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Multiperiodic multifractal martingale measures

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Abstract

A nonnegative 1-periodic multifractal measure on $\mathbb{R}$ is obtained as infinite random product of harmonics of a 1-periodic function $W(t)$. Such infinite products are statistically self-affine and generalize certain Riesz products with random phases. They are martingale structures, therefore converge. The criterion on W for nondegeneracy is provided. It differs completely from those for other known random measures constructed as martingale limits of multiplicative processes. In particular, it is very sensitive to small changes in $W(t)$. When these infinite products are interpreted in the framework of thermodynamic formalism for random transformations, $\log W$ is a potential function when $W > 0$. For regular enough potentials, in case of degeneracy, the natural normalization makes the sequence of measures converge. Moreover, this normalization is neutral for nondegenerate martingales. The multifractal analysis of the limit martingale measure is performed for a class of potential functions having a dense countable set of jump points.

Résumé

On construit sur $\mathbb{R}$ une mesure aléatoire positive 1-périodique comme limite d'une suite de mesures aléatoires dont les densités sont des produits d'harmoniques d'une fonction 1-périodique W. Les mesures « produits infinis » ainsi obtenues sont statistiquement auto-affines. Elles généralisent certains produits de Riesz avec phases. Elles existent parce que la suite des densités est une martingale. On obtient la CNS sur W pour que la limite soit non dégénérée. Cette condition est très différente de celle obtenue pour les autres mesures connues comme limites de processus multiplicatifs de nature martingale. En particulier, elle est très sensible à de petites perturbations de W. Plaçant ces produits infinis dans le contexte du formalisme thermodynamique pour des transformations aléa-

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toires, $\log W$ est un potentiel lorsque $W > 0$. Pour les potentiels assez réguliers donnant lieu à une limite dégénérée, la normalisation naturelle rend la suite de mesures convergente ; elle ne modifie pas les martingales non dégénérées. L'analyse multifractale des mesures limites non dégénérées est obtenue pour une classe de potentiels présentant un ensemble dense de points de saut.

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1. Introduction

The random statistically self-affine multifractal measures that this paper investigates are limits of martingales obtained as products of b-harmonics of a periodic function. Let W be a nonnegative 1-periodic measurable function satisfying:

$$\int_{[0,1]} W(t)\,\mathrm{d}t = 1.$$

Let $(\phi_n)_{n\geqslant 0}$ be a sequence of independent random phases distributed uniformly in $[0,1]$. Let $b \geqslant 2$ be an integer. For every $n \geqslant 1$, denote by μ_n the random measure whose density with respect to the Lebesgue measure ℓ on $\mathbb{R}$ is:

$$\frac{\mathrm{d}\mu_n}{\mathrm{d}\ell}(t) = \prod_{k=0}^{n-1} W\big(b^k(t+\phi_k)\big).$$

We let $n \to \infty$ and study the limits μ of such densities μ_n.

Unexpected mathematical results, practical motivations, and numerical calculations have helped one another in this study in particularly intimate fashion. Mandelbrot's original canonical cascade [21] and the mathematical theory it inspired [14] provoked two separate broad developments. One led to much more general and more abstract mathematics. But actual uses in science and engineering also demand the "invention" of very specific multifractals of ever increasing variety and versatility.

In particular, one needs stationary multifractal measures that are natural and simple to define and simulate numerically. The heuristics and the pictures in [7] suggested that these goals could be fulfilled by the measures μ studied in this paper. In fact, as we show, mathematics defeated this hope. But it also revealed a subtle phenomenon. It was not suspected, might have escaped brute-force numerics, and is of great mathematical interest. Its practical implications are also great but will be discussed elsewhere.

The article [7] was developed with no awareness of Riesz products [24], [27, Chapter V 7]. But those classical objects and the Riesz products with random phases [10–12] provide special examples of our sequences μ_n which do not vanish with positive probability when $n \to \infty$, i.e., are nondegenerate.

Our broadest result is “qualitative”: the Riesz products and other special examples are exceptional and unstable, in the sense that the nondegeneracy of the limit is destroyed by small changes in W. This new phenomenon invalidates the conjectures stated in [7].

Our second result is that, under suitable sufficient conditions, the normalized sequence $(\mu_n/\mu_n([0,1]))_{n\geqslant 1}$ converges weakly on compact subsets of $\mathbb{R}$. This normalization is necessary in concrete contexts therefore was used in [7,20]. As taken in [7] this step was mathematically unjustified yet had a very fortunate effect: it revealed the subtle phenomenon studied in this paper. As a result, all future applications will have to face a very important complicating issue. An observed measure that seems to be a multifractal limit may, instead, be a very different mathematical object, providing different insights into the generating mechanism.

The sequel characterizes the nondegeneracy of the limit measure as well as performs its multifractal analysis under weak assumptions on the regularity of W.

1.1. The limit measure

For every real t, the sequence $(\frac{\mathrm{d}\mu_n}{\mathrm{d}\ell}(t))_{n\geqslant 1}$ is a 1-mean nonnegative martingale with respect to the filtration $(\sigma(\phi_0,\ldots,\phi_{n-1}))_{n\geqslant 1}$. Therefore, the existence of the random multiplicative measure μ we seek follows from the theory in [14]. Throughout, weak convergence of measures on a locally compact Hausdorff set K means weak* convergence in the dual of $C(K)$, the space of real continuous functions on K. Our precise result is that, with probability one, the sequence $(\mu_n)_{n\geqslant 0}$ restricted to the compact interval $[0,1]$ converges weakly to a measure $\mu^{(0)}$, and the endpoints 0 and 1 are not atoms of $\mu^{(0)}$.

Consequently, by the 1-periodicity of W, there exists a unique measure μ on $\mathbb{R}$ such that $\mu^{(0)}(\cdot+k)$ is the restriction of μ to $[k,k+1]$ for every $k\in\mathbb{Z}$.

In the sequel, μ will denote $\mu^{(0)}$. Let us detail the contents of the paper.

1.2. Condition of nondegeneracy

The first question is whether or not the martingale limit μ is nondegenerate, meaning that $\mu\neq 0$ with positive probability. To answer, it is now necessary to go beyond the criterion. Theorem 1 reports the surprising fact that μ is nondegenerate if and only if the martingale $\mu_n([0,1])$ equals 1 almost surely. In particular μ has to be a probability measure, and can be characterized via the Fourier coefficients of W.

1.3. The measure μ is generically degenerate

The condition of nondegeneracy forces certain products of Fourier coefficients of W to vanish. Therefore degeneracy holds on an open and dense set of functions W. For example, μ is degenerate if $\widehat{W}(j)\widehat{W}(jb)\neq 0$ for some $j\in\mathbb{Z}^*$. To the contrary, as soon as $\widehat{W}(jb)=0$ for all $j\in\mathbb{Z}^*$, μ is nondegenerate.

The example of $W_1(t)=\frac{8}{35}(1-\cos(2\pi t))^4$ and $b=5$. The associated measure $\mu=\mu_{W_1}$ is nondegenerate because $\widehat{W}_1(5j)=0$ for all $j\in\mathbb{Z}^*$. For $t\in[0,1]$, the background of Fig. 1 shows the integral of the sinusoidal $W_1(t)$, that is, $t\mapsto\mu_1([0,t])$, and the foreground

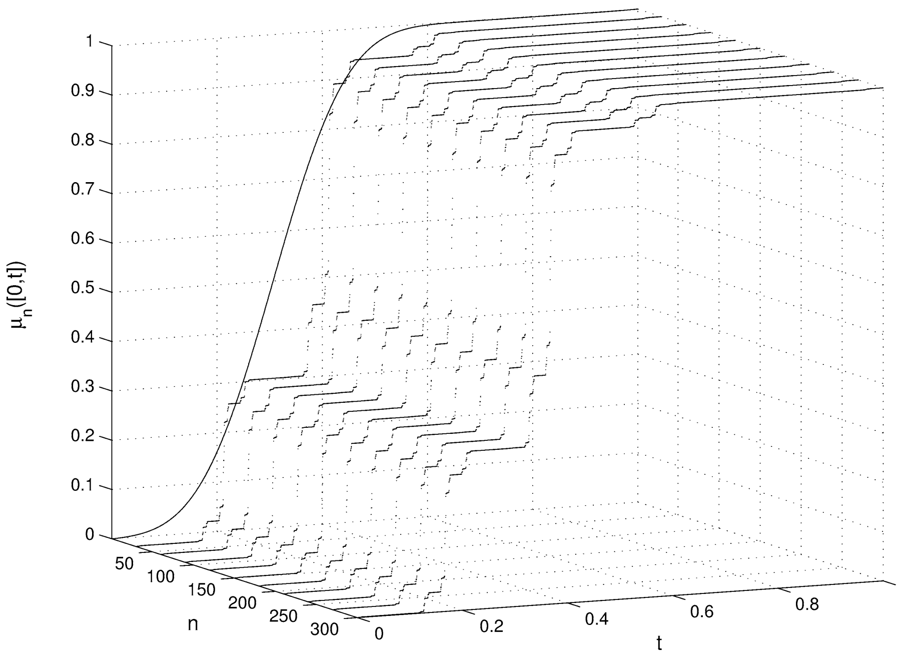

Fig. 1.

shows slightly translated samples of $t \mapsto \mu_n([0,t])$ for $n \in \{30k\colon\ 1 \leqslant k \leqslant 10\}$. We see a graphic confirmation that the sequence $(\mu_n)_{n\geqslant 1}$ converges to a probability measure.

The slightly perturbed example of $W_2(t) = \frac{80000}{353603}(1 - \cos(2\pi t) + 0.1\cos(10\pi t))^4$, *for which* $\widehat{W}_2(1)\widehat{W}_2(5) \neq 0$. Fig. 2 is plotted with the same set of phases as used in Fig. 1. It illustrates how a small perturbation of W_1 suffices to insure a degenerate $\mu = \mu_{W_2}$.

A completely different criterion is found for other random statistically self-affine measures generated by multiplicative martingales, for example, the canonical multifractal cascades (CCM) [15,21] and the multifractal products of pulses (MPCP) [3]. In terms of the multifractal function $\tau(q)$ (defined in Eq. (1)), the usual criterion is $\tau'(1) < 0$, which holds on an open set of parameters. The function $\tau(q)$ is not central here. Nevertheless, Proposition 2 shows that for a certain class of functions W the condition $\tau'(1) \geqslant 0$ suffices for degeneracy.

1.4. Rate of degeneracy

Assume that μ is degenerate and W is positive and satisfies the principle of bounded distortions (8) (for example if W is Hölder continuous), with probability one $\lim_{n\to\infty} \frac{1}{n} \log \|\mu_n\|$ exists and is equal to $\psi_W(1)$ (see Eq. (3)). Proposition 3 shows that this limit $\psi_W(1)$ is never equal to 0, so that μ_n converges exponentially fast to 0 almost surely.

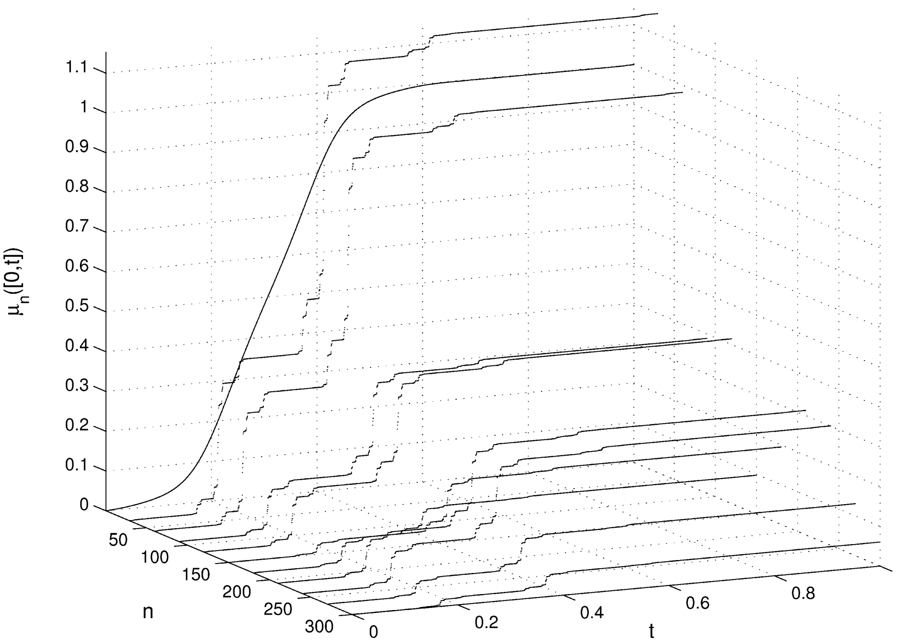

Fig. 2.

1.5. The natural normalization. The measure ν

When the sequence $(\mu_n)_{n\geqslant 1}$ is degenerate, it is natural to consider the normalized sequence of measures on $[0,1]$,

$$\nu_n = \frac{\mu_n}{\mu_n([0,1])}.$$

Moreover, it is important to compare what happens here with what is observed for other martingales generated by random multiplications. For example for the initial "lognormal" martingale model considered in [20], numerical simulations revealed that when the nonnormalized sequence converges to 0, the normalized sequence does not converge.

Only limits of subsequences of $(\nu_n)_{n\geqslant 1}$ are considered in [12]. We point out that the thermodynamic formalism for random transformations [17,18] insures the weak convergence of ν_n when W is positive and Hölder continuous.

Fig. 3 illustrates the convergence of the sequence ν_n obtained by normalization of μ_n in Fig. 2 (W_2 is positive).

