COVERING NUMBERS OF DIFFERENT POINTS IN DVORETZKY COVERING

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ABSTRACT. Consider the Dvoretzky random covering on the circle $\mathbb{T}$ with a decreasing length sequence $\{\ell_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty} \ell_n = \infty$. We study, for a given $\beta \geq 0$, the set F_β of points which are asymptotically covered by a number βL_n of the first n randomly placed intervals where $L_n = \sum_{k=1}^{n} \ell_k$. Three typical situations arise, delimited by two "phase transitions", according to $\overline{\alpha}$ is zero, positive-finite or infinite, where $\overline{\alpha} = \limsup_{n\to\infty} \frac{L_n}{-\log \ell_n}$. More precisely, if ℓ_n tends to zero rapidly enough so that $\overline{\alpha} = 0$ then, with probability one, $\dim_H F_\beta = 1$ for all $\beta \geq 0$; if ℓ_n is moderate so that $0 < \overline{\alpha} < +\infty$ then, with probability one, we have $\dim F_\beta = d_{\overline{\alpha}}(\beta)$ for $\beta \in J_{\overline{\alpha}}$ and $F_\beta = \varnothing$ for $\beta \notin J_{\overline{\alpha}}$ where $d_{\overline{\alpha}}(\beta) = 1+\overline{\alpha}(\beta-1-\beta\log\beta)$ and $J_{\overline{\alpha}}$ is the interval consisting of β's such that $d_{\overline{\alpha}}(\beta) \geq 0$; eventually, if ℓ_n is so slow that $\overline{\alpha} = \lim_{n\to\infty} \frac{L_n}{-\log \ell_n} = +\infty$ then, with probability one, $F_1 = \mathbb{T}$. This solves a problem raised by L. Carleson in a rather satisfactory fashion.

Analogous results are obtained for the Poisson covering of the line, which is studied as a tool.

1. INTRODUCTION

We consider the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ which is identified with the interval $[0, 1)$, a decreasing sequence $\{\ell_n\}_{n\geq 1}$ $(0 < \ell_n < 1)$ which tends to 0 at ∞ and such that $\sum_{n=1}^{\infty} \ell_n = \infty$, and a sequence of i.i.d. random variables $\{\omega_n\}_{n\geq 1}$ of the uniform distribution (i.e. Lebesgue distribution). We denote by $I_n = \omega_n + (0, \ell_n)$ the open interval of length ℓ_n with left end point ω_n. In this paper, we study how a given point $t \in \mathbb{T}$ is covered by these intervals I_n.

The Dvoretzky covering problem is to find necessary conditions and sufficient conditions on the length sequence $\{\ell_n\}$ for the whole circle $\mathbb{T}$ to be covered almost surely, or equivalently for $\mathbb{T}$ to be covered infinitely often. That is to say

$$\mathbb{P}\left(\mathbb{T} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} I_n\right) = 1 \tag{1.1}$$

where $\mathbb{P}$ is the probability measure of the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The problem was raised in 1956 by A. Dvoretzky [D]. It attracted attentions of P. Lévy, J.P. Kahane, P. Erdös, P. Billard, B. Mandelbrot who made significant contributions (see [K1]).

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We first observe that, with probability one, almost every point in $\mathbb{T}$ with respect to the Lebesgue measure is covered by an infinite number of intervals I_n. Furthermore, we have the following quantitative description of this infinity, i.e. with probability one for almost every $t \in \mathbb{T}$, we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} 1_{(0,\ell_k)}(t-\omega_k)}{\sum_{k=1}^{n} \ell_k} = 1 \tag{1.2}$$

where 1_A denotes the characteristic function of a set A. In fact, for any $t \in \mathbb{T}$ the series $\sum_{k=1}^{\infty} \frac{\mathbf{1}_{(0,\ell_k)}(t-\omega_k)-\ell_k}{L_k}$ converges almost surely, where $L_n = \sum_{k=1}^{n} \ell_k$, because the partial sums of the series form a martingale which is L^2-bounded by $\sum_{k=1}^{\infty} \frac{\ell_k(1-\ell_k)}{L_k^2} < \infty$ (the last series does converge and its verification is left to the reader). Hence (1.2) follows from this convergence, the Kronecker lemma and the Fubini theorem. However, the condition $\sum_{n=1}^{\infty} \ell_n = \infty$ is not sufficient for every point $t \in \mathbb{T}$ to be covered.

In 1972, after the works of the authors mentioned above, L. Shepp [S] obtained a complete solution to the problem by finding a necessary and sufficient condition for covering (i.e. for (1.1) to be realized):

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(\ell_1 + \cdots + \ell_n) = \infty. \tag{1.3}$$

To get more information on further developments and related topics of the subject, we may refer to Kahane's book [K1] and his survey papers [K4, K5, K6].

When Shepp's result is established, a natural problem, which was raised by L. Carleson (communication to J. P. Kahane who transmitted it to the second author), is how to describe the infinity of the set of intervals covering a given point. First works in this direction appeared in [F1, FK1].

We describe the Carleson problem in the following manner. Define, for $n \geq 1$, the nth *covering number* of $t \in \mathbb{T}$ by

$$N_n(t) = \mathrm{Card}\{1 \leq j \leq n : I_n \ni t\} = \sum_{k=1}^{n} 1_{(0,\ell_k)}(t-\omega_k)$$

which is the number of those intervals covering t among the first n intervals. Since the expectation $\mathbb{E}N_n(t)$ of $N_n(t)$ is equal to L_n, we are naturally led to compare the asymptotic behavior of $N_n(t)$ with that of L_n. Thus, for any $\beta \geq 0$, we define the (random) sets

$$\underline{F}_\beta = \left\{ t \in \mathbb{T} : \liminf_{n\to\infty} \frac{N_n(t)}{L_n} = \beta \right\},$$

$$\overline{F}_\beta = \left\{ t \in \mathbb{T} : \limsup_{n\to\infty} \frac{N_n(t)}{L_n} = \beta \right\},$$

$$F_\beta = \underline{F}_\beta \bigcap \overline{F}_\beta.$$

A previous work [F3] showed that, in the case $\ell_n = \frac{\alpha}{n}$ ($\alpha > 0$), these sets may be non-empty for a certain interval of β. In other words, points on the circle may be differently covered. As we shall prove, it is not the case for all length sequences $\{\ell_n\}$ ($\ell_n = \frac{\log n}{n}$ being a counter-example, see Theorem 1.3, i.e. in this case every point is covered in the "same" way).

In this paper, we will prove, under some regularity conditions on ℓ_n, that there exists a deterministic interval J of β such that with probability one, the sets F_β, $\underline{F}_\beta$ and $\overline{F}_\beta$ are non-empty for *every* $\beta \in J$. Furthermore, we determine the size of these sets by computing their Hausdorff dimensions, which are given by an explicit formula (Theorem 1.1, 1.2 and 1.3). Notice that the interval J may be the infinite interval $\mathbb{R}_+ = [0,\infty)$, a finite subinterval or a singleton.

As we have already pointed out, the asymptotic behavior of $\frac{N_n(t)}{L_n}$ was first investigated in [F1] and [FK1], especially in the case $\ell_n = \frac{\alpha}{n}$. In this case ($\ell_n = \frac{\alpha}{n}$), the Hausdorff dimension of F_β was calculated almost surely for a given β, but not almost surely simultaneously for all β in a nontrivial interval [F3]. A similar problem on $\{0,1\}^{\mathbb{N}}$ (in place of $\mathbb{T}$) was treated in [FK2].

In order to state our result, we define

$$\overline{\alpha} = \limsup_{n\to\infty} \frac{\sum_{j=1}^n \ell_j}{-\log\, \ell_n} \tag{1.4}$$

$$\widehat{\alpha} = \inf_{b\geq 2} \limsup_{n\to\infty} \sup_{m\geq 1} \frac{\sum_{\ell_j\in[b^{-(n+m)},b^{-n}]} \ell_j}{\log\, b^m} \tag{1.5}$$

where $b \geq 2$ is an integer. For $0 < \epsilon < 1$, let

$$\Lambda_\epsilon = \{j \geq 1;\ \epsilon \leq \ell_j < 1\}.$$

Our results concern three classes of sequences $\{\ell_n\}_{n\geq 1}$, roughly described as rapid sequences for which we have $\overline{\alpha} = 0$, moderate sequences for which $0 < \overline{\alpha} < \infty$ and slow sequences for which $\overline{\alpha} = +\infty$.

We will make one of the two following regularity assumptions (the first one is made for the classes qualified by $\overline{\alpha} = 0$ or $0 < \overline{\alpha} < \infty$, and the second one for the class qualified by $\overline{\alpha} = \infty$):

(H) $$\limsup_{n\to\infty} n\ell_n < \infty$$

(H$_\infty$) $$\lim_{n\to\infty} n\ell_n = \infty.$$

An equivalent formulation of **(H)** is

$$\limsup_{\epsilon\to 0} \epsilon \mathrm{Card}\ \Lambda_\epsilon < \infty.$$

An equivalent formulation of (**H**$_\infty$) is

$$\lim_{\epsilon\to 0} \epsilon \mathrm{Card}\ \Lambda_\epsilon = \infty.$$

The assumption (**H**) implies $0 \leq \widehat{\alpha} < \infty$. One always has $\overline{\alpha} \leq \widehat{\alpha}$. One also has $\overline{\alpha} = \widehat{\alpha} = 0$ as soon as $\lim_n n\ell_n = 0$. Some of these assertions are easy to check. Others will be checked in the last section (Appendix).

To state our results, we also need to introduce the function

$$d_\alpha(\beta) = 1 + \alpha(\beta - 1 - \beta \log\, \beta) \tag{1.6}$$

defined for $\alpha \geq 0$ and $\beta \geq 0$. In the following, $\dim F$ denotes the Hausdorff dimension of a set F.

Theorem 1.1 (Case $\overline{\alpha} = 0$). *Assume* (**H**), *i.e.* $\limsup_{n\to\infty} n\ell_n < \infty$. *Suppose* $\overline{\alpha} = 0$. *With probability one, for all* $\beta \geq 0$ *such that* $d_{\widehat{\alpha}}(\beta) > 0$ *we have*

$$\dim(F_\beta) = \dim(\underline{F}_\beta) = \dim(\overline{F}_\beta) = 1. \tag{1.7}$$

Theorem 1.2 (Case $0 < \overline{\alpha} < \infty$). *Assume* (**H**), *i.e.* $\limsup_{n\to\infty} n\ell_n < \infty$. *Suppose* $0 < \overline{\alpha} < \infty$. *With probability one, for all* $\beta \geq 0$ *such that* $d_{\widehat{\alpha}}(\beta) > 0$, *we have*

$$\dim(F_\beta) = d_{\overline{\alpha}}(\beta) \tag{1.8}$$

and

$$F_\beta = \varnothing \quad (\forall \beta \geq 0,\ d_{\overline{\alpha}}(\beta) < 0). \tag{1.9}$$

If, moreover, $\overline{\alpha}$ *is defined by a limit (not just a limsup), (1.8) and (1.9) hold for* $\underline{F}_\beta$ *and* $\overline{F}_\beta$ *instead of* F_β.

