HETEROGENEOUS UBIQUITOUS SYSTEMS IN $\mathbb{R}^d$ AND HAUSDORFF DIMENSION

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Abstract. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,1]^d$, $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to 0, and $\delta > 1$. The classical ubiquity results are concerned with the computation of the Hausdorff dimension of limsup-sets of the form $S(\delta) = \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} B(x_n, \lambda_n^\delta)$.

Let μ be a positive Borel measure on $[0,1]^d$, $\rho \in (0,1]$ and $\alpha > 0$. Consider the finer limsup-set

$$S_\mu(\rho,\delta,\alpha) = \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N:\, \mu(B(x_n,\lambda_n^\rho))\sim\lambda_n^{\rho\alpha}} B(x_n, \lambda_n^\delta).$$

We show that, under suitable assumptions on the measure μ, the Hausdorff dimension of the sets $S_\mu(\rho,\delta,\alpha)$ can be computed. Moreover, when $\rho < 1$, a yet unknown saturation phenomenon appears in the computation of the Hausdorff dimension of $S_\mu(\rho,\delta,\alpha)$. Our results apply to several classes of multifractal measures, and $S(\delta)$ corresponds to the special case where μ is a monofractal measure like the Lebesgue measure.

The computation of the dimensions of such sets opens the way to the study of several new objects and phenomena. Applications are given for the Diophantine approximation conditioned by (or combined with) b-adic expansion properties, by averages of some Birkhoff sums and branching random walks, as well as by asymptotic behavior of random covering numbers.

1. Introduction

Since the famous result of Jarnik [34] concerning Diophantine approximation and Hausdorff dimension, the following problem has been widely encountered and studied in various mathematical situations.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a compact metric space E and $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to 0. Let us define the limsup set

$$S = \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} B(x_n, \lambda_n),$$

and let D be its Hausdorff dimension. Let $\delta > 1$. What can be said about the Hausdorff dimension of the subset $S(\delta)$ of S defined by

$$S(\delta) = \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} B(x_n, \lambda_n^\delta) \ ?$$

Intuitively one would expect the Hausdorff dimension of $S(\delta)$ to be lower bounded by D/δ. This has been proved to hold in many cases which can roughly be separated into two classes:

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- when the sequence $\{(x_n, \lambda_n)\}_n$ forms a sort of "regular system" [3, 18, 19], which ensures a strong uniform repartition of the points $\{x_n\}_n$.
- when the sequence $\{(x_n, \lambda_n)\}_n$ forms an ubiquitous system [22, 23, 33, 15] with respect to a monofractal measure carried by the set S.

Let us mention that similar results are obtained in [47] when E is a Julia set. When $\dim S(\delta) < D$, such subsets $S(\delta)$ are often referred to as exceptional sets [21]. Another type of exceptional sets arises when considering the level sets of well-chosen functions:

- the function associating with each point $x \in [0,1]$ the frequency of the digit $i \in \{0, 1, \dots, b-1\}$ in the b-adic expansion of x,
- more generally the function associating with each point x the average of the Birkhoff sums related to some dynamical systems,
- the function $x \mapsto h_f(x)$, when f is either a function or a measure on $\mathbb{R}^d$ and $h_f(x)$ is a measure of the local regularity (typically an Hölder exponent) of f around x.

It is a natural question to ask whether these two approaches can be combined to obtain finer exceptional sets. Let us take an example to illustrate our purpose.

On one side, it is known since Jarnik's results [34] that if the sequence $\{(x_n, \lambda_n)\}_n$ is made of the rational pairs $\{(p/q, 1/q^2)\}_{p,q\in\mathbb{N}^{*2},\, p\leq q}$, then for every $\delta > 1$ the subset $S(\delta)$ of $[0,1]$ has a Hausdorff dimension equal to $1/\delta$. In the ubiquity's setting, this is a consequence of the fact that the family $\{(p/q, 1/q^2)\}_{p,q\in\mathbb{N}^{*2}}$ forms an ubiquitous systems associated with the Lebesgue measure [22, 23].

On the other side, given $(\pi_0, \pi_1, \dots, \pi_{b-1}) \in [0,1]^b$ such that $\sum_{i=0}^{b-1} \pi_i = 1$, Besicovitch and later Eggleston [24] studied the sets $E^{\pi_0,\pi_1,\dots,\pi_{b-1}}$ of points x such that the frequency of the digit $i \in \{0, 1, \dots, b-1\}$ in the b-adic expansion of x is equal to π_i. More precisely, for any $x \in [0,1]$, let us consider the b-adic expansion of $x = \sum_{m=1}^{\infty} x_m b^{-m}$, where $\forall m$, $x_m \in \{0, 1, \dots, b-1\}$. Let $\phi_{i,n}(x)$ be the mapping

$$x \mapsto \phi_{i,n}(x) = \frac{\#\{m \leq n : x_m = i\}}{n}. \tag{1}$$

Then $E^{\pi_0,\pi_1,\dots,\pi_{b-1}} = \{x : \forall i \in \{0, 1, \dots, b-1\}, \lim_{n\to+\infty} \phi_{i,n}(x) = \pi_i\}$. They found that $\dim E^{\pi_0,\pi_1,\dots,\pi_{b-1}} = \sum_{i=0}^{b-1} -\pi_i \log_b \pi_i$.

We address the problem of the computation of the Hausdorff dimension of the subsets $E_\delta^{\pi_0,\pi_1,\dots,\pi_{b-1}}$ of $[0,1]$ defined by

$$E_\delta^{\pi_0,\pi_1,\dots,\pi_{b-1}} = \left\{ x \in (0,1) : \left\{ \begin{array}{c} \exists\, (p_n, q_n)_n \in (\mathbb{N}^{*2})^{\mathbb{N}} \text{ such that } q_n \to +\infty, \\ |x - p_n/q_n| \leq 1/q_n^{2\delta} \ \text{ and } \forall i \in \{0, \dots, b-1\}, \\ \lim_{n\to+\infty} \phi_{i,[\log_b(q_n^2)]}\,(p_n/q_n) = \pi_i \end{array} \right. \right\}$$

($[x]$ denotes the integer part of x). In other words, we seek in this example for the Hausdorff dimension of the set of points of $[0,1]$ which are well-approximated by rational numbers fulfilling a given Besicovitch condition (i.e. having given digit frequencies in their b-adic expansion). This problem is not covered by the works mentioned above. The main reason is the heterogeneity of the repartition of the rational numbers satisfying the Besicovitch conditions. As a consequence of Theorems 2.1 and 2.2 of this paper, we obtain

$$\dim E_\delta^{\pi_0,\pi_1,\dots,\pi_{b-1}} = \frac{\sum_{i=0}^{b-1} -\pi_i \log_b \pi_i}{\delta}. \tag{2}$$

The key point to achieve this work is to see the Besicovitch condition as a scaling property derived from a multinomial measure. More precisely, the computation of the Hausdorff dimensions of the sets $E_\delta^{\pi_0,\pi_1,\dots,\pi_{b-1}}$ proves to be a particular case of the following problem: Let μ be a positive Borel measure on the compact metric space E considered above. Given $\alpha > 0$ and $\delta \geq 1$, what is the Hausdorff dimension of the set of points x of E that are well-approximated by points of $\{(x_n, \lambda_n)\}_n$ at rate δ, i.e. such that for an infinite number of integers n, $|x - x_n| \leq \lambda_n^\delta$, conditionally to the fact that the corresponding sequence of pairs (x_n, λ_n) satisfy

$$\lim_{n\to\infty} \frac{\log \mu\big(B(x_n, \lambda_n)\big)}{\log(\lambda_n)} = \alpha? \tag{3}$$

In other words, if $\varepsilon = (\varepsilon_n)_{n\geq 1}$ is a sequence of positive numbers converging to 0, what is the Hausdorff dimension of

$$S_\mu(\delta, \alpha, \varepsilon) = \bigcap_{N\geq 0} \bigcup_{n\geq N:\, \lambda_n^{\alpha+\varepsilon_n} \leq \mu(B(x_n,\lambda_n)) \leq \lambda_n^{\alpha-\varepsilon_n}} B(x_n, \lambda_n^\delta) \ ? \tag{4}$$

We study the problem in $\mathbb{R}^d$ ($d \geq 1$). An upper bound for the Hausdorff dimension of $S_\mu(\delta, \alpha, \varepsilon)$ is given by Theorem 2.1 for *weakly redundant systems* $\{(x_n, \lambda_n)\}_n$ (see Definition 2.1). Its proof uses ideas coming from multifractal formalism for measures [17, 43].

Theorem 2.2 (case $\rho = 1$) gives a precise lower bound for the Hausdorff dimension of $S_\mu(\delta, \alpha, \varepsilon)$ when the family $\{(x_n, \lambda_n)\}_n$ forms a 1*-heterogeneous ubiquitous system with respect to the measure* μ (see Definition 2.2 for this notion, which generalizes the notion of ubiquitous system mentioned above). It can be applied to measures μ that enjoy some statistical self-similarity property, and to any family $\{(x_n, \lambda_n)\}_n$ as soon as the support of μ is covered by $\limsup_{n\to\infty} B(x_n, \lambda_n)$.

To fix ideas, let us state a corollary of Theorems 2.1 and 2.2. This result uses the Legendre transform τ_μ^* of the "dimension" function τ_μ considered in the multifractal formalism studied in [17] (see Section 2.2 and Definition 8).

Theorem 1.1. *Let μ be a multinomial measure on $[0,1]^d$. Suppose that the family $\{(x_n, \lambda_n)\}_n$ forms a weakly redundant 1-heterogeneous ubiquitous system with respect to $\big(\mu, \alpha, \tau_\mu^*(\alpha)\big)$.*

There is a positive sequence $\varepsilon = (\varepsilon_n)_{n\geq 1}$ converging to 0 at ∞ such that

$$\forall\ \delta \geq 1, \quad \dim S_\mu(\delta, \alpha, \varepsilon) = \tau_\mu^*(\alpha)/\delta.$$

Examples of remarkable families $\{(x_n, \lambda_n)\}_n$ are discussed in Section 6, as well as examples of suitable statistically self-similar measures μ. There, the measures μ are chosen so that the property (3) has a relevant interpretation (for instance in terms of the b-adic expansion of the points x_n).

The formula (4) defining the set $S_\mu(\delta, \alpha, \varepsilon)$ naturally leads to the question of conditioned ubiquity into the following more general form: Let $\rho \in (0, 1]$. What is the Hausdorff dimension of

$$S_\mu(\rho, \delta, \alpha, \varepsilon) = \bigcap_{N\geq 0} \bigcup_{n\geq N:\, \lambda_n^{\rho(\alpha+\varepsilon_n)} \leq \mu(B(x_n,\lambda_n^\rho)) \leq \lambda_n^{\rho(\alpha-\varepsilon_n)}} B(x_n, \lambda_n^\delta) \ ? \tag{5}$$

We remark that, in (4) and (5), if μ equals the Lebesgue measure and if $\alpha = d$, the conditions on $B(x_n, \lambda_n^\rho)$ are empty, since they are independent of x_n, λ_n and ρ.

This remains true for a strictly monofractal measure μ of index α, that is such that $\exists\, C>0$, $\exists\, r_0$ such that $\forall\, x\in \operatorname{supp}(\mu)$, $\forall\, 0<r\le r_0$, $C^{-1}r^\alpha \le \mu(B(x,r))\le Cr^\alpha$.

Again, an upper bound for the Hausdorff dimension of $S_\mu(\rho,\delta,\alpha,\varepsilon)$ is found in Theorem 2.1 for weakly redundant systems.

Theorem 2.2 (case $\rho<1$) yields a lower bound for the Hausdorff dimension of $S_\mu(\rho,\delta,\alpha,\varepsilon)$ when $\rho<1$, as soon as the family $\{(x_n,\lambda_n)\}_n$ forms a *ρ-heterogeneous ubiquitous system with respect to μ* in the sense of Definition 2.3. The introduction of this dilation parameter ρ substantially modifies Definition 2.2 and the proofs of the results in the initial case $\rho=1$.

As a consequence of Theorem 2.2, a new saturation phenomenon occurs for systems that are both weakly redundant and ρ-heterogeneous ubiquitous systems when $\rho<1$. This points out the heterogeneity introduced when considering ubiquity conditioned by measures that are not monofractal. The following result is also a corollary of Theorems 2.1 and 2.2.

Theorem 1.2. *Let μ be a multinomial measure on $[0,1]^d$. Let $\rho\in(0,1)$. Suppose that $\{(x_n,\lambda_n)\}_n$ forms a weakly redundant ρ-heterogeneous ubiquitous system with respect to $\big(\mu,\alpha,\tau_\mu^*(\alpha)\big)$.*

There is a positive sequence $\varepsilon=(\varepsilon_n)_{n\ge1}$ converging to 0 at ∞ such that

$$\forall\ \delta\ge 1,\quad \dim\, S_\mu(\rho,\delta,\alpha,\varepsilon)=\min\Big(\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{\delta},\tau_\mu^*(\alpha)\Big).$$

Under the assumptions of Theorem 1.2, when $\tau_\mu^*(\alpha)<d$, although δ starts to increase from 1, $\dim\, S_\mu(\rho,\delta,\alpha,\varepsilon)$ remains constant until δ reaches the critical value $\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{\tau_\mu^*(\alpha)}>1$. When δ becomes larger than $\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{\tau_\mu^*(\alpha)}$, the dimension decreases. This is what we call a saturation phenomenon.

It turns out that conditioned ubiquity as defined in this paper is closely related to the local regularity properties of some new classes of functions and measures having dense sets of discontinuities. In particular, Theorem 2.2 is a crucial tool to analyze measures constructed as the measures $\nu_{\rho,\gamma,\sigma}$

$$\nu_{\rho,\gamma,\sigma}=\sum_{n\ge0}\lambda_n^\gamma\,\mu\big(B(x_n,\lambda_n^\rho)\big)^\sigma\delta_{x_n},$$

where δ_{x_n} is the probability Dirac mass at x_n, $\rho\in(0,1]$, and γ,σ are real numbers which make the series converge. Conditioned ubiquity is also essential to perform the multifractal analysis of Lévy processes in multifractal time. These objects have multifractal properties that were unknown until now. Their study is achieved in other works [9, 10, 11, 12].

The definitions of weakly redundant and ρ-heterogeneous ubiquitous systems are given in Section 2. The statements of the main results (Theorems 2.1 and 2.2) then follow. The proofs of Theorem 2.1, Theorem 2.2 (case $\rho=1$) and Theorem 2.2 (case $\rho<1$) are respectively achieved in Sections 3, 4 and 5. Finally, our results apply to suitable examples of systems $\{(x_n,\lambda_n)\}_n$ and measures μ that are discussed in Section 6.

2. Definitions and statement of results

It is convenient to endow $\mathbb{R}^d$ with the supremum norm $\|\cdot\|_\infty$ and with the associated distance $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d\mapsto\|x-y\|_\infty=\max_{1\le i\le d}(|x_i-y_i|)$. Throughout the paper, for a set S, $|S|$ denotes then the diameter of S.

We briefly recall the definition of the generalized Hausdorff measures and Hausdorff dimension in $\mathbb{R}^d$. Let ξ be a *gauge* function, i.e. a non-negative non-decreasing function on $\mathbb{R}_+$ such that $\lim_{x\to 0^+}\xi(x)=0$. Let S be a subset of $\mathbb{R}^d$. For $\eta>0$, let us define

$$\mathcal{H}^\xi_\eta(S)=\inf_{\{C_i\}_{i\in\mathcal{I}}:S\subset\bigcup_{i\in\mathcal{I}}C_i}\quad\sum_{i\in\mathcal{I}}\xi\left(|C_i|\right),\ \text{(the family } \{C_i\}_{i\in\mathcal{I}} \text{ covers } S)$$

where the infimum is taken over all countable families $\{C_i\}_{i\in\mathcal{I}}$ such that $\forall i\in\mathcal{I}$, $|C_i|\le\eta$. As η decreases to 0, $\mathcal{H}^\xi_\eta(S)$ is non-decreasing, and $\mathcal{H}^\xi(S)=\lim_{\eta\to 0}\mathcal{H}^\xi_\eta(S)$ defines a Borel measure on $\mathbb{R}^d$, called Hausdorff ξ-measure.

Defining the family $\xi_\alpha(x)=|x|^\alpha$ ($\alpha\ge 0$), there exists a unique real number $0\le D\le d$, called the Hausdorff dimension of S and denoted $\dim S$, such that $D=\sup\left\{\alpha\ge 0:\mathcal{H}^{\xi_\alpha}(S)=+\infty\right\}=\inf\left\{\alpha:\mathcal{H}^{\xi_\alpha}(S)=0\right\}$ (with the convention $\sup\emptyset=0$). We refer the reader to [40, 26] for instance for more details on Hausdorff dimensions.

Let μ be a positive Borel measure with a support contained in $[0,1]^d$. The analysis of the local structure of the measure μ in $[0,1]^d$ may be naturally done using a c-adic grid ($c\ge 2$). This is the case for instance for the examples of measures of Section 6. We shall thus need the following definitions.

Let c be an integer ≥ 2. For every $j\ge 0$, $\forall\,\mathbf{k}=(k_1,\dots,k_d)\in\{0,1,\dots,c^j-1\}^d$, $I^c_{j,\mathbf{k}}$ denotes the c-adic box $[k_1c^{-j},(k_1+1)c^{-j})\times\ldots\times[k_dc^{-j},(k_d+1)c^{-j})$. Then, $\forall x\in[0,1)^d$, $I^c_j(x)$ stands for the unique c-adic box of generation j that contains x, and $\mathbf{k}^c_{j,x}$ is the unique (multi-)integer such that $I^c_j(x)=I^c_{j,\mathbf{k}^c_{j,x}}$. If both $\mathbf{k}=(k_1,\dots,k_d)$ and $\mathbf{k}'=(k'_1,\dots,k'_d)$ belong to $\mathbb{N}^d$, $\|\mathbf{k}-\mathbf{k}'\|_\infty=\max_i|k_i-k'_i|$. The set of c-adic boxes included in $[0,1)^d$ is denoted by $\mathbf{I}$.

Finally, the lower Hausdorff dimension of μ, $\underline{\dim}(\mu)$, is defined, as usual, as $\inf\left\{\dim E: E\in\mathcal{B}([0,1]^d),\ \mu(E)>0\right\}$.

2.1. **Weakly redundant systems.**

Let $\{x_n\}_{n\in\mathbb{N}}$ be a family of points of $[0,1]^d$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ a non-increasing sequence of positive real numbers converging to 0. For every $j\ge 0$, let

$$T_j=\left\{n:\ 2^{-(j+1)}<\lambda_n\le 2^{-j}\right\}. \tag{6}$$

The following definition introduces a natural property from which an upper bound for the Hausdorff dimension of limsup-sets (4) and (5) can be derived. *Weak redundancy* is slightly more general than *sparsity* in [27].