Let $(\Omega,\mathcal{B},\mathbb{P}) = ((\mathbb{R}/\mathbb{Z})^{\otimes\mathbb{N}}, \mathcal{B}(\mathbb{R}/\mathbb{Z})^{\otimes\mathbb{N}}, \ell^{\otimes\mathbb{N}})$. For $\omega\in\Omega$, write $\omega = (\phi_i(\omega))_{i\geqslant 0}$. Define on $\mathbb{R}/\mathbb{Z}$ $f(t) = bt$ as well as the random Perron–Frobenius operator $\mathcal{L}_{\log W} = \{\mathcal{L}^{\omega}_{\log W}\colon \omega\in\Omega\}$ acting on the space $C(\mathbb{R}/\mathbb{Z})^{\Omega}$ of families $\{q_\omega\colon \omega\in\Omega\}$ of real-valued continuous functions on $\mathbb{R}/\mathbb{Z}$ by the formula:

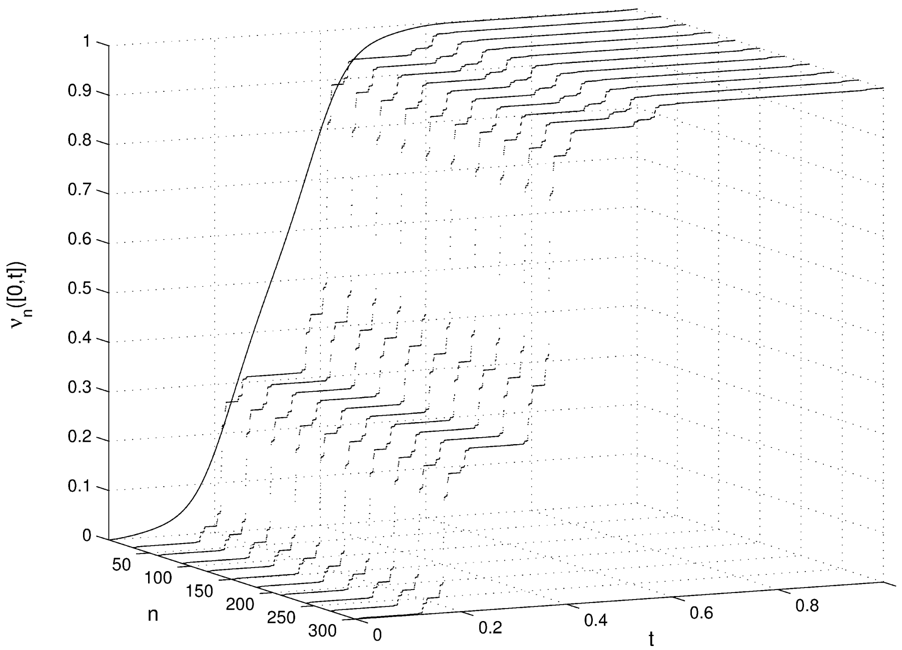

Fig. 3.

$$\mathcal{L}^{\omega}_{\log W} q_\omega(t) = \sum_{t' \in f^{-1}(t)} W(t' + \phi_0) q_\omega(t').$$

Let θ be the ergodic transformation on $(\Omega, \mathbb{P})$ defined by: $\theta(\omega) = (b\phi_{i+1}(\omega))_{i \geqslant 0}$. It is easily seen that for all $\omega \in \Omega$, $n \geqslant 2$ and $g \in C(\mathbb{R}/\mathbb{Z})$,

$$\int_{\mathbb{R}/\mathbb{Z}} g(t)\, \nu_n(\mathrm{d}t) = \frac{\int_{\mathbb{R}/\mathbb{Z}} \mathcal{L}^{\theta^{n-1}\omega}_{\log W} \circ \cdots \circ \mathcal{L}^{\theta\omega}_{\log W} \circ \mathcal{L}^{\omega}_{\log W}(g)(t) \ell(\mathrm{d}t)}{\int_{\mathbb{R}/\mathbb{Z}} \mathcal{L}^{\theta^{n-1}\omega}_{\log W} \circ \cdots \circ \mathcal{L}^{\theta\omega}_{\log W} \circ \mathcal{L}^{\omega}_{\log W}(1)(t) \ell(\mathrm{d}t)}$$

(here we identified $[0, 1)$ with $\mathbb{R}/\mathbb{Z}$ and ν_n with its restriction to $[0, 1)$). The almost sure weak convergence of ν_n is a consequence of Proposition 2.5 in [18]. Denote the almost sure limit by ν. To go back to $[0, 1]$, it is an exercise to show that with probability one, 0, as any fixed deterministic point, is not an atom of ν on $\mathbb{T}$.

Observe that under the previous assumptions, if μ is nondegenerate then it coincides with ν since $\mu_n([0, 1]) = 1$ almost surely.

1.6. The multifractal structure of μ and ν

If λ is a positive measure on $[0, 1]$, the multifractal function τ_λ of λ is defined here as in [12]. It is

$$\tau_\lambda : q \mapsto \limsup_{r\to 0} -\frac{1}{\log(r)} \int_{[0,1]} \lambda\big(I_r(t)\big)^{q-1}\,\lambda(\mathrm{d}t), \tag{1}$$

where $I_r(t) = [t - r/2, t + r/2] \cap [0, 1]$.

Adding the restrictive condition that the range of W is isolated from 0 and ∞, we show that for a large class of functions, the multifractal function τ_μ of μ takes the form:

$$\tau_\mu(q) = 1 - q + \psi_W(q), \tag{2}$$

where

$$\psi_W(q) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log_b \int_{[0,1]} \prod_{k=0}^{n-1} W\big(b^k(t+\phi_k)\big)^q\,\mathrm{d}t\right). \tag{3}$$

Section 5 shows that this class of functions strictly includes functions analogous to the exponential of potential of weak bounded variations recently introduced for the thermodynamic formalism [16,26]. In particular, this class includes functions W with a dense countable set of jump points.

The multifractal analysis of μ consists in the computation of the Hausdorff and packing dimension of level sets like

$$X_\alpha = \left\{t \in [0,1]\colon \lim_{r\to 0} \frac{\log \mu(I_r(t))}{\log r} = \alpha\right\} \quad (\alpha \geqslant 0).$$

Once (2) is established, Section 5.3 shows that those dimensions follow as in [12] using the Large Deviations theory. The main difficulty is to show that (2) holds under weak hypotheses. We also show that τ_μ is differentiable at 1. Hence the Hausdorff dimension of the measure μ, i.e., the smallest Hausdorff dimension of a Borel set of full μ-measure, is equal to $-\tau_\mu'(1)$ (this is also the case when μ is a CCM or a MPCP).

If W is positive and Hölder continuous, the multifractal function τ_ν of ν (recall that $\nu = \mu$ in case of nondegeneracy) takes the form already obtained in [12], namely,

$$\tau_\nu(q) = 1 - q\big(1 + \psi_W(1)\big) + \psi_W(q). \tag{4}$$

Also using [19] it will be seen that due to the ergodicity of θ on $(\Omega, \mathbb{P})$, τ_ν is strictly convex and analytic.

1.7. A natural question: does $\tau_\mu(q) = 1 - q + \log_b \int_{[0,1]} W(t)^q\,\mathrm{d}t$ on some nontrivial interval when μ is nondegenerate?

It is impossible to answer this question numerically by computing

$$1 - q + \frac{1}{n}\mathbb{E}\left(\log_b \int_{[0,1]} \prod_{k=0}^{n-1} W\big(b^k(t+\phi_k)\big)^q\,\mathrm{d}t\right)$$

for large values of n. This problem is raised in [11,12] (Section 7) under the form: does $\psi_W(q)$ simplify in $\log_b \int_{[0,1]} W(t)^q \,\mathrm{d}t$ on a nontrivial interval? Except when W is constant, Theorem 3 shows that if W is positive and $\log W$ satisfies the principle of bounded distortions (8), then $\psi_W(q) < \log_b \int_{[0,1]} W(t)^q \,\mathrm{d}t$ outside a discrete set. This follows from the condition for nondegeneracy. The equality holds on $\mathbb{R}$ when W is constant when restricted to each interval $(k/b, (k+1)/b)$, $0 \leqslant k \leqslant b-1$. We conjecture that the answer no except in this case.

Remark 1. W being a positive continuous 1-periodic function such that $\int_0^1 W(t)\,\mathrm{d}t = 1$, can our product construction be modified to yield a more familiar result, namely, a random measure m having the function $f : q \mapsto 1 - q + \log_b \int_0^1 W(t)^q \,\mathrm{d}t$ as its multifractal function on a nontrivial interval? Such a measure is indeed obtained as the almost sure weak limit of the sequence of measures $(m_n)_{n\geqslant 1}$ on $[0,1]$ whose densities with respect to ℓ are given by:

$$\frac{\mathrm{d}m_n}{\mathrm{d}\ell}(t) = \prod_{k=0}^{n-1} W\big(b^k(t + \phi_{k,l})\big) \quad \text{if } t \in \big[l/b^k, (l+1)/b^k\big),$$

where the random phases $\phi_{k,l}$ ($k \geqslant 0$, $0 \leqslant l \leqslant b^k - 1$) are independent and uniformly distributed in $[0,1]$. By using techniques developed for CCM and MPCP [1,3,15], one can show [4] that m is nondegenerate if and only if $f'(1^-) < 0$. Moreover, assuming that m is nondegenerate and defining J as the open interval of those q's such that $-f'(q)q + f(q) > 0$ we have: with probability one, both multifractal formalisms of [6,23] hold for m on $-f'(J)$ (the largest as possible open interval on which they could hold), and $\tau_m = f$ on J.

1.8. Relations with the properties of Riesz products

The simplest Riesz product with random phases is the special case $W(t) = 1 + a\cos(2\pi t)$ for some $a \in [0,1)$; in this case the restriction of μ_n to $[0,1]$ is clearly a probability measure for all $n \geqslant 1$. This and closely related "generalized" Riez products are considered in [10–12], which neither point out the martingale nature of some of these products, nor study nondegeneracy. While we consider μ_n, [12] typically considers on $[0,1]$ a weak limit of a subsequence of $(\nu_n = \mu_n/\mu_n([0,1]))_{n\geqslant 1}$. Our Theorem 1 exhibits all the functions W for which this normalization is not necessary for convergence to a nondegenerate limit.

For the simplest Riesz products, the approximate formula given in [10] for the Hausdorff dimension of μ is improved in Corollary 2 of this paper.

[11,12] (see also [13] for a closely related problem in the deterministic case) perform the multifractal analysis of limit of subsequences of ν_n when the terms of the infinite product are continuous and satisfy a principle of bounded variations. Both assumptions are relaxed in Theorem 4 and Remark 8 of this paper.

If W is positive and Hölder continuous, the multifractal analysis of the limit ν of ν_n is implicit in [19], but not complete. Section 6 collects both results of [12,19] to give a complete result for the multifractal spectrum of ν.

[11,12] also study infinite products where the random phases are not i.i.d. but satisfy a stationary ergodic property; the martingale structure disappears and it is necessary to consider weak limits of subsequences of $(\mu_n/\mu_n([0,1]))_{n\geqslant 1}$. If W is positive and Hölder continuous, [18] yields the almost sure convergence of the normalized sequence.

1.9. Summary

Section 2 introduces some definitions needed in the sequel, and says precisely in what μ is statistically self-similar (Proposition 1). Section 3 deals with the necessary and sufficient condition for nondegeneracy of μ_n. Section 4 provides a lower bound for the Hausdorff dimension of μ in the general case. Sections 5 and 6 perform the multifractal analysis of μ and ν, respectively. Section 7 briefly relates these measures with a kind of multiplicative cascades measure.

2. Some definitions and statistical self-affinity

Densities. For $0 \leqslant n < m$ and $t \in [0,1]$, let:

$$P_{n,m}(t) = \prod_{k=n}^{m-1} W\big(b^k(t+\phi_k)\big)$$

and $P_n = P_{0,n}$.

A^m. For every integer $m \geqslant 0$ we denote by A^m the set of finite words of length m on the alphabet $A = \{0,\ldots,b-1\}$ ($A^0 = \{\varepsilon\}$). Then for $a \in A^m$, $|a| = m$ and I_a denotes the closed b-adic subinterval of $[0,1]$ naturally encoded by a.

A^*. We denote $\bigcup_{m=0}^{\infty} A^m$ by A^* and $\{0,\ldots,b-1\}^{\mathbb{N}}$ by ∂A^*. The set A^* acts on the left on the disjoint union $A^* \cup \partial A^*$ by the concatenation operation. Thus, for every $a \in A^*$, let C_a denote $a\partial A^*$, namely the cylinder generated by a. Denote by $\mathcal{A}$ the σ-field generated by the C_a's in ∂A^*. ∂A^* is endowed with the standard ultrametric distance d defined by $d(a,b) = b^{-|a\wedge b|}$, where $|a \wedge b| = \sup\{n \geqslant 1\colon a_1\ldots a_n = b_1\ldots b_n\}$.

$\dim_H$ *and* $\dim_P$. The Hausdorff (respectively packing) dimension of a subset of $\mathbb{R}$ (respectively ∂A^*) is considered with respect to the usual distance (respectively d), and denoted by $\dim_H$ (respectively $\dim_P$). (See [9] for a detailed account.)

$I_n(t)$, $C_n(\tilde{t})$ *and* $I_r(t)$. For $t \in [0,1]$ (respectively $\tilde{t} \in \partial A^*$) and $n \geqslant 1$, $I_n(t)$ (respectively $C_n(\tilde{t})$) denotes the closure of the b-adic semi-open to the right interval (respectively the cylinder) of the nth generation which contains t (respectively $\tilde{t}$). For $r \in (0,1)$, $I_r(t)$ denotes the interval $[t - r/2, t + r/2] \cap [0,1]$.

Given a positive measure ν on $[0,1]$ and t a point in the closed support of ν, the "lower log-densities" $\underline{\alpha}_\nu(t)$ and $\underline{\beta}_\nu(t)$, and the "upper log-densities" $\overline{\alpha}_\nu(t)$ and $\overline{\beta}_\nu(t)$ of ν at t are defined by:

$$\begin{cases} \underline{\alpha}_\nu(t) = \liminf\limits_{r\to 0} \dfrac{\log \nu(I_r(t))}{\log r}, & \underline{\beta}_\nu(t) = \liminf\limits_{n\to\infty} \dfrac{\log \nu(I_n(t))}{-n\log b}, \\ \overline{\alpha}_\nu(t) = \limsup\limits_{r\to 0} \dfrac{\log \nu(I_r(t))}{\log r}, & \overline{\beta}_\nu(t) = \limsup\limits_{n\to\infty} \dfrac{\log \nu(I_n(t))}{-n\log b}. \end{cases}$$

If $\underline{\alpha}_\nu(t) = \overline{\alpha}_\nu(t)$ (respectively $\underline{\beta}_\nu(t) = \overline{\beta}_\nu(t)$) simply write $\alpha_\nu(t)$ (respectively $\beta_\nu(t)$).