Theorem 1.3 (Case $\overline{\alpha} = +\infty$). *Assume* ($\mathbf{H}_\infty$), *i.e.* $\lim_{n\to\infty} n\ell_n = \infty$. *Then almost surely we have*

$$\lim_{n\to\infty} \frac{N_n(t)}{L_n} = 1 \quad (\forall\ t \in \mathbb{T}).$$

Theorem 1.3 says that when ℓ_n tends slowly to zero (e.g. $\ell_n = \frac{\log n}{n}$), every point $t \in \mathbb{T}$ is covered by a same covering number of intervals. This is a new phenomenon, which was not known and which is not produced for moderate sequences like $\ell_n = \frac{\alpha}{n}$ (see Theorem 1.2). The quick sequences like $\frac{1}{n\log n}$ share another extreme property that all numbers are possible, according to Theorem 1.1. We may say that there are two "phase transitions", from quick sequences to moderate sequences and from moderate sequences to slow sequences.

Let us consider the following family of parameterized sequences

$$\ell_n = \frac{\alpha}{n \log^\gamma(n+1)}$$

where $\alpha > 0$ and $-\infty < \gamma \leq 1$ (remark that $\sum \ell_n < \infty$ if $\gamma > 1$). Then we have

1. if $0 < \gamma \leq 1$, then $\overline{\alpha} = \widehat{\alpha} = 0$ and the assumption (**H**) is satisfied.
2. if $\gamma = 0$, then $\overline{\alpha} = \widehat{\alpha} = \alpha > 0$ and the assumption (**H**) is satisfied;
3. if $\gamma < 0$, then $\overline{\alpha} = \widehat{\alpha} = \infty$ and the assumption ($\mathbf{H}_\infty$) is satisfied.

In this family we find representatives of all three cases.

Let us consider another family of sequences all of which tends quickly to zero. First notice that $\widehat{\alpha} = 0$ implies $\overline{\alpha} = 0$. So, when $\widehat{\alpha} = 0$, as corollary of Theorem 1.1, we get that with probability one the formula (1.7) holds for all $\beta \geq 0$. Here is a family of quick sequences satisfying (**H**): for n large enough we have

$$\ell_n = \frac{\alpha}{n\big(\log^{\circ\tau} n\big)^\gamma \prod_{j=1}^{\tau-1} \log^{\circ j} n}$$

where $\alpha > 0$, $\gamma \in (0,1)$, $\tau \geq 1$ is an integer and $\log^{\circ j} x$ means the j-fold composition of $\log x$. In this case, we have

$$L_n \sim \frac{\alpha}{1-\gamma}\big(\log^{\circ\tau} n\big)^{1-\gamma}.$$

The assumption $(\mathbf{H}_\infty)$ is satisfied by the following families of slow sequences (for n large enough):

$$\ell_n = \alpha \frac{\log^\gamma n}{n}$$

$(\alpha, \gamma > 0)$ for which $L_n \sim \frac{\alpha}{1+\gamma} \log^{1+\gamma} n$;

$$\ell_n = \alpha \frac{(\log^{\circ\tau} n)^\gamma}{n^\sigma}$$

$(\tau \geq 1, \alpha > 0,\ \gamma \geq 0,\ 0 < \sigma < 1)$ for which $L_n \sim \frac{\alpha}{1-\sigma} n^{1-\sigma} (\log^{\circ\tau} n)^\gamma$;

$$\ell_n = \frac{\alpha}{(\log^{\circ\tau} n)^\gamma}$$

$(\tau \geq 1,\ \alpha > 0, \gamma > 0)$ for which $L_n \sim \alpha n (\log^{\circ\tau} n)^{-\gamma}$.

The set F_0 (i.e. $\beta = 0$) contains the set $\mathcal{F} = \mathbb{T} \setminus \limsup I_n$ consisting of points which are only finitely covered. Points in $\mathcal{F}$ are described by $N_n(t) = O(1)$ and those in F_0 by $N_n(t) = o(L_n)$. The Shepp condition is an exact condition for $\mathcal{F} = \varnothing$. We don't know similar condition for $F_0 = \varnothing$. However, Theorem 1.1 and Theorem 1.2 show that for a regular sequence satisfying $\overline{\alpha} = \widehat{\alpha}$ we have

$$\overline{\alpha} < 1 \Rightarrow F_0 \neq \varnothing; \qquad \overline{\alpha} > 1 \Rightarrow F_0 = \varnothing.$$

It was known ([K1], p. 160) that $\dim \mathcal{F} = 1 - \overline{\alpha} > 0$ when $0 \leq \overline{\alpha} < 1$. Then, $F_0 \neq \varnothing$ and even $\dim F_0 \geq d_{\overline{\alpha}}(0) = 1 - \overline{\alpha} > 0$. So, new information provided by Theorem 1.2 for F_0 is that the preceding inequality is an equality. When $\overline{\alpha} = 1$, it is possible that $F_0 \neq \varnothing$ although $\dim F_0 = 0$. Indeed, it is the case for

$$\ell_n = \frac{1}{n} \left(1 - \frac{1+\delta}{\log n}\right)$$

with $\delta > 0$, for which the Shepp condition (1.3) is violated.

If $\widehat{\alpha} = \overline{\alpha} > 0$, as a corollary of Theorem 1.2 we get that with probability one, the formula (1.8) holds for all $\beta \geq 0$ such that $d_{\overline{\alpha}}(\beta) \geq 0$, and $F_\beta = \varnothing$ if $d_{\overline{\alpha}}(\beta) < 0$. It is the case when $\ell_n = \alpha/n$ with $\alpha > 0$. Recall that in this case $L_n \sim \alpha \log n$.

We treat the above Dvoretzky covering problem on the circle by a closely related Poisson covering of the real line which was introduced by B. Mandelbrot [M1, M2]. The idea was exploited in [K3] and [F3]. We point out that the idea of using Poisson processes was also used in [J1, J2] in a different context (covering with intervals of same size, or with sizes aX_n where $a > 0$ and $\{X_n\}$ is an i.i.d. sequence). Another idea comes from [B], which consists of simultaneously constructing a class of random measures, called Poisson multiplicative chaos, and simultaneously estimating their dimensions. Construction of single random measure is provided in [K2] in its full generality. Single measure corresponding to a fixed β was already introduced in [F3]. The main difference of the present paper from [F3] is that we are now able to prove that these single measures for different β's can be simultaneously constructed and their dimensions simultaneously computed.

We organize the paper as follows. In Section 2, we simultaneously construct Poisson multiplicative chaos, and we prove a lower bound for their

Hausdorff dimensions. Then we specify such a multiplicative chaos adapted to the study of Dvoretzky covering numbers. In Section 3, we prove that almost surely, each of multiplicative chaos is supported by one of the sets F_β. This, together with what we obtained in Section 2, yields the lower bounds for the Hausdorff dimensions of F_β's in Theorems 1.1 and 1.2. Section 4 is devoted to the study of upper bounds concerning Theorems 1.1 and 1.2. Section 5 proves Theorem 1.3. Section 6 states analogous results for covering numbers associated with the covering of real line by random Poisson intervals. The last Section 7 (appendix) discusses properties of the sequence $\{\ell_n\}$, which are useful throughout the paper.

2. Simultaneously constructed Poisson multiplicative chaos

The problem concerning the Dvoretzky covering will be converted into a similar problem related to a Poisson covering. That is to say, we will construct random measures using Poisson point processes. These measures are called Poisson multiplicative chaos (see [K2] for a general account of multiplicative chaos). We will calculate the Hausdorff dimensions of these random measures, because these measures are supported by the sets in questions, as we shall prove in Section 3.

2.1. **Dimensions of Poisson multiplicative chaos.** In this subsection, we show how to construct the needed random measures and state the results about their Hausdorff dimensions.

Let $\lambda = dt$ be the Lebesgue measure on $\mathbb{R}$ and let μ be a measure on $\mathbb{R}^+ = (0, +\infty)$ which is assumed finite on compact subsets and concentrated on the interval $(0,1)$. The product measure $\nu = \lambda \otimes \mu$ is defined on the upper plan $\mathbb{R} \times \mathbb{R}^+$. We consider the Poisson point process (X_n, Y_n) with intensity ν. For a Borel subset B of $\mathbb{R} \times \mathbb{R}^+$, define

$$N(B) = \text{Card}\left(\{(X_n, Y_n)\} \cap B\right).$$

For $t \in \mathbb{R}$ and $0 < \epsilon < 1$, denote

$$D_\epsilon(t) = \left\{(x,y) \in \mathbb{R} \times \mathbb{R}^+ \,:\, 1 > y \geq \epsilon, t - y < x < t\right\}.$$

For a fixed positive number $0 < a < \infty$, we define

$$P_\epsilon^a(t) = a^{N_\epsilon^P(t)} \exp[(1-a)\nu(D_\epsilon(t))] \qquad (t \in \mathbb{R}, \quad \epsilon > 0) \tag{2.1}$$

where

$$N_\epsilon^P(t) = N\big(D_\epsilon(t)\big)$$

is the number of points in the domain $D_\epsilon(t)$ of the Poisson process with intensity ν. In the setting of Poisson covering, $N_\epsilon^P(t)$ is also called the number of intervals $(X_n, X_n + Y_n)$ with $Y_n \geq \epsilon$ which cover t, i.e. $t \in (X_n, X_n + Y_n)$.

We use P_ϵ^a to denote the measure $P_\epsilon^a(t)dt$ restricted on the interval $[0,1]$. According to [K2], for every fixed $0 < a < \infty$ the random measure $P_\epsilon^a(t)dt$ converges almost surely to a limit random measure as $\epsilon \to 0$. We will prove that under some condition on ν there exists an interval J of a such that with probability one the random measure $P_\epsilon^a(t)dt$ converges for *every* $a \in J$. In order to give a precise statement, we need the following notation. We define

$$\overline{\alpha}^P = \limsup_{\epsilon\to 0} \frac{\nu(D_\epsilon(t))}{-\log\epsilon} \tag{2.2}$$

$$\widehat{\alpha}^P = \inf_{b\geq 2} \limsup_{n\to\infty} \sup_{m\geq 1} \frac{\nu(D_{b^{-(n+m)}}(t)\setminus D_{b^{-n}}(t))}{\log b^m}. \tag{2.3}$$

Notice that both $\nu(D_\epsilon(t))$ and $\nu(D_{b^{-(n+p)}}(t))\setminus D_{b^{-n}}(t))$ do not depend on t, so sometimes we will write $\nu(D_\epsilon)$ for $\nu(D_\epsilon(t))$. Also notice that $\overline{\alpha}^P \leq \widehat{\alpha}^P$.