Definition 2.1. *The family $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ is said to form a weakly redundant system if there exists a sequence of integers $(N_j)_{j\ge 0}$ such that*
(i) $\lim_{j\to\infty}\log N_j/j=0$.
(ii) for every $j\ge 1$, T_j can be decomposed into N_j pairwise disjoint subsets (denoted $T_{j,1},\dots,T_{j,N_j}$) such that for each $1\le i\le N_j$, the family $\left\{B(x_n,\lambda_n):\ n\in T_{j,i}\right\}$ is composed of disjoint balls.

We have $\bigcup_{i=1}^{N_j} T_{j,i} = T_j$. Since the $T_{j,i}$ are pairwise disjoint, any point $x \in [0,1]^d$ is covered by at most N_j balls $B(x_n, \lambda_n)$, $n \in T_j$. Moreover, for every i and j, the number of balls of $T_{j,i}$ is bounded by $C_d \cdot 2^{dj}$, where C_d is a positive constant depending only on d. Indeed, if two integers $n \neq n'$ are such that λ_n and $\lambda_{n'}$ belong to $T_{j,i}$, then $\|x_n - x_{n'}\|_\infty \geq 2^{-j}$.

2.2. **Upper bounds for Hausdorff dimensions of conditioned limsup sets.** Let μ be a finite positive Borel measure on $[0,1]^d$.

We let the reader verify that if supp $\mu = [0,1]^d$, then the concave function

$$\tau_\mu : q \mapsto \liminf_{j\to\infty} -j^{-1} \log_c \sum_{\mathbf{k}\in\{0,\dots,c^j-1\}^d} \mu(I^c_{j,\mathbf{k}})^q \tag{7}$$

does not depend on the integer $c \geq 2$. This function is often considered when performing the multifractal formalism for measures of [17]. Then, the Legendre transform of τ_μ at $\alpha \in \mathbb{R}_+$, denoted by τ_μ^*, is defined by

$$\tau_\mu^* : \alpha \mapsto \inf_{q\in\mathbb{R}} \big(\alpha q - \tau_\mu(q)\big) \in \mathbb{R} \cup \{-\infty\}. \tag{8}$$

Theorem 2.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a family of points of $[0,1]^d$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ a non-increasing sequence of positive real numbers converging to 0. Let μ be a positive finite Borel measure with a support equal to $[0,1]^d$. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a positive sequence converging to 0, $\rho \in (0,1]$, $\delta \geq 1$ and $\alpha \geq 0$. Let us define*

$$S_\mu(\rho,\delta,\alpha,\varepsilon) = \bigcap_{N\geq 1} \ \bigcup_{n\geq N:\, \lambda_n^{\rho(\alpha+\varepsilon_n)} \leq \mu(B(x_n,\lambda_n^\rho)) \leq \lambda_n^{\rho(\alpha-\varepsilon_n)}} B(x_n, \lambda_n^\delta).$$

Suppose that $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ forms a weakly redundant system. Then

$$\dim S_\mu(\rho,\delta,\alpha,\varepsilon) \leq \min\Big(\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{\delta}, \tau_\mu^*(\alpha) \Big). \tag{9}$$

Moreover, $S_\mu(\rho,\delta,\alpha,\varepsilon) = \emptyset$ if $\tau_\mu^*(\alpha) < 0$.

The result does not depend on the precise value of the sequence $\{\varepsilon_n\}_n$, as soon as $\lim_{n\to+\infty} \varepsilon_n = 0$. The proof of Theorem 2.1 is given in Section 3.

2.3. **Heterogeneous ubiquitous systems.** Let $\alpha > 0$ and $\beta \in (0,d]$ be two real numbers. They play the role respectively of the Hölder exponent of μ and of the lower Hausdorff dimension of an auxiliary measure m.

The upper bound obtained by Theorem 2.1 is rather natural. Here we seek for conditions that make the inequality (9) become an equality. The following Definitions 2.2 and 2.3 provide properties guarantying this equality.

The notion of *heterogeneous ubiquitous system* generalizes the notion of *ubiquitous system* in $\mathbb{R}^d$ considered in [22]. The abbreviation m-a.e. or μ-a.e. means as usual m- or μ-almost every or m- or μ-almost everywhere.

Definition 2.2. *The system $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ is said to form a 1-heterogeneous ubiquitous system with respect to (μ,α,β) if conditions* **(1-4)** *are fulfilled.*

(1) *There exist two non-decreasing continuous functions ϕ and ψ defined on $\mathbb{R}_+$ with the following properties:*
- $\varphi(0) = \psi(0) = 0$, $r \mapsto r^{-\varphi(r)}$ and $r \mapsto r^{-\psi(r)}$ are non-increasing near 0^+,
- $\lim_{r\to 0^+} r^{-\varphi(r)} = +\infty$, and $\forall\, \varepsilon > 0$, $r \mapsto r^{\varepsilon-\varphi(r)}$ is non-decreasing near 0^+,
- φ and ψ verify **(2), (3)** *and* **(4)**.

(2) *There is a measure m with support $[0,1]^d$ enjoying the following properties:*
- *m-a.e. $y \in [0,1]^d$ belongs to $\limsup_{n\to+\infty} B(x_n, \lambda_n/2)$, i.e.*

$$m\Big(\bigcap_{N\geq 1}\ \bigcup_{n\geq N} B\big(x_n,\lambda_n/2\big)\Big) = \|m\|. \tag{10}$$

- *We have:*

$$\begin{cases} \text{For } m\text{-a.e. } y \in [0,1]^d, \ \ \exists\, j(y), \ \ \forall j \geq j(y), \\ \forall\ \mathbf{k} \textit{ such that } \|\mathbf{k}-\mathbf{k}^c_{j,y}\|_\infty \leq 1, \ \mathcal{P}^1_1(I^c_{j,\mathbf{k}}) \textit{ holds,} \end{cases} \tag{11}$$

where $\mathcal{P}^1_M(I)$ is said to hold for the set I and for the real number $M \geq 1$ when

$$M^{-1}|I|^{\alpha+\psi(|I|)} \leq \mu\big(I\big) \leq M|I|^{\alpha-\psi(|I|)}. \tag{12}$$

- *We have:*

$$\begin{cases} \text{For } m\text{-a.e. } y \in [0,1]^d, \ \ \exists\, j(y), \ \ \forall j \geq j(y), \\ \forall\ \mathbf{k} \textit{ such that } \|\mathbf{k}-\mathbf{k}^c_{j,y}\|_\infty \leq 1, \ \mathcal{D}^m_1(I^c_{j,\mathbf{k}}) \textit{ holds,} \end{cases} \tag{13}$$

where $\mathcal{D}^m_M(I)$ is said to hold for the set I and for the real number $M > 0$ when

$$m(I) \leq M|I|^{\beta-\varphi(|I|)}. \tag{14}$$

(3) *(Self-similarity of m) For every c-adic box L of $[0,1)^d$, let f_L denote the canonical affine mapping from L onto $[0,1)^d$. There exists a measure m^L on L, equivalent to the restriction $m_{|L}$ of m to L (in the sense that $m_{|L}$ and m^L are absolutely continuous with respect to one another), such that property (13) holds for the measure $m^L\circ f_L^{-1}$ instead of the measure m.*

For every $J \geq 1$, let us then introduce the sets

$$E^L_J = \left\{ x \in L : \ \left\{ \begin{array}{c} \forall\ j \geq J + \log_c\big(|L|^{-1}\big), \ \forall\ \mathbf{k} \textit{ such that } \|\mathbf{k}-\mathbf{k}^c_{j,x}\|_\infty \leq 1, \\ \textit{we have:} \ \ m^L\big(I^c_{j,\mathbf{k}}\big) \leq \left(\frac{|I^c_{j,\mathbf{k}}|}{|L|}\right)^{\beta-\varphi\left(\frac{|I^c_{j,\mathbf{k}}|}{|L|}\right)} \end{array} \right. \right\}.$$

The sets E^L_J form a non-decreasing sequence in L, and by (13) and property **(3)**, *$\bigcup_{J\geq1} E^L_J$ is of full m^L-measure. We can thus consider the integer*

$$J(L) = \inf\left\{J \geq 1 : \ m^L(E^L_J) \geq \|m^L\|/2\right\}.$$

For every $x \in (0,1)^d$ and $j \geq 1$, let us define the set of balls

$$\mathcal{B}_j(x) = \Big\{B(x_n,\lambda_n) : x \in B\big(x_n,\lambda_n/2\big) \textit{ and } \lambda_n \in (c^{-(j+1)}, c^{-j}]\Big\}.$$

Notice that this set may be empty. When $\delta > 1$ and $B(x_n,\lambda_n) \in \mathcal{B}_j(x)$, consider $B(x_n,\lambda_n^\delta)$. This ball contains an infinite number of c-adic boxes. Among them, let $\mathbf{B}^\delta_n$ be the set of c-adic boxes of maximal diameter. Then define

$$B^\delta_j(x) = \bigcup_{B(x_n,\lambda_n)\in\mathcal{B}_j(x)} \mathbf{B}^\delta_n.$$

(4) *(Control of the growth speed $J(L)$ and of the mass $\|m^L\|$) There exists a subset $\mathcal{D}$ of $(1,\infty)$ such that for every $\delta \in \mathcal{D}$, for m-a.e. $x \in [0,1]^d$ (or equivalently, by (10) for m-a.e. $x \in \limsup_{n\to\infty} B\big(x_n,\lambda_n/2\big)$), there is an infinite number of generations j for which there exists $L \in \mathcal{B}^\delta_j(x)$ such that*

$$J(L) \leq \log_c\big(|L|^{-1}\big)\varphi(|L|) \quad \textit{and} \quad |L|^{\varphi(|L|)} \leq \|m^L\|. \tag{15}$$

Remark 2.1. 1. **(1)** *is a technical assumption. In* **(2)**, *(13) provides us with a lower bound for the lower Hausdorff dimension of the analyzing measure m. (11) yields a control of the local behavior of μ, m-a.e.. Then (10) is the natural condition on m to analyze ubiquitous properties of $\{(x_n,\lambda_n)\}_n$ conditioned by μ.* **(3)** *details a self-similar property for m, and* **(4)** *imposes a control of the growth speed in the level sets for the "copies" $m^L\circ f_L^{-1}$ of m. The combination of* **(3)** *and* **(4)** *supplies the monofractality property used in classical ubiquity results.*

2. *If μ is a strictly monofractal measure of exponent d (typically the Lebesgue measure), then* **(1-4)** *are always fulfilled with $\alpha=\beta=d$ and $\mu=m$ as soon as (10) holds. In fact, in this case,* **(1-4)** *imply the conditions required to be an ubiquitous system in the sense of* [22, 23].

3. *Property* **(4)** *can be weakened without affecting the conclusions of Theorem 2.2 below as follows:*

(weak 4) *There exists a subset $\mathcal{D}$ of $(1,\infty)$ such that for every $\delta\in\mathcal{D}$, for m-a.e. $x\in(0,1)$, there exists an increasing sequence $j_k(x)$ such that for every k, there exists $B(x_{n_k},\lambda_{n_k})\in\mathcal{B}_{j_k(x)}(x)$ as well as a c-adic box L_k included in $B(x_{n_k},\lambda_{n_k}^\delta)$ such that (15) holds with $L=L_k$; moreover $\lim_{k\to\infty}\frac{\log|L_k|}{\log\lambda_{n_k}}=\delta$.*

This weaker property, necessary in [11], *slightly complicates the proof and we decided to only discuss this point in this remark.*

In order to treat the case of the limsup-sets (5) defined with a dilation parameter $\rho<1$, conditions **(2)** and **(4)** are modified as follows.

Definition 2.3. *Let $\rho<1$. The system $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ is said to form a ρ-heterogeneous ubiquitous system with respect to (μ,α,β) if the following conditions are fulfilled.*

(1) *and* **(3)** *are the same as in Definition 2.2.*

(2(ρ)) *There exists a measure m with a support equal to $[0,1]^d$ such that:*

• *There exists a non-decreasing continuous function χ defined on $\mathbb{R}_+$ such that $\chi(0)=0$, $r\mapsto r^{-\chi(r)}$ is non-increasing near 0^+, $\lim_{r\to0^+}r^{-\chi(r)}=+\infty$, and $\forall\varepsilon,\theta,\gamma>0$, $r\mapsto r^{\varepsilon-\theta\varphi(r)-\gamma\chi(r)}$ is non-decreasing near 0.*

Moreover, for m-a.e. $y\in[0,1]^d$, there exists an infinite number of integers $\{j_i(y)\}_{i\in\mathbb{N}}$ with the following property: the ball $B(y,c^{-\rho j_i(y)})$ contains at least $c^{j_i(y)(d(1-\rho)-\chi(c^{-j_i(y)}))}$ points x_n such that the associated pairs (x_n,λ_n) all satisfy

$$\begin{array}{c}\lambda_n\in[c^{-j_i(y)+1},c^{-j_i(y)(1-\chi(c^{-j_i(y)}))}],\\ \text{for every } n'\neq n,\ B(x_n',\lambda_n')\bigcap B(x_n,\lambda_n)=\emptyset.\end{array}\tag{16}$$

• *(11) and (13) of assumption* **(2)** *are also supposed here.*

(4') *There exists J_m such that for every $j\geq J_m$, for every c-adic box $L=I^c_{j,\mathbf{k}}$, (15) holds. In particular,* **(4)** *holds with $\mathcal{D}=(1,+\infty)$.*

Remark 2.2. 1. *Heuristically, (16) ensures that for m-a.e. y, for infinitely many j, approximately $c^{jd(1-\rho)}$ "disjoint" pairs (x_n,λ_n) such that $\lambda_n\sim c^{-j}$ can be found in the neighborhood $B(y,c^{-\rho j})$ of y. This property is stronger than (10).*

2. *Condition* **(4')** *is stronger than* **(4)**, *in the sense that it implies* **(4)** *for any system $\{(x_n,\lambda_n)\}$ and $\mathcal{D}=(1,+\infty)$. It appears that* **(4')** *is often satisfied, for instance by the first two classes described in Section 6.2 (see* [13]*).*

Property **(4)** *is needed for the last two examples developed in Section 6.2 and for other measures constructed similarly (see* [14]*). Indeed, for these kinds of random* *measures, it was impossible for us to prove* **(4')**, *and we are only able to derive that, with probability 1,* **(4)** *holds with a dense countable set $\mathcal{D}$ (see* [14]*) .*

Before stating the results, a last property has to be introduced. Let $\rho < 1$. For every set I, for every constant $M > 1$, $\mathcal{P}^{\rho}_M(I)$ is said to hold if

$$M^{-1}|I|^{\alpha+\psi(|I|))+2\alpha\chi(|I|))} \le \mu\big(I\big) \le M|I|^{\alpha-\psi(|I|)-2\alpha\chi(|I|)}. \tag{17}$$

The dependence in ρ of $\mathcal{P}^{\rho}_M(I)$ is hidden in the function χ (see (16)).

It is convenient for a ρ-heterogeneous ubiquitous system $\{(x_n, \lambda_n)\}$ ($\rho \in (0,1]$) with respect to (μ, α, β) to introduce the sequences $\varepsilon^{\rho}_M = (\varepsilon^{\rho}_{M,n})_{n\ge1}$ defined for a constant $M \ge 1$ by $\varepsilon^{\rho}_{M,n} = \max(\varepsilon^{\rho,-}_{M,n}, \varepsilon^{\rho,+}_{M,n})$, where

$$\lambda_n^{\alpha\pm\varepsilon^{\rho,\pm}_{M,n}} = M^{\mp}(2\lambda_n)^{\alpha\pm\psi(2\lambda_n)\pm2\alpha\chi(2\lambda_n)} (\text{by convention } \chi \equiv 0 \text{ if } \rho = 1). \tag{18}$$

2.4. **Lower bounds for Hausdorff dimensions of conditioned limsup-sets.** The triplets (μ, α, β), together with the auxiliary measure m, have the properties required to study the exceptional sets we introduced before.
Let $\widehat{\delta} = (\delta_n)_{n\ge1} \in [1,\infty)^{\mathbb{N}^*}$, $\widetilde{\varepsilon} = (\varepsilon_n)_{n\ge1} \in (0,\infty)^{\mathbb{N}^*}$, $\rho \in (0,1]$, $M \ge 1$, and

$$\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \widetilde{\varepsilon}) = \bigcap_{N\ge1} \bigcup_{n\ge N:\ \mathcal{Q}(x_n,\lambda_n,\rho,\alpha,\varepsilon_n) \text{ holds}} B(x_n, \lambda_n^{\delta_n}), \tag{19}$$

where $\mathcal{Q}(x_n, \lambda_n, \rho, \alpha, \varepsilon_n)$ holds when $\lambda_n^{\rho(\alpha+\varepsilon_n)} \le \mu\big(B(x_n, \lambda_n^{\rho})\big) \le \lambda_n^{\rho(\alpha-\varepsilon_n)}$. So, when $\widehat{\delta}$ is a constant sequence equal to some $\delta \ge 1$, the set $\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \widetilde{\varepsilon})$ coincides with the set $S_\mu(\rho, \delta, \alpha, \widetilde{\varepsilon})$ defined in (4) and considered in Theorem 2.1.

Theorem 2.2. *Let μ be a finite positive Borel measure whose support is $[0,1]^d$, $\rho \in (0,1]$ and $\alpha, \beta > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,1]^d$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ a non-increasing sequence of positive real numbers converging to 0.*

Suppose that $\{(x_n, \lambda_n)\}_{n\in\mathbb{N}}$ forms a ρ-heterogeneous ubiquitous system with respect to (μ, α, β). *Let $\widehat{\mathcal{D}}$ be the set of points δ of $\mathbb{R}$ which are limits of a non-decreasing element of $\big(\{1\} \cup \mathcal{D}\big)^{\mathbb{N}^*}$ (in the case of $\rho < 1$, $\mathcal{D} = (1, +\infty)$).*

There exists a constant $M \ge 1$ such that for every $\delta \in \widehat{\mathcal{D}}$, we can find a non-decreasing sequence $\widehat{\delta}$ converging to δ and a positive measure $m_{\rho,\delta}$ which satisfy $m_{\rho,\delta}\left(\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon^{\rho}_M)\right) > 0$, and such that for every $x \in \widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon^{\rho}_M)$, (recall that $\chi \equiv 0$ if $\rho = 1$ and the definition of ε^{ρ}_M (18))

$$\limsup_{r\to0^+} \frac{m_{\rho,\delta}\big(B(x,r)\big)}{r^{D(\beta,\rho,\delta)-\xi_{\rho,\delta}(r)}} < \infty, \tag{20}$$

$$\text{where} \quad \begin{cases} \forall\, \rho \in (0,1], D(\beta, \rho, \delta) = \min\Big(\dfrac{d(1-\rho)+\rho\beta}{\delta}, \beta\Big) \\ \forall\, r > 0, \quad \xi_{\rho,\delta}(r) = (4+d)\varphi(r) + \chi(r). \end{cases}$$

$\widehat{\delta}$ can be taken equal to the constant sequence $(\delta)_{n\ge1}$ if $\delta \in \{1\} \cup \mathcal{D}$.

