E}(\mathcal{N}(Q_m)^p) = 0$, so

$$[\mathbb{E}(\mathcal{N}(Y_A - Y_{A'})^p)]^{1/p} \leq \sum_{l\geq 0}[\mathbb{E}(\mathcal{N}(R_l)^p]^{1/p} + [\mathbb{E}(\mathcal{N}(R'_l)^p]^{1/p}$$

and the conclusion results from (11), (16) and (17).

Proof of Theorem 5*ii*). First note that by assumption the sequence $t \mapsto Y_{A(t),n}$ is a martingale in $\mathcal{W}^{1,p}(\mathrm{Int}(I))$, which is a Banach algebra of type p which satisfies the Radon-Nikodym property (see Section 2.2). Moreover it is well known (see Brezis 1983 Ch.8) that continuous elements of $\mathcal{W}^{1,p}(\mathrm{Int}(I))$ are absolutely continuous. So by using the same imbedding as in the proof of Theorem 3, it is then enough to show that this martingale is bounded in L^p norm in $\mathcal{W}^{1,p}(\mathrm{Int}(I))$:

On the one hand by (14) and the computations made in the proof of Theorem 1 we have $\sup_{n\in\mathbb{N}_+,t\in I}\mathbb{E}(|Y_{A(t),n}|^p)<\infty$.

On the other hand it follows from the differentiability of the $t\mapsto A_i(t)$, (15), the Fatou Lemma, the inequality $\mathbb{E}(\mathcal{N}(Y_{A(t),n}-Y_{A(t'),n}))^p\le\mathbb{E}(\mathcal{N}(Y_{A(t)}-Y_{A(t')})^p$ which holds for every $n\ge 1$ and $t,t'\in I$, and Theorem 4 that

$$\sup_{n\in\mathbb{N}_+,t\in I}\mathbb{E}\left(|\frac{d}{dt}Y_{A(t),n}|^p\right)\le K.$$

Then by using the Fubini Theorem, $\sup_{n\in\mathbb{N}_+}\mathbb{E}\int_I[|Y_{A(t),n}|^p+|\frac{d}{dt}Y_{A(t),n}|^p]\,dt<\infty$.

5 A generalized construction relaxing the self-similarity.

In the construction of Section 1, we can decide that the distributions of the random variables $A(a)$ for $a\in\mathbb{N}_+^n$ depend on the word a itself. We denote it by $A^{(a)}$ and we continue to assume that the $A^{(a)}$'s are independent. Then for every $a\in T$

$$Y_{A,n}^{(a)}=\sum_{b_1\dots b_n\in\mathbb{N}_+^n}A_{b_1}^{(a)}A_{b_2}^{(ab_1)}\dots A_{b_n}^{(ab_1\dots b_{n-1})}$$

defines a martingale with mean I_B.

5.1 Extended results on L^p convergence and continuity.

For $x\in\mathbb{R}_+$, $n\in\mathbb{N}_+$, $(A^{(a)}=(A_i^{(a)})_{i\ge1})_{a\in T}$ is a sequence of a random elements of $B^{\mathbb{N}_+}$, and $a\in T$ define

$$S_{A,n}(x,a)=\sum_{b_1\dots b_n\in\mathbb{N}_+^n}\mathbb{E}[\mathcal{N}(A_{b_1}^{(a)}A_{b_2}^{(ab_1)}\dots A_{b_n}^{(ab_1\dots b_{n-1})})^x],\ S_{A,0}(x,a)=1,$$

and

$$\psi_A(x,a)=\mathbb{E}[\sum_{i\ge1}\mathcal{N}(A_i^{(a)})^x].$$

Theorem 1*i*) easily extends to:

Theorem 6 *Fix $p\in]1,2]$ and assume that B is of type p.*

i) If $a\in T$ and

$$\sum_{n\ge0}S_{A,n}^{1/p}(p,a)+\sup_{b\in T}\psi_A(p,ab)<\infty$$

then $(Y_{A,n}^{(a)})_{n\ge1}$ converges almost surely, and in L^p norm, to a r. v. $Y_A^{(a)}$ with mean I_B.

ii) If $\sup_{a\in T}\sum_{n\ge0}S_{A,n}^{1/p}(p,a)+\psi_A(p,a)<\infty$, then $\sup_{a\in T}\mathbb{E}(\mathcal{N}(Y_A^{(a)})^p)<\infty$.