Similarly, if $\tilde{\nu}$ is a positive measure on ∂A^* and $\tilde{t}$ is a point in the closed support of $\tilde{\nu}$, define:

$$\begin{cases} \underline{\beta}_{\tilde{\nu}}(\tilde{t}) = \liminf\limits_{n\to\infty} \dfrac{\log \tilde{\nu}(C_n(\tilde{t}))}{-n\log b}, \\ \overline{\beta}_{\tilde{\nu}}(\tilde{t}) = \limsup\limits_{n\to\infty} \dfrac{\log \tilde{\nu}(C_n(\tilde{t}))}{-n\log b}. \end{cases}$$

π is the mapping from ∂A^* to $[0,1]$ defined by $\tilde{t} = \tilde{t}_1 \ldots \tilde{t}_i \ldots \mapsto \sum_{i\geqslant 1} \tilde{t}_i / b^i$.

$\tilde{\ell}$ is the unique measure on $(\partial A^*, \mathcal{A})$ such that for all $a \in A^*$, $\tilde{\ell}(C_a) = b^{-|a|}$.

Now if ρ is a nonnegative measure on $(\partial A, \mathcal{A}^*)$, for $n \geqslant 1$ we define $P_n.\rho$ as the measure whose density with respect to $\tilde{\ell}$ is equal to

$$\frac{\mathrm{d}(P_n.\rho)}{\mathrm{d}\rho}(\tilde{t}) = P_n\big(\pi(\tilde{t})\big).$$

The arguments required for Proposition 1 also show that, with probability one, the sequence $(P_n.\rho)_{n\geqslant 1}$ converges weakly to a nonnegative random measure $P.\rho$. Moreover, since the random factors $W(b^k(\pi(\tilde{t}) + \phi_k))$, $k \geqslant 1$, are mutually independent, it follows from [14] that the operator $L : \rho \mapsto \mathbb{E}(P.\rho)$ on nonnegative measures is a projection (by definition if $f \in C(\partial T)$ then $\int_{\partial A^*} f(t)\mathbb{E}(P.\rho)(\mathrm{d}t) = \mathbb{E}(\int_{\partial A^*} f(t) P.\rho(\mathrm{d}t))$).

Let $\tilde{\mu}$ denote $P.\tilde{\ell}$. The following remark will be useful in the proof of Theorem 1. By construction $\mu = \tilde{\mu} \circ \pi^{-1}$. For $a \in A^*$ the probability distribution of $\tilde{\mu}(C_a)$ depends only on $|a|$. Moreover, since ∂A^* is totally disconnected, we have $\|\tilde{\mu}\| = \|\mu\| = \sum_{a\in A^m} \tilde{\mu}(C_a)$ for all $m \geqslant 0$. Consequently

$$\mathbb{E}(\tilde{\mu}) = \mathbb{E}\big(\|\mu\|\big)\tilde{\ell}. \tag{5}$$

We adopt the convention $0 \times \infty = 0$.

Given a nontrivial compact subinterval I of $[0,1]$, the affine increasing mapping from $[0,1]$ onto I is denoted by f_I. The length of I is denoted by $|I|$.

Given two random variables X and Y, identity in distribution is denoted by $X \overset{d}{\equiv} Y$.

Given a real x, $[x]$ stands for the largest integer less than or equal to x.

Self-affinity. The statistical self-affinity property of μ is made explicit now.

Proposition 1 (Statistical self-affinity). *Fix $n \geqslant 1$ and a nontrivial compact subinterval I of* $[0,1]$ *with length b^{-n}. Define the sequence of measures $(\mu_m^I)_{m\geqslant 1}$ on I by*:

$$\frac{\mathrm{d}\mu_m^I}{\mathrm{d}\ell}(t) = P_{n,m}(t).$$

For all $m > n$, the restriction of μ_m to I and the measure μ_{m-n}^I are related by:

$$\mu_m(\mathrm{d}t) = P_n(t)\,\mu_{m-n}^I(\mathrm{d}t) \tag{6}$$

and the following properties hold:

(i) *For all $f \in C(I)$ and $m \geqslant 1$, $\int_I f(t)\,\mu_m^I(\mathrm{d}t) \overset{d}{\equiv} |I| \int_{[0,1]} f \circ f_I(t)\,\mu_m(\mathrm{d}t)$; in particular $\|\mu_m^I\| \overset{d}{\equiv} |I|\|\mu_m\|$.*
(ii) *With probability one, $(\mu_m^I)_{m\geqslant 1}$ converges weakly to a measure μ^I as m tends to ∞ and for all $f \in C(I)$, $\int_I f(t)\,\mu^I(\mathrm{d}t) \overset{d}{\equiv} |I| \int_{[0,1]} f \circ f_I(t)\,\mu(\mathrm{d}t)$; in particular $\|\mu^I\| \overset{d}{\equiv} |I|\|\mu\|$.*
(iii) *The measures μ^{I_a}, $a \in A^n$, are deduced from one another by an horizontal translation.*

The verifications are left to the reader.

3. Nondegeneracy and rate of degeneracy

The characterization of the nondegeneracy of μ, i.e., when is μ positive with positive probability, is the first problem to be solved, and this phenomenon is expressed in Theorem 1 via the Fourier coefficients of W. Then, Proposition 2 completes this result by a different sufficient condition for degeneracy. Proposition 3 gives precisions on the rate of convergence to 0 in case of degeneracy.

For every $k \in \mathbb{Z}$, let $\widehat{W}(k)$ stand for $\int_{[0,1]} W(t)\mathrm{e}^{-2\mathrm{i}k\pi t}\,\mathrm{d}t$. By assumption $\widehat{W}(0) = 1$. For every $n \geqslant 1$ let Y_n stand for $\mu_n([0,1])$; $(Y_n, \sigma(\phi_0, \ldots, \phi_{n-1}))_{n\geqslant 1}$ is a martingale with expectation 1, which converges to $\|\mu\|$.

Theorem 1 (Nondegeneracy). *The following properties are equivalent*:

(i) $\mathbb{P}(\|\mu\| > 0) > 0$;
(ii) $(Y_n)_{n\geqslant 1}$ *is uniformly integrable*;
(iii) $\forall n \geqslant 1$, $Y_n = 1$ *almost surely*;
(iv) $\|\mu\| = 1$ *almost surely* (*μ is a probability measure*);
(v) $\forall n \geqslant 2\ \forall (j_0, \ldots, j_{n-1}) \in \mathbb{Z}^n \setminus \{0, \ldots, 0\}$, $\sum_{k=0}^{n-1} j_k b^k = 0 \Rightarrow \prod_{k=0}^{n-1} \widehat{W}(j_k) = 0$.

It follows from Theorem 1 that if property (v) is violated then Y_n vanishes almost surely, but $\mathbb{E}(Y_n^h) \uparrow_{n\to\infty} \infty$ for all $h > 1$.

Proposition 2 (A condition for degeneracy). *Suppose that $W > 0$ and $\log W$ satisfies the following weak principle of bounded distortions*:

$$\varphi(n)=\sum_{k=0}^{n}\sup_{t,s\in[0,1],\,|t-s|\leqslant b^{-k}}\big|\log W(t)-\log W(s)\big|=\mathrm{o}(n). \tag{7}$$

Let $D_W=1-\int_{[0,1]}W(t)\log_b W(t)$*. If* $D_W<0$ *then* μ *is degenerate. The same conclusion holds if* $D_W=0$ *and moreover* $\varphi(n)=\mathrm{o}(\sqrt{n\log\log n}\,)$.

Proposition 3 (Rate of degeneracy). *Suppose that* μ *is degenerate. Moreover, suppose that* W *is positive and that* $\log W$ *satisfies the principle of bounded distortions*:

$$C=\sum_{k=0}^{\infty}\sup_{t,s\in[0,1],\,|t-s|\leqslant b^{-k}}\big|\log W(t)-\log W(s)\big|<\infty. \tag{8}$$

Then, with probability one $\psi_W(1)=\lim_{n\to\infty}\frac{1}{n}\log\|\mu_n\|$ *exists and* $\psi_W(1)<0$.

Remark 2. (1) The nondegeneracy condition is algebraic. It forces certain $\widehat{W}(k)$ with $k\neq 0$ to be null, and at least one $\widehat{W}(kb)$ to be null. This characterization shows that nondegeneracy holds on a closed subset of functions W with empty interior in the set of nonnegative integrable functions on $[0,1]$ with mean 1.

(2) Here are two simple conditions under which nondegeneracy holds:

(a) There exists $p\geqslant 0$ such that $\widehat{W}(k)=0$ for all $k\notin b^p(\mathbb{Z}\setminus b\mathbb{Z})$.
(b) W is a trigonometric polynomial of the form

$$W(t)=1+\sum_{k\in K}a_k\cos\big(2\pi m_k b^{p_k}t\big)+b_k\sin\big(2\pi m_k b^{p_k}t\big),$$

where K is a finite set, the a_k and b_k are so that $\sum_{k\in K}\sqrt{a_k^2+b_k^2}<1$ in order to insure that W is nonnegative, the p_k are nonnegative integers, and the m_k are positive distinct integers so that: for all $(\varepsilon_k)_{k\in K}\in\{-1,0,1\}^K\setminus\{(0,\dots,0)\}$, b does not divide $\sum_{k\in K}\varepsilon_k m_k$.

For instance, if $b=5$ and $K=\{1,3\}$ then the choice $m_1=1$, $m_3=3$ yields the functions

$$\begin{aligned}W(t)=1&+a_1\cos\big(2\pi\times 5^{p_1}t\big)+b_1\sin\big(2\pi\times 5^{p_1}t\big)\\&+a_3\cos\big(2\pi\times 3\times 5^{p_3}t\big)+b_3\sin\big(2\pi\times 3\times 5^{p_3}t\big),\end{aligned}$$

where p_1 and p_3 are arbitrary nonnegative integers.

(3) Let T be the operator on the 1-periodic functions of $L^1_{\mathrm{loc}}(\mathbb{R})$ defined by:

$$f\mapsto Tf:t\mapsto\frac{1}{b}\sum_{j=0}^{b-1}f\left(\frac{t}{b}+\frac{j}{b}\right).$$

It is immediate that for every $k \in \mathbb{Z}$, $\widehat{Tf}(k) = \hat{f}(kb)$. So if $Tf = 0$, f is of mean 0, and if the function W defined by $W = 1 + f$ is nonnegative, then the function W satisfies the condition for nondegeneracy since $\widehat{W}(kb) = 0$ if $k \neq 0$. Conversely, all the functions W satisfying the condition for nondegeneracy and such that $\widehat{W}(kb) = 0$ if $k \neq 0$ are of the form $W = 1 + g$ for some 1-periodic $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ with $Tg = 0$.

This remark will be useful to construct explicit examples of functions with a dense countable set of jump points satisfying the "weakened" weak principle of bounded distortions in Section 5.1.

The proof of Theorem 1 begins with the following lemma, which explains the origin of property (v).

Lemma 1. *Assume that* $\sum_{k\in\mathbb{Z}} |\widehat{W}(k)| < \infty$. *Properties* (iii) *and* (v) *in Theorem* 1 *are equivalent.*

Proof. Notice that $Y_1 = 1$ almost surely. Since $\sum_{k\in\mathbb{Z}} |\widehat{W}(k)| < \infty$, $t \mapsto \sum_{k\in\mathbb{Z}} \widehat{W}(k)\mathrm{e}^{2\mathrm{i}\pi kt}$ is a continuous version of W. Therefore, for every $n \geqslant 2$,

$$\begin{aligned}
1 = Y_n = Y_n(\phi_0, \ldots, \phi_{n-1}) &= \int_{[0,1]} \prod_{k=0}^{n-1} W\big(b^k(t + \phi_k)\big)\,\mathrm{d}t \\
&= \int_0^1 \prod_{k=0}^{n-1} \sum_{j\in\mathbb{Z}} \widehat{W}(j)\mathrm{e}^{2\mathrm{i}\pi j b^k (t+\phi_k)}\,\mathrm{d}t \\
&= \int_0^1 \sum_{(j_0,\ldots,j_{n-1})\in\mathbb{Z}^n} \prod_{k=0}^{n-1} \widehat{W}(j_k)\mathrm{e}^{2\mathrm{i}\pi \sum_{k=0}^{n-1} j_k b^k (t+\phi_k)}\,\mathrm{d}t \\
&= \sum_{(j_0,\ldots,j_{n-1})\in\mathbb{Z}^n,\ \sum_{k=0}^{n-1} j_k b^k = 0} \prod_{k=0}^{n-1} \widehat{W}(j_k)\mathrm{e}^{2\mathrm{i}\pi \sum_{k=0}^{n-1} j_k b^k \phi_k}.
\end{aligned}$$

Since $\phi_0, \ldots, \phi_{n-1}$ are mutually independent and uniformly distributed, this holds almost surely if and only if the function of n variables

$$Y_n : (u_0, \ldots, u_{n-1}) \in [0,1]^n \mapsto \sum_{(j_0,\ldots,j_{n-1})\in\mathbb{Z}^n,\ \sum_{k=0}^{n-1} j_k b^k = 0} \prod_{k=0}^{n-1} \widehat{W}(j_k)\mathrm{e}^{2\mathrm{i}\pi \sum_{k=0}^{n-1} j_k b^k u_k}$$

is identically equal to 1. This is equivalent to (v). □

Proof of Theorem 1. To see that (i) and (ii) are equivalent, recall that the mapping L defined in Section 1 is a projection. Moreover, it follows from (5) that $L(\tilde{\ell}) = \mathbb{E}(\|\mu\|)\tilde{\ell}$. Consequently, the equality $L \circ L(\ell) = L(\ell)$ yields $\mathbb{E}(\|\tilde{\mu}\|) = (\mathbb{E}(\|\tilde{\mu}\|))^2$ and

$\mathbb{E}(\|\mu\|) \in \{0,1\}$. Since $(Y_n)_{n\geqslant 1}$ is a 1-mean martingale, $\mathbb{E}(\|\mu\|) = 1$ is equivalent to the uniform integrability of the martingale. The same argument shows that (iv) implies (ii).