We will need an analog of the assumption (**H**) involved for the Dvoretzky covering, namely :

(**HP**) $$\limsup_{\epsilon\to 0} \epsilon\mu([\epsilon,1)) < \infty.$$

Under (**HP**), Fubini Theorem shows that both $\overline{\alpha}^P$ and $\widehat{\alpha}^P$ are finite; moreover, when

$$\int_{[0,1]} \exp\left\{\int_{(t,1)} \mu((s,1))\,ds\right\} dt < \infty$$

(for instance when $\overline{\alpha}^P < 1$), the Hausdorff dimension of the uncovered set $\mathcal{R} = \mathbb{R}_+ \setminus \cup_n(X_n, X_n+Y_n)$ is equal to $1-\overline{\alpha}^P$. It is actually equal to the lower index of the Laplace exponent associated with the subordinator range $\mathbb{R}\setminus\cup_{n,X_n\geq 0}(X_n, X_n+Y_n)$ (see [Be] or [FiFrS]).

For a measure σ, $\dim\sigma$ denotes the Hausdorff dimension of the measure, or more precisely the lower Hausdorff dimension in the terminology of [F2]. That is to say, there is no charge on any Borel set with Hausdorff dimension strictly smaller than $\dim\sigma$ but some Borel set of dimension $\dim\sigma$ is charged by the measure. See also [Mat].

Theorem 2.1. *Suppose $\overline{\alpha}^P = 0$. Then*

(*i*) *With probability one, for all $a > 0$, the measure P_ϵ^a converges, as $\epsilon \to 0$, to a positive measure P^a whose support is $[0,1]$.*

(*ii*) *Assume moreover that* (**HP**) *is satisfied. With probability one, for all $a > 0$ such that $d_{\widehat{\alpha}^P}(a) > 0$, we have $\dim P^a = 1$.*

Theorem 2.2. *Suppose $0 < \overline{\alpha}^P < \infty$. Then*

(*i*) *With probability one, for all $a > 0$ such that $d_{\overline{\alpha}^P}(a) > 0$, the measure P_ϵ^a converges, as $\epsilon \to 0$, to a positive measure P^a whose support is $[0,1]$.*

(*ii*) *Assume moreover that* (**HP**) *is satisfied. With probability one, for all $a > 0$ such that $d_{\widehat{\alpha}^P}(a) > 0$ we have $\dim P^a \geq d_{\overline{\alpha}^P}(a)$.*

The parts (i) of these theorems will be proved in the subsection 2.3. and the parts (ii) in the subsection 2.4. In the next subsection 2.2., we mainly show how to construct simultaneously the measures P^a.

2.2. **Simultaneously constructed b-adic multiplicative chaos.** In order to prove Theorem 2.1 and Theorem 2.2, we convert our problem to one on a b-adic tree.

Fix an integer $b \geq 2$. For any integer $m \geq 0$ we denote by A^m the set of finite words of length m on the alphabet $\{0,\dots,b-1\}$ (by convention, $A^0 = \{\varnothing\}$). We use $|w|$ to denote the length m of $w \in A^m$ and I_w to denote the closed b-adic subinterval $[\sum_{i=1}^m w_i b^{-i}, b^{-m}+\sum_{i=1}^m w_i b^{-i}]$ of $[0,1]$ naturally encoded by $w = w_1 \cdots w_m$. Let $A^* = \bigcup_{m=0}^{\infty} A^m$ and $\partial A^* = \{0,\dots,b-1\}^{\mathbb{N}}$. The set $A \cup \partial A$ is equipped with the concatenation operation. For $w \in A^*$, $C_w = w\partial A$ denotes the cylinder determined by w, i.e. $C_w = \{ww' : w' \in \partial A^*\}$. Let $\mathcal{A}$ be the σ-field of ∂A^* generated by all cylinders.

Let π be the mapping from ∂A^* into $[0,1]$ defined by

$$\pi(\tilde{t}) = \sum_{i=1}^{\infty} \frac{\tilde{t}_i}{b^i} \qquad (\tilde{t} = \tilde{t}_1 \dots \tilde{t}_i \dots \in \partial A^*).$$

Let $\tilde{\lambda}$ be the natural measure on $(\partial A^*, \mathcal{A})$ defined by $\tilde{\lambda}(C_w) = b^{-|w|}$ for all $w \in A^*$. Notice that λ, the restriction on $[0,1]$ of the Lebesgue measure, is the image of $\tilde{\lambda}$ under π, i.e. $\lambda = \tilde{\lambda} \circ \pi^{-1}$.

For $0 < a < \infty$ and $\epsilon > 0$, we denote by $\widetilde{P}_{\epsilon}^a$ the measure on $(\partial A^*, \mathcal{A})$ whose density with respect to $\tilde{\lambda}$ is equal to $P_{\epsilon}^a(\pi(\tilde{t}))$.

The $\widetilde{P}_{\epsilon}^a$-mass of the cylinder C_w will be denoted by

$$Y_{\epsilon}(w,a) = \widetilde{P}_{\epsilon}^a(C_w)$$

and it can be written as

$$Y_{\epsilon}(w,a) = \int_{C_w} P_{\epsilon}^a(\pi(\tilde{t}))d\tilde{t} = \int_{I_w} P_{\epsilon}^a(t)dt \tag{2.4}$$

where $I_w = \pi(C_w)$.

The essential point of Theorem 2.1 (i) and Theorem 2.2 (i) is the following proposition that we will prove by studying the family indexed by w of functional martingales $\{Y_{b^{-|w|+m}}(w,\cdot)\}_{m\geq 1}$.

Proposition 2.3. *Let $b \geq 2$ be an integer and let K be a compact subinterval of $\mathbb{R}_+^*$. Suppose $\overline{\alpha}^P < \infty$ and $\inf_{a\in K} d_{\overline{\alpha}^P}(a) > 0$. Then, with probability one, for all $w \in A^*$, the function $Y_{\epsilon}(w,\cdot)$ converges uniformly on K, as $\epsilon \to 0$, to a positive analytic function $Y(w,\cdot)$.*

Corollary 2.4. *We make the same assumptions as in Proposition 2.3. With probability one, for all $a \in K$, the measure $\widetilde{P}_{\epsilon}^a$ converges weakly, as $\epsilon \to 0$, to a measure $\widetilde{P}^a$ such that $\widetilde{P}^a(C_w)=Y(w,a)$ for every $w \in A^*$. Moreover, the support of $\widetilde{P}^a$ is ∂A^*. Consequently, the measure P_{ϵ}^a converges weakly, as $\epsilon \to 0$, to the measure $P^a = \widetilde{P}^a \circ \pi^{-1}$, whose support is $[0,1]$.*

For $w \in A^*$, the restriction of $\widetilde{P}_{b^{-|w|}\epsilon}^a$ to C_w can be written as

$$d\widetilde{P}_{b^{-|w|}\epsilon}^a = \widetilde{P}_{b^{-|w|}}^a \cdot d\widetilde{P}_{\epsilon}^{a,C_w} \tag{2.5}$$

where $\widetilde{P}_{\epsilon}^{a,C_w}$ is the measure on $(C_w, w\mathcal{A})$ whose density with respect to $\tilde{\lambda}$ is

$$\frac{d\widetilde{P}_{\epsilon}^{a,C_w}}{d\tilde{\lambda}}(\tilde{t}) = \frac{\widetilde{P}_{b^{-|w|}\epsilon}^a(\tilde{t})}{\widetilde{P}_{b^{-|w|}}(\tilde{t})}.$$

Notice that

$$\frac{\widetilde{P}_{b^{-|w|}\epsilon}^a(\tilde{t})}{\widetilde{P}_{b^{-|w|}}(\tilde{t})} = a^{N_{b^{-|w|}\epsilon}^P(\pi(\tilde{t})) - N_{b^{-|w|}}^P(\pi(\tilde{t}))} \exp\left[(1-a)\nu\left(D_{b^{-|w|}\epsilon} \setminus D_{b^{-|w|}}\right)\right]. \tag{2.6}$$

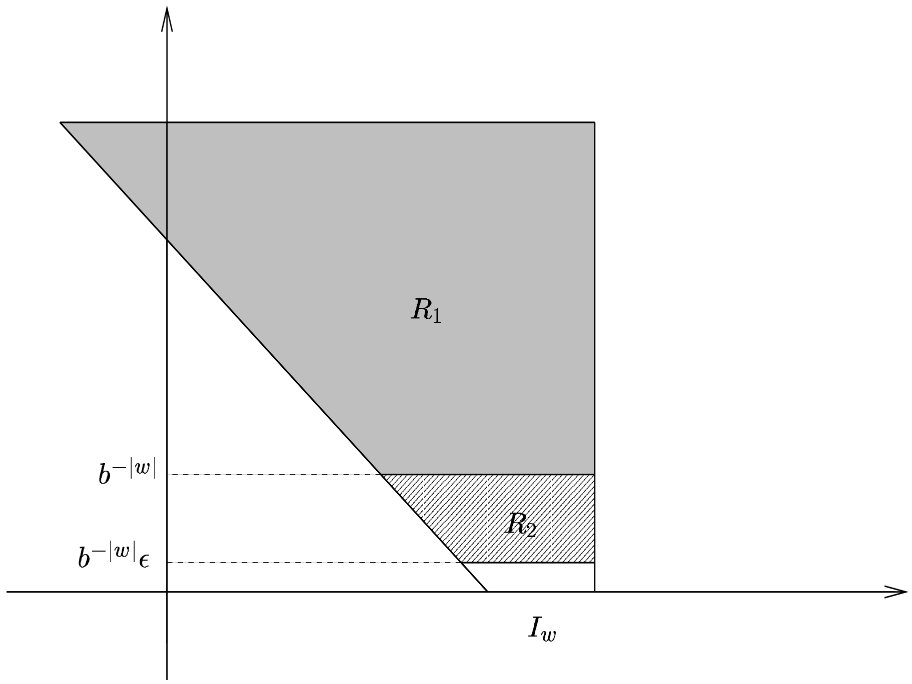

FIGURE 1. *The regions defining $\tilde{P}^a_{b^{-|w|}}$ and $\tilde{P}^{a,C_w}_\epsilon$*

This is a consequence of the following decomposition: for any $0<\epsilon,\epsilon'<1$ we have

$$D_{\epsilon'\epsilon}(t)=(D_{\epsilon'\epsilon}(t)\setminus D_{\epsilon'}(t))\cup D_{\epsilon'}(t).$$

The $\widetilde{P}^{a,C_w}_\epsilon$-mass of C_w, magnified by $b^{|w|}$, will be denoted by

$$Z_\epsilon(w,a)=b^{|w|}\widetilde{P}^{a,C_w}_\epsilon(C_w)$$

and it can be written as

$$Z_\epsilon(w,a)=b^{|w|}\int_{C_w}\frac{\widetilde{P}^a_{b^{-|w|}\epsilon}(\tilde{t})}{\widetilde{P}_{b^{-|w|}}(\tilde{t})}d\tilde{t}=b^{|w|}\int_{I_w}\frac{P^a_{b^{-|w|}\epsilon}(t)}{P_{b^{-|w|}}(t)}dt. \tag{2.7}$$

Fix an integer $b\geq 2$. Define

$$\overline{\alpha}^P_b=\limsup_{n\to\infty}\sup_{m\geq 1}\frac{\nu(D_{b^{-(n+m)}}(t)\setminus D_{b^{-n}}(t))}{\log b^m}.$$

The following proposition will be useful for proving Theorem 2.1 (ii) and Theorem 2.2 (ii). We will prove the proposition by studying the family indexed by w of functional martingales $\{Z_{b^{-m}}(w,\cdot)\}_{m\geq 1}$.