Now, we attach to each $a \in T$ a pair $(A^{(a)}, A'^{(a)})$ of random elements of $B^{\mathbb{N}_+}$ in such a way that the $A^{(a)}$'s and the $A'^{(a)}$'s belong to $\mathcal{E}_B$ and the $(A^{(a)}, A'^{(a)})$'s are independent, and we define for $p \in]1,2]$, if B is of type p,

$$\tilde{S}(A, A', p) = \sup_{a \in T} \max(\sum_{n \geq 0} (S_{A,n}(p,a))^{1/p}, \sum_{n \geq 0} (S_{A',n}(p,a))^{1/p}),$$

$$\tilde{\psi}(A, A', p) = \sup_{a \in T} \max(\psi_{A-\bar{I}_k}(p,a), \psi_{A'-\bar{I}_k}(p,a))$$

and

$$\tilde{\alpha}_{A,A',p} = \max[C_p^{2/p} \tilde{S}(A, A', p)^2 \tilde{\psi}^{1/p}(A, A', p), C_p^{1/p} \tilde{S}(A, A', p)].$$

Theorem 4 extends to:

Theorem 7 *Fix $p \in]1,2]$ and assume that B is of type p and*

$$\sup_{a \in T} \{\psi_A(p,a) + \psi_{A'}(p,a) + \sum_{n \geq 0} S_{A,n}^{1/p}(p,a) + \sum_{n \geq 0} S_{A',n}^{1/p}(p,a)\} < \infty.$$

Then

$$\begin{aligned} & \sup_{a \in T} [\mathbb{E}(\mathcal{N}(Y_A^{(a)} - Y_{A'}^{(a)})^p)]^{1/p} \\ \leq \quad & \tilde{\alpha}_{A,A',p} \sup_{a \in T} \{\mathbb{E}(\sum_{i \geq 1} \mathcal{N}(A_i^{(a)} - A_i'^{(a)})^p)]^{1/p} + [\mathbb{E}(\mathcal{N}(\sum_{i \geq 1} A_i^{(a)} - A_i'^{(a)})^p)]^{1/p}\}. \end{aligned}$$

The proof of this result is deduced from the computations in the proof of Theorem 4.

Remark 5 As Theorem 4, Theorem 7 is a result of continuity; in particular it gives a control of the impact on the martingale limit when one perturbs a self-similar cascade by a non self-similar one.

5.2 Dimension of the related measure in the non-negative case.

We assume that $B = \mathbb{R}$ and the $A_i^{(a)}$'s are non-negative, and we define a measure on the boundary of the tree T, related to the construction of Y_A. This measure is an extension of the generalized Mandelbrot measure considered in Liu and Rouault (1996) and Liu (2000) for the statistically self-similar case (which corresponds to the construction of section 1):

It follows from our assumptions that with probability one, for every $a \in T$, the non-negative martingale $Y_{A,n}^{(a)}$ converges to a non-negative random variable $Y_A^{(a)} = Y^{(a)}$ with mean ≤ 1 and the limits again satisfy the relations

$$Y^{(a)} = \sum_{i \geq 1} A_i^{(a)} Y^{(ai)}. \qquad (E_a)$$

Denote by ∂T the set $\mathbb{N}_+^{\mathbb{N}_+}$. For $n \geq 1$ and $a \in \mathbb{N}_+^n$ define $C(a) = \{x \in \partial T; x_1 \ldots x_n = a\}$. Fix $c > 1$ and define the standard ultrametric distance on ∂T by $d(x,y) = c^{-|x \wedge y|}$ where

$|x \wedge y|$ denotes the length of the maximal common sequence of x and y, that is $|x \wedge y| = \sup\{n \geq 1;\, x_1 \dots x_n = y_1 \dots y_n\}$.