It is clear that (iii) implies (ii) and that (iii) implies (iv). It remains to show that (v) implies (iii) and (ii) implies (v).

To prove that (v) implies (iii), notice that property (v) means that certain Fourier coefficients of W are null. It is then standard that W is the limit in $L^1([0,1])$ of a sequence $(f_p)_{p\geqslant 1}$ of nonnegative trigonometric polynomials with mean 1 such that $\widehat{W}(k) = 0 \Rightarrow \hat{f}_p(k) = 0$ for all $k \in \mathbb{Z}^*$ and $p \geqslant 1$: $f_p = W * g_p$ where

$$g_p : t \mapsto \big(1+\cos(2\pi t)\big)^p \Big/ \int\limits_{[0,1]} \big(1+\cos(2\pi t)\big)^p \,\mathrm{d}t$$

so that $\hat{f}_p(k) = \widehat{W}(k)\hat{g}_p(k)$ for all $k \in \mathbb{Z}$. In particular each f_p satisfies property (v), as well as the assumption of Lemma 1, so for every $p, n \geqslant 1$ almost surely

$$\int\limits_{[0,1]} \prod_{k=0}^{n-1} f_p\big(b^k(t+\phi_k)\big)\,\mathrm{d}t = 1.$$

Therefore, for every $p, n \geqslant 1$,

$$\begin{aligned} |1-Y_n| &\leqslant \int\limits_{[0,1]} \left| \prod_{k=0}^{n-1} f_p\big(b^k(t+\phi_k)\big) - \prod_{k=0}^{n-1} W\big(b^k(t+\phi_k)\big) \right| \mathrm{d}t \\ &\leqslant \sum_{k=0}^{n-1} \big| f_p\big(b^k(t+\phi_k)\big) - W\big(b^k(t+\phi_k)\big)\big| \\ &\quad \times \prod_{0\leqslant k'<k} f_p\big(b^{k'}(t+\phi_{k'})\big) \prod_{k<k'\leqslant n-1} W\big(b^{k'}(t+\phi_{k'})\big) \end{aligned}$$

and

$$\mathbb{E}\big(|1-Y_n|\big) \leqslant \|f_p - W\|_{L^1} \sum_{k=0}^{n-1} \|f_p\|_{L^1}^k \|W\|_{L^1}^{n-1-k} = n\|f_p - W\|_{L^1}.$$

By our choice of $(f_p)_{p\geqslant 1}$ we get (iii).

Now suppose (ii) holds but (v) fails. Fix $n_0 \geqslant 2$ and $(l_0, \ldots, l_{n_0-1}) \in \mathbb{Z}^{n_0} \setminus \{0, \ldots, 0\}$ such that $\sum_{k=0}^{n_0-1} l_k b^k = 0$ and $\prod_{k=0}^{n_0-1} \widehat{W}(l_k) \neq 0$. Then, for every $n \geqslant 1$, choose $(j_0, \ldots, j_{n+n_0-1})$ such that $j_0 = \cdots = j_{n-1} = 0$ and $(j_n, \ldots, j_{n+n_0-1}) = (l_0, \ldots, l_{n_0-1})$. By using the Fubini lemma together with the 1-periodicity of W and the independences we get:

$$
\begin{aligned}
&\mathbb{E}\Big([Y_{n+n_0}-Y_n]\mathrm{e}^{-2\mathrm{i}\pi\sum_{k=0}^{n+n_0-1} j_k b^k\phi_k}\Big)\\
&\quad=\mathbb{E}\Big(Y_{n+n_0}\mathrm{e}^{-2\mathrm{i}\pi\sum_{k=0}^{n+n_0-1} j_k b^k\phi_k}\Big)\\
&\quad=\int\limits_{[0,1]}\prod_{k=0}^{n-1}\mathbb{E}\big(W\big(b^k(t+\phi_k)\big)\big)\prod_{k=n}^{n+n_0-1}\mathbb{E}\Big(W\big(b^k(t+\phi_k)\big)\mathrm{e}^{-2\mathrm{i}\pi j_k b^k\phi_k}\Big)\,\mathrm{d}t\\
&\quad=\int\limits_{[0,1]}\prod_{k=n}^{n+n_0-1}\mathbb{E}\Big(W\big(b^k(t+\phi_k)\big)\mathrm{e}^{-2\mathrm{i}\pi j_k b^k\phi_k}\Big)\,\mathrm{d}t\\
&\quad=\int\limits_{[0,1]}\prod_{k=0}^{n_0-1}\mathrm{e}^{2\mathrm{i}\pi l_k b^{n+k}t}\int\limits_{[0,1]}W\big(b^{n+k}u\big)\mathrm{e}^{-2\mathrm{i}\pi l_k b^{n+k}u}\,\mathrm{d}u\,\mathrm{d}t\\
&\quad=\int\limits_{[0,1]}\exp\Bigg(2\mathrm{i}\pi b^n t\sum_{k=0}^{n_0-1} l_k b^k\Bigg)\prod_{k=0}^{n_0-1} b^{-(n+k)}\int\limits_{[0,b^{n+k}]}W(u)\mathrm{e}^{-2\mathrm{i}\pi l_k u}\,\mathrm{d}u\,\mathrm{d}t\\
&\quad=\prod_{k=0}^{n_0-1}\widehat{W}(l_k).
\end{aligned}
$$

On the other hand, $\mathbb{E}(|Y_{n+n_0}-Y_n|)$ has to converge to 0 as n tends to ∞ since by (ii) the martingale $(Y_n)_{n\geqslant 1}$ is uniformly integrable, a contradiction. □

Proof of Proposition 2. We proceed as in [25] to obtain the necessary condition of nondegeneracy for CCM, via a size-biasing approach.

For every $t\in[0,1]$ and $n\geqslant 1$, define on $(\Omega,\sigma(\phi_0,\dots,\phi_{n-1}))$ the probability measure $\mathbb{P}_{t,n}$ whose density with respect to $\mathbb{P}$ is given by:

$$
\frac{\mathrm{d}\mathbb{P}_{t,n}}{\mathrm{d}\mathbb{P}}(\omega)=P_n(t).
$$

The sequence $(P_n(t))_{n\geqslant 1}$ is a 1-mean positive martingale with respect to the filtration $(\sigma(\phi_0,\dots,\phi_{n-1}))_{n\geqslant 1}$. This allows us to consider $\mathbb{P}_t$, the Kolmogorov extension of $(\mathbb{P}_{t,n})_{n\geqslant 1}$ to $(\Omega,\sigma(\phi_n,\ n\geqslant 1))$. Following [25, Theorem 4.1(i)], to conclude, it suffices to show that for all $t\in[0,1]$, $\mathbb{P}_t(\limsup_{n\to\infty}\mu_n(I_n(t))=\infty)=1$. To see this, notice that under our assumptions, it is straightforward that with probability one, for all $n\geqslant 1$, for all $t,s\in[0,1]$ such that $|t-s|\leqslant b^{-n}$,

$$
\mathrm{e}^{-\varphi(n)}\leqslant\frac{P_n(t)}{P_n(s)}\leqslant\mathrm{e}^{\varphi(n)}.
$$

It follows that

$$\log \mu_n\big(I_n(t)\big) \geqslant -\varphi(n) + \sum_{k=0}^{n-1} -\log(b) + \log W\big(b^k(t+\phi_k)\big).$$

The random variables $-\log(b) + \log W(b^k(t+\phi_k))$, $k \geqslant 0$, are i.i.d. with respect to $\mathbb{P}_t$, with $\mathbb{P}_t$ expectation $-D_W \log b$ and positive variance (otherwise W is constant equal to b, contradicting $\int_0^1 W(t)\,\mathrm{d}t = 1$). Consequently, if $D_W < 0$ then

$$\mathbb{P}_t\Big(\limsup_{n\to\infty} \mu_n\big(I_n(t)\big) = \infty\Big) = 1$$

follows from the strong law of large numbers and the property $\varphi(n) = \mathrm{o}(n)$, and if $D_W = 0$, the same follows from the law of the iterated logarithm and the property $\varphi(n) = \mathrm{o}(\sqrt{n \log\log n}\,)$. $\square$

Proof of Proposition 3. It follows from the computations done in the proof of Theorem 2 in Section 5 (see also [12], Section 7) that, almost surely, $\psi_W(1) = \lim_{n\to\infty} \frac{1}{n} \log_b \|\mu_n\|$ exists. Moreover, $\psi_W(1)$ is also the limit of $\frac{1}{n} X_n$, where $X_n = \mathbb{E}(\log_b \|\mu_n\|)$, and for all $m, n \geqslant 1$,

$$X_{n+m} \leqslant 2C + X_m + X_n.$$

It follows that the sequence $X_n + 2C$ is sub-additive and $\psi_W(1) = \inf_{n\geqslant 1}(X_n + 2C)/n$. Moreover, $\lim_{n\to\infty} X_n = -\infty$ since $\sup_{n\geqslant 1} \mathbb{E}(\|\mu_n\|) < \infty$ and $\lim_{n\to\infty} \|\mu_n\| = 0$. This yields $\psi_W(1) < 0$. $\square$

4. A lower bound for $\dim_H(\mu)$

When the measure μ is nondegenerate, it is natural to ask for a lower bound estimate of its dimension. Under suitable assumptions this bound will prove in Section 5.3 to be the exact value of this dimension.

Proposition 4. *Suppose that μ is nondegenerate and that $\int_{[0,1]} W^p(t)\,\mathrm{d}t < \infty$ for some $p > 1$. With probability one, for $\tilde{\mu}$-almost every $\tilde{t} \in \partial A^*$,*

$$\underline{\beta}_{\tilde{\mu}}(\tilde{t}) \geqslant D_W = 1 - \int_{[0,1]} W(t) \log_b W(t)\,\mathrm{d}t \geqslant 0.$$

The Hausdorff dimension of μ, $\dim_H(\mu)$, was defined is Section 1.

Corollary 1 (Lower bound for $\dim(\mu)$). *Suppose that μ is nondegenerate and that $\int_{[0,1]} W^p(t)\,\mathrm{d}t < \infty$ for some $p > 1$. With probability one, $0 \leqslant D_W \leqslant \dim_H(\mu) \leqslant 1$. In particular μ is atomless when $D_W > 0$.*

Corollary 1 is simply a consequence of Proposition 4, the relation $\mu = \tilde{\mu} \circ \pi^{-1}$ and a Billingsley lemma (cf. [5, pp. 136–145]).

Proof of Proposition 4. For $n \geqslant 1$, $\varepsilon > 0$ and $\eta > 0$, the Chebychev inequality applied to the probability measure $\tilde{\mu}$ and the random variables $\tilde{\mu}(C_n(\tilde{t}))^\eta$ yields

$$\tilde{\mu}\big(\big\{\tilde{t} \in \partial A^*\colon\ \tilde{\mu}\big(C_n(\tilde{t})\big)^\eta b^{n\eta(D_W-\varepsilon)} \geqslant 1\big\}\big) \leqslant \sum_{a\in A^n} \tilde{\mu}(C_a)^{1+\eta} b^{n\eta(D_W-\varepsilon)} = f_{n,\varepsilon}(\eta).$$

Applying successively Proposition 1, the Fatou lemma, and the Jensen inequality to $(\int_{I_a} P_n(t)\,\mu_{m-n}^{I_a}(\mathrm{d}t))^{1+\eta}$ yields

$$\begin{aligned}
\mathbb{E}\big(f_{n,\varepsilon}(\eta)\big) &\leqslant b^{n\eta(D_W-\varepsilon)} \sum_{a\in A^n} \liminf_{m\to\infty} \mathbb{E}\left(\left(\int_{I_a} P_n(t)\,\mu_{m-n}^{I_a}(\mathrm{d}t)\right)^{1+\eta}\right) \\
&\leqslant b^{n\eta(D_W-\varepsilon)} \sum_{a\in A^n} \liminf_{m\to\infty} \mathbb{E}\left(\big\|\mu_{m-n}^{I_a}\big\|^\eta \int_{I_a} P_n(t)^{1+\eta}\,\mu_{m-n}^{I_a}(\mathrm{d}t)\right) \\
&= b^{n\eta(D_W-1-\varepsilon)}\left(\int_{[0,1]} W(t)^{1+\eta}\,\mathrm{d}t\right)^n
\end{aligned}$$

(we also used the independences and the property: since μ is nondegenerate, it follows from Theorem 1 and Proposition 1 that $\|\mu_{m-n}^{I_a}\| = b^{-n}$). This yields $\mathbb{E}(f_{n,\varepsilon}(\eta)) \leqslant b^{n\eta(-\varepsilon+\mathrm{o}(\eta))}$ so $\sum_{n\geqslant 1}\mathbb{E}(f_{n,\varepsilon}(\eta)) < \infty$ if η is small enough. Finally, for every $\varepsilon > 0$, with probability one $\sum_{n\geqslant 1}\tilde{\mu}(\{\tilde{t} \in \partial A^*\colon\ \tilde{\mu}(C_n(\tilde{t}))^\eta b^{n\eta(D_W-\varepsilon)} \geqslant 1\}) < \infty$. One concludes with Borel–Cantelli lemma. □

To see that $D_W \geqslant 0$ we proceed as follows: on the one hand, we learn from Proposition 2 that $D_W > 0$ when W is a positive trigonometric polynomial satisfying the condition for nondegeneracy. On the other hand, for every $p > 1$, the set of these polynomials is dense in the set of functions of $L^p([0,1])$ satisfying the condition for nondegeneracy.

5. Multifractal analysis of μ

We have to assume some restrictions on the function W.

(H_1) Property (v) of Theorem 1 holds for W (i.e., μ is nondegenerate).
(H_2) $0 < \underline{w} < W < \overline{w} < \infty$ for some real numbers $\underline{w}$ and $\overline{w}$.