Proposition 2.5. *Let $b\geq 2$ be an integer and let K be a compact subinterval of $\mathbb{R}_+^*$.*

(i) Suppose $\overline{\alpha}^P<\infty$ and $\inf_{a\in K} d_{\overline{\alpha}^P}(a)>0$. Then, with probability one, for all $w\in A^*$, the function $Z_\epsilon(w,\cdot)$ converges uniformly on K, as $\epsilon\to 0$, to a positive analytic function $Z(w,\cdot)$.

(ii) Suppose moreover that $\inf_{a\in K} d_{\widehat{\alpha}^P}(a)>0$ and choose b such that $\overline{\alpha}^P_b$

is close enough to $\widehat{\alpha}^P$ *to insure* $\inf_{a\in K} d_{\overline{\alpha}_b^P}(a) > 0$. *There exists* $p > 1$ *such that*

$$\sup_{w\in A^*, a\in K} \mathbb{E}(Z(w,a))^p < \infty, \qquad \sup_{w\in A^*, a\in K} \mathbb{E}\left(\left|\frac{dZ(w,a)}{da}\right|^p\right) < \infty.$$

It is important to point out that the restrictions on C_w of the measures $\tilde{P}^a_{b^{-|w|}}$ and $\tilde{P}^{a,C_w}_\epsilon$ are independent since they involve respectively Poisson points in two disjoint regions. See Figure 1: the bigger region R_1 for $\tilde{P}^a_{b^{-|w|}}$ and the smaller one R_2 for $\tilde{P}^{a,C_w}_\epsilon$.

2.3. **Proofs of Propositions 2.3 and 2.5.** We give here the proofs of Propositions 2.3 and 2.5 the first of which allows us to construct simultaneously the measures $\widetilde{P}^a$ and the second will be used in the proofs of Theorems 2.1 (ii) and 2.2 (ii).

Proof of Proposition 2.3 We shall consider K as a compact subset in the complex plan.

It is clear that $P^z_\epsilon(t)$ is well defined and is an analytic function of $z \in \mathbb{C}$. For any $w \in A^*$ and any $m \geq 0$, consider the function $\widehat{Y}_m(w,z)$ of z defined by

$$\widehat{Y}_m(w,z) = \int_{I_w} P^z_{b^{-|w|-m}}(t)dt.$$

By writing

$$\widehat{Y}_m(w,z) = \int_{I_w} z^{N^P_{b^{-|w|-m}}(t)} \exp[(1-z)\nu(D_{b^{-|w|-m}})]\, dt,$$

we see that it is an analytic extension into the complex plan of $Y_{b^{-|w|-m}}(w,a)$ as function of $a > 0$.

First step. We first prove that there exist $1 < p \leq 2$, a bounded complex neighborood D of K and $\varepsilon_D > 0$ such that

$$\sup_{z\in D} \mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p) \leq Cb^{-(|w|+m+1)\varepsilon_D}. \tag{2.8}$$

where C is a constant independent of m.

In order to prove (2.8), we write

$$\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z) = \int_{I_w} U(t)V(t)\, dt$$

with $U(t) = P^z_{b^{-|w|-m}}(t)$ and $V(t) = P^z_{b^{-|w|-m-1}}(t)/P^z_{b^{-|w|-m}}(t) - 1$. Let

$$\epsilon_m = b^{-|w|-m-1}.$$

Then we can write

$$\begin{aligned} U(t) &= z^{N^P_{\epsilon_{m-1}}(t)} \exp[(1-z)\nu(D_{\epsilon_{m-1}})] \\ V(t) &= z^{N^P_{\epsilon_m}(t) - N^P_{\epsilon_{m-1}}(t)} \exp[(1-z)\nu(D_{\epsilon_m} \setminus D_{\epsilon_{m-1}})] - 1. \end{aligned}$$

We divide I_w into b^m equal subintervals and denote by J_w the first one from the left. For $t \in J_w$ and $0 \leq k \leq b^m - 1$, define

$$U_k(t) = U(t + kb^{-|w|-m}), \qquad V_k(t) = V(t + kb^{-|w|-m}).$$

Then for $i \in \{0,1\}$ define

$$S_i(t) = \sum_{0\le 2k+i\le b^m-1} U_{2k+i}(t)V_{2k+i}(t).$$

By changes of variables, we get the following expression

$$\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z) = \int_{J_w} (S_1(t)+S_2(t))\ dt.$$

Now by using the Jensen inequality and the elementary inequality

$$|x+y|^p \le 2^{p-1}(|x|^p+|y|^p),$$

we get

$$\begin{aligned} & |\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p \\ \le\ & |J_w|^p \int_{J_w} |S_0(t)+S_1(t)|^p \frac{dt}{|J_w|} \\ \le\ & 2^{p-1}|J_w|^{p-1} \int_{J_w} (|S_0(t)|^p + |S_1(t)|^p)\ dt. \end{aligned} \tag{2.9}$$

We are then led to estimate $\mathbb{E}|S_i(t)|^p$. For the sake of convenience, we introduce the following function

$$\varphi(p,z) = (1-\Re(z))p + |z|^p - 1 \qquad (p\in\mathbb{R}, z\in\mathbb{C}). \tag{2.10}$$

Since the measure ν is invariant by horizontal translation, all (U_k,V_k) have the same distribution. Since $\mathbb{E}z^X = e^{v(z-1)}$ holds for any complex number z and any Poisson variable X with parameter v, a simple computation yields that for $p\in\mathbb{R}$

$$\mathbb{E}(|U_k(t)|^p) = \exp\left[\nu(D_{\epsilon_{m-1}})\varphi(p,z)\right] \tag{2.11}$$

and for $p>1$

$$\begin{aligned} \mathbb{E}(|V_k(t)|^p) &\le 2^{p-1}\left(1+\exp\left[\nu(D_{\epsilon_m}\setminus D_{\epsilon_{m-1}})\varphi(p,z)\right]\right) \\ &\le 2^p \exp\left[\nu(D_{\epsilon_m}\setminus D_{\epsilon_{m-1}})\varphi(p,z)\right], \end{aligned} \tag{2.12}$$

where for the first inequality we used once more the above elementary inequality and for the second one we used the fact that the mapping $\varphi(p,z)$ is a convex function of p, null at $p=0$ and non negative at $p=1$, so non-negative on $[1,\infty)$.

Moreover, by construction, $\sigma(U_k;\ 0\le k\le b^m-1)$ and $\sigma(V_k;\ 0\le k\le b^m-1)$ are independent, and the V_{2k}'s are mutually independent, as well as the V_{2k+1}'s. Indeed, if t and t' are two points in I_w having a distance at least $b^{-|w|+m}$, then

$$(D_{\epsilon_m}(t)\setminus D_{\epsilon_{m-1}}(t)) \cap (D_{\epsilon_m}(t')\setminus D_{\epsilon_{m-1}}(t')) = \varnothing.$$

This implies the independence.

Now we can apply the following lemma to estimate $\mathbb{E}|S_0(t)|^p$ and $\mathbb{E}|S_1(t)|^p$.

Lemma 2.6 (Von Bahr-Esseen [vBaH]). *Let $(U_i)_{i\ge0}$ and $(V_i)_{i\ge0}$ be two sequences of complex random variables such that $\sigma(U_i;\, i\ge0)$ and $\sigma(V_i;\, i\ge 0)$ are independent and that the V_i's are mutually independent. Assume that*

$\sum_{i\geq 0} U_i V_i$ *is almost surely defined and that* V_i *is integrable with mean* 0 *for all* $i \geq 0$*. Then for every* $p \in [1,2]$

$$\mathbb{E}\left|\sum_{i\geq 0} U_i V_i\right|^p \leq 2^p \sum_{i\geq 0} \mathbb{E}|U_i|^p \mathbb{E}|V_i|^p.$$

It follows from this lemma and (2.11) and (2.12) that for $p \in (1,2]$ we have

$$\mathbb{E}(|S_1(t)|^p) + \mathbb{E}(|S_2(t)|^p) \leq 2^{2p} b^m \exp\left[\nu(D_{\epsilon_m})\varphi(p,z)\right].$$

This, together with (2.9) yields

$$\mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p) \leq C\epsilon_m^{p-1} \exp\left[\nu(D_{\epsilon_m})\varphi(p,z)\right] \tag{2.13}$$

where $C = 2^{3p-1} b^{-|w|+p-1}$. We now compare $\nu(D_{\epsilon_m})$ with $-\log \epsilon_m$:

$$\nu(D_{\epsilon_m}) = (-\log \epsilon_m)\left(\overline{\alpha}^P + \left(\frac{\nu(D_{\epsilon_m})}{-\log \epsilon_m} - \overline{\alpha}^P\right)\right).$$

This allows us to rewrite the right hand side of (2.13) as follows

$$C\epsilon_m^{\left(p-1-\overline{\alpha}^P\varphi(p,z)\right)} \cdot \epsilon_m^{-\left(\frac{\nu(D_{\epsilon_m})}{-\log\epsilon_m} - \overline{\alpha}^P\right)\varphi(p,z)}. \tag{2.14}$$

Consider

$$\Phi(p,z) = p - 1 - \overline{\alpha}^P \varphi(p,z).$$

We have $\Phi(1,z) = 0$ whenever $z \in K$. Moreover, we have $\frac{\partial\Phi(1,z)}{\partial p} = d_{\overline{\alpha}^P}(z) \quad (z \in K)$; hence our assumption is

$$\inf_{z\in K} \frac{\partial \Phi(1,z)}{\partial p} > 0.$$

So we can choose $1 < p \leq 2$ close enough to 1 such that

$$3\varepsilon_D := \inf_{z\in K} \Phi(p,z) > 0. \tag{2.15}$$

Now, by continuity of $\Phi(p,z)$ in $z \in \mathbb{C}$ we can choose a bounded complex neighborhood D of K such that

$$\inf_{z\in D} \Phi(p,z) \geq 2\varepsilon_D.$$

On the other hand, by the definition of $\overline{\alpha}^P$, the fact $\varphi(p,z) \geq 0$ and the boundedness of D, for large m we have

$$\left(\frac{\nu(D_{\epsilon_m})}{-\log\epsilon_m} - \overline{\alpha}^P\right)\varphi(p,z) \leq \varepsilon_D. \tag{2.16}$$

Therefore it follows from (2.13)–(2.16) that for large m we have

$$\sup_{z\in D} \mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p) \leq C\epsilon_m^{\varepsilon_D} = Cb^{-(|w|+m+1)\varepsilon_D}.$$

This inequality holds for all $m \geq 1$ if we change C to be a suitable constant.

Second step. We follow the idea of Biggins [Bi]. Apply the Cauchy formula to get the uniform convergence of $\widehat{Y}_m(w,\cdot)$ on the compact subsets of D as $m \to \infty$.