For the two first classes of measures of Section 6.2 (Gibbs measures and products of multinomial measures), **(4')** holds instead of **(4)** and $\mathcal{D} = (1, +\infty)$, and thus Theorem 2.2 applies with any $\rho \in (0,1]$. For the last two classes of measures of Section 6.2 (independent multiplicative cascades and compound Poisson cascades), Theorem 2.2 cannot be applied when $\rho < 1$.

Corollary 2.3. *Under the assumptions of Theorem 2.2, there exists $M \geq 1$ such that for every $\delta \in \widehat{\mathcal{D}}$, there exists a non-decreasing sequence $\widehat{\delta}$ converging to δ such that $\mathcal{H}^{\xi_{\rho,\delta}}(\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon_M^\rho)) > 0$. Moreover, $\widehat{\delta} = (\delta)_{n\geq 1}$ if $\delta \in \{1\} \cup \mathcal{D}$.*

In particular, $\dim \widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon_M^\rho) \geq D(\beta, \rho, \delta)$.

When $\rho < 1$, $D(\beta, \rho, \delta)$ remains constant and equal to β when δ ranges in $[1, \frac{d(1-\rho)+\rho\beta}{\beta}]$. This is what we call a saturation phenomenon. Then, as soon as $\frac{d(1-\rho)+\rho\beta}{\beta} < \delta$, we are back to a "normal" situation where $D(\beta, \rho, \delta)$ decreases as $1/\delta$ when δ increases.

When $\rho = 1$, $D(\beta, \rho, \delta) = \beta/\delta$, thus there is no saturation phenomenon.

Corollary 2.4. *Fix $\widetilde{\varepsilon} = (\varepsilon_n)_{n\geq 1}$ a positive sequence converging to 0. Assume that $\{(x_n, \lambda_n)\}_{n\in\mathbb{N}}$ forms a weakly redundant and a ρ-heterogeneous ubiquitous system with respect to $(\mu, \alpha, \tau_\mu^*(\alpha))$. Under the assumptions of Theorem 2.1 and Theorem 2.2, there exists a constant $M \geq 1$ such that for every $\delta \in [\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{\tau_\mu^*(\alpha)}, +\infty) \cap \widehat{\mathcal{D}}$, there exists a non-decreasing sequence $\widehat{\delta}$ converging to δ such that*

$$\begin{aligned} \dim\big(\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon_M^\rho)\big) &= \dim\big(\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon_M^\rho) \setminus \bigcup_{\delta'>\delta} S_\mu(\rho, \delta', \alpha, \widetilde{\varepsilon})\big) \\ &= D(\tau_\mu^*(\alpha), \rho, \delta). \end{aligned}$$

Moreover, $\widehat{\delta}$ can be taken equal to $(\delta)_{n\geq 1}$ if $\delta \in \{1\} \cup \mathcal{D}$.

Remark 2.3. 1. *Corollary 2.3 is an immediate consequence of Theorem 2.2.*

2. *In order to prove Corollary 2.4, observe first that when $\delta > 1$ and $\widehat{\delta}$ is a non-decreasing sequence converging to δ, $\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon_M^\rho) \subset S_\mu(\rho, \delta', \alpha, \varepsilon_M^\rho)$ for all $\delta' < \delta$. Theorem 2.1 gives the optimal upper bound for* $\dim\big(\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon_M^\rho)\big)$. *Again by Theorem 2.1, when $\delta \geq \frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{\tau_\mu^*(\alpha)}$, for $\delta' > \delta$, the sets $S_\mu(\rho, \delta', \alpha, \widetilde{\varepsilon})$ form a non-increasing family of sets of Hausdorff dimension $< D(\tau_\mu^*(\alpha), \rho, \delta)$. This implies $\mathcal{H}^{\xi_{\rho,\delta}}\big(\bigcup_{\delta'>\delta} S_\mu(\rho, \delta', \alpha, \widetilde{\varepsilon})\big) = 0$. Finally the lower bound for the dimension* $\dim\big(\widehat{S}_\mu(\rho, \widehat{\delta}, \alpha, \varepsilon_M^\rho) \setminus \bigcup_{\delta'>\delta} S_\mu(\rho, \delta', \alpha, \widetilde{\varepsilon})\big)$ *is given by Corollary 2.3. This holds for any sequence $\widetilde{\varepsilon}$ converging to zero.*
When $\delta = \rho = 1$ and $\widehat{\delta} = (1)_{n\geq 1}$, the arguments are similar to those used for $\delta > 1$.

3. Upper bound for the Hausdorff dimension of conditioned limsup-sets: Proof of Theorem 2.1

The sequence $\{(x_n, \lambda_n)\}_n$ is fixed, and is supposed to form a weakly redundant system (Definition 2.1). We shall need the functions defined for every $j \geq 1$ by

$$\tau_{\mu,\rho,j}(q) = -j^{-1} \log_2 \sum_{n\in T_j} \mu\big(B(x_n, \lambda_n^\rho)\big)^q \text{ and } \tau_{\mu,\rho}(q) = \liminf_{j\to\infty} \tau_{\mu,\rho,j}(q),$$

with the convention that the empty sum equals 0 and $\log(0) = -\infty$.

In the sequel, the Besicovitch's covering theorem is used repeatedly

Theorem 3.1. *(Theorem 2.7 of* [40]*) Let d be an integer greater than 1. There is a constant $Q(d)$ depending only on d with the following properties. Let A be a bounded subset of $\mathbb{R}^d$ and $\mathcal{F}$ a family of closed balls such that each point of A is the center of some ball of $\mathcal{F}$.*

There are families $\mathcal{F}_1, ..., \mathcal{F}_{Q(d)} \subset \mathcal{F}$ covering A such that each $\mathcal{F}_i$ is disjoint, i.e.

$$A \subset \bigcup_{i=1}^{Q(d)} \bigcup_{F \in \mathcal{F}_i} F \text{ and } \forall F, F' \in \mathcal{F}_i \text{ with } F \neq F',\ F \cap F' = \emptyset.$$

Let $(N_j)_{j\geq 1}$ be a sequence as in Definition 2.1, and consider for every $j \geq 1$ the associated partition $\{T_{j,1}, \dots, T_{j,N_j}\}$ of T_j. For every subset S of T_j, for every $1 \leq i \leq N_j$, Theorem 3.1 is used to extract from $\{B(x_n, \lambda_n^\rho) : n \in T_{j,i} \cap S\}$ $Q(d)$ disjoint families of balls denoted by $T_{j,i,k}(S)$, $1 \leq k \leq Q(d)$, such that

$$\bigcup_{n \in T_{j,i} \cap S} B(x_n, \lambda_n^\rho) \subset \bigcup_{k=1}^{Q(d)} \bigcup_{n \in T_{j,i,k}(S)} B(x_n, \lambda_n^\rho). \tag{21}$$

Let us then introduce the functions

$$\widehat{\tau}_{\mu,\rho,j}(q) = -j^{-1} \log_2 \sup_{S \subset T_j} \sum_{n \in \bigcup_{i=1}^{N_j} \bigcup_{k=1}^{Q(d)} T_{j,i,k}(S)} \mu\big(B(x_n, \lambda_n^\rho)\big)^q \quad (j \geq 1)$$

and $\widehat{\tau}_{\mu,\rho}(q) = \liminf_{j\to\infty} \widehat{\tau}_{\mu,\rho,j}(q)$. Recall that τ_μ is defined in (7).

Lemma 3.2. *Under the assumptions of Theorem 2.1, one has*

$$\tau_{\mu,\rho} \geq d(1-\rho) + \rho\tau_\mu \quad \text{and} \quad \widehat{\tau}_{\mu,\rho} \geq \rho\tau_\mu. \tag{22}$$

Proof. • Let us show the first inequality of (22).

First suppose that $q \geq 0$. Fix $j \geq 1$ and $1 \leq i \leq N_j$. For every $n \in T_{j,i}$, $B(x_n, \lambda_n^\rho) \cap [0,1]^d$ is contained in the union of at most 3^d distinct dyadic boxes of generation $j_\rho := [j\rho] - 1$ denoted $B_1(n), \dots, B_{3^d}(n)$. Hence

$$\mu\big(B(x_n, \lambda_n^\rho)\big)^q \leq \Big(\sum_{i=1}^{3^d} \mu\big(B_i(n)\big)\Big)^q \leq 3^{dq} \sum_{i=1}^{3^d} \mu\big(B_i(n)\big)^q.$$

Moreover, since the balls $B(x_n, \lambda_n)$ $(n \in T_{j,i})$ are pairwise disjoint and of diameter larger than $2^{-(j+1)}$, there exists a universal constant C_d depending only on d such that each dyadic box of generation j_ρ meets less than $C_d 2^{d(1-\rho)j}$ of these balls $B(x_n, \lambda_n^\rho)$. Hence when summing over $n \in T_{j,i}$ the masses $\mu\big(B(x_n, \lambda_n^\rho)\big)^q$, each dyadic box of generation j_ρ appears at most $C_d 2^{d(1-\rho)j}$ times. This implies that

$$\sum_{n \in T_{j,i}} \mu\big(B(x_n, \lambda_n^\rho)\big)^q \leq 3^{dq} C_d 2^{d(1-\rho)j} \sum_{\mathbf{k} \in \{0,\dots,2^{j_\rho}-1\}^d} \mu(I_{j,\mathbf{k}})^q \tag{23}$$

$$\text{and} \sum_{n \in T_j} \mu\big(B(x_n, \lambda_n^\rho)\big)^q \leq 3^{dq} C_d N_j 2^{d(1-\rho)j} \sum_{\mathbf{k} \in \{0,\dots,2^{j_\rho}-1\}^d} \mu(I_{j,\mathbf{k}})^q. \tag{24}$$

Since $\log N_j = o(j)$, we obtain $\tau_{\mu,\rho}(q) \geq d(1-\rho) + \rho\tau_\mu(q)$.

Now suppose that $q < 0$. Let us fix $j \geq 1$ and $1 \leq i \leq N_j$. For every $n \in T_{j,i}$, $B(x_n, \lambda_n^\rho)$ contains a dyadic box $B(n)$ of generation $[j\rho] + 1$, and $\mu\big(B(x_n, \lambda_n^\rho)\big)^q \leq \mu\big(B(n)\big)^q$. The same arguments as above also yield $\tau_{\mu,\rho}(q) \geq d(1-\rho) + \rho\tau_\mu(q)$.

• We now prove the second inequality of (22).

Suppose that $q \geq 0$. Fix $j \geq 1$ and S a subset of T_j, as well as $1 \leq i \leq N_j$ and $1 \leq k \leq Q(d)$. We use the decomposition (21). Since the balls $B(x_n, \lambda_n^\rho)$ $(n \in T_{j,i,k}(S))$ are pairwise disjoint and of diameter larger than $2^{-(j+1)\rho}$, there exists a universal constant C'_d, depending only on d, such that each dyadic box of generation j_ρ meets less than C'_d of these balls. Consequently, the arguments used to get (23) yield here

$$\sum_{n\in T_{j,i,k(S)}} \mu\big(B(x_n,\lambda_n^\rho)\big)^q \leq 3^{dq}C'_d \sum_{\mathbf{k}\in\{0,\dots,2^{j\rho}-1\}^d} \mu(I_{j,\mathbf{k}})^q$$

and
$$\sum_{n\in\bigcup_{i=1}^{N_j}\bigcup_{k=1}^{Q(d)} T_{j,i,k}(S)} \mu\big(B(x_n,\lambda_n^\rho)\big)^q \leq 3^{dq}C'_dQ(d)N_j \sum_{\mathbf{k}\in\{0,\dots,2^{j\rho}-1\}^d} \mu(I_{j,\mathbf{k}})^q.$$

The right hand side in the previous inequality does not depend on S, hence

$$\sup_{S\subset T_j} \sum_{n\in\bigcup_{i=1}^{N_j}\bigcup_{k=1}^{Q(d)} T_{j,i,k}(S)} \mu\big(B(x_n,\lambda_n^\rho)\big)^q \leq 3^{dq}C'_dQ(d)N_j \sum_{\mathbf{k}\in\{0,\dots,2^{j\rho}-1\}^d} \mu(I_{j,\mathbf{k}})^q,$$

and the conclusion follows. The case $q<0$ is left to the reader. □

Proof of Theorem 2.1. Let $0\leq\alpha\leq\tau'_\mu(0^-)$. We have $\tau^*_\mu(\alpha)=\inf_{q\geq0}(\alpha q-\tau_\mu(q))$. We first prove that $\dim S_\mu(\rho,\delta,\alpha)\leq \frac{d(1-\rho)+\rho\tau^*_\mu(\alpha)}{\delta}$. For this, we fix $\eta>0$ and $N\geq1$ so that $\varepsilon_n<\eta$ for $n\geq N$. Then we introduce the set $S_\mu(N,\eta,\rho,\delta,\alpha)=\bigcup_{n\geq N:\,\lambda_n^{\rho(\alpha+\eta)}\leq\mu\left(B(x_n,\lambda_n^\rho)\right)} B(x_n,\lambda_n^\delta)$, which can be written as

$$S_\mu(N,\eta,\rho,\delta,\alpha)=\bigcup_{j\geq\inf_{n\geq N}\log_2(\lambda_n^{-1})}\ \bigcup_{n\in T_j:\,\lambda_n^{\rho(\alpha+\eta)}\leq\mu\left(B(x_n,\lambda_n^\rho)\right)} B(x_n,\lambda_n^\delta).$$

We remark that $S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})\subset S_\mu(N,\eta,\rho,\delta,\alpha)$ and use $S_\mu(N,\eta,\rho,\delta,\alpha)$ as covering of $S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})$ in order to estimate the D-dimensional Hausdorff measure of $S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})$ for a fixed $D\geq0$.

Let $q\geq0$ such that $\tau_\mu(q)>-\infty$. Let j_q be an integer large enough so that $j\geq j_q$ implies $\tau_{\mu,\rho,j}(q)\geq\tau_{\mu,\rho}(q)-\eta$. Also let $j_N=\max\big(j_q,\inf_{n\geq N}\log_2(\lambda_n^{-1})\big)$. For some constant C depending on $D,\delta,\alpha,\eta,\rho$ and q only, we have

$$\begin{aligned}
\mathcal{H}^{\xi_D}_{2\cdot2^{-j_N\delta}}\big(S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})\big) &\leq \sum_{j\geq j_N}\ \sum_{n\in T_j:\,\lambda_n^{\rho(\alpha+\eta)}\leq\mu\left(B(x_n,\lambda_n^\rho)\right)} \big|B(x_n,\lambda_n^\delta)\big|^D\\
&\leq \sum_{j\geq j_N}\sum_{n\in T_j} \big|B(x_n,\lambda_n^\delta)\big|^D \lambda_n^{-q\rho(\alpha+\eta)}\mu\big(B(x_n,\lambda_n^\rho)\big)^q\\
&\leq \sum_{j\geq j_N}(22^{-j\delta})^D 2^{(j+1)q\rho(\alpha+\eta)}2^{-j\tau_{\mu,\rho,j}(q)}\\
&\leq C\sum_{j\geq j_N} 2^{-j(D\delta-q\rho(\alpha+\eta)+\tau_{\mu,\rho}(q)-\eta)}.
\end{aligned}$$

Therefore, if $D>\frac{\rho(\alpha+\eta)-\tau_{\mu,\rho}(q)+\eta}{\delta}$, $\mathcal{H}^{\xi_D}_{2\cdot2^{-j_N\delta}}\big(S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})\big)$ converges to 0 as $N\to\infty$, and $\dim S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})\leq D$. This yields $\dim S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})\leq\frac{q\rho(\alpha+\eta)-\tau_{\mu,\rho}(q)+\eta}{\delta}$, which is less than $\frac{d(1-\rho)+\rho(\alpha q-\tau_\mu(q))+(q\rho+1)\eta}{\delta}$ by Lemma 3.2. This holds for every $\eta>0$ and for every $q\geq0$ such that $\tau_\mu(q)>-\infty$. Finally, $\dim S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})\leq \frac{d(1-\rho)+\rho\inf_{q\geq0}\alpha q-\tau_\mu(q)}{\delta}=\frac{d(1-\rho)+\rho\tau^*_\mu(\alpha)}{\delta}$.

Let us now show that dim $S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon}) \leq \tau_\mu^*(\alpha)$. This time, for $j \geq 1$ we define $S_j = \{n \in T_j : \lambda_n^{\rho(\alpha+\eta)} \leq \mu\big(B(x_n,\lambda_n^\rho)\big)\}$. By (21), we remark that

$$S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon}) \subset \bigcup_{j\geq j_N} \bigcup_{i=1}^{N_j} \bigcup_{k=1}^{Q(d)} \bigcup_{n\in T_{j,i,k}(S_j)} B(x_n,\lambda_n^\rho).$$

By definition of $\widehat{\tau}_{\mu,\rho}(q)$, a computation mimicking the previous one yields

$$\mathcal{H}^{\xi_D}_{2\cdot 2^{-\rho j_N}}\big(S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon})\big) \quad \leq \quad C\sum_{j\geq j_N} 2^{-j(D\rho - q\rho(\alpha+\eta)+\widehat{\tau}_{\mu,\rho}(q)-\eta)}.$$

Hence dim $S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon}) \leq \frac{q\rho(\alpha+\eta)-\widehat{\tau}_{\mu,\rho}(q)+\eta}{\rho}$, for every $\eta>0$ and every $q\geq 0$ such that $\tau_\mu(q) > -\infty$. The conclusion follows from Lemma 3.2.

Finally, when $\tau_\mu^*(\alpha)<0$ and $S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon}) \neq \emptyset$, the previous estimates show that $\mathcal{H}^{\xi_D}_{2\cdot 2^{-\rho j_N}}(S_\mu(\rho,\delta,\alpha,\widetilde{\varepsilon}))$ is bounded for $D \in (\tau_\mu^*(\alpha),0)$ (we can formally extend the definition of $\mathcal{H}^{\xi_D}$ to the case $D<0$). This is a contradiction.