Our third assumption allows certain functions W to have a dense countable set of jump points. This assumption includes a condition inspired from the weak principle of bounded distortions (see Remark 3(1)) recently considered in the thermodynamic formalism (see [16,26]), but it is less restrictive than this principle:

(H_3) *"Weakened" weak principle of bounded distortions for* $\log W$: there exists a sequence $(S_n)_{n\geqslant 1}$ of finite subsets of $[0,1]$, all including $\{0,1\}$, such that

$$h_n = \sum_{k=0}^{n} \sup_{\substack{t,s\in[0,1],\ |t-s|\leqslant b^{-k},\\ S_n\cap[t,s]=\emptyset}} \left|\log W(t) - \log W(s)\right| = \mathrm{o}(n)$$

and

$$m_n = \min\left\{k\in\mathbb{N}\colon\ b^{-k} \leqslant \inf_{t,s\in S_n,\ t\neq s} |t-s|\right\} = \mathrm{o}(n).$$

Remark 3. (1) The weak principle of bounded distortions, for example in the deterministic context of [26] (see also [16] and [12, Theorem 3]), would assume the more restrictive condition that there exists $n_0 \geqslant 1$ such that $S_n = S_{n_0}$ for all $n \geqslant n_0$, i.e., W should be piecewise continuous. Even in this case, if W is not continuous, the fact that we consider random phases creates complications that, to be circumvented, necessitate the new ideas we develop in the case of an infinite number of jump points.

(2) We adapt the approach of [12] to find τ_μ. The main difficulty is located in the impossibility, under (H_3), to directly applying the (key) Kingman sub-multiplicative ergodic theorem involved in [12].

Before beginning the study of the multifractal structure of μ, we exhibit some nontrivial examples of functions W satisfying the above assumptions.

5.1. Nontrivial examples of functions W

We shall use Remark 2(3) in Section 3, where the operator T was defined.

Functions W (*with a dense countable set of jump points*) *of the form* $1+\sum_{p\geqslant 1} g_p$ *where the* g_p *are piecewise Hölder continuous with at least two jump points and* $Tg_p = 0$.

Fix $(\widetilde{m}_n)_{n\geqslant 1}$ a nondecreasing sequence of integers such that $\widetilde{m}_n = \mathrm{o}(n)$ and $\lim_{n\to\infty} \widetilde{m}_n = \infty$.

Fix a sequence $(\alpha_p)_{p\geqslant 1} \in (0,1]^{\mathbb{N}^*}$.

For every $p \geqslant 1$, construct a 1-periodic function $f_p \in L^1_{\mathrm{loc}}(\mathbb{R})$ with the following properties:

(i) f_p is given on $[0,1/b)$ by $t \mapsto -\sum_{j=1}^{b-1} f_p(t+j/b)$.
(ii) The set of jump points of f_p in $(1/b,1)$ is nonempty and finite, and f_p is α_p-Hölder continuous between two consecutive jump points.

Due to (i) we have $Tf_p = 0$ so $\hat{f}_p(kb) = 0$ for all $k\in\mathbb{Z}$.

Then denote by D_p the set containing 0 and 1 and all the points where the function f_p jumps. Denote by $\|f_p\|_\infty$ the supremum of $|f_p|$ and by C_p a positive real number such that for all $t,s\in[0,1]$ such that $[t,s]\subset[0,1]\setminus D_p$,

$$\left|f_p(t) - f_p(s)\right| \leqslant C_p|t-s|^{\alpha_p}.$$

Assume that the sets $D_p \setminus \{0,1,1/b\}$ are pairwise disjoint. For $j \geqslant 1$, define $R_j = \bigcup_{p=1}^{j} D_p$. Fix (it is easy to construct one) a nondecreasing sequence $(j_n)_{n\geqslant 1}$ of integers such that for every $n \geqslant 1$ large enough, $b^{-\widetilde{m}_n} \leqslant \inf_{t,s\in R_{j_n},\, t\neq s} |t-s|$, and $\lim_{n\to\infty} R_{j_n} = \bigcup_{p=1}^{\infty} D_p$. Choose $S_n = R_{j_n}$. It follows that $m_n \leqslant \widetilde{m}_n = \mathrm{o}(n)$.

Finally, choose a sequence of real numbers $(\beta_p)_{p\geqslant 1}$ such that

$$\begin{cases} \displaystyle\sum_{p\geqslant 1} |\beta_p| \|f_p\|_\infty < \frac{1}{2}, \\ \displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{j_n} \frac{|\beta_p| C_p}{1-b^{-\alpha_p}} = 0. \end{cases}$$

Then define

$$W = 1 + \sum_{p\geqslant 1} \beta_p f_p.$$

By construction W jumps at every point of $\bigcup_{p\geqslant 1} D_p \setminus \{0,1,1/b\}$, $W \geqslant 1/2$, W is bounded, $\int_{[0,1]} W(t)\,\mathrm{d}t = 1$ and W satisfies the condition for nondegeneracy since $TW = 1$.

It is clear that we can force $\bigcup_{p\geqslant 1} D_p$ to be dense in $[0,1]$.

Now, if $n \geqslant 1$ is large enough and $[t,s] \subset [0,1] \setminus S_n$ is such that $|t-s| \leqslant b^{-k}$ for some $\widetilde{m}_n \leqslant k \leqslant n$, then by construction all the f_p, $1 \leqslant p \leqslant j_n$, are continuous on $[t,s]$, so

$$\bigl|\log W(t) - \log W(s)\bigr| \leqslant 2\bigl|W(t) - W(s)\bigr| \leqslant 2\sum_{p=1}^{j_n} |\beta_p| C_p b^{-\alpha_p k} + 4\sum_{p>j_n} |\beta_p| \|f_p\|_\infty.$$

Consequently

$$\begin{aligned} \frac{h_n}{n} &\leqslant 2\frac{\widetilde{m}_n}{n} \sup_{t\in[0,1]} W(t) + \frac{2}{n}\sum_{p=1}^{j_n} |\beta_p| C_p \sum_{k=0}^{n} b^{-\alpha_p k} + 4\sum_{p>j_n} |\beta_p| \|f_p\|_\infty \\ &\leqslant 2\frac{\widetilde{m}_n}{n} \sup_{t\in[0,1]} W(t) + \frac{2}{n}\sum_{p=1}^{j_n} \frac{|\beta_p| C_p}{1-b^{-\alpha_p}} + 4\sum_{p>j_n} |\beta_p| \|f_p\|_\infty. \end{aligned}$$

It follows that $\lim_{n\to\infty} h_n/n = 0$.

5.2. *The multifractal function of μ*

As in [12], we begin with the identification of a natural candidate to be the multifractal function of μ. Proposition 2 provides sufficient conditions on W for D_W to be positive (here μ is nondegenerate). In this case, Corollary 1 says that μ is atomless. Without these information in general, we have to consider the case $D_W = 0$ in our statements and proofs.

Theorem 2 (Multifractal function τ_μ). *Assume* (H_1), (H_2) *and* (H_3).

(1) *Suppose that* $0 \leqslant D_W < 1$.
  (i) *With probability one, the limit* τ_μ *as* $r \to 0^+$ *of*

$$q \in \mathbb{R} \mapsto \tau_r(q) = -\frac{1}{\log r} \log \int_{[0,1]} \mu\big(I_r(t)\big)^{q-1}\, \mu(\mathrm{d}t)$$

  exists and it is equal to

$$q \in \mathbb{R} \mapsto 1 - q + \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\log_b \int_{[0,1]} P_n(t)^q \,\mathrm{d}t\right).$$

  If $D_W > 0$ *then the function* τ_μ *is convex and decreasing, and if* $D_W = 0$ *then* τ_μ *is convex and decreasing on* $(-\infty, 1)$ *and null on* $[1, \infty)$.
  (ii) τ_μ *is differentiable at* 0 *and* 1 *with* $\tau_\mu'(0) = -1 + \int_{[0,1]} \log_b W(t)\,\mathrm{d}t$ *and* $-\tau_\mu'(1) = D_W$; τ_μ *is not affine on* $[0, 1]$.

(2) $D_W = 1$ *if and only if* $W = 1$ *almost everywhere*; *that is* μ *is the Lebesgue measure and* $\tau_\mu(q) = 1 - q$.

Theorem 3. *Assume* (H_1), (H_2) *and* (H_3).

(i) $\tau_\mu(q) \leqslant 1 - q + \log_b \int_0^1 W(t)^q\,\mathrm{d}t$ *for all* $q \in \mathbb{R}$, *with equality for* $q \in \{0, 1\}$.

(ii) *Suppose* W *is positive and* $\log W$ *satisfies the principle of bounded distortions* (8). *Then, either* W *is constant, or*

$$\tau_\mu(q) < 1 - q + \log_b \int_0^1 W(t)^q\,\mathrm{d}t$$

for every $q \in \mathbb{R} \setminus S$, *where* S *is a discrete set that contains* $\{0, 1\}$. *Moreover, if* $\sup_{t\in[0,1]} W(t) > b$ *then* S *is upper bounded.*

(iii) *If* W *is equal to a positive constant* w_k *on every interval* $(k/b, (k+1)/b)$ $(0 \leqslant k \leqslant b-1)$ *then for all* $q \in \mathbb{R}$,

$$\tau_\mu(q) = 1 - q + \log_b \int_0^1 W(t)^q\,\mathrm{d}t = 1 - q + \log_b \sum_{k=0}^{b-1} w_k^q.$$

Remark 4. In the proof of Theorem 3(ii), we show that if W is nonconstant, positive, and $\log W$ satisfies (8), then $\psi_W(q) < \log_b \int_0^1 W^q(t)\,\mathrm{d}t$ for all $q \in \mathbb{R}$ except on a discrete set that contains $\{0,1\}$. The proof is valid even is W does not satisfy the condition for nondegeneracy.

The proof of Theorem 2 needs two lemmas, namely Lemmas 2 and 3, both consequences of (H_3). The proofs of these lemmas are postponed until after the one of Theorem 2 and the statement of Lemma 4. The proof of Theorem 3 ends this section.

Lemma 2. *There exists a positive function $\varphi(n) = \mathrm{o}(n)$ such that, with probability one, for n large enough, for all $t, s \in [0,1]$ with $|t-s| \leqslant b^{-n}$,*

$$\mathrm{e}^{-\varphi(n)} \leqslant \frac{\prod_{k=0}^{n-1} W(b^k(t+\phi_k))}{\prod_{k=0}^{n-1} W(b^k(s+\phi_k))} \leqslant \mathrm{e}^{\varphi(n)}.$$

Remark 5. Because of the assumption (H_3) on $\log W$, the set of integers n for which the inequalities in Lemma 2 hold depends on $\omega \in \Omega$. Consequently, it is not possible to obtain the first part of Theorem 2 as directly as the corresponding result in [12, Theorem 4].

We also need Lemma 3 which involves new definitions.

Fix $\gamma \in (1/2, 1)$. For every j and $p \geqslant 0$, denote by $\varepsilon_{j,p}$ the finite word written with $p \times j$ times the letter 0 ($\varepsilon_{j,0} = \varepsilon$), and then for $n \geqslant 1$ denote by $E_{j,n}$ the event

$$\begin{aligned} E_{j,n} = \bigl\{&\forall a \in \varepsilon_{j,n-1} A^j, \\ &\#\bigl\{0 \leqslant k \leqslant j - m_j\colon\ S_j \cap \bigl[b^{(n-1)j+k}(I_a + \phi_{(n-1)j+k}) \bmod 1\bigr] \neq \emptyset\bigr\} \leqslant j^\gamma\bigr\}. \end{aligned}$$

Then define $M_{j,n}(\omega) = \#\{1 \leqslant l \leqslant n\colon\ \omega \notin E_{j,l}\}$.

Lemma 3. *There exists a sequence $(\beta_j)_{j\geqslant 1}$ tending to 0 at ∞ such that for every $j \geqslant 1$ large enough, with probability one, for n large enough $M_{j,n} \leqslant \beta_j n$.*

Proof of Theorem 2. (1)(i). We proceed in four steps.

Step 1. We show that for every $q \in \mathbb{R}$, $\lim_{r\to 0^+} \tau_r(q)$ exists almost surely if and only if $\lim_{n\to\infty} 1 - q + \frac{1}{n}\log_b \int_{[0,1]} \prod_{k=0}^{n-1} P_n(t)^q \,\mathrm{d}t$ exists almost surely. Moreover, these limits are equal whenever they exist.

Notice that it suffices to establish this property when r tends to 0 along the sequence $(b^{-n})_{n\geqslant 1}$. We distinguish two cases.

First case: $q - 1 \geqslant 0$. For every $n \geqslant 1$ and $a \in A^n$, define I_a^- as being the closed b-adic interval of the nth generation immediately on the left side of I_a if $I_a \subset (0,1]$ and $\emptyset$ otherwise; also define I_a^+ as being the closed b-adic interval of the nth generation immediately on the right side of I_a if $I_a \subset [0,1)$ and $\emptyset$ otherwise.