Fix an arbitrary non-empty closed disk $D(z_0, 2\rho) \subset D$. For $z \in D(z_0, \rho)$ and $m \geq 0$, the Cauchy formula yields

$$\begin{aligned}&|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)| \\ &\leq \frac{1}{\pi} \int_0^{2\pi} |\widehat{Y}_{m+1}(w, z_0 + 2\rho e^{it}) - \widehat{Y}_m(w, z_0 + 2\rho e^{it})| dt.\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E} &\sup_{z\in D(z_0,\rho)} |\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)| \\ &\leq 2 \sup_{z\in D(z_0,2\rho)} \mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|) \\ &\leq 2 \sup_{z\in D(z_0,2\rho)} (\mathbb{E}|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p)^{1/p}.\end{aligned}$$

By the estimate (2.8) that we got in the *first step*, we obtain

$$\mathbb{E} \sum_{m=1}^{\infty} \sup_{z\in D(z_0,\rho)} |\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)| = O\left(\sum_{m=1}^{\infty} b^{-(|w|+m+1)\frac{\varepsilon_D}{p}}\right) < \infty.$$

This implies that almost surely $\widehat{Y}_m(w,\cdot)$ converges uniformly on $D(z_0,\rho)$. It follows that almost surely $\widehat{Y}_m(w,\cdot)$ converges uniformly on some neighborhood of K to an analytic function $Y(w,\cdot)$.

Third step. Now we prove that almost surely the function $Y_\epsilon(w,\cdot)$ converges uniformly on K to $Y(w,\cdot)$ as $\epsilon \to 0$ continuously. What we proved in the *second step* is the convergence as $\epsilon \to 0$ along a discrete sequence.

As in the second step, we apply the Cauchy formula to estimate the derivative $\frac{d\widehat{Y}_m(w,z)}{dz}$. In fact,

$$\begin{aligned}\mathbb{E}\left(\sup_{z\in D(z_0,\rho)} \left|\frac{d\widehat{Y}_m(w,z)}{dz}\right|^p\right)^{1/p} &\leq \frac{2}{\rho} \sup_{z\in D(z_0,\rho)} \Big(\mathbb{E}(|\widehat{Y}_m(w,z)|^p)\Big)^{1/p} \\ \leq \frac{2}{\rho} \sup_{z\in D(z_0,\rho)} \Big(\mathbb{E}(|\widehat{Y}_0(w,z)|^p)\Big)^{1/p} & \\ +\frac{2}{\rho} \sum_{m=0}^{\infty} \sup_{z\in D(z_0,\rho)} &\Big(\mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p)\Big)^{1/p}.\end{aligned}$$

Consequently, by the Fatou lemma we have

$$\mathbb{E}\left(\sup_{a\in K} \left|\frac{Y(w,a)}{da}\right|^p\right) < \infty.$$

From this, the fact $\mathbb{E}|Y(w,a)|^p < \infty$ ($\forall a \in \mathbb{R}$) and the mean value theorem, it follows that

$$\mathbb{E}(\sup_{a\in K} |Y(w,a)|^p) < \infty \tag{2.17}$$

(N.B. It is possible to obtain (2.17) without using the above estimates of derivative. However this approach of derivative estimation will be indispensable in the proof of Proposition 2.5).

For $t \geq 1$, denote by $\mathbb{F}_t$ the sub-σ-field of the Borel σ-field of $\big(C(K,\mathbb{R}), \| \|_\infty\big)$ generated by the random continuous functions

$$a \in K \mapsto Y_{1/t'}(w,a), \qquad (1 < t' \leq t).$$

Define

$$M_t(\cdot) = \mathbb{E}(Y(w,\cdot)|\mathbb{F}_t)$$

(which is well defined by (2.17)). It is clear that $(M_t(\cdot), \mathbb{F}_t)_{t\geq 1}$ is a martingale taking values in $C(K,\mathbb{R})$. It follows from Proposition V-2-6 of [N] that if the martingale $M_t(\cdot)$ is right continuous, then it converges almost surely in $C(K,\mathbb{R})$ to $Y(w,\cdot)$. But this is indeed the case since we learn from the second step that for every $m \geq 0$, we have

$$\mathbb{E}(Y(w,\cdot)|\mathbb{F}_{b^m}) = Y_{b^{-m}}(w,\cdot)$$

and that $\big(M_t(\cdot) = Y_{1/t}(w,\cdot), \mathbb{F}_t\big)_{t\geq 1}$ is a right continuous martingale.

Fourth step. We prove that almost surely $Y(w,a) > 0$ for all $a \in K$. We assume $K = [0,1]$ without loss of generality.

For any subinterval J of K and any $w \in A^*$, let

$$S_J^w = \{\omega \in \Omega : \ \exists\, a \in J \ \text{ such that } \ Y(w,a) = 0\}.$$

It is straightforward to verify that the event S_J^w belongs to $\bigcap_{n\geq 1} \mathcal{A}_n$ where $\mathcal{A}_n$ is the σ-field generated by the Poisson process restricted in the strip $\mathbb{R} \times (0, 1/n]$. The Kolmogorov zero-one law shows that the probability of the tail event S_J^w is equal to 0 or 1. We claim that $\mathbb{P}(S_K^w) = 0$.

Otherwise, $S_{[0,1]}^w$ has probability one. Then either $S_{[0,1/2]}^w$ or $S_{[1/2,1]}^w$ has positive probability. As we have seen above, this positive probability must be 1. Assume, for example, $S_{[0,1/2]}^w$ has probability one. Then, either $S_{[0,1/4]}^w$ or $S_{[1/4,1]}^w$ has probability one. Consequently, there exists a decreasing sequence $(J_n)_{n\geq 0}$ of dyadic intervals such that $\mathbb{P}(S_{J_n}^w) = 1$ for all $n \geq 0$. Let a_0 be the unique point in $\bigcap_{n\geq 0} J_n$. By the continuity of $Y(w,\cdot)$, we have $\mathbb{P}(Y(w,a_0) = 0) = 1$. However $Y(w,a_0)$ is the limit of a positive mean L^p-bounded martingale (see Second step). So, $Y(w,a_0)$ cannot be zero with probability one. This contradiction proves the claim.

Since A^* is countable, all the previous results hold almost surely and simultaneously for all $w \in A^*$. □

Proof of Corollary 2.4. It follows from Proposition 2.3 that with probability one, for any $a \in K$ and for any cylinder C_w we have

$$\lim_{\epsilon\to 0} \widetilde{P}_\epsilon^a(C_w)) = Y(w,a)$$

(the convergence is uniform on $a \in K$ for any w). Since ∂A^* is totally disconnected, it follows that with probability one, for any $a \in K$, the measure $\widetilde{P}_\epsilon^a$ converges weakly to a measure $\widetilde{P}^a$ such that

$$\widetilde{P}^a(C_w) = Y(w,a) \qquad (\forall w \in A^*).$$

Consequently, with probability one, for all $a \in K$, P_ϵ^a converges weakly to $P^a = \widetilde{P}^a \circ \pi^{-1}$, since $P_\epsilon^a = \widetilde{P}_\epsilon^a \circ \pi^{-1}$. The support of these limit measures is $[0,1]$ because of the positivity of $a \in K \mapsto Y(w,a)$ (see Proposition 2.3). □

Proof of Proposition 2.5.

(i) We proceed as in the proof of Proposition 2.3. Choose the same bounded deterministic complex neighborhood D of K as in the proof of Proposition 2.3. For $w \in A^*$ and $m \geq 0$, consider the function

$$\widehat{Z}_m(w,\cdot) := Z_{b^{-m}}(w,\cdot).$$

Its analytic extension, denoted by $\widehat{Z}_m(w,z)$, has the following expression:

$$b^{|w|}\int_{I_w} z^{N^P_{b^{-|w|-m}}(t)-N^P_{b^{-|w|}}(t)} \exp[(1-z)\nu(D_{b^{-|w|-m}}(t)\setminus D_{b^{-|w|}}(t))]\,dt.$$

It follows from computations similar to those in the first step of the proof of Proposition 2.3 that there exist $1 < p \leq 2$ and $C > 0$ such that for $z \in D$ we have

$$\begin{aligned}&\mathbb{E}(|\widehat{Z}_{m+1}(w,z)-\widehat{Z}_m(w,z)|^p)\\ &\leq Cb^{-(m+1)[p-1-\overline{\alpha}^P\varphi(p,z)]}b^{(m+1)\left(\frac{\nu(D_{b^{-|w|-m-1}})-\nu(D_{b^{-|w|}})}{\log b^{m+1}}-\overline{\alpha}^P\right)\varphi(p,z)}.\end{aligned}$$

We notice that

$$\overline{\alpha}^P \geq \limsup_{m\to\infty}\frac{\nu(D_{b^{-|w|-m-1}})-\nu(D_{b^{-|w|}})}{\log b^{m+1}}.$$

So, we can conclude as in the proof of Proposition 2.3.

(ii) For $m \geq 0$, $z \in \mathbb{C}$ and $p \in (1,2]$ we also have

$$\begin{aligned}&\mathbb{E}(|\widehat{Z}_{m+1}(w,z)-\widehat{Z}_m(w,z)|^p)\\ &\leq Cb^{-(m+1)[p-1-\overline{\alpha}^P_b\varphi(p,z)]}b^{(m+1)\left(\frac{\nu(D_{b^{-|w|-m-1}})-\nu(D_{b^{-|w|}})}{\log b^{m+1}}-\overline{\alpha}^P_b\right)\varphi(p,z)}\end{aligned}$$

(where $\overline{\alpha}^P$ is replaced by $\overline{\alpha}^P_b$). Since our assumption is $\inf_{a\in K} d_{\overline{\alpha}^P_b}(a) > 0$, the same arguments as those used in proving Proposition 2.3 allow to choose p close enough to 1 as well as a bounded complex neighborhood D of K such that

$$2\varepsilon_D := \inf_{z\in D}\left[p-1-\overline{\alpha}^P_b\varphi(p,z)\right] > 0.$$

By the definition of $\overline{\alpha}^P_b$ and the boundedness of D, we can fix $n_0 \geq 1$ such that for all $w \in A^*$ with $|w| \geq n_0$ and all $m \geq 0$, we have

$$\left(\frac{\nu(D_{b^{-|w|-m-1}})-\nu(D_{b^{-|w|}})}{\log b^{m+1}}-\overline{\alpha}^P_b\right)\varphi(p,z) \leq \varepsilon_D.$$

It follows that for all $w \in A^*$ with $|w| \geq n_0$ and all $m \geq 0$ we have

$$\mathbb{E}(|\widehat{Z}_{m+1}(w,z)-\widehat{Z}_m(w,z)|^p) \leq Cb^{-(m+1)\varepsilon_D}$$

for some suitable constant $C > 0$. Then, the conclusion follows from computations similar to those used to get (2.17) in the proof of Proposition 2.3, together with the fact that $\widehat{Z}_0(w,\cdot) \equiv 1$. $\square$

2.4. **Proofs of Theorem 2.1 and 2.2.** Theorem 2.1 and Theorem 2.2 will be proved at the same time.