The proof when $\alpha \geq \tau_\mu'(0^-)$ follows similar lines. □

4. Conditioned ubiquity. Proof of Theorem 2.2 (case $\rho = 1$)

We assume that a 1-heterogeneous ubiquitous system is fixed. With each pair (x_n,λ_n) is associated the ball $I_n = B(x_n,\lambda_n)$. For every $\delta\geq 1$, $I_n^{(\delta)}$ denotes the contracted ball $B(x_n,\lambda_n^\delta)$. The following property is useful in the sequel. Because of the assumption **(1)** on φ and ψ, we have

$$\exists C>1,\ \forall\, 0<r\leq s\leq 1,\ s^{-\varphi(s)} \leq Cr^{-\varphi(r)} \text{ and } s^{-\psi(s)} \leq Cr^{-\psi(r)}. \tag{25}$$

We begin with a simple technical lemma

Lemma 4.1. *Let $y\in[0,1]^d$, and assume that there exists an integer $j(y)$ such that for some integer $c\geq 2$, (11) and (13) hold for y and every $j\geq j(y)$.*

There exists a constant M independent of y with the following property: for every n such that $y \in B(x_n,\lambda_n/2)$ and $\log_c \lambda_n^{-1} \geq j(y)+4$, $\mathcal{D}_M^m(B(y,2\lambda_n))$ and $\mathcal{P}_M^1(B(x_n,\lambda_n))$ hold.

Proof. Assume that $y\in B(x_n,\lambda_n/2)$ with $\lambda_n \leq c^{-j(y)-4}$. Let j_0 be the smallest integer j such that $c^{-j}\leq \lambda_n/2$, and j_1 the largest integer j such that $c^{-j}\geq 2\lambda_n$. We have $j_0 \geq -\log_c\lambda_n \geq j_1 \geq j(y)$. We thus ensured by construction that $j_0 - 4 \leq -\log_c \lambda_n \leq j_1+4$.

Recall that $I_j(y)$ is the unique c-adic box of scale j containing y, and that $\mathbf{k}_{j,y}$ is the unique $k\in\mathbb{N}^d$ such that $y \in I^c_{j,\mathbf{k}} = I_j(y)$. We have $I^c_{j_0}(y) \subset B(x_n,\lambda_n) \subset \bigcup_{\|\mathbf{k}-\mathbf{k}^c_{j_1,y}\|_\infty\leq 1} I^c_{j_1,\mathbf{k}}$, which yields $\mu(I^c_{j_0}(y)) \leq \mu(B(x_n,\lambda_n)) \leq \sum_{\|\mathbf{k}-\mathbf{k}^c_{j_1,y}\|_\infty\leq 1} \mu(I^c_{j_1,\mathbf{k}})$.

Applying (11) and (12) yields

$$|c^{-j_0}|^{\alpha+\psi(|c^{-j_0}|)} \leq \mu(B(x_n,\lambda_n)) \leq 3^d |c^{-j_1}|^{\alpha-\psi(|c^{-j_1}|)}.$$

Combining the fact that $j_0-4\leq -\log_c\lambda_n \leq j_1+4$ with (25) and (18) gives

$$\lambda_n^{\alpha+\varepsilon^{1,+}_{M,n}} = M^{-1}|2\lambda_n|^{\alpha+\psi(2\lambda_n)} \leq \mu(B(x_n,\lambda_n)) \leq M|2\lambda_n|^{\alpha-\psi(2\lambda_n)} = \lambda_n^{\alpha-\varepsilon^{1,-}_{M,n}}$$

for some constant M that does not depend on y.

Similarly, we get from (13) and (14) that $\mathcal{D}_M^m\big(B(y,2\lambda_n)\big)$ holds for some constant $M>0$ that does not depend on y. $\square$

Proof of Theorem 2.2 in the case $\rho=1$. Throughout the proof, C denotes a constant which depends only on c, α, β, δ, φ and ψ.

The case $\delta=1$ follows immediately from the assumptions (here $m_\delta=m_1=m$).

Now let $M\geq 1$ be the constant given by Lemma 4.1. Let $\delta\in\widehat{\mathcal{D}}\cap(1,+\infty)$, and let $\{d_n\}_{n\geq 1}$ be a non-decreasing sequence in $\mathcal{D}$ converging to δ (if $\delta\in\mathcal{D}$, $d_n=\delta$ for every n). For every $k\geq 1$, $j\geq 1$ and $y\in[0,1]^d$, let

$$n_{j,y}^{(d_k)}=\inf\left\{n:\lambda_n\leq c^{-j},\ \exists j'\geq j:\left\{\begin{array}{l} B(x_n,\lambda_n)\in\mathcal{B}_{j'}(y)\ \text{ and}\\ \exists\ L\in\mathbf{B}_n^{d_k},\ \ (15)\text{ holds}\end{array}\right\}\right\}. \tag{26}$$

We shall find a sequence $\widehat{\delta}=(\delta_n)_{n\geq 1}$, converging to δ, to construct a generalized Cantor set K_δ in $\widehat{S}_\mu(1,\widehat{\delta},\alpha,\varepsilon_M^1)$ and simultaneously the measure m_δ on K_δ. The successive generations of c-adic boxes involved in the construction of K_δ, namely G_n, are obtained by induction.

- First step: The first generation of boxes defining K_δ is taken as follows.

Let $L_0=[0,1]^d$. Consider the first element d_1 of $\mathcal{D}$ of the sequence converging to δ. We first impose that $\delta_n:=d_1$, for every $n\geq 1$. The values of the sequence $\widehat{\delta}$ will be modified in the next steps of the construction so that $\widehat{\delta}$ will become a non-decreasing sequence satisfying $\lim_{n\to+\infty}\delta_n=\delta$.

Due to assumptions **(2)**, **(3)** and **(4)**, there exist $E^{L_0}\subset E_{J(L_0)}^{L_0}$ such that $m(E^{L_0})\geq\|m\|/4$ and an integer $J'(L_0)\geq J(L_0)$ such that for all $y\in E^{L_0}$:

- $y\in\bigcap_{N\geq 1}\ \bigcup_{n\geq N}B(x_n,\lambda_n/2)$,
- for every $j\geq J'(L_0)$, both (11) and (13) hold,
- there are infinitely many integers j such that (15) holds for some $L\in\mathcal{B}_j^{d_1}(y)$.

In order to construct the first generation of balls of the Cantor set, we invoke the Besicovitch's covering Theorem 3.1. We are going to apply it to $A=E^{L_0}$ and to several families $\mathcal{F}_1(j)$ of balls constructed as follows.

For $y\in E^{L_0}$, we denote $n_{j,y}^{(d_1)}$ by $n_{j,y}$. Then for every $j\geq J'(L_0)+4$, we define $\mathcal{F}_1(j)=\{B\left(y,2\lambda_{n_{j,y}}\right):\ y\in E^{L_0}\}$.

The family $\mathcal{F}_1(j)$ fulfills the conditions of Theorem 3.1. Thus, for every $j\geq J'(L_0)+4$, $Q(d)$ families of disjoint balls $\mathcal{F}_1^1(j),...,\mathcal{F}_1^{Q(d)}(j)$, can be extracted from $\mathcal{F}_1(j)$. Therefore, since $m(A)=m(E^{L_0})\geq\|m\|/4$, for some i we have $m\Big(\bigcup_{B\in\mathcal{F}_1^i(j)}B\Big)\geq\|m\|/(4\,Q(d))$.

Again, we extract from $\mathcal{F}_1^i(j)$ a finite family of pairwise disjoint balls $\widetilde{G}_1(j)=\{B_1,B_2,\dots,B_N\}$ such that

$$m\Big(\bigcup_{B_k\in\widetilde{G}_1(j)}B_k\Big)\geq\frac{\|m\|}{8\,Q(d)}. \tag{27}$$

By construction, with each B_k can be associated a point $y_k\in E^{L_0}$ so that $B_k=B(y_k,2\lambda_{n_{j,y_k}})$. Moreover, by construction (see (26)), $I_{n_{j,y_k}}=B(x_{n_{j,y_k}},\lambda_{n_{j,y_k}})\subset$

$B(y_k, 2\lambda_{n_{j,y_k}}) = B_k$. Thus $I^{(d_1)}_{n_{j,y_k}} = B(x_{n_{j,y_k}}, \lambda^{d_1}_{n_{j,y_k}})$ is included in B_k. Finally, Lemma 4.1 yield $\mathcal{P}^1_M(B(x_{n_{j,y_k}}, \lambda_{n_{j,y_k}}))$ and $\mathcal{D}^m_M(B_k)$.

Let F_k be the closure of one of the c-adic boxes of maximal diameter included in $I^{(d_1)}_{n_{j,y_k}}$, and such that both (15) holds for F_k. Such a box exists by (26). Moreover, by construction we have $|F_k| \le |I^{(d_1)}_{n_{j,y_k}}| \le C|F_k|$ for some universal constant C.

We write $\underline{B_k} = F_k$. Conversely, if a c-adic box F can be written $\underline{B}$ for some larger ball B, we write $B = \overline{F}$. Therefore, for every closed box F constructed above we can ensure by construction that

$$C^{-1}|F| \le |\overline{F}|^{d_1} \le C|F|, \tag{28}$$

where C depends only on the fixed given sequence $\{d_n\}_n$. We eventually set

$$G_1(j) = \{\underline{B_k} : B_k \in \widetilde{G}_1(j)\}. \tag{29}$$

We notice the following property that will be used in the last step: By construction, if F_1 and F_2 are two distinct elements of $G_1(j)$ then their distance is at least $\max_{i\in\{1,2\}}(|\overline{F_i}|/2 - \left(|\overline{F_i}|/2\right)^{d_1})$, which is larger than $\max_{i\in\{1,2\}} |\overline{F_i}|/3$ for j large enough ($d_1 > 1$ by our assumption).

On the algebra generated by the elements of $G_1(j)$, a probability measure m_δ is defined by

$$m_\delta(F) = \frac{m(\overline{F})}{\sum_{F_k\in G_1(j)} m(\overline{F_k})}.$$

Let $F \in G_1(j)$. By construction, $\mathcal{D}^m_M(\overline{F})$ holds. Using consecutively this fact, (28) and (25), we obtain

$$m(\overline{F}) \le M|\overline{F}|^{\beta-\varphi(|\overline{F}|)} \le C|F|^{\beta/d_1}|\overline{F}|^{-\varphi(|\overline{F}|)} \le C|F|^{\beta/d_1}|F|^{-\varphi(|F|)}.$$

Moreover, by (27), and recalling the definition of $G_1(j)$ (29), we obtain

$$\sum_{F_k\in G_1(j)} m(\overline{F_k}) = \sum_{B_k\in \widetilde{G}_1(j)} m(B_k) \ge \frac{\|m\|}{8\,Q(d)}.$$

As a consequence, $\forall\, F \in G_1(j),\ m_\delta(F) \le 8\,Q(d)C\|m\|^{-1}|F|^{\beta/d_1}|F|^{-\varphi(|F|)}$.

By our assumption **(1)**, we can fix j_1 large enough so that

$$\forall\, F \in G_1(j_1),\ 8\,Q(d)C\|m\|^{-1} \le |F|^{-\varphi(|F|)}.$$

We choose the c-adic elements of the first generation of the construction of K_δ as being those of $G_1 := G_1(j_1)$. By construction

$$\forall\, F \in G_1,\ m_\delta(F) \le |F|^{\beta/d_1 - 2\varphi(|F|)}. \tag{30}$$

We know that by construction, for every $F \in G_1$, there exists $y_k \in E^{L_0}$ such that $B(x_{n_{j_1,y_k}}, \lambda_{n_{j_1,y_k}}) \subset \overline{F} = B(y_k, 2\lambda_{n_{j_1,y_k}})$.

As a consequence, for every $y \in \bigcup_{F\in G_1} F$, there exists an integer n such that $\lambda_n \le c^{-4}$, $|x_n - y| \le \lambda_n^{\delta_n}$, and $\mathcal{P}^1_M(I_n) = \mathcal{P}^1_M(B(x_n, \lambda_n))$ holds.

- Second step: The second generation of boxes is obtained as follows. Let n_1 be the largest integer among the $n^{(d_1)}_{j_1,y_k}$, where the y_k are the points naturally associated with the balls $I \in G_1$ above.

Consider d_2, the second element of the sequence $\{d_n\}_n$ converging to δ. We modify the sequence $\widehat{\delta}$: for every $n > n_1$, we impose $\delta_n := d_2$.

Let us focus on one of the c-adic boxes $L \in G_1$. The selection procedure is the same as in the first step. Due to assumptions **(2)**, **(3)** and **(4)**, we can find a subset E^L of $E^L_{J(L)}$ such that $m^L\big(E^L\big) \geq \|m^L\|/4$ and an integer $J'(L) \geq J(L)$ such that for all $y \in E^L$:

- $y \in \bigcap_{N\geq 1} \bigcup_{n\geq N} B(x_n, \lambda_n/2)$,
- $\forall\, j \geq J'(L) + \log_c\big(|L|^{-1}\big)$,

$$\forall\, \mathbf{k},\ \|\mathbf{k}-\mathbf{k}^c_{j,y}\|_\infty \leq 1,\ \mathcal{D}_1^{m^L\circ f_L^{-1}}\big(f_L(I^c_{j,\mathbf{k}})\big) \text{ and } \mathcal{P}^1_1(I^c_{j,\mathbf{k}}) \text{ hold.} \tag{31}$$

- There are infinitely many integers j such that (15) holds for some $L \in \mathcal{B}^{d_2}_j(y)$.

We again apply Theorem 3.1 to $A = E^L$ and to families $\mathcal{F}_2(j)$ of balls constructed as above. Hence, for every $j \geq J'(L) + \log_c\big(|L|^{-1}\big) + 4$, $\mathcal{F}_2(j) = \Big\{B(y, 2\lambda_{n^{(d_2)}_{j,y}}) : y \in E^L\Big\}$ ($n^{(d_2)}_{j,y}$ is defined in (26)). We set $n_{j,y} := n^{(d_2)}_{j,y}$.

The family $\mathcal{F}_2(j)$ fulfills the conditions of Theorem 3.1 and covers E^L. By Theorem 3.1, for every $j \geq J'(L)+\log_c\big(|L|^{-1}\big)+4$, $Q(d)$ families of pairwise disjoint boxes $\mathcal{F}^1_2(j), \dots, \mathcal{F}^{Q(d)}_2(j)$, whose union covers E^L, can be extracted from $\mathcal{F}_2(j)$. Since $m^L(A) = m^L(E^L) \geq \|m^L\|/4$, there exists i such that $m^L\Big(\bigcup_{B\in\mathcal{F}^i_2(j)} B\Big) \geq \|m^L\|/4Q(d)$.

As in the first step, we extract from $\mathcal{F}^i_2(j)$ a finite family of disjoint balls $\widetilde{G}^L_2(j) = \{B_1, B_2, \dots, B_N\}$ such that

$$m^L\Big(\bigcup_{B_k\in\widetilde{G}^L_2(j)} B_k\Big) \geq \frac{\|m^L\|}{8\,Q(d)}. \tag{32}$$

As above, with each B_k is associated a point $y_k \in E^L$ so that $B_k = B(y_k, 2\lambda_{n_{j,y_k}})$, and $I^{(d_2)}_{n_{j,y_k}} \subset I_{n_{j,y_k}} \subset B_k$. Now, notice that Lemma 4.1 applies with $m^L \circ f_L^{-1}$ instead of m and with the same constant M. It follows that $\mathcal{D}_M^{m^L\circ f_L^{-1}}\big(f_L(B_k)\big)$ and $\mathcal{P}^1_M(I_{n_{j,y_k}})$ hold. Let F_k be the closure of one of the c-adic balls of maximal diameter included in $I^{(d_2)}_{n_{j,y_k}}$ such that (15) holds for F_k.

We then define the notation $\underline{B_k} = F_k$, and conversely $B_k = \overline{F_k}$. We also have (28) (for the same constant C). We eventually define

$$G^L_2(j) = \{\underline{B_k} : B_k \in \widetilde{G}^L_2(j)\}. \tag{33}$$

On the algebra generated by the elements F of $G^L_2(j)$, an extension of the restriction to the ball L of the measure m_δ is defined by

$$m_\delta(F) = \frac{m^L(\overline{F})}{\sum_{F_k\in G^L_2(j)} m^L(\overline{F_k})}\, m_\delta(L).$$

Let $F \in G^L_2(j)$. Since $\mathcal{D}_M^{m^L\circ f_L^{-1}}\big(f_L(\overline{F})\big)$ holds, we have

$$\begin{aligned} m^L(\overline{F}) &\leq M\left(\frac{|\overline{F}|}{|L|}\right)^{\beta-\varphi\left(\frac{|\overline{F}|}{|L|}\right)} \leq C|F|^{\beta/d_2}|L|^{-\beta}\left(\frac{|\overline{F}|}{|L|}\right)^{-\varphi\left(\frac{|\overline{F}|}{|L|}\right)} \\ &\leq C|F|^{\beta/d_2}|L|^{-\beta}|F|^{-\varphi(|F|)}, \end{aligned}$$

where (25) has been used. Moreover, by (32) and (33),

$$\sum_{F_k\in G_2^L(j)} m^L(\overline{F_k}) = \sum_{B_k\in\widetilde{G}_2^L(j)} m^L(B_k) \geq \|m^L\|/8\,Q(d).$$

Consequently, since $m_\delta(L)$ can be bounded using (30), we obtain

$$\begin{aligned} m_\delta(F) &\leq 8m_\delta(L)Q(d)\|m^L\|^{-1}C|F|^{\beta/d_2}|L|^{-\beta}|F|^{-\varphi(|F|)} \\ &\leq 8\,Q(d)\|m^L\|^{-1}C|L|^{\beta/d_1-\beta-2\varphi(|L|)}|F|^{\beta/d_2-\varphi(|F|)}. \end{aligned}$$

By **(1)**, we can choose $j_2(L)$ large enough so that for every integer $j\geq j_2(L)$, for every c-adic ball $F\in G_2^L(j)$, $8\,Q(d)C\|m^L\|^{-1}|L|^{\beta/d_1-\beta-2\varphi(|L|)}\leq |F|^{-\varphi(|F|)}$. Then, taking $j_2=\max\left\{j_2(L):L\in G_1\right\}$, and defining

$$G_2=\bigcup_{L\in G_1} G_2^L(j_2),$$

this yields an extension of m_δ to the algebra generated by the elements of $G_1\bigcup G_2$ and such that for every $F\in G_1\bigcup G_2$, $m_\delta(F)\leq |F|^{\beta/d_2-2\varphi(|F|)}$ (indeed if $F\in G_1$ $|F|^{\beta/d_1}\leq|F|^{\beta/d_2}$ because $d_2\geq d_1$).

Notice that by construction, for every $F\in G_2$, $|F|\leq \max_{F\in G_1} 2(c^{-4}|F|)^{d_2}$.