Fix $n \geqslant 1$ and $a \in A^n$. For every $t \in I_a$, we have $I_{b^{-n}}(t) \subset I_a^- \cup I_a \cup I_a^+$. Due to the fact that $q \geqslant 1$, this implies that

$$\mu\bigl(I_{b^{-n}}(t)\bigr)^{q-1} \leqslant 3^{q-1}\bigl(\mu(I_a^-)^{q-1} + \mu(I_a)^{q-1} + \mu\bigl(I_a^+\bigr)^{q-1}\bigr)$$

and then

$$\int_{[0,1]} \mu\big(I_{b^{-n}}(t)\big)^{q-1}\,\mu(\mathrm{d}t) \leqslant 3^{q-1} \sum_{a\in A^n} \big(\mu(I_a^-)^{q-1} + \mu(I_a)^{q-1} + \mu\big(I_a^+\big)^{q-1}\big)\mu(I_a). \tag{9}$$

On the other hand, if $a \in A^{n+1}$, $I_a \subset I_{b^{-n}}(t)$ for every $t \in I_a$ so

$$\mu(I_a)^q = \int_{I_a} \mu(I_a)^{q-1}\,\mu(\mathrm{d}t) \leqslant \int_{I_a} \mu\big(I_{b^{-n}}(t)\big)^{q-1}\,\mu(\mathrm{d}t)$$

and

$$\sum_{a\in A^{n+1}} \mu(I_a)^q \leqslant \int_{[0,1]} \mu\big(I_{b^{-n}}(t)\big)^{q-1}\,\mu(\mathrm{d}t). \tag{10}$$

Now, we use the following important remark. Eventhough we do not know that μ is atomless, the theory in [14] tells us that, with probability one, the b-adic points are not atoms of μ. It follows that with probability one, for every $a \in A^*$,

$$\mu(I_a) = \lim_{m\to\infty} \mu_{|a|+m}(I_a) = \lim_{m\to\infty} \int_{I_a} P_{|a|}(t) P_{|a|,|a|+m}(t)\,\mathrm{d}s. \tag{11}$$

Moreover, by Lemma 2, with probability one, for n large enough, for all $a \in A^n$ and $m \geqslant 1$,

$$\mathrm{e}^{-\varphi(n)} \leqslant \frac{\int_{I_a} P_n(s) P_{n,n+m}(s)\,\mathrm{d}s}{P_n(t_a)\int_{I_a} P_{n,n+m}(s)\,\mathrm{d}s} \leqslant \mathrm{e}^{\varphi(n)},$$

where $t_a = \inf(I_a)$. But due to Proposition 1(i) and Theorem 1(iii) we have:

$$\int_I P_{n,n+m}(s)\,\mathrm{d}s = b^{-n}$$

for every interval I of length b^{-n}. Consequently

$$\mathrm{e}^{-\varphi(n)} \leqslant \frac{\int_{I_a} P_n(s) P_{n,n+m}(s)\,\mathrm{d}s}{b^{-n} P_n(t_a)} \leqslant \mathrm{e}^{\varphi(n)}$$

and by (11),

$$\mathrm{e}^{-\varphi(n)} \leqslant \frac{\mu(I_a)}{b^{-n} P_n(t_a)} \leqslant \mathrm{e}^{\varphi(n)}. \tag{12}$$

Now, if $I \in \{I_a, I_a^-, I_a^+\}$ is nonempty, applying Lemma 2 with $(t,s) = (\inf(I), \inf(I_a))$ in (12) written with I yields

$$e^{-2\varphi(n)} \leqslant \frac{\mu(I)}{b^{-n} P_n(t_a)} \leqslant e^{2\varphi(n)}.$$

So

$$\exp\bigl(-h(q)\varphi(n)\bigr) \leqslant \frac{\mu(I)^{q-1}\mu(I_a)}{b^{-n(q-1)}b^{-n} P_n(t_a)^q} \leqslant \exp\bigl(h(q)\varphi(n)\bigr), \tag{13}$$

where $h(q) = 1 + 2|q-1|$. A last application of Lemma 2 yields

$$e^{-|q|\varphi(n)} \leqslant \frac{\int_{I_a} P_n(s)^q \,\mathrm{d}s}{b^{-n} P_n(t_a)^q} \leqslant e^{|q|\varphi(n)}$$

and we deduce from (13) that with probability one, for n large enough, for all $a \in A^n$ and I a nonempty element of $\{I_a, I_a^-, I_a^+\}$,

$$\exp\bigl(-\bigl[|q| + h(q)\bigr]\varphi(n)\bigr) \leqslant \frac{\mu(I)^{q-1}\mu(I_a)}{b^{-n(q-1)} \int_{I_a} P_n(s)^q \,\mathrm{d}s} \leqslant \exp\bigl(\bigl[|q| + h(q)\bigr]\varphi(n)\bigr). \tag{14}$$

Finally, the conclusion is a consequence of (9), (10) and (14).

Second case: $q - 1 < 0$. Fix $n \geqslant 1$ and $a \in A^{n+1}$. We saw that $I_a \subset I_{b^{-n}}(t)$ for every $t \in I_a$. Consequently

$$\int_{[0,1]} \mu\bigl(I_{b^{-n}}(t)\bigr)^{q-1} \mu(\mathrm{d}t) \leqslant \sum_{a \in A^{n+1}} \mu(I_a)^q. \tag{15}$$

On the other hand, if $a \in A^n$, fix $a' \in A^{n+2}$ such that $I'_a := I_{a'} \subset I_a$ and I'_a does not contain any endpoint of I_a. We have $I_{b^{-(n+2)}}(t) \subset I_a$ for all $t \in I'_a$ so

$$\mu(I_a)^{q-1}\mu\bigl(I'_a\bigr) \leqslant \int_{I'_a} \mu\bigl(I_{b^{-(n+2)}}(t)\bigr)^{q-1} \mu(\mathrm{d}t).$$

This yields

$$\sum_{a \in A^n} \mu(I_a)^{q-1}\mu\bigl(I'_a\bigr) \leqslant \int_{[0,1]} \mu\bigl(I_{b^{-(n+2)}}(t)\bigr)^{q-1} \mu(\mathrm{d}t). \tag{16}$$

By using Lemma 2 we get, with probability one, for all n large enough and $a \in A^n$,

$$b^{-2}\underline{w}^2 e^{-(\varphi(n)+\varphi(n+2))} \leqslant \frac{\mu(I'_a)}{\mu(I_a)},$$

so by (16)

$$\sum_{a\in A^n}\mu(I_a)^q\leqslant b^2\underline{w}^{-2}\mathrm{e}^{\varphi(n)+\varphi(n+2)}\int\limits_{[0,1]}\mu\big(I_{b^{-(n+2)}}(t)\big)^{q-1}\,\mu(\mathrm{d}t) \tag{17}$$

and the proof ends like in the first case by using (15), (17) and (14).

Step 2. We use the notations introduced with Lemma 3. For all j and $n\geqslant 1$ and all $q\in\mathbb{R}$, define:

$$Y_{j,n}(q)=b^{(n-1)j}\int\limits_{I_{\varepsilon_{j,n-1}}}P_{(n-1)j,nj}(t)^q\,\mathrm{d}t$$

($Y_{j,1}(q)=\int_{[0,1]}P_j(t)^q\,\mathrm{d}t$). Define $C_W=\max(|\log\underline{w}|,|\log\overline{w}|)$. We use the notations of Lemma 3 and prove the following property:

($\mathcal{P}$) For every j large enough, with probability one, for all $n\geqslant 1$ large enough, $0\leqslant i\leqslant j-1$ and $q\in\mathbb{R}$,

$$\exp\big(-\tilde{h}(j,n,q)\big)\leqslant\frac{Y_{nj+i,1}(q)}{\prod_{l=1}^n Y_{j,l}(q)}\leqslant\exp\big(\tilde{h}(j,n,q)\big),$$

where $\tilde{h}(j,n,q)=2|q|h_jn+C_W|q|(2\beta_j jn+2(j^\gamma+m_j)n+i)$.

It follows from the definition of $M_{j,n}$ and the inequality $W^q\leqslant\exp(C_W|q|)$ that

$$\exp\big(-C_W|q|(M_{j,n}\,j+i)\big)\leqslant\frac{Y_{nj+i,1}(q)}{Z}\leqslant\exp\big(C_W|q|(M_{j,n}\,j+i)\big)$$

with

$$Z=\int\limits_{[0,1]}\prod_{\substack{1\leqslant l\leqslant n,\\ \omega\in E_{j,l}}}P_{(l-1)j,lj}(t)^q\,\mathrm{d}t.$$

Moreover, again because of $W^q\leqslant\exp(C_W|q|)$, we have $\mathrm{e}^{-C_W|q|j}\leqslant Y_{j,l}(q)\leqslant\mathrm{e}^{C_W|q|j}$ for each $0\leqslant l\leqslant n-1$. So

$$\begin{aligned}\exp\big(-C_W|q|(2M_{j,n}\,j+i)\big)&\leqslant\frac{Y_{nj+i,1}(q)}{Z\prod_{1\leqslant l\leqslant n,\ \omega\notin E_{j,l}}Y_{j,l}(q)}\\&\leqslant\exp\big(C_W|q|(2M_{j,n}\,j+i)\big).\end{aligned} \tag{18}$$

Define $l_1(\omega) = \min\{1 \leqslant l \leqslant n\colon\ \omega \in E_{j,l}\}$. By construction we have:

$$Z = \sum_{a \in A^{(l_1-1)j}} \int_{I_a} \prod_{\substack{l_1 \leqslant l \leqslant n,\\ \omega \in E_{j,l}}} P_{(l-1)j,lj}(t)^q \,\mathrm{d}t.$$

By the 1-periodicity of W, the integral

$$\int_{I_a} \prod_{\substack{l_1 \leqslant l \leqslant n,\\ \omega \in E_{j,l}}} P_{(l-1)j,lj}(t)^q \,\mathrm{d}t$$

does not depend on $a \in A^{(l_1-1)j}$. It follows that

$$Z = b^{(l_1-1)j} \int_{I_{\varepsilon_{j,l_1-1}}} \prod_{\substack{l_1 \leqslant l \leqslant n,\\ \omega \in E_{j,l}}} P_{(l-1)j,lj}(t)^q \,\mathrm{d}t.$$

Now, by using the definition of E_{j,l_1} and computations similar to those used in the first step and in the proof of Lemma 2, we get:

$$\exp\bigl(-2|q|h_j - 2C_W|q|(j^\gamma + m_j)\bigr) \leqslant \frac{Z}{Y_{j,l_1}(q)\,Z_1} \leqslant \exp\bigl(2|q|h_j + 2C_W|q|(j^\gamma + m_j)\bigr)$$

with

$$Z_1 = b^{l_1 j} \int_{I_{\varepsilon_{j,l_1}}} \prod_{\substack{l_1+1 \leqslant l \leqslant n,\\ \omega \in E_{j,l}}} P_{(l-1)j,lj}(t)^q \,\mathrm{d}t.$$

Repeating the same argument until the last l for which $\omega \in E_{j,l}$ we get:

$$\exp\bigl(-\hat{h}(j,n,q)\bigr) \leqslant \frac{Z}{\prod_{1 \leqslant l \leqslant n,\ \omega \in E_{j,l}} Y_{j,l}(q)} \leqslant \exp\bigl(\hat{h}(j,n,q)\bigr), \tag{19}$$

where $\hat{h}(j,n,q) = (2|q|h_j + 2C_W|q|(j^\gamma + m_j))(n - M_{j,n})$. Then property $(\mathcal{P})$ follows from Lemma 3, (18) and (19).

Step 3. Fix $q \in \mathbb{R}$. We show that the limit in Step 1 exists almost surely and is equal to $1 - q + \psi_W(q)$.

By construction, for every $j \geqslant 1$ the random variables $Y_{j,l}(q)$, $l \geqslant 1$ are i.i.d. and integrable. It then follows from Step 2 and the law of large numbers that for every j large enough, with probability one,

$$\begin{aligned}
&-2|q|\frac{h_j}{j} - 2C_W|q|\left(\beta_j + \frac{j^\gamma + m_j}{j}\right) + \frac{1}{j}\mathbb{E}\big(\log Y_{j,1}(q)\big) \\
&\leqslant \liminf_{N\to\infty} \frac{\log Y_{N,1}(q)}{N} \leqslant \limsup_{N\to\infty} \frac{\log Y_{N,1}(q)}{N} \\
&\leqslant 2|q|\frac{h_j}{j} + 2C_W|q|\left(\beta_j + \frac{j^\gamma + m_j}{j}\right) + \frac{1}{j}\mathbb{E}\big(\log Y_{j,1}(q)\big)
\end{aligned}$$

and the conclusion follows by letting j tend to ∞.

Step 4. We show that with probability one, the convergence as $r \to 0^+$ of $\tau_r(q)$ holds for all $q \in \mathbb{R}$, and $\lim_{r\to 0^+} \tau_r(q) = 1 - q + \psi_W(q)$.

It suffices to notice that almost surely, for $n \geqslant 1$ and $q, q' \in \mathbb{R}$,

$$\left|\frac{1}{n}\log Y_{n,1}(q) - \frac{1}{n}\log Y_{n,1}(q')\right| \leqslant C_W|q - q'|,$$

and then to use Step 3, together with (9), (10), (15) and (17). The property of the limit function τ_μ to be convex nonincreasing is inherited from the τ_r. The fact that τ_μ is decreasing if $D_W > 0$ and decreasing on $(-\infty, 1)$ and null on $[1, \infty)$ if $D_W = 0$ will be explained in Remark 7 (Section 5.3).

(1)(ii). It follows from the proof of (i) (Step 3) that the function τ_μ is the limit of the sequence of convex functions $f_n = \mathbb{E}(\tau_{b^{-n}})$. Moreover, due to the concavity of the logarithm, for all $n \geqslant 1$ and $q \in \mathbb{R}$, $f_n(q) \leqslant f(q) = 1 - q + \log_b \int_{[0,1]} W(t)^q\,\mathrm{d}t$, so $\tau_\mu(q) \leqslant f(q)$. Then, the differentiability of τ_μ at 0 and 1 results from the equalities $f_n(0) = f(0) = 1$, $f_n(1) = f(1) = 0$, $f_n'(0) = f'(0) = -1 + \int_{[0,1]} \log_b W(t)\,\mathrm{d}t$ and $f_n'(1) = f'(1) = -D_W$ for all $n \geqslant 1$. τ_μ is not affine on $[0, 1]$ because of the values of $\tau_\mu(0)$, $\tau_\mu(1)$ and $\tau_\mu'(1)$.

(2)(ii). We have $D_W = 1$ if and only if the derivative of the convex function $f : q \mapsto \int_{[0,1]} W(t)^q\,\mathrm{d}t$ at 1 is null. Since $f(0) = f(1) = 1$, this yields $W = 1$ almost everywhere. In this case μ is the Lebesgue measure and $\tau_\mu(q) = 1 - q$ for all $q \in \mathbb{R}$. □

To prove Lemmas 2 and 3, we need the:

Lemma 4. *For $\gamma \in (1/2, 1)$ and $n \geqslant 1$ define $p_n = p_n(\gamma)$ the probability that there exists $a \in A^n$ for which $\#\{0 \leqslant k \leqslant n - m_n\colon S_n \cap [b^k(I_a + \phi_k) \bmod 1] \neq \emptyset\} \geqslant n^\gamma$.*

The series $\sum_{n\geqslant 1} p_n$ converge.