The relations (E_a) make it possible to define almost surely a unique measure $\mu = \mu_\omega$ on ∂T equipped with $\mathcal{T}$, the σ-algebra generated by the C_a's, by

$$\mu(C(a)) = A_{a_1}^{(\epsilon)} A_{a_2}^{(a_1)} \dots A_{a_n}^{(a_1 \dots a_{n-1})} Y^{(a)},\ \forall\, n \geq 1,\ \forall\, a \in \mathbb{N}_+^n.$$

Recall that the Hausdorff dimension of μ is

$$\dim(\mu) = \inf\{\dim(B);\ B \in \mathcal{T},\ \mu(B) = \|\mu\|\},$$

where $\dim(B)$ denotes the Hausdorff dimension of B with respect to the distance d.

Now we obtain a result which generalizes those of Peyrière (1977) and Liu and Rouault (1996) and shows that $\dim(\mu)$ depends only on the behaviour of the $A^{(a)}$'s when the length of a tends to ∞:

Theorem 8 *Assume that the hypothesis of Theorem 6ii) is satisfied for some $p \in]1, 2]$ and that*

$$\sup_{a \in T} \mathbb{E}[\sum_{i \in \mathbb{N}_+} A_i^{(a)} \log^2 A_i^{(a)}] < \infty. \tag{18}$$

For every $n \geq 1$ define

$$D_n = -\sum_{a = a_1 \dots a_{n-1} \in \mathbb{N}_+^{n-1}} \mathbb{E}[\prod_{k=1}^{n-1} A_{a_k}^{(a_1 \dots a_{k-1})}] \mathbb{E}[\sum_{a_n \in \mathbb{N}_+} A_{a_n}^{(a)} \log_c A_{a_n}^{(a)}]$$

Then $\mathbb{E}(\|\mu\|) = 1$ and with probability one, conditionally to $\mu \neq 0$, for μ-almost every $x \in \partial T$

$$\liminf_{n \to \infty} \frac{\log(\mu(C(x_1 \dots x_n))}{\log|C(x_1 \dots x_n)|} = D_- = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} D_k$$

and

$$\limsup_{n \to \infty} \frac{\log(\mu(C(x_1 \dots x_n))}{\log|C(x_1 \dots x_n)|} = D_+ = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} D_k,$$

where $|C(x_1 \dots x_n)| = c^{-n}$, the diameter of $C(x_1 \dots x_n)$; consequently $D_- \leq \dim(\mu) \leq D_+$. In particular when $D_- = D_+$, the exact value of $\dim(\mu)$ is obtained without a self-similarity hypothesis.

Remark 6 Theorem 8 yields very precise information on the local dimension of one given random measure, and it is easily seen that it makes it possible to derive such information for countable families of such measures defined simultaneously. But the approach used in the proof does not apply for uncountable families. We shall see in Theorem 9 how Theorem 3 makes it possible to remedy this for certain of these families in the self-similar case.

Proof of Theorem 8. We have $\mathbb{E}(\|\mu\|) = 1$ by Theorem 6, and the result about $\dim(\mu)$ is a consequence of the estimates on the logarithmic density and a generalization of a Billingsley lemma (1965, p. 136-145) in Peyrière (1977).

Now we prove the results on the logarithmic density of μ by noting that the approach in Peyrière (1976, 1977), which is simplified in Liu and Rouault (1996) for the self-similar case, is the right one in the present context.

Define on $(\Omega \times \partial T, \mathcal{A} \otimes \mathcal{T})$ the probability measure $\mathcal{Q}$ given for every $A \in \mathcal{A} \otimes \mathcal{T}$ by

$$\mathcal{Q}(A) = \mathbb{E}\left(\int_{\partial T} \mathbf{1}_A(\omega, x)\mu_\omega(dx) \right).$$