Finally we define n_2 as the largest integer among the $n^{(d_2)}_{j_2(L),y_k}$, where the y_k are the points naturally associated with the balls $F\in G_2$ above.

- Third step: We end the induction. Assume that N generations of closed c-adic boxes $G_1,\dots,G_N$ are found for some integer $N\geq 2$. Assume also that a probability measure m_δ on the algebra generated by $\bigcup_{1\leq p\leq N} G_p$ is defined and that the following properties hold (the fact that this holds for $N=2$ comes from the two previous steps):

(i) For every $1\leq p\leq N$, the elements of G_p are closed pairwise disjoint c-adic boxes, and for $2\leq p\leq N$, $\max_{F\in G_p}|F|\leq 2c^{-4d_p}\max_{F\in G_{p-1}}|F|^{d_p}$.

For $1\leq p\leq N$, with each $F\in G_p$ is associated a ball $\overline{F}$ enjoying the properties:

- $F\subset\overline{F}$,
- there is a constant $C>0$ which depends only on δ such that (28) holds,
- if $F_1\neq F_2$ belong to G_p, their distance is at least $\max_{i\in\{1,2\}}\overline{F}_i/3$,
- the $\overline{F}$'s ($F\in G_p$) are pairwise disjoint.
- $\overline{F}$ satisfies the next parts **(ii)**, **(iii)**, **(iv)**, **(v)** and **(vi)**.

(ii) For every $2\leq p\leq N$, each element F of G_p is included in an element L of G_{p-1}. Moreover, $\overline{F}\subset L$, $\log_c\left(|\overline{F}|^{-1}\right)\geq J(L)+\log_c\left(|L|^{-1}\right)$ and $\overline{F}\cap E^L_{J(L)}\neq\emptyset$.

(iii) There exists a sequence $\widehat{\delta}=\{\delta_q\}_{q\geq1}$ such that:

- $\widehat{\delta}$ is non-decreasing, and $\forall\, q\geq 1$, $\delta_q\leq\delta$,
- for every $1\leq p\leq N$ and $F\in G_p$, there is an integer q such that $F\subset I_q^{(\delta_q)}=B(x_q,\lambda_q^{\delta_q})\subset\overline{F}$, $\mathcal{P}^1_M(I_q)$ holds, and $\delta_q=d_p$.
- for every $1\leq p\leq N-1$, we found an integer n_p such that for every $q\in\{n_{p-1}+1,n_{p-1}+2,\dots,n_p\}$, $\delta_q=\delta_p$ (with the convention that $n_0=0$).

(iv) For every $F\in\bigcup_{1\leq p\leq N} G_p$, $m_\delta(F)\leq|F|^{\beta/d_N-2\varphi(|F|)}$.

(v) For every $1 \leq p \leq N-1$, $L \in G_p$, and $F \in G_{p+1}$ such that $F \subset L$,

$$m_\delta(F) \leq 8\, Q(d) m_\delta(L) \frac{m^L(\overline{F})}{\|m^L\|}.$$

(vi) Every $L \in \bigcup_{1 \leq p \leq N} G_p$ satisfies (15).

The constructions of a generation G_{N+1} of c-adic balls and an extension of m_δ to the algebra generated by the elements of $\bigcup_{1 \leq p \leq N+1} G_p$ such that properties **(i)** to **(vi)** hold for $N+1$ are done in the same way as when $N=1$.

By induction, and because of the separation property **(i)**, we get:
- a sequence $(G_N)_{N \geq 1}$ and a non-decreasing sequence $\widehat{\delta}$ converging to δ,
- a probability measure m_δ on $\sigma\big(F : F \in \bigcup_{N \geq 1} G_N\big)$
such that properties **(i)** to **(vi)** hold for every $N \geq 1$. We now define

$$K_\delta = \bigcap_{N \geq 1} \bigcup_{F \in G_N} F.$$

By construction, $m_\delta(K_\delta) = 1$ and because of property **(iii)**, we have $K_\delta \subset \widehat{S}_\mu(1, \widehat{\delta}, \alpha, \varepsilon_M^1)$. The measure m_δ can be extended to $\mathcal{B}([0,1]^d)$ by the usual way: $m_\delta(B) := m_\delta(B \cap K_\delta)$ for $B \in \mathcal{B}([0,1]^d)$. Finally, since $\delta_n \leq \delta$ for every $n \geq 1$, property **(iv)** implies that for every $F \in \bigcup_{N \geq 1} G_N$,

$$m_\delta(F) \leq |F|^{\beta/\delta - 2\varphi(|F|)}. \tag{34}$$

- Last step: Proof of (20). If $F \in G_N$, we set $g(F) = N$.

Let us fix B an open ball of $[0,1]^d$ of length less than the one of the elements of G_1, and assume that $B \cap K_\delta \neq \emptyset$. Let L be the element of largest diameter in $\bigcup_{N \geq 1} G_N$ such that B intersects at least two elements of $G_{g(L)+1}$ included in L. We remark that this implies that B does not intersect any other element of $G_{g(L)}$, and as a consequence $m_\delta(B) \leq m_\delta(L)$.

Let us distinguish three cases:

• When $|B| \geq |L|$: we have by (34)

$$m_\delta(B) \leq m_\delta(L) \leq |L|^{\beta/\delta - 2\varphi(|L|)} \leq C |B|^{\beta/\delta - 2\varphi(|B|)}. \tag{35}$$

• When $|B| \leq c^{-J(L)-3}|L|$: let $L_1, \ldots, L_p$ be the elements of $G_{g(L)+1}$ that intersect B. We use property **(v)** to get

$$m_\delta(B) = \sum_{i=1}^p m_\delta(B \cap L_i) \leq m_\delta(L) \frac{8\, Q(d)}{\|m^L\|} \sum_{i=1}^p m^L(\overline{L}_i). \tag{36}$$

Let j_0 be the unique integer such that $c^{-j_0} \leq |B| < c^{-j_0+1}$. Assume that B intersects for instance the boxes L_{i_1} and L_{i_2}. Then, by **(i)**, we have $|B| \geq \max(|\overline{L}_{i_1}|, |\overline{L}_{i_2}|)/3$ when j_0 is large enough. Consequently, when $|B|$ is small enough, we get $|B| \geq (\max_{i=1,\ldots,p} |\overline{L}_i|)/3$ and the scale of the boxes $\overline{L}_i$ (defined as $[-\log_c |\overline{L}_i|]$) is always larger than $j_0 - [\log_c 3] \geq j_0 - 2$.

By property **(ii)**, for each $i \in \{1, \ldots d\}$, we have $E^L_{J(L)} \cap \overline{L}_i \neq \emptyset$. Let $y \in E^L_{J(L)} \cap \overline{L}_i$ for some i, and let us consider the c-adic box $I^c_{j_0-2, \mathbf{k}_{j_0-2,y}}$. For every $z \in \overline{L}_i$, $|y - z| \leq c^{-(j_0-2)}$. This yields

$$\overline{L}_i \subset \bigcup_{\mathbf{k}:\ \|\mathbf{k} - \mathbf{k}_{j_0-2,y}\|_\infty \leq 1} I^c_{j_0-2, \mathbf{k}}.$$

The ball B intersects L_i, thus the distance between y and B is at most $c^{-(j_0-2)}$. As a consequence, if $L_{i'} \neq L_i$, the distance between y and $L_{i'}$ is lower than $c^{-(j_0-3)}$. This implies that

$$\bigcup_{i=1}^{p} \overline{L}_i \subset \bigcup_{\mathbf{k}:\ \|\mathbf{k}-\mathbf{k}_{j_0-3,y}\|_\infty \le 1} I^c_{j_0-3,\mathbf{k}}. \tag{37}$$

Since $y \in E^L_{J(L)}$ and $j_0 \ge -\log_c |L| + J(L) + 3$, assumption **(3)** ensures the control of the m-mass of the unions of all the balls that appear on the left hand-side of (37) by the sum of the masses of the 3^d c-adic boxes $I^c_{j_0-3,\mathbf{k}}$, $\|\mathbf{k}-\mathbf{k}_{j_0-3,y}\|_\infty \le 1$. These boxes all satisfy

$$m^L(I^c_{j_0-3,\mathbf{k}}) \le \left(\frac{|I^c_{j_0-3,\mathbf{k}}|}{|L|}\right)^{\beta-\varphi\left(\frac{|I^c_{j_0-3,\mathbf{k}}|}{|L|}\right)} \le C\left(\frac{|B|}{|L|}\right)^{\beta}\left(\frac{|B|}{|L|}\right)^{-\varphi\left(\frac{|B|}{|L|}\right)}$$

where C depends only on β. Injecting this in (36) and using that the $\overline{L}_i$ are pairwise disjoint, we obtain that for $|B|$ small enough

$$\begin{aligned} m_\delta(B) &\le m_\delta(L)\frac{8\,Q(d)}{\|m^L\|}\sum_{i=1}^{p} m^L(\overline{L}_i) \\ &\le m_\delta(L)\frac{8\,Q(d)}{\|m^L\|}3^d\,C\left(\frac{|B|}{|L|}\right)^{\beta}\left(\frac{|B|}{|L|}\right)^{-\varphi\left(\frac{|B|}{|L|}\right)} \\ &\le m_\delta(L)\frac{C}{\|m^L\|}\left(\frac{|B|}{|L|}\right)^{\beta}|B|^{-\varphi(B)}, \end{aligned}$$

where C takes into account all the constant factors. We then use consecutively two facts. First, by (34), $m_\delta(L) \le |L|^{\beta/\delta}|L|^{-2\varphi(|L|)} \le C|L|^{\beta/\delta}|B|^{-2\varphi(|B|)}$, which implies, since $r \mapsto r^{\beta(1-1/\delta)}$ is bounded near 0,

$$m_\delta(B) \le \frac{C}{\|m^L\|}|B|^{\beta/\delta}|B|^{-3\varphi(|B|)}\left(\frac{|B|}{|L|}\right)^{\beta(1-1/\delta)} \le \frac{C}{\|m^L\|}|B|^{\beta/\delta}|B|^{-3\varphi(|B|)}.$$

Second, **(vi)** allows to upper bound $\|m^L\|^{-1}$ by $|L|^{-\varphi(L)}$, which yields

$$m_\delta(B) \le C|L|^{-\varphi(|L|)}|B|^{\beta/\delta}|B|^{-3\varphi(|B|)} \le C|B|^{\beta/\delta}|B|^{-4\varphi(|B|)}. \tag{38}$$

• $c^{-J(L)-3}|L| < |B| \le |L|$: we need at most $c^{d(J(L)+4)}$ contiguous boxes of diameter $c^{-J(L)-3}|L|$ to cover B. For these boxes, the estimate (38) can be used. Also we know by **(vi)** that $c^{J(L)} \le |L|^{-\varphi(L)}$, so for $|B|$ small enough

$$\begin{aligned} m_\delta(B) &\le Cc^{d(J(L)+4)}\left(c^{-J(L)-3}|L|\right)^{\beta/\delta-4\varphi(c^{-J(L)-3}|L|)} \le Cc^{dJ(L)}|B|^{\beta/\delta-4\varphi(|B|)} \\ &\le C|L|^{-d\varphi(|L|)}|B|^{\beta/\delta-4\varphi(|B|)} \le C|B|^{\beta/\delta-(4+d)\varphi(|B|)}. \end{aligned}$$

Combining (35) and (38) with assumption **(1)**, we obtain a universal constant C such that for every non-trivial ball B of $[0,1]^d$ small enough, we have $m_\delta(B) \le C|B|^{\beta/\delta}|B|^{-(4+d)\varphi(|B|)}$. This yields (20). □

5. Dilation and Saturation. Proof of Theorem 2.2 (Case $\rho < 1$)

The introduction of the condition (16) induces a modification in the construction of the Cantor set with respect to the case $\rho = 1$, in the selection of the pairs (x_n, λ_n). The following lemma is comparable with Lemma 4.1

Lemma 5.1. *Let $y \in [0,1]^d$, and assume that (11) and (13) hold for y when $j \geq j(y)$ for some integer $j(y)$. There exists a constant M independent of y with the following property: for every integer j such that $j(1-\chi(c^{-j})) \geq \frac{j(y)+5}{\rho}$, for every integer n such that $\lambda_n \in [c^{-j+1}, c^{-j(1-\chi(c^{-j}))}]$ and*

$$B(y, (c^\rho - 1)c^{-j\rho}) \subset B(x_n, \lambda_n^\rho) \subset B(y, c^{-j\rho(1-\chi(c^{-j}))}), \tag{39}$$

then $\mathcal{P}_M^\rho(B(x_n, \lambda_n^\rho))$ holds. Moreover, the same constant M can be chosen so that $\mathcal{D}_M^m(B(y,r))$ holds for $r \in (0, c^{-j(y)-1})$.

Proof. Let us fix j such that (39) holds, and let us denote j_1 the integer $[j\rho] + 2$ and j_2 the integer $[j\rho(1-\chi(c^{-j}))] - 2$. By definition of j_1 and j_2, (39) implies that $I^c_{j_1}(y) \subset B(x_n, \lambda_n^\rho) \subset \bigcup_{\|\mathbf{k}-\mathbf{k}^c_{j_2,y}\|_\infty \leq 1} I^c_{j_2,\mathbf{k}}$. Combining this with (11) yields

$$(c^{-j_1})^{\alpha+\psi(c^{-j_1})} \leq \mu(B(x_n, \lambda_n^\rho)) \leq 3^d (c^{-j_2})^{\alpha-\psi(c^{-j_2})}. \tag{40}$$

We have $c^{-j_1} \leq 2\lambda_n^\rho = |B(x_n, \lambda_n^\rho)| \leq 2c^{-j_2}$, but by (39) we also have

$$C^{-1}(2c^{-j_2})^{\frac{1}{1-\chi(c^{-j})}} \leq 2\lambda_n^\rho \leq C(2c^{-j_1})^{1-\chi(c^{-j})} \tag{41}$$

for some constant C independent of y and j. Hence, using the monotonicity of $r \mapsto r^{-\psi(r)}$, (40) and (41) yields the two inequalities

$$\begin{aligned} M^{-1}(2\lambda_n^\rho)^{\frac{\alpha}{1-\chi(c^{-j})}} (2\lambda_n^{\frac{\rho}{1-\chi(c^{-j})}})^{\psi\left(2\lambda_n^{\frac{\rho}{1-\chi(c^{-j})}}\right)} &\leq \mu(B(x_n, \lambda_n^\rho)), \\ (2\lambda_n^\rho)^{\psi(2\lambda_n^\rho)} &\leq \left(2\lambda_n^{\frac{\rho}{1-\chi(c^{-j})}}\right)^{\psi\left(2\lambda_n^{\frac{\rho}{1-\chi(c^{-j})}}\right)} \end{aligned}$$

for some constant $M \geq 1$ also independent of y and j. Eventually, since $\chi(r) \to 0$ when $r \to 0$, we have $\frac{1}{1-\chi(c^{-j})} \leq 1 + 2\chi(c^{-j})$ for j large enough. As a consequence, for the same constant M we can write

$$M^{-1}(2\lambda_n^\rho)^{\alpha+2\alpha\chi(2\lambda_n^\rho)+\psi(2\lambda_n^\rho)} \leq \mu(B(x_n, \lambda_n^\rho)).$$

The upper bound of (40) is treated with the same arguments, and we obtain $\mu(B(x_n, \lambda_n^\rho)) \leq M(2\lambda_n^\rho)^{\alpha-\alpha\chi(2\lambda_n^\rho)-\psi(2\lambda_n^\rho)}$. Hence $\mathcal{P}_M^\rho(B(x_n, \lambda_n^\rho))$ holds.

To prove that $\mathcal{D}_M^m(B(y,r)$ holds for some $M > 0$ independent of y and $r \in (0, c^{-j(y)-1})$ it is enough to write that $B(y,r) \subset \bigcup_{\|\mathbf{k}-\mathbf{k}^c_{j,y}\|_\infty \leq 1} I^c_{j,\mathbf{k}}$, where j is the largest integer such that $r \leq c^{-j}$, and then to use (13). □

If y, j and (x_n, λ_n) satisfy (16), then they also satisfy (39). This ensures that the Cantor set we are going to build is included in $S_\mu(\rho, \delta, \alpha, \varepsilon_M^\rho)$.

Proof of Theorem 2.2 in the case $\rho < 1$. Here again, the case $\delta = 1$ is obvious and left to the reader. Since $\mathcal{D} = (1, \infty)$, we deal with the sets $\widehat{S}_\mu(\rho, (\delta)_{n\geq 1}, \alpha, \varepsilon_M^\rho)$, which are equal to the sets $S_\mu(\rho, \delta, \alpha, \varepsilon_M^\rho)$.

Let $\delta > 1$. As in the proof of Theorem 2.2, we construct a generalized Cantor set K_δ in $S_\mu(\rho, \delta, \alpha, \varepsilon_M^\rho)$ and a measure $m_{\rho,\delta}$ on K_δ.

- First step: The first generation in the construction of K_δ is as follows:

Let $L_0=[0,1]^d$. Using assumption $\mathbf{(2(\rho))}$, there exist a subset E^{L_0} of $E^{L_0}_{J(L_0)}$ of m-measure larger than $\|m\|/4$ and an integer $J'(L_0)\geq J(L_0)$ such that $\forall y\in E^{L_0}$, $\forall j\geq J'(L_0)$, (11) and (13) hold. There is a subset $\widetilde{E}^{L_0}$ of E^{L_0} of m-measure greater than $\|m\|/8$ such that for every $y\in \widetilde{E}^{L_0}$, (16) holds.

Once again we are going to apply Theorem 3.1 to $A=\widetilde{E}^{L_0}$ and to families $\mathcal{B}_1(j)$ of balls built as follows. Let $y\in\widetilde{E}^{L_0}$. We define

$$n_{j,y,\rho}=\inf\left\{n: c^{-n(1-\chi(c^{-n}))}\leq c^{-\frac{j+5}{\rho}} \text{ and (16) holds with } j_i(y)=n\right\}. \tag{42}$$

Then for every $j\geq J'(L_0)$, let us introduce the family

$$\mathcal{B}_1(j)=\left\{B(y,3c^{-\rho n_{j,y,\rho}}):\ y\in\widetilde{E}^{L_0}\right\}.$$

For every $j\geq J'(L_0)$, the family $\mathcal{B}_1(j)$ fulfills conditions of Theorem 3.1.