Proof of Lemma 2. Fix $\gamma > 1/2$. By Lemma 4 and the Borel–Cantelli lemma, for almost every $\omega \in \Omega$, there exists $n_0(\omega) \geqslant 1$ such that for $n \geqslant n_0$, for all $a \in A^n$,

$$\#\big\{0 \leqslant k \leqslant n - m_n\colon S_n \cap \big[b^k(I_a + \phi_k) \bmod 1\big] \neq \emptyset\big\} < n^\gamma.$$

This implies that for $n \geqslant n_0(\omega)$, $a \in A_n$ and $t, s \in I_a$, we have:

$$\begin{aligned}
&\left|\sum_{k=0}^{n} \log\bigl[W\bigl(b^k(t+\phi_k)\bigr)\bigr] - \log\bigl[W\bigl(b^k(s+\phi_k)\bigr)\bigr]\right| \\
&\leqslant \sum_{\substack{0\leqslant k\leqslant n-m_n,\\ S_n\cap[b^k(I_a+\phi_k) \bmod 1]=\emptyset}} \bigl|\log\bigl[W\bigl(b^k(t+\phi_k)\bigr)\bigr] - \log\bigl[W\bigl(b^k(s+\phi_k)\bigr)\bigr]\bigr| \\
&\quad + (n^\gamma + m_n)\bigl(\log(\overline{w}) - \log(\underline{w})\bigr) \\
&\leqslant h_n + (n^\gamma + m_n)\bigl(\log(\overline{w}) - \log(\underline{w})\bigr)
\end{aligned}$$

by definition of h_n. So the conclusion follows if we take:

$$\varphi(n) = 2\bigl[h_n + (n^\gamma + m_n)\bigl(\log(\overline{w}) - \log(\underline{w})\bigr)\bigr]. \qquad \Box$$

Proof of Lemma 3. By definition, for j and $n \geqslant 1$,

$$M_{j,n}(\omega) = \sum_{l=1}^{n} \mathbf{1}_{\Omega\setminus E_{j,l}}(\omega),$$

where the random variables $\mathbf{1}_{\Omega\setminus E_{j,l}}$, $1 \leqslant l \leqslant n$, are independent copies of a Bernoulli random variable with parameter p_j (defined in Lemma 4).

Define $\beta_j = 2p_j/(1+p_j)$ (β_j tends to 0 at ∞). Then, the estimate of $\mathbb{P}(M_{j,n} \geqslant [\beta_j n])$ is standard and one has $\sum_{n\geqslant 1} \mathbb{P}(M_{j,n} \geqslant [\beta_j n]) < \infty$. $\Box$

Proof of Lemma 4. Fix $\gamma > 1/2$. For every $n \geqslant 1$, denote by $N_n + 1$ the number of elements of S_n. Notice that $N_n b^{-m_n} \leqslant 1$. The ϕ_k being uniformly distributed, for every $0 \leqslant k \leqslant n - m_n$ and $a \in A^n$,

$$\mathbb{P}\bigl(S_n \cap \bigl[b^k(I_a+\phi_k) \bmod 1\bigr] \neq \emptyset\bigr) = N_n b^{k-n}.$$

So the probability that $b^k(I_a+\phi_k) \bmod 1$ meets S_n for at least n^γ values of k in $[0, n-m_n]$ is bounded by (we use the independences between the ϕ_k):

$$a_n = \sum_{l=n^\gamma}^{n-m_n} \sum_{0\leqslant k_1<\cdots<k_l\leqslant n-n_m} \prod_{i=1}^{l} N_n b^{k_i-n} = \sum_{l=n^\gamma}^{n-m_n} N_n^l b^{-nl} \sum_{0\leqslant k_1<\cdots<k_l\leqslant n-n_m} b^{\sum_{i=1}^l k_i}.$$

By bounding every term of the form $b^{\sum_{i=1}^l k_i}$ by $b^{\sum_{i=0}^{l-1} n-m_n-i}$ and the number of terms in $\sum_{0\leqslant k_1<\cdots<k_l\leqslant n-n_m} b^{\sum_{i=1}^l k_i}$ by n^l, we get:

$$a_n \leqslant \sum_{l=n^\gamma}^{n-m_n} N_n^l b^{-nl} n^l b^{(n-m_n)l-(l^2-l)/2} \leqslant \sum_{l=n^\gamma}^{n-m_n} n^l b^{-(l^2-l)/2} \leqslant n^{n+1} b^{-(n^{2\gamma}-n^\gamma)/2}$$

(we used $N_n b^{-m_n} \leqslant 1$). As $\gamma > 1/2$, an elementary study shows that $\sum_{n\geqslant 1} b^n a_n < \infty$. Since $p_n \leqslant b^n a_n$, we have the conclusion. $\square$

Proof of Theorem 3. According to the notations of the introduction, denote by f the function $q \mapsto 1 - q + \log_b \int_0^1 W(t)^q \,\mathrm{d}t$.

(i) This is shown in the proof of Theorem 2(ii) or Proposition 10 in [12].

(ii) Suppose that W is not constant. Let S be the set of those points $q \in \mathbb{R}$ such that $\tau_\mu(q) = f(q)$. Suppose that there exists $p_0 \in S$ and $(q_n)_{n\geqslant 1}$ a sequence of pairwise distinct points in S such that $q_n \to p_0$ as $n \to \infty$.

For every $q \in \mathbb{R}$, writing $\tau_\mu(q) = f(q)$ is equivalent to $\psi_W(q) = \log_b \int_0^1 W(t)^q \,\mathrm{d}t$, i.e., $\psi_{W_q}(1) = 0$, where $W_q = W^q / \int_0^1 W(t)^q \,\mathrm{d}t$. Since W^q also satisfies the assumptions of Proposition 3, it follows from this proposition that $\tau_\mu(q) = f(q)$ is equivalent to the nondegeneracy of the measure μ_q associated with W_q like μ with W. By Theorem 1(v), the nondegeneracy of μ_q implies that for every $j \in \mathbb{Z}^*$, $\widehat{W}_q(j)\widehat{W}_q(bj) = 0$, or equivalently $\widehat{W^q}(j)\widehat{W^q}(bj) = 0$. Now suppose that $\widehat{W^{p_0}}(b) \neq 0$. The same holds for $\widehat{W^q}(b)$ in a neighborhood of p_0, so we can assume without loss of generality that $\widehat{W^{q_n}}(b^2) = 0$ for all $n \geqslant 1$. Since the mapping $q \mapsto \widehat{W^q}(b^2)$ has an analytic extension to $\mathbb{C}$ ($\underline{w} \leqslant W \leqslant \overline{w}$), this yields $\widehat{W^q}(b^2) = 0$ for all $q \in \mathbb{R}$. On the other hand, since W is not constant, $\ell(\{t \in [0,1]\colon\ W(t) > 1\}) > 0$ and either $\lim_{q\to\infty} |\int_{[0,1]} W(t)^q \cos(2\pi b^2 t)\,\mathrm{d}t| = \infty$ or $\lim_{q\to\infty} |\int_{[0,1]} W(t)^q \sin(2\pi b^2 t)\,\mathrm{d}t| = \infty$, a contradiction.

Supposing that $\widehat{W^{p_0}}(b^2) \neq 0$ leads to a similar contradiction. Consequently, the set S is discrete. If $\sup_{t\in[0,1]} W(t) > b$ then $f(q) > 0$ for q large enough. Since $\tau_\mu(q) \leqslant 0$ for $q \geqslant 1$, it follows that the discrete set S is upper bounded.

(iii) The function $W_q = W^q / \int_0^1 W(t)^q \,\mathrm{d}t$ is of the same kind as W. In particular, $\widehat{W}_q(bj) = 0$ for all $j \in \mathbb{Z}^*$. Consequently, property (v) of Theorem 1 is fulfilled by W_q, hence the associated measure μ_{W_q} nondegenerate. It follows that $\|\mu_{W_q,n}\| = 1$ for all $n \geqslant 1$ and $q \in \mathbb{R}$. This yields the conclusion. $\square$

5.3. *The multifractal spectrum of μ*

We denote τ_μ by τ in this section.
If $\alpha \geqslant 0$, define:

$$\begin{cases} \underline{X}_\alpha = \{t \in [0,1]\colon\ \underline{\alpha}_\mu(t) = \alpha\}, \quad \overline{X}_\alpha = \{t \in [0,1]\colon\ \overline{\alpha}_\mu(t) = \alpha\}, \quad X_\alpha = \underline{X}_\alpha \cap \overline{X}_\alpha, \\ V_\alpha = \{t \in [0,1]\colon\ \underline{\alpha}_\mu(t) \geqslant \alpha\}, \quad V^\alpha = \{t \in [0,1]\colon\ \overline{\alpha}_\mu(t) \leqslant \alpha\}. \end{cases}$$

We exclude the case where W is almost everywhere equal to 1. It follows from Theorem 2 that we have $\alpha_{\inf} < \alpha_{\sup}$, where $\alpha_{\inf} = \inf\{-\tau'_+(q)\colon q \geqslant 0\}$ and $\alpha_{\sup} = \sup\{-\tau'_-(q)\colon q \leqslant 0\}$ ($\alpha_{\inf} = 0$ if $D_W = 0$).

Theorem 4. *Assume (H_1), (H_2) and (H_3).*

(i) *With probability one, for every $q \geqslant 0$ such that $-\tau'_+(q) > \alpha_{\inf}$ and $L \in \{H, P\}$,*

$$0 < -\tau'_+(q)q + \tau(q) \leqslant \dim_L V_{-\tau'_+(q)} \cap V^{-\tau'_-(q)} \leqslant -\tau'_-(q)q + \tau(q)$$

and for every $q \leqslant 0$ such that $-\tau'_-(q) < \alpha_{\sup}$ and $L \in \{H, P\}$,

$$0 < -\tau'_-(q)q + \tau(q) \leqslant \dim_L V_{-\tau'_+(q)} \cap V^{-\tau'_-(q)} \leqslant -\tau'_+(q)q + \tau(q).$$

Moreover, at each q where the convex function τ is differentiable and $-\tau'(q) \in (\alpha_{\inf}, \alpha_{\sup})$, for every $E \in \{X, \underline{X}, \overline{X}\}$ and $L \in \{H, P\}$,

$$\dim_L E_{-\tau'(q)} = -\tau'(q)q + \tau(q) > 0.$$

(ii) *With probability one, $V_\alpha \cap V^\beta = \emptyset$ for all (α, β) such that $\alpha \leqslant \beta$ and $[\alpha, \beta] \not\subset [\alpha_{\inf}, \alpha_{\sup}]$.*

Remark 6. (1) Theorem 4 concludes as Theorem 1 in [12] for μ, the difference being that now W satisfies the weak assumption (H_3).

(2) In the proof of Theorem 4(i), we deal with atomless measures μ_q in order to compute some Laplace transform and use the Large Deviations theory to show that μ_q is carried by $V_{-\tau'_+(q)} \cap V^{-\tau'_-(q)}$. When $D_W = 0$, we are not able to prove that $\mu_1 = \mu$ is atomless since we only know that $\dim_H \mu = D_W = 0$ (Corollary 2). This is why we cannot claim that X_0 is not empty. If we could prove that μ is atomless, this would yield $X_0 \neq \emptyset$ and $\dim_H X_0 = 0$.

(3) One also could derive similar results in the framework of "box" multifractal analysis [6]. Also notice that when W satisfies (8), μ is a kind of random version of quasi-Bernoulli measures considered in [6].

Theorem 4 will be obtained by using a convenient family of auxiliary measures. Our approach is a slight modification of the one of [12]. Instead of constructing these measures directly on $[0, 1]$, we obtain them as projections of measures defined on ∂A^*.

Let Ω^* be a subset of Ω such that $\mathbb{P}(\Omega^*) = 1$ and for all $\omega \in \Omega^*$ the martingale limit measure $\tilde{\mu}$ exists. Fix $\omega \in \Omega^*$. Then for $q \in \mathbb{R}$, let $\tilde{\mu}_{q,n}$, $n \geqslant 1$, be the sequence of measures on ∂A^*, defined by:

$$\frac{\mathrm{d}\tilde{\mu}_{q,n}}{\mathrm{d}\tilde{\ell}}(\tilde{t}) = \frac{P_n(\pi(\tilde{t}))^q}{\int_{[0,1]} P_n(\pi(\tilde{t}))^q \,\mathrm{d}t}.$$

It possesses a subsequence $\tilde{\mu}_{q,n_j(q)}$ which converges to a probability measure $\tilde{\mu}_q$ with the following property:

Proposition 5. *For $\mathbb{P}$-almost every ω in Ω^*, for all $q \in \mathbb{R}$, for $\tilde{\mu}_q$-almost every $\tilde{t} \in \partial A^*$: if $q \geqslant 0$ then*

$$-\tau'_+(q)q + \tau(q) \leqslant \underline{\beta}_{\tilde{\mu}_q}(\tilde{t}) \leqslant \overline{\beta}_{\tilde{\mu}_q}(\tilde{t}) \leqslant -\tau'_-(q)q + \tau(q);$$

if $q \leqslant 0$ then

$$-\tau'_-(q)q + \tau(q) \leqslant \underline{\beta}_{\tilde{\mu}_q}(\tilde{t}) \leqslant \overline{\beta}_{\tilde{\mu}_q}(\tilde{t}) \leqslant -\tau'_+(q)q + \tau(q).$$

Corollary 2. *Due to the differentiability of τ_μ at 1, with probability one the Hausdorff dimension of μ is exactly D_W.*

Remark 7. (1) It follows from Proposition 5 that $-\tau'_{\mathrm{sgn}(q)}(q)q + \tau(q) \geqslant 0$ for all $q \in \mathbb{R}$, because the logarithmic density of a measure cannot tend to $-\infty$. This forces $-\tau'_{\mathrm{sgn}(q)}(q)q + \tau(q)$ to be positive if $-\tau'_{\mathrm{sgn}(q)}(q) \in (\alpha_{\inf}, \alpha_{\sup})$.