(i) Let $J = \{a > 0;\ d_{\overline{\alpha}^P}(a) > 0\}$ (N.B. $J = (0, \infty)$ if $\overline{\alpha} = 0$). The interval J can be approximated by an increasing sequence of compact subintervals $(K_n)_{n\geq 1}$ of J. Since $d_{\overline{\alpha}^P}(\cdot)$ is continuous and is positive on J, we have $\inf_{a\in K_n} d_{\overline{\alpha}^P}(a) > 0$ for all $n \geq 1$. So we can apply Corollary 2.4 to get (i) in both Theorems 2.1 and 2.2.

(ii) Let $(K_n)_{n\geq 1}$ be an increasing sequence of compact subintervals of $\widehat{J} = \{a > 0;\ d_{\widehat{\alpha}^P}(a) > 0\}$ such that $\widehat{J} = \cup_{n\geq 1} K_n$.

Fix $K = K_n$. Take $p > 1$ and $b \geq 2$ as in Proposition 2.5 (ii).

Since $P^a = \widetilde{P}^a \circ \pi^{-1}$, by the Billingsley Lemma (see also [F2]), it suffices to show that

$$\mathbb{P}\text{-}a.s.\ \ \forall a \in K \ \text{ we have } \ \widetilde{P}^a(E(a)) > 0 \tag{2.18}$$

where

$$E(a) = \left\{ \tilde{t} \in \partial A^* : \liminf_{n\to\infty} \frac{\log \widetilde{P}^a(C_{\tilde{t}_1\ldots\tilde{t}_n})}{\log b^{-n}} \geq d_{\overline{\alpha}^P}(a) \right\}.$$

Even, it suffices to show, for any $\varepsilon > 0$, that

$$\mathbb{P}\text{-}a.s.\ \ \forall a \in K \ \text{ we have } \ \widetilde{P}^a\left(\liminf_n E_{n,\varepsilon}(a)\right) > 0 \tag{2.19}$$

where

$$E_{n,\varepsilon}(a) = \left\{ \tilde{t} \in \partial A^* : \ \frac{\log \widetilde{P}^a(C_{\tilde{t}_1\ldots\tilde{t}_n})}{\log b^{-n}} \geq d_{\overline{\alpha}^P}(a) - \varepsilon \right\}.$$

In order to prove (2.19), by the Borel-Cantelli Lemma, it suffices to show that for every $\varepsilon > 0$ we have

$$\mathbb{P}\text{-}a.s.\ \ \forall a \in K \ \text{ we have } \ \sum_{n\geq 1} \widetilde{P}^a(E^c_{n,\varepsilon}(a)) < \infty. \tag{2.20}$$

(So, $\liminf_n E_{n,\varepsilon}(a)$ has full $\widetilde{P}^a$-measure).

Consider $\tilde{t} \mapsto \widetilde{P}^a(C_{\tilde{t}_1\ldots\tilde{t}_n})$ as a random variable with respect to the probability measure $\widetilde{P}^a/\|\widetilde{P}^a\|$. The formula (2.20) means that the variable takes large values, i.e.

$$\widetilde{P}^a(C_{\tilde{t}_1\ldots\tilde{t}_n}) > b^{-n(d_{\overline{\alpha}^P}(a)-\varepsilon)}$$

with small probability.

For any positive number $\eta > 0$, the Tchebychev inequality leads to

$$\begin{aligned} \widetilde{P}^a(E_{n,\varepsilon}(a)^c) &\leq b^{\eta n(d_{\overline{\alpha}^P}(a)-\varepsilon)} \int_{\partial A^*} \left(\widetilde{P}^a(C_{\tilde{t}_1\ldots\tilde{t}_n})\right)^\eta \widetilde{P}^a(d\tilde{t}) \\ &= b^{\eta n(d_{\overline{\alpha}^P}(a)-\varepsilon)} \sum_{w\in A^n} (\widetilde{P}^a(C_w))^{1+\eta} \end{aligned} \tag{2.21}$$

where the last equality is due to the fact that the variable is constant on each n-cylinder.

We are now led to estimate the $(1+\eta)$-moment of $\widetilde{P}^a(C_w)$. For a single parameter a, the estimation is rather easy. But what we have to do is an estimation which is uniform on $a \in K$. This is more difficult, as we shall see now.

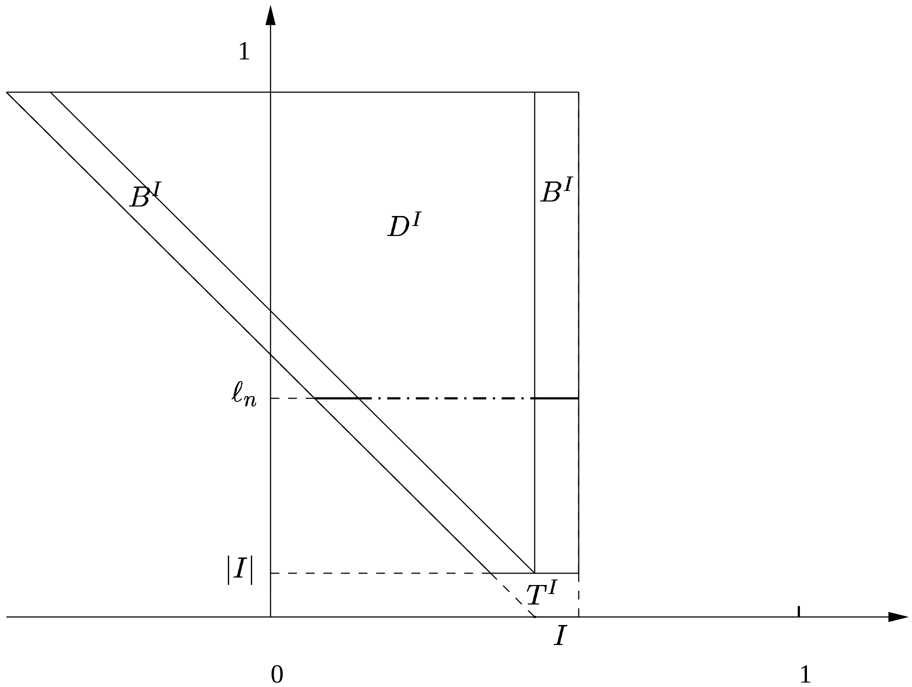

FIGURE 2. *The regions D^I, B^I and T^I*

Take $M \geq \max(1, \sup K)$. For any interval I we define

$$D^I = D_{|I|}(\inf I) \bigcap D_{|I|}(\sup I), \qquad B^I = \bigcup_{t \in I} D_{|I|}(t) \setminus D^I.$$

See Figure 1. Keep in mind that B^I is much smaller than D^I in the sense that $\frac{\nu(B^I)}{\nu(D^I)} \to 0$ as the length $|I| \to 0$. For every $w \in A^*$ and $t \in I_w$, we have $D^{I_w} \subset D_{b^{-m}}(t)$ and $D_{b^{-m}}(t) \setminus D^{I_w} \subset B^{I_w}$. It follows from (2.5) that almost surely, for every $a \in K$, and $w \in A^*$

$$\widetilde{P}^a(C_w) \leq M^{N(B^{I_w})} a^{N(D^{I_w})} \exp[(1-a)\nu(D_{b^{-|w|}})] b^{-|w|} Z(w,a). \qquad (2.22)$$

This, together with (2.21), shows that for an arbitrary $\eta > 0$ we have

$$\widetilde{P}^a(E_{n,\epsilon}(a)^c) \leq f_{n,\eta}(a) := F_1(a) F_2(a) \qquad (2.23)$$

with

$$\begin{aligned} F_1(a) &= b^{n[-1+\eta(d_{\overline{\alpha}^P}(a)-\varepsilon-1)]} \exp[(1-a)(1+\eta)\nu(D_{b^{-n}})] \\ F_2(a) &= \sum_{w \in A^n} \left(M^{N(B^{I_w})} a^{N(D^{I_w})} Z(w,a) \right)^{1+\eta}. \end{aligned}$$

The positive number $\varepsilon > 0$ being fixed, the problem is reduced to find a positive number η such that

$$\mathbb{P}\text{-}a.s. \ \ \forall a \in K \quad \sum_{n \geq 1} f_{n,\eta}(a) < \infty.$$

(recall that $f_{n,\eta}(a)$ is defined in (2.23)). This will be done if we find $\eta > 0$ such that

(I) There exists a constant $C = C(K, \eta) > 0$ such that for all $n \geq 1$,

$$\sup_{a \in K} \mathbb{E}(|f'_{n,\eta}(a)|) \leq Cnb^{-n\frac{\eta\varepsilon}{2}}. \tag{2.24}$$

(II) Let $a_0 = \inf K$. We have

$$\mathbb{P}\text{-}a.s. \quad \sum_{n=1}^{\infty} f_{n,\eta}(a_0) < \infty \tag{2.25}$$

Indeed, if (I) holds, by the Fubini Theorem

$$\mathbb{E} \int_K \sum_{n=1}^{\infty} |f'_{n,\eta}(a)|\, da = \int_K \sum_{n=1}^{\infty} \mathbb{E}|f'_{n,\eta}(a)|\, da < \infty.$$

Therefore $\mathbb{P}$-almost surely $\int_K \sum_{n=1}^{\infty} |f'_{n,\eta}(a)| da < \infty$. Then by the mean value theorem, $\mathbb{P}$-almost surely for all $a \in K$ we have

$$\sum_{n=1}^{\infty} |f_{n,\eta}(a) - f_{n,\eta}(a_0)| \leq \sum_{n=1}^{\infty} \int_K |f'_{n,\eta}(u)|\, du < \infty.$$

This, together with (II), allows us to conclude:

$$\mathbb{P}\text{-}a.s. \quad \sup_{a \in K} \sum f_{n,\eta}(a) < \infty.$$

We prove now (I) and (II). Since F_1 is a deterministic function, we have

$$\mathbb{E}|f'_{n,\eta}(a)| \leq |F'_1(a)| \mathbb{E}F_2(a) + F_1(a)\mathbb{E}|F'_2(a)|.$$

However

$$\begin{aligned} F'_1(a) &= F_1(a)\left[d'_{\overline{\alpha}}(a) n\eta \log b - (1+\eta)\nu(D_{b^{-n}})\right] \\ F'_2(a) &= (1+\eta) \sum_{w \in A^n} M^{(1+\eta)N(B^{I_w})} \cdot N(D^{I_w}) a^{(1+\eta)N(D^{I_w})-1} \cdot Z(w,a)^{1+\eta} \\ &\quad + (1+\eta) \sum_{w \in A^n} M^{(1+\eta)N(B^{I_w})} \cdot a^{(1+\eta)N(D^{I_w})} \cdot Z(w,a)^{\eta} Z(w,a)' \end{aligned}$$

Before estimating $\mathbb{E}|f'_{n,\eta}(a)|$, we remark the following facts

(R1) For all $w \in A^*$ and $a \in K$, the random variables $N(D^{I_w})$, $N(B^{I_w})$ and $Z(w,a)$ are independent;
(R2) If $\eta > 0$ is small enough, $\mathbb{E}\left(Z(w,a)^{1+\eta}\right)$ and $\mathbb{E}\left(Z(w,a)^{\eta}|Z'(w,a)|\right)$ are uniformly bounded over $a \in K$ and $w \in A^*$;
(R3) The function $d'_{\overline{\alpha}^P}(a)$ is bounded over K;
(R4) If $\nu(B) < \infty$ and $r > 0$ one has

$$\mathbb{E}(r^{N(B)}) = e^{\nu(B)(r-1)}$$

and then by differentiation with respect to r we get

$$\mathbb{E}\big(N(B) r^{N(B)-1}\big) = \nu(B) e^{\nu(B)(r-1)}$$

(R5) $\nu(B^{I_w})$ is bounded for all $n \geq 1$ and all $w \in A^n$;
(R6) $\nu(D^{I_w}) - \nu(D_{b^{-n}}) = O(1)$ for all $n \geq 1$ and all $w \in A^n$, and $\nu(D_{b^{-n}}) = O(n)$.