Hence, $\forall j\geq J'(L_0)$, $Q(d)$ families of disjoint balls $\mathcal{B}_1^1(j),...,\mathcal{B}_1^{Q(d)}(j)$ can be extracted from $\mathcal{B}_1(j)$. The same procedure as in Theorem 2.2 allows us to extract from these new families a finite family of disjoint balls $\widetilde{G}_1(j)=\{B_1,B_2,\dots,B_N\}$ such that

$$m\Big(\bigcup_{B_k\in\widetilde{G}_1(j)} B_k\Big)\geq\frac{\|m\|}{16\,Q(d)}. \tag{43}$$

Recall that with each B_k can be associated a point $y_k\in\widetilde{E}^{L_0}$ so that $B_k=B(y_k,3c^{-\rho n_{j,y_k,\rho}})$. Let us fix one of the balls $B_k=B(y_k,3c^{-\rho n_{j,y_k,\rho}})$. By construction, we can find $[c^{n_{j,y_k,\rho}(d(1-\rho)-\chi(c^{-n_{j,y_k,\rho}}))}]$ points x_n in the ball $B(y_k,c^{-\rho n_{j,y_k,\rho}})$ such that (16) holds. We denote $\mathcal{S}(B_k)$ the set of these points x_n. The corresponding balls $B(x_n,\lambda_n)$ are pairwise disjoint. By construction, for each of these points $x_n\in\mathcal{S}(B_k)$, we have

$$B(y_k,(c^\rho-1)c^{-\rho n_{j,y_k,\rho}})\subset B(x_n,\lambda_n^\rho)\subset B\big(y_k,c^{-\rho n_{j,y_k,\rho}(1-\chi(c^{-n_{j,y_k,\rho}}))}\big). \tag{44}$$

Therefore each point $x_n\in\mathcal{S}(B_k)$ such that (16) holds verifies the conditions of Lemma 5.1. Thus $\mathcal{P}_M^\rho(B(x_n,\lambda_n^\rho))$ and $\mathcal{D}_M^m(B_k)$ hold for some constant M independent of the scale and of x. This constant M is the one chosen to define $S_\mu(\rho,\delta,\alpha,\varepsilon_M^\rho)$.

Let us now consider $I_n^{(\delta)}=B(x_n,\lambda_n^\delta)$. Let $F_{n,k}$ be the closure of one of the c-adic box of maximal diameter included in $I_n^{(\delta)}$. Since $|B_k|=6c^{-\rho n_{j,y_k,\rho}}$, we have $|B_k|\leq C|F_{n,k}|^{\rho/\delta}$ for some constant C depending only on δ.

We write $\underline{B_k}=F_{n,k}$. Conversely, if a closed c-adic box F can be written $\underline{B}$ for some larger ball B, we write $B=\overline{F}$. Pay attention to the fact that a number equal to $\#\mathcal{S}(B_k)\geq[c^{n_{j,y_k,\rho}(d(1-\rho)-\chi(c^{-n_{j,y_k,\rho}}))}]$ of c-adic boxes $F_{n,k}$ can be written as $\underline{B_k}$ for the same ball B_k. For every c-adic box F such that there exists k with $B_k=\overline{F}$, we ensured by construction

$$|\overline{F}|\leq C|F|^{\rho/\delta} \tag{45}$$

for some constant C depending on δ. Moreover, the c-adic box F is included in a contracted ball $I_n^{(\delta)}=B(x_n,\lambda_n^\delta)$ such that $\mathcal{P}_M^\rho(B(x_n,\lambda_n^\rho))$ holds.

Since $|B_k|=6c^{-\rho n_{j,y_k,\rho}}$, there is $C>0$ independent of k and ρ such that

$$\#\mathcal{S}(B_k)\geq[c^{n_{j,y_k,\rho}(d(1-\rho)-\chi(c^{-n_{j,y_k,\rho}}))}]\geq C^{-1}|B_k|^{-\frac{d(1-\rho)}{\rho}}|B_k|^{\chi(|B_k|)}. \tag{46}$$

We eventually define

$$G_1(j) = \{F_{n,k} : \overline{F_{n,k}} \in \widetilde{G}_1(j)\}. \tag{47}$$

We notice that F_1 and F_2 belong to $G_1(j)$ and $\overline{F_1} \neq \overline{F_2}$ then the distance between F_1 and F_2 is by construction at least $\max_{i\in\{1,2\}} \overline{F_i}/3$.

On the algebra generated by the elements of $G_1(j)$, a probability measure $m_{\delta,\rho}$ is defined by

$$m_{\rho,\delta}(F) = \frac{\frac{m(\overline{F})}{\#\,\mathcal{S}(\overline{F})}}{\sum_{B_k\in\widetilde{G}_1(j)} m(B_k)}.$$

Since $\mathcal{D}^m_M(\overline{F})$ holds for the measure m, by (45) and (25), we have

$$m(\overline{F}) \leq M|\overline{F}|^{\beta-\varphi(|\overline{F}|)} \leq C|F|^{\rho\beta/\delta}|\overline{F}|^{-\varphi(|\overline{F}|)} \leq C|F|^{\rho\beta/\delta}|F|^{-\varphi(|F|)}.$$

Then, we also have by (46) and (44)

$$(\#\,\mathcal{S}(\overline{F}))^{-1} \quad\leq\quad C|\overline{F}|^{\frac{d(1-\rho)}{\rho}}|\overline{F}|^{-\chi(|\overline{F}|)} \leq C|F|^{\frac{\rho}{\delta}\frac{d(1-\rho)}{\rho}}|F|^{-\chi(|F|)} \leq C|F|^{\frac{d(1-\rho)}{\delta}}|F|^{-\chi(|F|)}.$$

Moreover, by (43) and the definition of $G_1(j)$ (29), we get

$$\sum_{B_k\in\widetilde{G}_1(j)} m(B_k) \geq \frac{\|m\|}{16\,Q(d)}.$$

Thus, $\forall\ F \in G_1(j),\ m_{\rho,\delta}(F) \leq 16\,Q(d)C\|m\|^{-1}|F|^{-\varphi(|F|)}|F|^{-\chi(|F|)}|F|^{\frac{d(1-\rho)+\rho\beta}{\delta}}$. By our assumption **(1)**, we can fix j_1 large enough so that

$$\forall\ F \in G_1(j_1),\ 16\,Q(d)C\|m\|^{-1} \leq |F|^{-\varphi(|F|)}.$$

We choose the c-adic elements of the first generation of the construction of K_δ as being those of $G_1 := G_1(j_1)$. By construction

$$\forall\ F \in G_1,\ m_{\rho,\delta}(F) \leq |F|^{\frac{d(1-\rho)+\rho\beta}{\delta}-2\varphi(|F|)-\chi(|F|)}, \tag{48}$$

and for every $x \in \bigcup_{F\in G_1} F$, there exists an integer n so that $\lambda_n \leq c^{-5/\rho}$, $\|x_n - x\|_\infty \leq \lambda_n^\delta$, and $\mathcal{P}^\rho_M(B(x_n,\lambda_n^\rho))$ holds. Moreover, $\max_{F\in G_1}|F| \leq 2c^{-5\delta/\rho}$.

- Second step: The second generation is built as in the case $\rho = 1$, by focusing on one c-adic box L of the first generation. We give the essential clues to obtain this second generation.

Using assumption $\mathbf{2(\rho)}$, there exist a subset E^L of $E^L_{J(L)}$ of m^L-measure larger than $\|m^L\|/4$ and an integer $J'(L) \geq J(L)$ such that for all $y \in E^L$, for every $j \geq J'(L) + \log_c\left(|L|^{-1}\right)$, (31) holds. Then, there exists a subset $\widetilde{E}^L$ of E^L of m^L-measure greater than $\|m^L\|/8$ such that for every $y \in \widetilde{E}^L$, (16) holds.

One more time we apply Theorem 3.1 to $A = \widetilde{E}^L$ and to families of balls $\mathcal{B}_2(j)$. Let $y \in \widetilde{E}^L$. For every $j \geq J'(L) + \log_c\left(|L|^{-1}\right)$, we define the family

$$\mathcal{B}_2(j) = \left\{B(y, 3c^{-\rho n_{j,y,\rho}}) :\ y \in \widetilde{E}^L\right\}.$$

The family $\widetilde{\mathcal{B}}_2(j)$ fulfills conditions of Theorem 3.1. Hence, $Q(d)$ families of disjoint balls $\mathcal{B}^1_2(j), ..., \mathcal{B}^{Q(d)}_2(j)$ can be extracted from $\mathcal{B}_2(j)$. Moreover, we can also extract from these families one finite family of disjoint balls $\widetilde{G}_2^L(j) = \{B_1, B_2, \dots, B_N\}$ such that

$$m^L\Big(\bigcup_{B_k\in\widetilde{G}_2(j)} B_k\Big) \geq \frac{\|m^L\|}{16\, Q(d)}. \tag{49}$$

Each of these balls B_k can be written $B(y_k, 3c^{-\rho n_{j,y_k,\rho}})$ for some point $y_k \in \widetilde{E}^L$ and some integer $n_{j,y_k,\rho}$. Moreover, by (16), with each B_k can be associated $[c^{n_{j,y_k,\rho}(d(1-\rho)-\chi(c^{-n_{j,y_k,\rho}}))}]$ points x_n in $B(y_k, c^{-\rho n_{j,y_k,\rho}})$ such that (16) holds. As above, $\mathcal{S}(B_k)$ denotes the set of these points x_n. The corresponding balls $B(x_n,\lambda_n)$ are pairwise disjoint.

By construction, (44) holds for each of these points $x_n \in \mathcal{S}(B_k)$. Moreover, Lemma 5.1 holds with the measure $m^L\circ f_L^{-1}$ instead of m and with the same constant M. Consequently, each point $x_n\in\mathcal{S}(B_k)$ such that (16) holds is such that $\mathcal{P}_M^\rho(B(x_n,\lambda_n^\rho))$ and $\mathcal{D}_M^{m^L\circ f_L^{-1}}\big(f_L(B_k)\big)$ hold.

We then consider $I_n^{(\delta)} = B(x_n,\lambda_n^\delta)$, and we denote by $F_{n,k}$ the closure of one c-adic box of maximal diameter included in $I_n^{(\delta)}$. Again we have (45).

We write $\underline{B_k} = F_{n,k}$. Conversely, if a closed c-adic box F can be written $\underline{B}$ for some larger ball B, we write $B = \overline{F}$. We eventually set

$$G_2^L(j) = \{F_{n,k} : \overline{F_{n,k}} \in \widetilde{G}_2^L(j)\}. \tag{50}$$

On the algebra generated by the elements of $G_2^L(j)$, an extension of the probability measure $m_{\rho,\delta}$ is defined by

$$m_{\rho,\delta}(F) = m_{\rho,\delta}(L)\frac{\frac{m^L(\overline{F})}{\#\mathcal{S}(\overline{F})}}{\sum_{B_k\in\widetilde{G}_2^L(j)} m^L(B_k)}.$$

Since $\mathcal{D}_M^{m^L\circ f_L^{-1}}\big(f_L(B_k)\big)$ and (45) hold, we get

$$m^L(\overline{F}) \leq \left(\frac{|\overline{F}|}{|L|}\right)^{\beta-\varphi\left(\frac{|\overline{F}|}{|L|}\right)} \leq C|F|^{\frac{\rho\beta}{\delta}}|L|^{-\beta}\left(\frac{|\overline{F}|}{|L|}\right)^{-\varphi\left(\frac{|\overline{F}|}{|L|}\right)} \leq C|F|^{\frac{\rho\beta}{\delta}}|L|^{-\beta}|F|^{-\varphi(|F|)},$$

where the monotonicity of $x\mapsto x^{-\varphi(x)}$ of assumption **(1)** is used. Then (46) applied to $\overline{F}$ and (49) yield

$$m_{\rho,\delta}(F) \quad\leq\quad m_{\rho,\delta}(L)\frac{16\,Q(d)C}{\|m^L\|}|F|^{\frac{\rho\beta}{\delta}}|L|^{-\beta}|F|^{-\varphi(|F|)}|F|^{\frac{d(1-\rho)}{\delta}}|F|^{-\chi(|F|)},$$

and using (48) finally gives

$$m_{\rho,\delta}(F) \quad\leq\quad \frac{16\,Q(d)C|L|^{\frac{d(1-\rho)+\rho\beta}{\delta}-\beta-2\varphi(|L|)-\chi(|L|)}}{\|m^L\|}|F|^{\frac{d(1-\rho)+\rho\beta}{\delta}-\varphi(|F|)-\chi(|F|)}$$

By assumption **(1)** we can choose $j_2(L)$ large enough so that for every integer $j\geq j_2(L)$, for every $I\in G_2^L(j)$,

$$16\,Q(d)C\|m^L\|^{-1}|L|^{\frac{d(1-\rho)+\rho\beta}{\delta}-\beta-2\varphi(|L|)-\chi(|L|)} \leq |F|^{-\varphi(|F|)}.$$

Then, taking $j_2 = \max\big\{j_2(L):\ L\in G_1\big\}$ and defining $G_2 = \bigcup_{L\in G_1} G_2^L(j_2)$, this yields an extension of $m_{\rho,\delta}$ to the algebra generated by the elements of $G_1\bigcup G_2$. We have for every $F\in G_1\bigcup G_2$, $m_{\rho,\delta}(F)\leq |F|^{\frac{d(1-\rho)+\rho\beta}{\delta}-2\varphi(|F|)-\chi(|F|)}$.

We remark that by construction if $J \in G_1$ and $F \in G_2$ verify $F \subset J$ we have

$$\sum_{F' \in G_2,\ \overline{F}' = \overline{F}} m_{\rho,\delta}(F') \leq 16\, Q(d) m_{\rho,\delta}(J) \frac{m^J(\overline{F})}{\|m^J\|}.$$

Also notice that by construction, $|F| \leq \max_{J \in G_1} 2(c^{-5}|J|)^{\delta/\rho} \leq (2c^{-5\delta/\rho})^2$ for every $F \in G_2$. Moreover, F is contained in some $I_n^{(\delta)}$ such that $|I_n^{(\delta)}| \leq C|F|$, where C is a constant which depends only on c.

- Third step: Assume that N generations of closed c-adic boxes $G_1, \dots, G_N$ have already been found for some integer $N \geq 2$. Assume also that a probability measure $m_{\rho,\delta}$ on the algebra generated by $\bigcup_{1 \leq p \leq N} G_p$ is defined and that:

(i) The elements of G_p are pairwise disjoint closed c-adic boxes, and for $1 \leq p \leq N$, $\max_{I \in G_p} |I| \leq \left(2c^{-5\delta/\rho}\right)^p$.

For $1 \leq p \leq N$, with each $F \in G_p$ is associated a ball $\overline{F}$ enjoying the properties:

- $F \subset \overline{F}$,
- there is a constant $C > 0$ which depends only on δ such that (45) holds,
- if $F_1 \neq F_2$ belong to G_p, their distance is at least $\max_{i \in \{1,2\}} \overline{F}_i/3$,
- the $\overline{F}$'s ($F \in G_p$) are pairwise disjoint,
- $\overline{F}$ satisfies the next parts **(ii)**, **(iii)**, **(iv)** and **(v)**.

(ii) For every $2 \leq p \leq N$, each element F of G_p is a subset of an element L of G_{p-1}. Moreover, $\overline{F} \subset L$, $\log_c\left(|\overline{F}|^{-1}\right) \geq J(L) + \log_c\left(|L|^{-1}\right)$ and $\overline{F} \cap E^L_{J(L)} \neq \emptyset$.

(iii) For every $1 \leq p \leq N$ and $F \in G_p$, there exists an integer q such that $F \subset B(x_q, \lambda_q^\delta) = I_q^{(\delta)} \subset \overline{F}$ and $\mathcal{P}^\rho_M(B(x_q, \lambda_q^\rho))$ holds, and $|I_q^{(\delta)}| \leq C|F|$ for some constant C which depends only on c.

(iv) For every $F \in \bigcup_{1 \leq p \leq N} G_p$, $m_{\rho,\delta}(F) \leq |F|^{\frac{d(1-\rho)+\rho\beta}{\delta} - 2\varphi(|F|) - \chi(|F|)}$.

(v) For every $1 \leq p \leq N-1$, $L \in G_p$, and $F \in G_{p+1}$ such that $F \subset L$,

$$\sum_{F' \in G_{p+1},\ \overline{F'} = \overline{F}} m_{\rho,\delta}(F') \leq 16\, Q(d) m_{\rho,\delta}(L) \frac{m^L(\overline{F})}{\|m^L\|}.$$

The construction of a generation G_{N+1} of c-adic boxes and an extension of $m_{\rho,\delta}$ to the algebra generated by the elements of $\bigcup_{1 \leq p \leq N+1} G_p$ such that properties **(i)** to **(v)** hold for $N+1$ are done as when $N = 1$.

Then, by induction, we get a sequence $(G_N)_{N \geq 1}$ and a probability measure on $\sigma\big(F : F \in \bigcup_{N \geq 1} G_N\big)$ such that properties **(i)** to **(v)** hold for every $N \geq 1$, and $K_{\rho,\delta} = \bigcap_{N \geq 1} \bigcup_{I \in G_N} F$. By construction, $m_{\rho,\delta}(K_{\rho,\delta}) = 1$ and because of **(iii)** $K_{\rho,\delta} \subset S_\mu(\rho, \delta, \alpha, \varepsilon^\rho_M)$. Finally, the measure $m_{\rho,\delta}$ is extended to $\mathcal{B}([0,1]^d)$ in the usual way: $m_{\rho,\delta}(B) := m_{\rho,\delta}(B \cap K_{\rho,\delta})$ for every $B \in \mathcal{B}([0,1]^d)$.

- Last step: Proof of (20). If $F \in G_N$, recall that we set $g(F) = N$.

Fix B an open ball of $[0,1]$ of diameter less than the one of the elements of G_1 such that $B \cap K_{\rho,\delta} \neq \emptyset$. Let L be the element of largest diameter in $\bigcup_{N \geq 1} G_N$ such that B intersects at least two balls $\overline{L}_i$ such that L_i belongs to $G_{g(L)+1}$ and L_i is included in L (hence $m_{\rho,\delta}(B) \leq m_{\rho,\delta}(L)$).

• When $|B| \geq |L|$:

$$m_{\rho,\delta}(B) \leq m_{\rho,\delta}(L) \leq |L|^{\frac{d(1-\rho)+\rho\beta}{\delta}-2\varphi(|L|)-\chi(|L|)} \leq C|B|^{\frac{d(1-\rho)+\rho\beta}{\delta}-2\varphi(|B|)-\chi(|B|)}.$$

• When $|B| < c^{-J(L)-3}|L|$: let $L_1, \ldots, L_p$ be the c-adic boxes in $G_{g(L)+1}$ such that $\forall i$ $\overline{L}_i$ intersects B. Property **(v)** yields

$$m_{\rho,\delta}(B) = \sum_{i=1}^{p} \sum_{L\in G_{g(L)+1},\ \overline{L}=\overline{L}_i} m_{\rho,\delta}(B\cap L) \leq \sum_{i=1}^{p} m_{\rho,\delta}(L) \frac{16\,Q(d)}{\|m^L\|} m^L(\overline{L}_i).$$

Let j_0 be the unique integer so that $c^{-j_0} \leq |B| < c^{-j_0+1}$. Because of **(i)**, we have $|B| \geq \max_i |\overline{L}_i|/3$. As a consequence $-\log_c |\overline{L}_i| \geq j_0 - [\log_c 3] \geq j_0 - 2$.