By definition "for $\mathcal{Q}$-almost every $(\omega, x) \in \Omega \times \partial T$" means "for $\mathbb{P}$-almost every $\omega \in \Omega$, for μ_ω-almost every $x \in \partial T$". Then for every $n \geq 1$ and $(\omega, x) \in \Omega \times \partial T$ define (with the convention $0 \times \infty = 0$)

$$W_n(\omega, x) = \mathbf{1}_{\{A_{x_n}^{(x_1 \dots x_{n-1})} > 0\}}(\omega) \log_c \ A_{x_n}^{(x_1 \dots x_{n-1})}(\omega)$$

and

$$\tilde{Y}_n(\omega, x) = Y^{(x_1 \dots x_n)}(\omega).$$

The same computations as those of Peyrière in Kahane and Peyrière (1976) show that

$$\mathbb{E}_\mathcal{Q}(W_n) = -D_n$$

and

$$\mathbb{E}_\mathcal{Q}(W_n^2) = \sum_{a = a_1 \dots a_{n-1} \in \mathbb{N}_+^{n-1}} \mathbb{E}[\prod_{k=1}^{n-1} A_{a_k}^{(a_1 \dots a_{k-1})}]\mathbb{E}[\sum_{a_n \in \mathbb{N}_+} A_{a_n}^{(a)} \log_c^2 \ A_{a_n}^{(a)}]$$

for every $n \geq 1$, that the W_n's are independent of one another, and finally, that the martingale (with respect to $\mathcal{Q}$) $\sum_{k=1}^n (W_k + D_k)/k$ is bounded in $L^2(\Omega \times \partial T, \mathcal{Q})$ by the assumption (18) of the theorem. So the Kronecker Lemma yields

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n W_k + \frac{1}{n} \sum_{k=1}^n D_k = 0 \ \mathcal{Q}\text{-almost surely.} \tag{19}$$

Now fix $\eta \in]0, p-1]$ and $\delta \in \{-1, 1\}$. A similar computation yields for every $n \geq 1$,

$$\mathbb{E}_\mathcal{Q}(\mathbf{1}_{\{\tilde{Y}_n > 0\}} \tilde{Y}_n^{\delta\eta}) = \sum_{a = a_1 \dots a_n \in \mathbb{N}_+^n} \mathbb{E}[\prod_{k=1}^n A_{a_k}^{(a_1 \dots a_{k-1})}]\mathbb{E}[(Y^{(a)})^{1+\delta\eta}].$$

By Theorem 6 we have $\sup_{a \in T} \mathbb{E}[(Y^{(a)})^p] < \infty$. So $\sup_{n \geq 1} \mathbb{E}_\mathcal{Q}(\mathbf{1}_{\{\tilde{Y}_n > 0\}} \tilde{Y}_n^{\delta\eta}) < \infty$ and then $\mathcal{Q}$-almost surely

$$\sum_{n \geq 1} \frac{1}{n^2} (\mathbf{1}_{\{\tilde{Y}_n > 0\}} \tilde{Y}_n^{-\eta} + \tilde{Y}_n^\eta) < \infty,$$

implying that $\lim_{n\to\infty} \mathbf{1}_{\{\tilde{Y}_n>0\}} \frac{\log \tilde{Y}_n}{n} = 0$. This, together with (19), yields the conclusion.

Now consider the functions $\lambda \in \Lambda \mapsto A_i(\lambda)$ of Section 2.2 and assume that they satisfy the assumptions of Theorem 3. Assume moreover that for every $\lambda \in \Lambda$ a. s. the $A_i(\lambda)$'s, $1 \leq i \leq N_{A(\lambda)}$ are positive. Then we are given a family $(\mu_\lambda)_{\lambda\in\Lambda}$ of measures on ∂T defined with probability one by

$$\mu_\lambda(C(a)) = A_{a_1}(\lambda)(\epsilon)A_{a_2}(\lambda)(a_1)\dots A_{a_n}(\lambda)(a_1\dots a_{n-1})Y_{A(\lambda)}(a),$$

$\forall\, n \geq 1$, $\forall\, a \in \mathbb{N}_+^n$, $\lambda \in \Lambda$, and using the same approach as in Barral (2000 b) Corollary $5(ii)(\beta)$ yields $\mu_\lambda \neq 0$ almost surely for all $\lambda \in \Lambda$.

The following theorem gives conditions under which $\dim(\mu_\lambda)$ is determined almost surely for the λ's in a subset of Λ of full Lebesgue measure, and it reveals, up to a set of null Lebesgue measure, a relation between the regularity of the initial $\lambda \in \Lambda \mapsto A_i(\lambda)$'s and the one of $\lambda \in \Lambda \mapsto \dim(\mu_\lambda)$, namely that they are the same.