In fact, (R1) follows from the construction because the three variables in question depend on the Poisson process restricted on three disjoint domains in $\mathbb{R}\times\mathbb{R}^+$, namely D^I, B^I and T^I (see Figure 1); (R2) is a consequence of Proposition 2.5 (ii) and the Hölder inequality; (R3) is obvious; (R4) is explained by itself; (R5) is due to the fact $\nu(B^{I_w}) \leq 2b^{-n}\mu([b^{-n},1))$ $(\forall w\in A^n)$ and the hypothesis (**HP**); the first assertion of (R6) is deduced from (R5) and the second one is a consequence of (R5) and the fact

$$\nu(D_{b^{-n}}) - \nu(B^{I_w}) \leq \nu(D^{I_w}) \leq \nu(D_{b^{-n}}) = O(\log b^n) = O(n).$$

All these remarks together imply if η is small enough then there exists a constant $C = C(K,\eta) > 0$ such that for all $w\in A^n$ and all $a\in K$ we have

$$\begin{aligned}\mathbb{E}\left(\left|f'_{n,\eta}(a)\right|\right) &\leq CnF_1(a)\cdot b^n e^{\nu(D^I)(a^{1+\eta}-1)}\\ &\leq Cnb^{n\eta[d_{\overline{\alpha}^P}(a)-\varepsilon-1]}e^{[(1+\eta)(1-a)+a^{1+\eta}-1]\nu(D_{b-n})}\\ &= Cnb^{-n(\eta\varepsilon+\eta_n(a))}\end{aligned}\tag{2.26}$$

where

$$\eta_n(a) = \eta(1-d_{\overline{\alpha}^P}(a)) + [(1+\eta)(a-1)-(a^{1+\eta}-1)]\frac{\nu(D_{b^{-n}})}{\log b^n}.$$

Let

$$H_a(\eta) = (1+\eta)(a-1)-(a^{1+\eta}-1).$$

We write

$$\eta_n(a) = \eta(1-d_{\overline{\alpha}^P}(a)) + H_a(\eta)\overline{\alpha}^P + H_a(\eta)\left(\frac{\nu(D_{b^{-n}})}{\log b^n} - \overline{\alpha}^P\right).$$

Notice that

$$H_a(0)=0,\quad H'_a(0) = a-1-a\log a,\quad H''_a(\eta) = -a^{1+\eta}\log^2 a.$$

So, we have

$$H_a(\eta) = H'_a(0)\eta + O(\eta^2)$$

where the constant involved in $O(\eta^2)$ is independent of a. Recall that $d_{\overline{\alpha}^P}(a) = 1+\overline{\alpha}^P H'_a(0)$. Thus we get

$$\eta_n(a) = O(\eta^2) + H_a(\eta)\left(\frac{\nu(D_{b^{-n}})}{\log b^n} - \overline{\alpha}^P\right).$$

Since $H_a(\eta) = O(\eta)$ and $\limsup\frac{\nu(D_{b^{-n}})}{\log b^n} = \overline{\alpha}^P$, for fixed $\varepsilon>0$, some small η and all large $n\geq 1$ we have

$$|\eta_n(a)| \leq \frac{\varepsilon\eta}{2}\qquad (\forall a\in K).\tag{2.27}$$

Finally, from (2.26) and (2.27), we get (I), i.e.

$$\sup_{a\in K}\mathbb{E}\left(\left|f'_{n,\eta}(a)\right|\right) \leq Cnb^{-n\frac{\eta\varepsilon}{2}}.$$

The fact (II) is easier to obtain, by similar computations showing that

$$\sup_{a\in K}\mathbb{E}\left(f_{n,\eta}(a)\right) \leq Cb^{-n\frac{\varepsilon\eta}{2}}.$$

□

2.5. **Poisson process associated with the Dvoretzky covering.** We have been working exclusively with the Poisson process. Now we show how the Dvoretzky covering is associated with a Poisson covering. In other words, we will construct a special Poisson process closely related to the Dvoretzky covering, as was done in [K3, F3].

Define two new sequences $(\ell'_n)_{n\geq 1}$ and $(\ell''_n)_{n\geq 1}$ built from $(\ell_n)_{n\geq 1}$ as follows

$$\ell'_{\frac{m(m-1)}{2}+1} = \cdots = \ell'_{\frac{m(m+1)}{2}} = \lambda'_m \quad \text{with } \lambda'_m = \ell_{\frac{m(m+1)}{2}}$$

and

$$\ell''_{\frac{m(m-1)}{2}+1} = \cdots = \ell''_{\frac{m(m+1)}{2}} = \lambda''_m \quad \text{with } \lambda''_m = \ell_{\frac{m(m-1)}{2}+1}.$$

For $t \in \mathbb{T}$ and $n \geq 1$ define

$$N'_n(t) = \sum_{k=1}^{n} 1_{(0,\ell'_k)}(t-\omega_k), \qquad N''_n(t) = \sum_{k=1}^{n} 1_{(0,\ell''_k)}(t-\omega_k) \tag{2.28}$$

We define the quantities $\{L'_n\}$, $\{\Lambda'_\epsilon\}$, $\overline{\alpha}'$ and $\widehat{\alpha}'$ associated to $\{\ell'_n\}$, as we define $\{L_n\}$, $\{\Lambda_\epsilon\}$, $\overline{\alpha}$ and $\widehat{\alpha}$ associated to $\{\ell_n\}$. Similarly, we define $\{\Lambda''_\epsilon\}$, $\overline{\alpha}''$ and $\widehat{\alpha}''$ associated to $\{\ell''_n\}$. Clearly, we have

$$L'_n \leq L_n \leq L''_n, \qquad N'_n(t) \leq N_n(t) \leq N''_n(t).$$

The following lemma shows that both sequences (ℓ'_n) and (ℓ''_n) are not significantly different from (ℓ_n). It is a consequence of Propositions 7.4 in the Appendix.

Lemma 2.7. *Assume* (**H**). *Then* $\sum(\ell''_n - \ell'_n) < \infty$. *Consequently* $L'_n \sim L_n \sim L''_n$ *as* $n \to \infty$. *Moreover,* (ℓ'_n) *and* (ℓ''_n) *obey* (**H**) *and*

$$\operatorname{Card}\Lambda'_{b^{-k}} \sim \operatorname{Card}\Lambda_{b^{-k}} \sim \operatorname{Card}\Lambda''_{b^{-k}}; \quad \overline{\alpha}' = \overline{\alpha} = \overline{\alpha}''; \quad \widehat{\alpha}' = \widehat{\alpha} = \widehat{\alpha}''.$$

Let μ be the measure defined by

$$\mu = \sum_{n=1}^{\infty} \delta_{\ell'_n} = \sum_{m=1}^{\infty} m\delta_{\lambda'_m}. \tag{2.29}$$

As in [F3], we construct a Poisson point process closely related to $\{\omega_n\}_{n\geq 1}$, whose intensity is given by $\nu = \lambda \otimes \mu$. Such a Poisson process with intensity ν is constructed as follows.

Fix the segment $J_{r,n} = [r, r+1] \times \{\ell'_n\}$ $(r \in \mathbb{Z}, n \geq 1)$. Let $N_{r,n}$ be a Poisson variable with mean value 1. A Poisson process with intensity $\lambda \otimes \delta_{\ell'_n}|_{J_{r,n}}$ is a set of the points $\{(r+\eta^{(j)}_{r,n}, \ell'_n)\}_{1\leq j\leq N_{r,n}}$ where $\{\eta^{(j)}_{r,n}\}_{j\geq 1}$ is an i.i.d. sequence variables uniformly distributed in $[0,1]$, which is independent of $N_{r,n}$. The union of all such random sets, assumed independent, is a Poisson process with intensity ν.

We identify $[0,1]$ with $\mathbb{T}$ and use the i.i.d. sequence $\{\omega_j\}$ as part of $\{\eta^{(j)}_{0,n}\}$. We could say that we modify the preceding Poisson point process to get a new one. Let

$$N_m = \sum_{n=m(m-1)/2+1}^{m(m+1)/2} N_{0,n}$$

which is a Poisson variable with mean value m. We modify the preceding Poisson point process on $[0,1] \times \{\lambda'_m\}$ as follows: if $N_m \leq m$, we take the first N_m variables in $\{\omega_{m(m-1)/2+j}\}$ $(1 \leq j \leq m)$ to be the variables $\eta_{0,n}^{(j)}$; if $N_m > m$, we take all variables in $\{\omega_{m(m-1)/2+j}\}$ $(1 \leq j \leq m)$ and keep the other $N_m - m$ supplementary variables $\eta_{0,n}^{(j)}$.

By the Lemma 2.7, the assumptions (**H**) made on the sequence $\{\ell_n\}$ implies that the assumption (**HP**) is satisfied by the measure $\nu = \lambda \otimes \mu$ with $\mu = \sum_{n=1} \delta_{\ell'_n}$. Moreover, we have $\widehat{\alpha} = \widehat{\alpha}^P$ and $\overline{\alpha} = \overline{\alpha}^P$ (see Proposition 7.2).

3. Proofs of Theorems 1.1 and 1.2: Lower bounds

3.1. **Lower bounds.** Without loss of generality, we assume that $\ell_n \leq \delta$ for some $\delta \in (0,1/2)$. Then, if $t \in [\delta, 1-\delta]$, any arc of the form $(\omega_n, \omega_n + \ell)$ containing t with $\ell \in \{\ell_n, \ell'_n, \ell''_n\}$ can be identified as a subinterval of $(0,1)$ (i. e. it contains neither 0 nor 1). Moreover, when (ω_n, ℓ'_n) is a point (X_p, Y_p) of the (modified) Poisson point process, a point $t \in [\delta, 1-\delta]$ is covered by $(\omega_n, \omega_n + \ell'_n)$ in the Dvoretzky covering if and only if it is covered by $(X_p, X_p + Y_p)$ in the Poisson covering.