The same arguments as in the proof of Theorem 2.2 (Case $\rho = 1$) yield that there exists an index i_0 and a point $y \in E^L_{J(L)} \cap \overline{L}_{i_0}$ such that $\bigcup_{i=1}^{p} \overline{L}_i$ is included in $\bigcup_{\mathbf{k}:\ \|\mathbf{k}-\mathbf{k}_{j_0-3,y}\|_\infty \leq 1} I^c_{j_0-3,\mathbf{k}}$. Hence

$$\sum_{i=1}^{p} m^L(\overline{L}_i) \leq \sum_{\mathbf{k}:\ \|\mathbf{k}-\mathbf{k}_{j_0-3,y}\|_\infty \leq 1} m^L(I^c_{j_0-3,\mathbf{k}}), \tag{51}$$

and by definition of $E^L_{J(L)}$, we can bound $m^L(I^c_{j_0-3,\mathbf{k}})$ by

$$m^L(I^c_{j_0-3,\mathbf{k}}) \leq \left(\frac{|I^c_{j_0-3,\mathbf{k}}|}{|L|}\right)^{\beta-\varphi\left(\frac{|I^c_{j_0-3,\mathbf{k}}|}{|L|}\right)} \leq C\left(\frac{|B|}{|L|}\right)^{\beta}\left(\frac{|B|}{|L|}\right)^{-\varphi\left(\frac{|B|}{|L|}\right)}.$$

There are 3^d such pairwise disjoint boxes in the sum (51), hence

$$\begin{aligned} m_{\rho,\delta}(B) &\leq \frac{16\,Q(d)}{\|m^L\|} m_{\rho,\delta}(L) 3^d C \left(\frac{|B|}{|L|}\right)^{\beta}\left(\frac{|B|}{|L|}\right)^{-\varphi\left(\frac{|B|}{|L|}\right)} \\ &\leq \frac{16\,Q(d)3^d C}{\|m^L\|} m_{\rho,\delta}(L)\left(\frac{|B|}{|L|}\right)^{\beta} |B|^{-\varphi(|B|)}. \end{aligned}$$

By **(iv)**, we obtain

$$m_{\rho,\delta}(L) \leq |L|^{\frac{d(1-\rho)+\rho\beta}{\delta}} |L|^{-2\varphi(|L|)-\chi(|L|)} \leq |L|^{\frac{d(1-\rho)+\rho\beta}{\delta}} |B|^{-2\varphi(|B|)-\chi(|B|)},$$

which yields

$$m_{\rho,\delta}(B) \leq \frac{16\,Q(d)3^d C}{\|m^L\|} |L|^{\frac{d(1-\rho)+\rho\beta}{\delta}} \left(\frac{|B|}{|L|}\right)^{\beta} |B|^{-3\varphi(|B|)-\chi(|B|)}.$$

Then, the second property of (15) in assumption **(4)** allows to upper bound $\|m^L\|^{-1}$ by $|L|^{-\varphi(|L|)}$, which is lower than $|B|^{-\varphi(|B|)}$, and thus

$$m_{\rho,\delta}(B) \leq C|L|^{\frac{d(1-\rho)+\rho\beta}{\delta}} \left(\frac{|B|}{|L|}\right)^{\beta} |B|^{-4\varphi(|B|)-\chi(|B|)}. \tag{52}$$

Finally, if $\beta > \frac{d(1-\rho)+\rho\beta}{\delta}$, (52) yields

$$\begin{aligned} m_{\rho,\delta}(B) &\leq C|B|^{\frac{d(1-\rho)+\rho\beta}{\delta}} \left(\frac{|B|}{|L|}\right)^{\beta-\frac{d(1-\rho)+\rho\beta}{\delta}} |B|^{-4\varphi(|B|)-\chi(|B|)} \\ &\leq C|B|^{\frac{d(1-\rho)+\rho\beta}{\delta}} |B|^{-4\varphi(|B|)-\chi(|B|)}; \end{aligned}$$

If $\beta \leq \frac{d(1-\rho)+\rho\beta}{\delta}$, (52) yields

$$m_{\rho,\delta}(B) \quad \leq \quad C|B|^{\beta}|L|^{\frac{d(1-\rho)+\rho\beta}{\delta}-\beta}|B|^{-4\varphi(|B|)-\chi(|B|)} \leq C|B|^{\beta}|B|^{-4\varphi(|B|)-\chi(|B|)}.$$

In both cases, if $D(\beta,\rho,\delta)=\min(\beta,\frac{1-\rho+\rho\beta}{\delta})$,

$$m_{\rho,\delta}(B) \leq C|B|^{D(\beta,\rho,\delta)}|B|^{-4\varphi(|B|)-\chi(|B|)}. \tag{53}$$

• $c^{-J(L)-3}|L| \leq |B| \leq |L|$: we need at most $c^{d(J(L)+4)}$ contiguous c-adic boxes of diameter $c^{-J(L)-3}|L|$ to cover B. For these boxes, (53) can be used to get

$$\begin{aligned} m_{\rho,\delta}(B) &\leq Cc^{d(J(L)+4)}\big(c^{-J(L)-3}|L|\big)^{D(\beta,\rho,\delta)-4\varphi(c^{-J(L)-3}|L|)-\chi(c^{-J(L)-3}|L|)} \\ &\leq Cc^{dJ(L)}|B|^{D(\beta,\rho,\delta)}|B|^{-4\varphi(|B|)-\chi(|B|)} \\ &\leq C|L|^{-d\varphi(|L|)}|B|^{D(\beta,\rho,\delta)}|B|^{-4\varphi(|B|)-\chi(|B|)} \\ &\leq C|B|^{D(\beta,\rho,\delta)}|B|^{-(4+d)\varphi(|B|)-\chi(|B|)}. \end{aligned}$$

This shows (20) and ends the proof of Theorem 2.2 when $\rho<1$. □

6. Examples

Section 6.1 exhibits several families $\{(x_n,\lambda_n)\}_n$ which satisfy (10) or (16) for any measure m, and form weakly redundant systems. Then Section 6.2 provides examples of triplets $\big(\mu,\alpha,\tau_\mu^*(\alpha)\big)$ leading to ρ-heterogeneous ubiquitous systems. It also gives relevant interpretations to property $\mathcal{P}_M^\rho$.

6.1. **Examples of families** $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$**.** Let us notice first that, to ensure (10), it suffices that

$$\bigcap_{N\geq 1}\bigcup_{n\geq N} B\big(x_n,\lambda_n/2\big)=[0,1]^d. \tag{54}$$

• The family of the b-adic numbers.

Fix b an integer ≥ 2. Let us consider the sequence $\{(\mathbf{k}b^{-j},2b^{-j})\}$, for $j\in\mathbb{N}$ and $\mathbf{k}=(k_1,k_2,\dots,k_d)\in\{0,\dots,b^j-1\}^d$. By construction, for every $j\geq 2$, $\bigcup_{\mathbf{k}\in\{0,\dots,b^j-1\}^d} B\big(\mathbf{k}b^{-j},b^{-j}\big)=[0,1]^d$. Hence (54) is satisfied, (16) holds for any measure m and the family is weakly redundant.

• The family of the rational numbers.

By Theorem 200 of [30], any point $x=(x_1,\dots,x_d)\in[0,1]^d$ such that at least one of the x_i is an irrational number satisfies for infinitely many $\mathbf{p}=(p_1,p_2,\dots,p_d)$ and q the inequality $\|x-\mathbf{p}/q\|_\infty \leq q^{-(1+1/d)}$. As a consequence, the sequence $\big\{\big(\mathbf{p}/q,2q^{-(1+1/d)}\big)\big\}$ for $q\in\mathbb{N}^*$ and $\mathbf{p}=(p_1,p_2,\dots,p_d)\in\{0,\dots,q-1\}^d$ fulfills (54). Here again, (16) holds for any measure m.

To ensure the weak redundancy, we must select only the rational numbers $\big\{\big(\mathbf{p}/q,2q^{-(1+1/d)}\big)\big\}$ such that at least one fraction p_i/q is irreducible. But (54) is no more satisfied. Indeed, the rational numbers $\mathbf{p}/q$ themselves do not belong to the corresponding limsup-set (each rational number belongs only to a finite number of balls $B\big(\mathbf{p}/q,2q^{-(1+1/d)}\big)$. Nevertheless, as soon as the rational points are not atoms of m (for instance if $\underline{\dim}(m)>0$), both (10) and (16) hold. In this case, by Theorem 193 of [30], the same holds with $\big\{\big(p/q,2/\sqrt{5}q^2\big)\big\}$ when $d=1$. This family is used to prove (2).

• The family $\big\{(\{n\alpha\},1/n)\big\}_{n\in\mathbb{N}}$.

Let us focus on the case $d = 1$ to introduce another family. Let α be an irrational number. For every $n \in \mathbb{N}$, we denote by $\{n\alpha\}$ the fractional part of $n\alpha$. If $x \notin \mathbb{Z}+\alpha\mathbb{Z}$, we have $|n\alpha - x| < 1/2n$ for an infinite number of integers n (see Theorem II.B in [20] for instance). Hence

$$\mathbb{R}\setminus(\mathbb{Z}+\alpha\mathbb{Z}) \subset \bigcap_{N\geq 1}\bigcup_{n\geq N} B(\{n\alpha\}, 1/2n).$$

As soon as $m(\mathbb{Z}+\alpha\mathbb{Z}) = 0$, (10) is satisfied for the family $\{(\{n\alpha\}, 1/n)\}_{n\geq 1}$. We do not know the measures m for which (16) holds. However the following property concerning the redundancy holds:

Proposition 6.1. $\{(\{n\alpha\}, 1/n)\}_{n\geq 1}$ *forms a weakly redundant system if and only if* $\inf\left\{\xi : \ \#\left\{(p,q) \in \mathbb{N}\times\mathbb{N}^* : \ |\alpha - p/q| \leq q^{-\xi}\right\} = \infty\right\} = 2$.

We know that every irrational number is approximated at rate $\xi \geq 2$ by the rational numbers. But the system $\{(\{n\alpha\}, 1/n)\}_n$ is weakly redundant if and only if the approximation rate by rational numbers of α is exactly equals 2.

Proof. Notations of Definition 2.1 are used.

We remark that T_j (defined by (6)) contains exactly 2^j integers.

Suppose that the family is not weakly redundant. For every partition of T_j into N_j subsets, we have $\limsup_{j\to+\infty} j^{-1}\log N_j > 0$. Let us fix such a partition. There exists $\varepsilon > 0$ such that for infinitely many integers j, we can find a real number $x \in [0,1]$ such that more than $2^{\varepsilon j}$ among the $\{B(x_n, \lambda_n)\}_{n\in T_j}$ contain x. Since these integers n belong to T_j, the corresponding λ_n belong to $(2^{-(j+1)}, 2^{-j}]$. Consequently, these $2^{\varepsilon j}$ integers n all verify $|\{n\alpha\} - x| \leq 2^{-j}$.

By a classical argument, there are two integers n and n' of T_j such that

$$n \neq n', \ |n-n'| \leq 2^j \text{ and } |\{n\alpha\} - \{n'\alpha\}| \leq 2\cdot 2^{-j(1+\varepsilon)}. \tag{55}$$

We deduce from (55) that there exists $p \in \mathbb{N}$ such that $\big||n-n'|\,\alpha - p\big| \leq 2\cdot 2^{-j(1+\varepsilon)} \leq 2|n-n'|^{-(1+\varepsilon)}$. Hence $\big|\alpha - p/|n-n'|\big| \leq 2|n-n'|^{-(2+\varepsilon)}$. Since (55) holds for infinitely many j, $|n-n'|$ cannot be bounded as j goes to ∞. This yields $\xi_\alpha := \inf\left\{\xi : \ \#\left\{(p,q) \in \mathbb{N}\times\mathbb{N}^* : \ \big|\alpha - p/q\big| \leq q^{-\xi}\right\} = \infty\right\} > 2$.

Conversely, if $\xi_\alpha > 2$, fix $\varepsilon \in (0, \xi_\alpha - 2)$. For infinitely many $(p,q) \in \mathbb{N}\times\mathbb{N}^*$, we have $|\alpha - p/q| \leq q^{-(2+\varepsilon)}$. For such an integer q, we have $\{nq\alpha\} \leq 1/qn$ for every $n \in \left[1, q^{\varepsilon/2}\right]$. For q large enough, let j_q be the largest integer j so that $[j, j+1] \subset [\log_2(q), (1+\varepsilon/2)\log_2(q)]$. Consider then T_{j_q}. By construction, the point 0 belongs to at least $2^{\frac{\varepsilon}{4}j_q}$ balls $B(x_n, \lambda_n)$ such that $n \in T_{j_q}$. Hence $N_{j_q} \geq 2^{j_q\varepsilon/4}$. Since this holds for infinitely many j's, the conclusion follows. □

• Poisson point processes.

Let S be a Poisson point process with intensity $\lambda\otimes\nu$ in the square $[0,1]\times(0,1]$, where λ denotes the Lebesgue measure on $[0,1]$ and ν is a positive locally finite Borel measure on $(0,1]$ (see [38] for the construction of a Poisson process). Let us take the family $\{(x_n, \lambda_n)\}_n$ equal to the set S. Let c be an integer ≥ 2. Then for $j \geq 1$, let us introduce the quantities $T_j^c = \{n : \ c^{-(j+1)} < \lambda_n \leq c^{-j}\}$, as well as

$$\beta_j = j^{-1}\log_c \nu((c^{-(j-1)}, c^{-(j-2)}]) \ \text{ and } \ \beta = \limsup_{j\to\infty} \beta_j.$$

We have $\beta = \limsup_{j\to\infty} j^{-1}\log_b \mathbb{E}(\# T_{j-2})$ for $b \in \{2, c\}$, but we use a basis c rather than 2 in order to discuss property (16). In fact, it is a general property that the number $\limsup_{j\to\infty} j^{-1}\log_c \# T_j^c$ itself does not depend on c. We group the information concerning (10), (16) and weak redundancy:

Proposition 6.2. (1) *Suppose* $\int_{[0,1]} \exp\Big(2\int_{[t,1]} \nu((2y,1))\,dy\Big)dt = +\infty$. *This implies in particular* $\beta \geq 1$. *With probability 1, (54) holds.*

(2) *Fix* $\rho \in (0,1)$. *Let* χ *be a function defined as in Definition 2.3. If there exists an increasing sequence* $(j_n)_{n\geq 1}$ *such that* $\beta_{j_n} \geq 1 - \chi(c^{-j_n}) + 4/j_n$, *then with probability 1, (16) holds for any measure* m.

(3) $\{(x_n, \lambda_n)\}_n$ *is weakly redundant almost surely if and only if* $\beta \leq 1$.

As a consequence, if $\nu(d\lambda) = \gamma d\lambda/\lambda^2$ with $\gamma > 1/2$, with probability 1, the system S is weakly redundant and (54) holds. In addition, if γ is large enough, with probability 1, (16) holds for any measure m.

Proof. (i) It is a consequence of Shepp's theorem (see [46] and [16]).

(ii) We shall need the following lemma.

Lemma 6.3. *Let* $\gamma \in \big(1,2,1\big)$. *Let* N *be a Poisson random variable with parameter* M. *For all* $p \geq 1$, *we have* $\mathbb{P}(N \leq M - M^\gamma) = O(M^{-p})\ \ (M \to \infty)$.

The proof of Lemma 6.3 uses the identity $\sum_{k=0}^n \exp(-M)\frac{M^k}{k!} = \int_M^\infty \frac{u^n}{n!}e^{-u}\,du$ $(M > 0,\ n \in \mathbb{N})$ as well as Laplace's method for equivalents of integrals.

For $j \geq 1$ and $0 \leq k \leq c^{[j\rho]} - 1$, let $\widehat{I}^c_{[j\rho],k}$ be the subset of $I^c_{[j\rho],k}$ obtained by keeping one over c of the consecutive c-adic subintervals of $I_{[j\rho],k}$ of generation $j-2$, that is $\widehat{I}^c_{[j\rho],k} = \bigcup_{k'=0,\ldots,c^{j-[j\rho]-3}-1} I^c_{j-2,c^{j-2-[j\rho]}k+ck'}$. Let us also define the random sets $S_{j,k} = \big\{n:\ \lambda_n \in (c^{-(j-1)}, c^{-(j-2)}],\ x_n \in \widehat{I}^c_{[j\rho],k}\big\}$, and the random variables $N_{j,k} = \# S_{j,k}$. The $N_{j,k}$'s are mutually independent Poisson random variables with parameter M_j equal to the product of $\nu\big((c^{-(j-1)}, c^{-(j-2)}]\big)$ with $\big|\widehat{I}^c_{[j\rho],k}\big|$, that is $M_j = c^{j\beta_j}\cdot c^{-[j\rho]-1}$.

Fix $\gamma \in (1/2, 1)$ and let $E_j = \big\{\forall\ 0 \leq k \leq c^{[j\rho]} - 1,\ N_{j,k} \geq M_j - M_j^\gamma\big\}$ for $j \geq 1$. We have $\mathbb{P}(E_j) = \big(\mathbb{P}(N_{j,0} \geq M_j - M_j^\gamma)\big)^{c^{[j\rho]}}$. Moreover, by definition of j_n, we have $\lim_{n\to\infty} M_{j_n} = \infty$. Consequently, using the form of M_j and Lemma 6.3, we have $\lim_{n\to\infty}\mathbb{P}(E_{j_n}) = 1$. Since the events E_{j_n} are independent, by the Borel-Cantelli lemma we have $\mathbb{P}(\limsup_{n\to\infty} E_{j_n}) = 1$.

A computation shows that $M_{j_n} - M_{j_n}^\gamma \geq c^{(\beta_{j_n}-\rho)j_n-4}$ for n large enough. It follows that with probability 1, there exist infinitely many j_n such that for all $0 \leq k \leq c^{[j_n\rho]} - 1$, $N_{j_n,k} \geq c^{j_n(1-\rho-\chi(c^{-j_n}))}$. Moreover, by construction, the balls $B(x_n, \lambda_n)$ for $n \in S_{j,k}$ are pairwise disjoint, and if $y \in [0,1]$, $B(y, c^{-j_n\rho})$ contains at least one of the $\widehat{I}_{[j_n\rho],k}$'s. The conclusion follows.