Theorem 9 *Assume that for every compact subset K of Λ, for every $\lambda \in K$*

$$\log \mathbb{E}(\sum_{i\geq 1}(A_i(\lambda))^x) = (x-1)\mathbb{E}(\sum_{i\geq 1} A_i(\lambda)\log A_i(\lambda)) + (x-1)\varepsilon_\lambda(x-1), \tag{20}$$

with $\lim_{x\to 1}\sup_{\lambda\in K}|\varepsilon_\lambda(x-1)| = 0$. *Then with probability one there exists $\Lambda' \subset \Lambda$ with $\ell(\Lambda') = \ell(\Lambda)$ such that for every $\lambda \in \Lambda'$, for μ_λ-almost every $x \in \partial T$*

$$\lim_{n\to\infty} \frac{\log(\mu_\lambda(C(x_1\dots x_n))}{\log|C(x_1\dots x_n)|} = D_\lambda = \mathbb{E}[\sum_{i\geq 1} A_i(\lambda)\log_c A_i(\lambda)];$$

consequently $\dim(\mu_\lambda) = D_\lambda$.

Remark 7 The completely satisfactory result would yield $\Lambda' = \Lambda$ a. s. This is achieved, only when $k = 1$, in Barral (2000 b), which deals with a particular family of measures and develops a different approach.

Proof of Theorem 9. We prove that the lim inf of the logarithmic density of μ_λ is at least D_λ; the proof that its lim sup is at most D_λ is similar.

Fix $\varepsilon > 0$. For $n \geq 1$ and $\lambda \in \Lambda$ define

$$F_{n,\lambda,\varepsilon} = \{x \in \partial T;\ \frac{\log(\mu_\lambda(C(x_1\dots x_n))}{\log|C(x_1\dots x_n)|} \leq D_\lambda - \varepsilon\}.$$

As ε is arbitrary, it is enough to prove that almost surely there exists $\Lambda_\varepsilon \subset \Lambda$ with $\ell(\Lambda_\varepsilon) = \ell(\Lambda)$ such that for every $\lambda \in \Lambda_\varepsilon$, $\sum_{n\geq 1}\mu_\lambda(F_{n,\lambda,\varepsilon}) < \infty$.

A simple computation using the definition of $F_{n,\lambda,\varepsilon}$ shows that for every $n \geq 1$, $\lambda \in \Lambda$ and $\eta > 0$

$$\mu_\lambda(F_{n,\lambda,\varepsilon}) \leq Q_{n,\lambda,\varepsilon,\eta} = \sum_{a\in\mathbb{N}_+^n} \mu_\lambda^{1+\eta}(C(a))c^{n\eta(D_\lambda-\varepsilon)}.$$

So the result will be established if for every compact subset K of Λ, we have for some $\eta > 0$

$$\sum_{n\geq 1} \mathbb{E}(\int_K Q_{n,\lambda,\varepsilon,\eta}\,\ell(d\lambda)) < \infty. \tag{21}$$

By the definition of μ_λ

$$\mathbb{E}(Q_{n,\lambda,\varepsilon,\eta}) = c^{n\eta(D_\lambda-\varepsilon)}[\mathbb{E}(\sum_{i\geq 1}(A_i^{1+\eta}(\lambda))]^n \mathbb{E}(Y_{A(\lambda)}^{1+\eta}). \tag{22}$$

Moreover if K is a fixed compact subset of Λ, by assumption (20), if η is small enough, for every $\lambda \in K$ and $n \geq 1$

$$c^{n\eta(D_\lambda-\varepsilon)}[\mathbb{E}(\sum_{i\geq 1}(A_i^{1+\eta}(\lambda))]^n \leq c^{-n\eta\varepsilon/2}. \tag{23}$$

Moreover $\mathbb{E}(\mathcal{N}^p_{\mathcal{W}^{m,p}}(Y_A)) < \infty$ by Theorem 3. So it follows from the Fubini Theorem that if η is small enough then $\mathbb{E}(\int_K Y_{A(\lambda)}^{1+\eta}\ell(d\lambda)) < \infty$. Then (21) follows from (22), (23), and the Fubini Theorem again.

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