(2) Since $\tau(1) = 0$ and τ is convex nonincreasing, it is decreasing on $(-\infty, 1)$. Moreover, if $D_W > 0$, i.e., $\tau'(1) < 0$, τ becomes negative on $(1, \infty)$. Consequently, it is also decreasing on $[1, \infty)$, otherwise $-\tau'_{\mathrm{sgn}(q)}(q)q + \tau(q) < 0$ for some $q > 1$, contradicting Proposition 5. If $D_W = 0$, i.e., $\tau'(1) = 0$, since τ is convex nonincreasing, $\tau(q) = 0$ for all $q \geqslant 1$. This completes the proof of Theorem 2(1)(i).

The proofs of Proposition 5 and Corollary 2 are postponed.

Proof of Theorem 4. (i) As a consequence of Proposition 5 and a Billingsley lemma [5, pp. 136–145], for $\mathbb{P}$-almost every $\omega \in \Omega^*$, for every $q \in \mathbb{R}$ such that $-\tau'_{\mathrm{sgn}(q)}(q)q + \tau(q) > 0$, the measure defined on $[0,1]$ by $\mu_q = \tilde{\mu}_q \circ \pi^{-1}$ is of Hausdorff dimension at least $-\tau'_{\mathrm{sgn}(q)}(q)q + \tau(q)$. In particular, it is atomless. Moreover, this measure is the weak limit of the sequence $\mu_{q,n_j(q)} = \tilde{\mu}_{q,n_j(q)} \circ \pi^{-1}$. So, for $n \geqslant 1$ and $a \in A^n$,

$$\mu_q(I_a) = \lim_{n_j(q)\to\infty} \frac{\int_{I_a} P_n(t)^q P_{n,n_j(q)}(t)^q \,\mathrm{d}t}{\int_{[0,1]} P_n(t)^q P_{n,n_j(q)}(t)^q \,\mathrm{d}t}.$$

The fact that $\int_{I_a} P_{n,n_j(q)}(t)^q \,\mathrm{d}t$ does not depend on $a \in A^n$ together with the same use of Lemma 2 as in the proof of Theorem 2 yield for n large enough, $a \in A^n$ and $s \in I_a$,

$$\mathrm{e}^{-|q|\varphi(n)} \frac{b^{-n} P_n(s)^q}{\int_{[0,1]} P_n(t)^q \,\mathrm{d}t} \leqslant \mu_q(I_a) \leqslant \mathrm{e}^{|q|\varphi(n)} \frac{b^{-n} P_n(s)^q}{\int_{[0,1]} P_n(t)^q \,\mathrm{d}t}.$$

Now, proceeding as in the proof of Theorem 2, we obtain for $\mathbb{P}$-almost every $\omega \in \Omega^*$, for every $q \in \mathbb{R}$ such that $-\tau'_{\mathrm{sgn}(q)}(q)q + \tau(q) > 0$, for all $\beta \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \log_b \int_{[0,1]} \mu\big(I_{b^{-n}}(t)\big)^\beta \mu_q(\mathrm{d}t) = \tau(\beta + q) - \tau(q).$$

Then mimicking the proof of Theorem 1 in [12] or the one of Theorem 2.18 in [23] (they use a standard Large Deviations theorem (see [8])) we obtain that μ_q is carried by $V_{-\tau'_+(q)} \cap V^{-\tau'_-(q)}$. This yields the lower bound for the dimensions.

The upper bounds for the dimensions are obtained as in [12, Theorem 1]. An alternative approach is to use Theorem 2.24, Propositions 2.5 and 2.6, and Lemma 4.4 in [23]. Notice that to make use of [23], it is nevertheless necessary to replace (it is immediate) the property of the measure in [23] to be a doubling measure by the following: via Lemma 2, with probability one, there exists a constant $C > 0$ such that for all r small enough, for all $t \in [0,1]$,

$$\frac{\mu(I_{2r}(t)}{\mu(I_r(t))} \leqslant C\mathrm{e}^{C\varphi([-\log(r)])}$$

with $\lim_{r\to 0}\varphi([-\log(r)])/\log r = 0$.

(ii) It is a consequence of Lemma 4.4 in [23]. □

Proof of Proposition 5. Since ∂A^* is totally disconnected, for all $\omega \in \Omega^*$, for all $q \in \mathbb{R}$, for all $a \in A^*$,

$$\tilde{\mu}_q(C_a) = \lim_{n_j(q)\to\infty} \frac{\int_{I_a} P_n(t)^q P_{n,n_j(q)}(t)^q \,\mathrm{d}t}{\int_{[0,1]} P_n(t)^q P_{n,n_j(q)}(t)^q \,\mathrm{d}t}.$$

Then, computations similar to those performed in the proof of Theorem 2 yield for $\mathbb{P}$-almost every $\omega \in \Omega^*$, for every $q \in \mathbb{R}$, for all $\beta \in \mathbb{R}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\log_b \sum_{a\in A^n} \tilde{\mu}_q(C_a)^{\beta+1} &= \lim_{n\to\infty} \frac{1}{n}\log_b \int\limits_{\partial A^*} \tilde{\mu}_q\big(C_n(\tilde{t})\big)^{\beta}\tilde{\mu}_q(\mathrm{d}\tilde{t}) \\ &= \tau\big((\beta+1)q\big) - (\beta+1)\tau(q).\end{aligned}$$

Here again, the Large Deviations theory yields the conclusion on the logarithmic density. □

Proof of Corollary 2. Consequence of Proposition 5 applied at $q = 1$, the existence of $\tau'(1)$, together with the Billingsley lemma [5, pp. 136–145]. □

6. Multifractal function and spectrum of ν

If W is Hölder continuous, we consider the measure ν obtained in Section 1: $\nu = \mu$ if μ is nondegenerate and ν is the weak limit of $\mu_n/\|\mu_n\|$ otherwise. Due to Theorems 3.1 and 3.2 in [17], the measure ν is almost surely equivalent to a probability measure $\mu^{\omega}_{\log W}$ such that the probability measure defined on $\mathbb{R}/\mathbb{Z} \times \Omega$ by

$$\mu_{\log W}(\mathrm{d}t, \mathrm{d}\omega) := \mu^{\omega}_{\log W}(\mathrm{d}t)\mathbb{P}(\mathrm{d}\omega)$$

is ergodic with respect to the skew product $(t,\omega) \mapsto (bt, \theta(\omega))$. It follows that, almost surely, ν and $\mu^{\omega}_{\log W}$ have the same multifractal nature. The results on multifractal analysis of Gibbs measures in [19] would provide the Hausdorff dimension of the level sets X_α only for all α almost surely instead of almost surely for all α. But we keep from the approach in [19] (Section 5) the following information: with probability one (with the notations of Section 1) the limit function

$$q \in \mathbb{R} \mapsto \lim_{n\to\infty} \frac{1}{n} \log_b \int_{\mathbb{R}/\mathbb{Z}} \mathcal{L}^{\theta^{n-1}\omega}_{\log W^q} \circ \cdots \circ \mathcal{L}^{\theta\omega}_{\log W^q} \circ \mathcal{L}^{\omega}_{\log W^q}(1)(t)\,\mathrm{d}t \tag{20}$$

exists and is strictly convex, and analytic; moreover, by definition it is equal to $q \mapsto \psi_W(q)$.

Define for ν and $\alpha \geqslant 0$ the sets X^ν_α, $\underline{X}^\nu_\alpha$, $\overline{X}^\nu_\alpha$, V^ν_α and $V^{\nu,\alpha}$ as X_α, $\underline{X}_\alpha$, $\overline{X}_\alpha$, V_α and V^α were for μ.

Theorem 5. *With probability one*:

(i) *The multifractal function of ν is strictly convex and analytic, and is almost surely given by $\tau_\nu(q) = 1 - q(1+\psi_W(1)) + \psi_W(q)$.*

(ii) *For all $q \in \mathbb{R}$, $E \in \{X, \underline{X}, \overline{X}\}$ and $L \in \{H, P\}$,* $\dim_L E^\nu_{-\tau'_\nu(q)} = -\tau'_\nu(q)q + \tau_\nu(q)$.

(iii) $V^\nu_\alpha \cap V^{\nu,\beta} = \emptyset$ *for all (α, β) such that $\alpha \leqslant \beta$ and $[\alpha, \beta] \not\subset \overline{-\tau'_\nu(\mathbb{R})}$.*

Proof. The existence of the limit function $\tau_\nu(q)$ is obtained as in Section 5.2 for μ. The multifractal spectrum of ν is derived like the one of μ in Section 5.3. The new point here is only the strict convexity and the analyticity of τ_ν which follows from the existence of the limit in (20). □

Remark 8. If W satisfies only satisfies (H_2) and (H_3), after replacing τ by τ_ν, the conclusions of Theorem 4 are true almost surely for any limit ν of a subsequence of ν_n. This holds for a larger choice of function W, since W does not necessarily satisfy property (v) of Theorem 1. In particular, given a dense countable subset S of $[0, 1]$, it is easy to construct W jumping at every point of S and satisfying (H_3).

7. A multiplicative cascade counterpart

The measures studied in previous sections deserve to be compared to those obtained by a multiplicative cascade construction.

Let $(W_0, \ldots, W_{b-1})$ be a nonnegative random vector in $\mathbb{R}^b$ such that $b^{-1}\sum_{j=0}^{b-1} W_j = 1$ almost surely. Let $((W_0, \ldots, W_{b-1})(n))_{n\geqslant 1}$ be a sequence of independent copies of $(W_0, \ldots, W_{b-1})$. Then let μ be the almost sure weak limit of the sequence of probability measures μ_n on $[0, 1]$ given by:

$$\frac{\mathrm{d}\mu_n}{\mathrm{d}\ell}(t) = \prod_{k=1}^{n} W_{a_k}(k) \quad \text{if } t \in I_{a_1\ldots a_n}$$

for every $a = a_1 \ldots a_n \in A^n$. This sequence is a martingale which converges almost surely weakly to a measure μ on $[0, 1]$.

The parallel with the measure studied in the previous sections is now easy to make by using Proposition 1: define for $n \geqslant 1$ and $a \in A^n$ the sequence $(\mu_m^{I_a})_{m\geqslant 1}$ by:

$$\frac{\mathrm{d}\mu_m^{I_a}}{\mathrm{d}\ell}(t) = \prod_{k=1}^{m} W_{a'_k}(n+k) \quad \text{if } t \in I_{aa'_1\ldots a'_m}$$

for every $a' = a'_1 \ldots a'_m \in A^m$. Then Proposition 1 holds if one specifies that I is one of the I_a and if (6) is replaced by the simpler relation

$$\mu_m(\mathrm{d}t) = \prod_{k=1}^{n} W_{a_k}(k)\, \mu_{m-n}^{I_a}(\mathrm{d}t).$$

The reader will adapt the approach used in Section 4 to obtain the following result, in this construction, the computations are easier, because the auxiliary measures have the simple expression

$$\mu_q(I_{a_1\ldots a_n}) = \frac{\prod_{k=1}^{n} W_{a_k}^q(k)}{\prod_{k=1}^{n}(\sum_{j=0}^{b-1} W_j^q(k))}.$$

For $\beta \geqslant 0$ define:

$$\begin{cases} \underline{E}_\beta = \{t \in [0,1]\colon \underline{\beta}_\mu(t) = \beta\}, \quad \overline{E}_\beta = \{t \in [0,1]\colon \overline{\beta}_\mu(t) = \beta\}, \quad E_\beta = \underline{E}_\beta \cap \overline{E}_\beta, \\ U_\beta = \{t \in [0,1]\colon \underline{\beta}_\mu(t) \geqslant \beta\}, \quad U^\beta = \{t \in [0,1]\colon \overline{\beta}_\mu(t) \leqslant \beta\}. \end{cases}$$

Theorem 6. *Assume that* $\sum_{k=0}^{b-1} \mathbb{E}(\mathbf{1}_{\{W_k>0\}}|\log W_k|) < \infty$. *Define the analytic decreasing convex function* $\tau_\mu : q \in \mathbb{R} \mapsto -q + \mathbb{E}(\log_b \sum_{k=0}^{b-1} \mathbf{1}_{\{W_k>0\}} W_k^q)$. *With probability one*:

(i) *for all* $q \in \mathbb{R}$, $F \in \{E, \underline{E}, \overline{E}\}$ *and* $L \in \{H, P\}$, $\dim_L F_{-\tau'_\mu(q)} = -\tau'_\mu(q)q + \tau_\mu(q)$;
(ii) $U_\alpha \cap U^\beta = \emptyset$ *for all* (α, β) *such that* $\alpha \leqslant \beta$ *and* $[\alpha, \beta] \not\subset \overline{-\tau'_\mu(\mathbb{R})}$.

Remark 9. (1) The level sets considered in Theorem 6 are those of the multifractal formalism developed in [6]. Indeed, because of the tree structure in the construction here, the Large Deviations theory can be used directly in the spirit of Section 5 only in this formalism. To get the same information for level sets involving centered intervals, it is possible to use the general approach of [2].

(2) The measure considered in this section is a version, with stronger correlations, of the microcanonical cascade measure m [21] obtained as follows: each node a of A^* is equipped with its own copy of $(W_0, \ldots, W_{b-1})$, $(W_0, \ldots, W_{b-1})(a)$, and these copies are mutually independent; the probability measure m is the almost sure weak limit of the sequence of probability measures $(m_n)_{n\geqslant 1}$ given by:

$$\frac{\mathrm{d}m_n}{\mathrm{d}\ell}(t) = \prod_{k=1}^{n} W_{a_k}(a_1 \ldots a_{k-1}) \quad \text{if } t \in I_{a_1\ldots a_n}.$$

Let $f : q \mapsto -q + \log_b \mathbb{E}(\sum_{k=0}^{b-1} \mathbf{1}_{\{W_k>0\}} W_k^q)$. Let J be the largest interval such that $-f'(q)q + f(q)$ is defined and positive for all $q \in J$. With probability one, the multifractal formalism in the sense of [6] or [23] holds for m on $-f'(J)$ and $\tau_m = f$ on J (cf. [1,2] for details). So in general, $\tau_\mu(q) < \tau_m(q)$ on J except for $q = 1$ where τ_μ and τ_m always coincide. It is exactly the same phenomenon as for μ and m in Section 1 (Remark 1).

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