(iii) If $\beta \leq 1$, the fact that $\{(x_n, \lambda_n)\}_n$ forms almost surely a weakly redundant system is a consequence of the estimates obtained in the proofs of Lemma 5 and 8 of [32] for the numbers $\widetilde{N}_{j,k} = \#\{n \in T_j:\ x_n \in [k2^{-j}, (k+1)2^{-j}]\}$.

If $\beta > 1$, computations patterned after those performed in proving (ii) show that if $\varepsilon \in (0, \beta - 1)$, with probability 1, there are infinitely many integers j such that for all $k \in \{0, \ldots, c^j - 1\}$, $\#\{n \in T_j:\ x_n \in I^c_{j,k}\} \geq c^{j\varepsilon}$. $\square$

• Random family based on uniformly distributed points.

Let $\{x_n\}_n$ be a sequence of points independently and uniformly distributed in $[0,1]^d$ and $\{\lambda_n\}_n$ a non-increasing sequence of positive numbers.

We do not know conditions ensuring that (16) holds for some non-trivial measure m. The following Proposition concerns (10) and weak redundancy.

Proposition 6.4. *Let $\beta = \limsup_{j\to\infty} j^{-1}\log_2 \#T_j$.*

1. *Suppose that $\limsup_{n\to+\infty} \left(\sum_{p=1}^n \lambda_p/2\right) - d\log n = +\infty$. This implies $\beta \geq 1$. With probability 1 (54) holds.*
2. *Suppose that $\beta \leq 1$. With probability 1, $\{(x_n,\lambda_n)\}_n$ is weakly redundant.*

As a consequence, if $\lambda_n = \gamma/n$ for some $\gamma > 2d$ then, with probability 1, $\{(x_n,\lambda_n)\}_n$ is weakly redundant and (54) holds.

Proof. (i) It is Proposition 9 of [35].

(ii) The estimates of [32] invoked in the proof of Proposition 6.2(iii) also concern $\widehat{N}_{j,k} = \#\{n \in T_j : \; x_n \in [k2^{-j},(k+1)2^{-j}]\}$ for the example we are dealing with (i.e. (x_n) is a sequence of i.i.d. uniform variables) when $d=1$. In particular, when $d=1$, a sufficient condition for the system to be weakly redundant is $\beta \leq 1$. Since a random variable with uniform distribution in $[0,1]^d$ is a random vector in $\mathbb{R}^d$ which components are independent uniform random variables in $[0,1]$, the same property holds in dimension d if $\beta \leq 1$. □

6.2. **Examples of measures μ and m, Interpretations of the property $\mathcal{P}_M^\rho$.** We give interpretations only for $\mathcal{P}_M^1$, since $\mathcal{P}_M^\rho$ contains similar information.

Given the measure μ and the exponent $\alpha>0$, there is typically an uncountable family of values of $\beta>0$ such that properties (11), (13), **(3)** and **(4)** of Definition 2.2 hold for many systems $\{(x_n,\lambda_n)\}_n$. Consequently, we seek for the largest value of β. It follows from the study of the multifractal nature of statistically self-similar (including the deterministic) measures we deal with that, in general, this optimal value is given by $\beta = \tau_\mu^*(\alpha)$ (see formulas (7) and (8)).

We select four classes of measures to which Theorem 2.2 is applicable. Other examples can be found in [28, 7, 2, 8, 14]. We keep in mind part *3.* of Remark 2.1.

For the rest of this section the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ are fixed, and we assume that $(0,1)^d \subset \limsup_{n\to\infty} B(x_n,\lambda_n/2)$.

For $C,\kappa,r>0$ and $\gamma>1/2$, let $\varphi_C(r) = C|\log(r)|^{-1/2}\big(\log\log|\log(r)|\big)^{1/2}$, $\widetilde{\varphi}_\kappa(r) = \big(\log|\log(r)|\big)^{-\kappa}$, and $\psi_\gamma(r) = C|\log(r)|^{-1/2}\big(\log|\log(r)|\big)^{\gamma}$.

• Product of d multinomial measures and frequencies of digits

Let $(\pi_0^{(i)},\dots,\pi_{c-1}^{(i)})$, $1\leq i\leq d$, be d probability vectors with positive components such that $\sum_{l=0}^{c-1}\pi_j^{(i)} = 1$, $\forall\, 1\leq i\leq d$. For $1\leq i\leq d$ let $\mu^{(i)}$ be the multinomial measure on $[0,1]$ associated with $(\pi_0^{(i)},\dots,\pi_{c-1}^{(i)})$, and $\mu = \mu^{(1)}\otimes\cdots\otimes\mu^{(d)}$ the product measure of the $\mu^{(i)}$ on $[0,1]^d$. We have $\tau_{\mu^{(i)}}(q) = -\log_c\sum_{k=0}^{c-1}(\pi_k^{(i)})^q$ and $\tau_\mu(q) = \sum_{i=1}^d \tau_{\mu^{(i)}}(q)$. It is convenient to take $\alpha = \tau_\mu'(q)$ for some given $q\in\mathbb{R}$. Let us then define $\beta = \tau_\mu^*(\alpha) = q\tau_\mu'(q) - \tau_\mu(q)$, and $\mu_q = \mu_q^{(1)}\otimes\cdots\otimes\mu_q^{(d)}$, where $\mu_q^{(i)}$ is the multinomial measure associated with the vector $\big(c^{\tau_{\mu^{(i)}}(q)}(\pi_0^{(i)})^q,\dots,c^{\tau_{\mu^{(i)}}(q)}(\pi_{c-1}^{(i)})^q\big)$.

It is proved in [13] that each measure $\mu^{(i)}$ satisfies properties (11), (13), **(3)** and **(4')** with the exponents $\alpha_i = \tau'_{\mu^{(i)}}(q)$ and $\beta_i = q\tau'_{\mu^{(i)}}(q) - \tau_{\mu^{(i)}}(q)$, and with m equal to $\mu_q^{(i)}$. This requires some work, because the masses of the c-adic boxes and of their immediate neighbors need to be controlled. We can choose $m^I \circ f_I^{-1} = m = \mu_q^{(i)}$, and **(3)** and **(4')** do not matter. Moreover, (φ, ψ) is of the form (φ_C, ψ_γ).

Now, in terms of conditioned ubiquity, it is interesting to recall the well-known interpretation of the conditions (11) and (13), which hold for each $\mu^{(i)}$, in terms of c-adic expansions (recall Section 1 and the definition (1) of $\phi_{k,j}$): For $\mu^{(i)}$-almost every point $x_i \in [0,1]$, for every $0 \le k \le c-1$, for all $y \in I_{j,k_{x_i}-1} \cup I_{j,k_{x_i}} \cup I_{j,k_{x_i}+1}$, $\lim_{j\to\infty} \phi_{k,j}(y) = c^{\tau_{\mu^{(i)}}(q)}(\pi_k^{(i)})^q$.

The previous remarks yield the following result, which implies (2).

Proposition 6.5. *Let $q \in \mathbb{R}$. The measure μ satisfies properties (11), (13),* **(3)** *and* **(4')** *with $\alpha = \tau'_\mu(q)$, $\beta = \tau^*_\mu(\alpha)$, (φ, ψ) of the form (φ_C, ψ_γ), and $m^I \circ f_I^{-1} = m = \mu_q$ for all $I \in \mathbf{I}$.*

Moreover, there exists a sequence $\varepsilon_n \searrow 0$ such that, when applying Theorem 2.2, property $\mathcal{Q}(x_n, \lambda_n, 1, \alpha, \varepsilon^1_{M,n})$ in (19) can be replaced by the following condition in terms of c-adic expansion: for every $1 \le i \le d$, for every $0 \le k \le c-1$, $\left|\phi_{k,[\log_c(\lambda_n^{-1})]}(x_{n,i}) - c^{\tau_{\mu^{(i)}}(q)}(\pi_k^{(i)})^q\right| \le \varepsilon_n$, where $x_n = (x_{n,1}, \ldots, x_{n,d})$.

- Gibbs measures and average of Birkhoff sums

Let ϕ be a $(1,\ldots,1)$-periodic Hölder continuous function on $\mathbb{R}^d$. Let T be the transformation of $[0,1)^d$ defined by $T\big((x_1,\ldots,x_d)\big) = (cx_1 \mod 1, \ldots, cx_d \mod 1)$. For $k \in \mathbb{N}$, let T^k denote the k^{th} iteration of T ($T^0 = \mathrm{Id}_{[0,1)^d}$). For every $x \in [0,1)^d$ and $n \ge 1$, let us also define the n^{th} Birkhoff sum of x, $S_n(\phi)(x) = \sum_{k=0}^{n-1} \phi\big(T^k(x)\big)$ as well as $D_n(\phi)(x) = \exp\big(S_n(\phi)(x)\big)$.

The Ruelle Perron-Frobenius theorem (see [44]) ensures that the probability measures μ_n given on $[0,1]^d$ by $\mu_n(dx) = D_n(\phi)(x)\,dx/\int_{[0,1)^d} D_n(\phi)(u)\,du$ converges weakly to a probability measure μ which is a Gibbs state with respect to the potential ϕ and the dynamical system $([0,1)^d, T)$. The multifractal analysis of μ is performed in [28, 29] for instance. With ϕ is also associated the analytic function $L:\ q \in \mathbb{R} \mapsto d\log(c) + \lim_{n\to\infty} j^{-1} \log \int_{[0,1)^d} D_n(q\phi)(u)\,du$, which is the topological pressure of $q\phi$. We have $\tau_\mu(q) = \frac{qL(1)-L(q)}{\log(c)}$. For $q \in \mathbb{R}$, let μ_q be the Gibbs measure defined as μ, but with the potential $q\phi$.

Then, the structure of μ combined with the Hölder regularity of ϕ and the law of the iterated logarithm (see Chapter 7 of [45]) yield

Proposition 6.6. *Let $q \in \mathbb{R}$. The measure μ satisfies properties (11), (13),* **(3)** *and* **(4')** *with $\alpha = \tau'_\mu(q)$, $\beta = \tau^*_\mu(\alpha)$, both φ and ψ of the form φ_C, and $m^I \circ f_I^{-1} = m = \mu_q$ for all $I \in \mathbf{I}$.*

There exists $C > 0$ such that, applying Theorem 2.2, in (19) the property $\mathcal{Q}(x_n, \lambda_n, 1, \alpha, \varepsilon^1_{M,n})$ can be replaced in terms of average of Birkhoff sums by: $\left|L'(q) - A_{[|\log_c(\lambda_n)|]}(x_n)\right| \le \varphi_C(\lambda_n)$, where $A_p(x) = S_p(\phi)(x)/p$.

- Independent multiplicative cascades, average of branching random walks

For these random measures, the situation is subtle. Indeed, the study achieved in [14] concludes that property **(4)** can be satisfied for some systems $\{(x_n,\lambda_n)\}_{n\geq 1}$, while the strong property **(4')** fails because of the unavoidable large values of $J(L)$ for some c-adic boxes L.

Let us recall that these measures μ are constructed as follows. Let X be a real valued random variable. Let us define $L:\ q\in\mathbb{R}\mapsto d\log(c)+\log\mathbb{E}(e^{qX})$, and assume that $L(1)<\infty$. For every c-adic box J included in $[0,1]^d$, let X_J be a copy of X. Moreover, assume that the X_J's are mutually independent. The branching random walk is then

$$\forall\, x\in[0,1)^d,\ \forall\, n\geq 1,\ S_n(x)=\sum_{J\in\mathbf{I},\ c^{-n}\leq|J|\leq c^{-1},\ x\in J} X_J. \tag{56}$$

The measure μ is obtained as the almost sure weak limit of the sequence μ_n on $[0,1]^d$ given by $\mu_n(dx)=\big(\mathbb{E}(e^X)\big)^{-n}e^{S_n(x)}\,dx$.

Let $\theta:\ q\in\mathbb{R}\mapsto\frac{qL(1)-L(q)}{\log(c)}$. In [39, 37], it is shown that $\theta'(1^-)>0$ is a necessary and sufficient condition for μ to be almost surely a positive measure with support equal to $[0,1]^d$. The multifractal nature of μ or of variants of μ has been investigated in many works [36, 31, 25, 42, 1, 41, 4]. We need to consider the interior $\mathcal{J}$ of the interval $\{q\in\mathbb{R}:\ \theta'(q)q-\theta(q)>0\}$.

For every $q\in\mathcal{J}$ and every c-adic box I in $[0,1)^d$, let us introduce the sequences of measures $\mu_{q,n}$ and $m^I_{q,n}$ defined as follows: $\mu_{q,n}$ is defined as μ_n but using $X_J(q):=qX_J$ instead of X_J in (56), and $m^I_{q,n}$ is defined as $\mu_{q,n}$ but with $qX_{f_I^{-1}(J)}$ instead of $X_J(q)$ in (56).

It is shown in [4] that, with probability 1, $\forall\, q\in\mathcal{J}$, the measures $\mu_{q,n}$ converge weakly to a positive measure μ_q on $[0,1]^d$; In addition, $\forall\, q\in\mathcal{J}$, for every c-adic box I of generation ≥ 1, the sequence of measures $m^J_{q,n}$ converges weakly to a measure m^I_q on $[0,1]^d$, and $\tau_\mu(q)=\theta(q)$ on $\mathcal{J}$.

The following result is a consequence of Theorem 4.1 in [14].

Proposition 6.7. *Suppose that* $\limsup_{n\to\infty}B(x_n,\lambda_n/4)\supset(0,1)^d$.

For every $q\in\mathcal{J}$, with probability 1 (and also with probability 1, for almost every $q\in\mathcal{J}$), μ satisfies properties (11), (13), **(3)** *and* **(4)** *with the exponents $\alpha=\tau'_\mu(q)$ and $\beta=\tau^*_\mu(\alpha)$, (φ,ψ) of the form $(\widetilde{\varphi}_\kappa,\psi_\gamma)$, $m=\mu_q$, $m^I\circ f_I^{-1}=m^I_q$ for all $I\in\mathbf{I}$, and $\mathcal{D}=\mathbb{Q}\cap(1,\infty)$.*

There exists $\gamma>1/2$ such that, applying Theorem 2.2, in (19) the property $\mathcal{Q}(x_n,\lambda_n,1,\alpha,\varepsilon^1_{M,n})$ can be replaced in terms of average of branching random walks by: $\big|L'(q)-A_{[|\log_c(\lambda_n)|]}(x_n)\big|\leq\psi_\gamma(2\lambda_n)$, where $A_p(x)=S_p(x)/p$.

- Poisson cascades and average of covering numbers in the case $d=1$.

Let $\xi>0$ and S a Poisson point process in $\mathbb{R}\times(0,1)$ with intensity Λ given by $\Lambda(ds\,d\lambda)=\xi ds d\lambda/2\lambda^2$. For every c-adic box I of $[0,1]$, define $S_I=\big\{(f_I^{-1}(t),|I|^{-1}\lambda):\ (t,\lambda)\in S,\ \lambda<|I|\big\}$. The point process S_I is a copy of S.

For every $t\in[0,1]$ and $\varepsilon\in(0,1]$, the covering number of t at height ε by the Poisson intervals $\{(s-\lambda,s+\lambda):(s,\lambda)\in S\}$ is defined by

$$N^S_\varepsilon(t)=\sum_{(t,\lambda)\in S,\ \lambda\geq\varepsilon}\mathbf{1}_{\{(s-\lambda,s+\lambda)\}}(t)=\#\big\{(s,\lambda)\in S:\ \lambda\geq\varepsilon,\ t\in(s-\lambda,s+\lambda)\big\}.$$

The measure μ on $[0,1]$ is the almost sure weak limit, as $\varepsilon \to 0$, of

$$\mu_\varepsilon(dt) = \big(\mathbb{E}(e^{N_\varepsilon^S(t)})\big)^{-1} e^{N_\varepsilon^S(t)}\, dt = \varepsilon^{\xi(e-1)} e^{N_\varepsilon^S(t)}\, dt. \tag{57}$$

Let $L : q \in \mathbb{R} \mapsto \xi^{-1} + e^q - 1$, and let $\theta :\ q \in \mathbb{R} \mapsto \xi\big(qL(1) - L(q)\big)$.

In [7], it is shown that $\theta'(1^-) > 0$ is a necessary and sufficient condition for μ to be almost surely a positive measure supported by $[0,1)^d$. Let $\mathcal{J} = \{q \in \mathbb{R} : \theta'(q)q - \theta(q) > 0$. It is also shown in [7] that, with probability 1, for all $q \in \mathcal{J}$, the measures $\mu_{q,\varepsilon}$ on $[0,1]$ given by $\mu_{q,\varepsilon}(dt) = \varepsilon^{\xi(e^q-1)} e^{qN_\varepsilon^S(t)}\, dt$ converge weakly, as $\varepsilon \to 0$, to a positive measure μ_q on $[0,1]$; moreover, for every $q \in \mathcal{J}$, for every c-adic interval I of generation ≥ 1, the family of measures $m^I_{q,\varepsilon}$ constructed as $\mu_{q,\varepsilon}$ but with $N_\varepsilon^{S_I}(t)$ instead of $N_\varepsilon^S(t)$ in (57) converges weakly, as $\varepsilon \to 0$, to a measure m^I_q on $[0,1]$; finally, we have $\tau_\mu(q) = \theta(q)$ on $\mathcal{J}$.

The same conclusions as in Proposition 6.7 hold if $\mathcal{Q}(x_n, \lambda_n, 1, \alpha, \varepsilon^1_{M,n})$ is replaced by $\left|L'(q) + \frac{1}{\xi \log(\lambda_n)} N_{\lambda_n}(x_n)\right| \leq \psi_\gamma(2\lambda_n)$.

More on covering numbers and related questions can be found in [5, 6].

6.3. **Example where** $\dim\big(\limsup_{n\to\infty} B(x_n, \lambda_n/2)\big) < d$**.** Let us return to the example of Gibbs measures μ in Section 6.2. Let $q_0 > 0$. Fix $\mathcal{K}$ a subset of $\mathbb{R}$ such that $\tau'_\mu(\mathcal{K}) \cap (\tau'_\mu(q_0), \tau'_\mu(-q_0)) = \emptyset$. Define the system

$$\{(x_n, \lambda_n)\} = \Big\{ \big((\mathbf{k} + \mathbf{1/2})\, c^{-j}, c^{-j}\big) : \frac{\log \mu\left(B\left((\mathbf{k}+\mathbf{1/2}), c^{-j}\right)\right)}{-j\log(c)} \in \mathcal{K} \Big\}.$$

Let $S = \limsup_{n\to\infty} B(x_n, \lambda_n/2)$. For every $q \in \mathcal{K}$, we have $\mu_q(S) = 1$ and $\dim S \leq \max\big(\tau^*_\mu(\tau'_\mu(-q_0)), \tau^*_\mu(\tau'_\mu(q_0))\big) < d$.

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