The case $\beta = 1$ was discussed in the introduction. Let

$$\widehat{J} = \{\beta > 0 : \ d_{\widehat{\alpha}}(\beta) > 0\} \setminus \{1\}.$$

For $b \geq 2$ and $k \geq 1$, define

$$m_k^{(b)} = \min\{j : \nu([2^{-j}, 1]) \geq k \log b\} = \min\{j : \sum_{n:\, \ell'_n \geq 2^{-j}} \ell'_n \geq k \log b\} \quad (3.1)$$

$$n_k^{(b)} = \mathrm{Card}\{n \geq 1 : \ b^{-m_k^{(b)}} \leq \ell'_n < 1\} = \mathrm{Card}\Lambda'_{b^{-m_k^{(b)}}}. \quad (3.2)$$

The following proposition involves the Poisson multiplicative chaos introduced in the Section 2.

Proposition 3.1. *Assume* (**H**). *Let K be a compact subinterval of $\widehat{J}$ and let $b \geq 2$ be an integer such that $\overline{\alpha}_b^P$ is close enough to $\widehat{\alpha}$ so that $\inf_{\beta \in K} d_{\overline{\alpha}_b^P}(\beta) > 0$. With probability one, for all $\beta \in K$, for P^β-almost every $t \in [\delta, 1-\delta]$, we have*

$$\liminf_{k \to \infty} \frac{N'_{n_k^{(b)}}(t)}{L'_{n_k^{(b)}}} \geq \beta, \qquad \limsup_{k \to \infty} \frac{N''_{n_k^{(b)}}(t)}{L'_{n_k^{(b)}}} \leq \beta.$$

Proposition 3.1, Theorems 2.1 and 2.2 immediately lead to the desired lower estimates.

Corollary 3.2 (Lower bound). *Under the assumption* (**H**), *with probability one, for all $\beta \geq 0$ such that $d_{\widehat{\alpha}}(\beta) > 0$, we have*

$$\dim(F_\beta) \geq 1 + \overline{\alpha}(\beta - 1 - \beta \log \beta).$$

3.2. **Proof of lower bounds.** We give here a proof of the Proposition 3.1. As we shall see, the Corollary 3.2 is an easy consequence of the Proposition 3.1.

Both the case $\beta = 0$ and the case $\beta = 1$ were discussed in the Introduction. Since the integer $b \geq 2$ is fixed, we write

$$m_k = m_k^{(b)}, \qquad n_k = n_k^{(b)}.$$

Without loss of generality, assume that $\delta = b^{-m_0}$. For every $\beta \in K$, $k \geq 0$ and $\varepsilon > 0$, define

$$E_{k,\varepsilon}^{-}(\beta) = \left\{ t \in [b^{-m_0}, 1 - b^{-m_0}] : \frac{N'_{n_k}(t)}{L'_{n_k}} \leq \beta - \varepsilon \right\}$$

$$E_{k,\varepsilon}^{+}(\beta) = \left\{ t \in [b^{-m_0}, 1 - b^{-m_0}] : \frac{N''_{n_k}(t)}{L'_{n_k}} \geq \beta + \varepsilon \right\}$$

By the Borel-Cantelli Lemma, it suffices to show that for every $\varepsilon > 0$, we have

$$\mathbb{P}-a.s. \ \ \forall \beta \in K \qquad \sum_{k \geq 0} P^{\beta}(E_{k,\varepsilon}^{-}(\beta)) < \infty, \quad \sum_{k \geq 0} P^{\beta}(E_{k,\varepsilon}^{+}(\beta)) < \infty. \tag{3.3}$$

We will only prove (3.3) when $K \subset \widehat{J} \cap (0,1)$. The case $K \subset \widehat{J} \cap (1, \infty)$ may be similarly treated.

Fix $M = (\min(K))^{-1}$ (so, $M > 1 \geq \max(K)$) and $\eta \in (0,1)$. For $\beta \in K \subset (0,1)$ and $k \geq 0$, the Tchebychev inequality gives

$$P^{\beta}(E_{k,\varepsilon}^{-}(\beta)) \ \leq \ \beta^{-\eta(\beta-\epsilon)L'_{n_k}} \int_{[b^{-m_0}, 1-b^{-m_0}]} \beta^{\eta N'_{n_k}(t)} P^{\beta}(dt), \tag{3.4}$$

$$P^{\beta}(E_{k,\varepsilon}^{+}(\beta)) \ \leq \ \beta^{\eta(\beta+\epsilon)L'_{n_k}} \int_{[b^{-m_0}, 1-b^{-m_0}]} \beta^{-\eta N''_{n_k}(t)} P^{\beta}(dt). \tag{3.5}$$

For $w \in A^{m_k}$ such that $I_w \subset [b^{-m_0}, 1 - b^{-m_0}]$, define

$$s'_k(w) = \sup_{t \in I_w} |N'_{n_k}(t) - N^P_{b^{-m_k}}(t)|,$$

$$s''_k(w) = \sup_{t \in I_w} |N''_{n_k}(t) - N^P_{b^{-m_k}}(t)|.$$

Notice that

$$N^P_{b^{-m_k}}(t) \leq N(B^{I_w}) + N(D^{I_w}) \qquad (\forall t \in I_w).$$

This, together with (2.22) and the equality $L'_{n_k} = \nu(D_{b^{-m_k}})$, shows that for all $\beta \in K$ we have

$$\int_{I_w} \beta^{\eta N'_{n_k}(t)} P^{\beta}(dt) \leq b^{-m_k} \exp\left[(1-\beta)\nu(D_{b^{-m_k}})\right] Z(w,\beta) A_{k,\eta}(w,\beta) \tag{3.6}$$

$$\int_{I_w} \beta^{-\eta N''_{n_k}(t)} P^{\beta}(dt) \leq b^{-m_k} \exp\left[\left((1-\beta)\nu(D_{b^{-m_k}})\right] Z(w,\beta) B_{k,\eta}(w,\beta) \tag{3.7}$$

where $A_{k,\eta}(w,\beta)$ and $B_{k,\eta}(w,\beta)$ are two random variables defined by

$$A_{k,\eta}(w,\beta) = M^{(1+\eta)N(B^{I_w}) + \eta s'_k(w)} \beta^{(1+\eta)N(D^{I_w})} \tag{3.8}$$

$$B_{k,\eta}(w,\beta) = M^{(1+\eta)N(B^{I_w}) + \eta s''_k(w)} \beta^{(1-\eta)N(D^{I_w})} \tag{3.9}$$

Take respectively summation of (3.6) and (3.7) over w. It follows from (3.4) and (3.5) that

$$P^{\beta}(E^{-}_{k,\varepsilon}(\beta)) \leq g_{k,\eta}(\beta), \tag{3.10}$$

$$P^{\beta}(E^{+}_{k,\varepsilon}(\beta)) \leq h_{k,\eta}(\beta) \tag{3.11}$$

with

$$g_{k,\eta}(\beta) = b^{-m_k}\beta^{-\eta(\beta-\epsilon)L'_{n_k}} \exp\left[(1-\beta)\nu(D_{b^{-m_k}})\right]\sum Z(w,\beta)A_{k,\eta}(w,\beta)$$

$$h_{k,\eta}(\beta) = b^{-m_k}\beta^{\eta(\beta+\epsilon)L'_{n_k}} \exp\left[(1-\beta)\nu(D_{b^{-m_k}})\right]\sum Z(w,\beta)B_{k,\eta}(w,\beta)$$

where both sums are taken over $w \in A^{m_k}$ such that $I_w \subset [b^{-m_0}, 1-b^{-m_0}]$. So, in order to prove (3.3), we have only to find a small $\eta > 0$ such that

$$\mathbb{P}\text{−}a.s.\ \ \forall \beta \in K \quad \sum_{k\geq 0} g_{k,\eta}(\beta) < \infty, \qquad \sum_{k\geq 0} h_{k,\eta}(\beta) < \infty. \tag{3.12}$$

The functions $g_{k,\eta}(\beta)$ and $h_{k,\eta}(\beta)$ are continuous functions of β. We are going to show the uniform convergence of the first series in (3.12). That of the second one may be proved in the same way. We will follow the same approach as in the proofs of Theorems 2.1 and 2.2. Since $L'_{n_k} \sim k\log b$ by the construction of $n_k = n_k^{(b)}$ (see (3.2)), we have only to show that there exist $\eta > 0$ and $C = C(K,\eta) > 0$ such that for all $k \geq 0$

$$\sup_{\beta\in K} \mathbb{E}|g'_{k,\eta}(\beta)| \leq C L'^{2}_{n_k} \exp\left[-\frac{\eta\varepsilon\alpha}{2}L'_{n_k}\right], \tag{3.13}$$

where $\alpha = \inf_{\beta\in K}|\log(\beta)|$, and that

$$\mathbb{P}\text{−}a.s. \qquad \sum_{k\geq 0} g_{k,\eta}(\beta_0) < \infty \tag{3.14}$$

where $\beta_0 = \inf(K)$.

Let us prove (3.13) and (3.14). Notice that the variable $Z(w,\beta)$ is independent of $N(B^{I_w})$, $s'_k(w)$ and $N(D^{I_w})$. It is then independent of $A_{k,\eta}(\beta)$. Thus, the same arguments as in the proofs of Theorems 2.1 and 2.2 show that there exists a constant $C = C(K,\eta)$ such that for all $\beta \in K$ and $k \geq 1$,

$$\begin{aligned}\mathbb{E}|g'_{k,\eta}(\beta)| \leq{}& C\beta^{-\eta(\beta-\epsilon)L'_{n_k}} \exp\left[(1-\beta)\nu(D_{b^{-m_k}})\right] \\ &\times \mathbb{E}\left((\nu(D_{b^{-k}}) + N(D^{I_w}))A_{k,\eta}(w,\beta)\right)\end{aligned} \tag{3.15}$$

where w is a typical element of A^{m_k} such that $I_w \subset [b^{-m_0}, 1-b^{-m_0}]$ and the expectation is independent of w. We estimate the expectation at the right hand side of (3.15) by using the Hölder inequality with the conjugate exponent (p,q), i.e. $p^{-1}+q^{-1} = 1$, such that

$$p = 1+\eta^2 < 2, \quad q = \frac{1+\eta^2}{\eta^2}.$$

Thus we get

$$\begin{aligned}&\mathbb{E}\left([\nu(D_{b^{-m_k}}) + N(D^{I_w})]A_{k,\eta}(w,\beta)\right) \\ &\quad\leq \left(\mathbb{E}[\nu(D_{b^{-m_k}}) + N(D^{I_w})]^p \beta^{p(1+\eta)N(D^{I_w})}\right)^{1/p} \\ &\qquad \times \left(\mathbb{E}M^{q(1+\eta)[N(B^{I_w})+s'_k(w)]}\right)^{1/q}\end{aligned} \tag{3.16}$$

In order to estimate the last two expectations, we will use the following lemmas, whose proofs are postponed in the next subsection.

Lemma 3.3. *Let N be a Poisson variable with parameter ξ. For any positive number $a > 0$ and $b > 0$ we have*

$$\mathbb{E}(a+N)^2 b^N = \left[(a+b\xi)^2 + b\xi\right] e^{\xi(b-1)}.$$

Lemma 3.4. *For any $r > 0$, there exist $C = C(r) > 0$ such that for all $k \geq 1$ we have*

$$\mathbb{E}e^{rs'_k(w)} \leq C, \qquad \mathbb{E}e^{rs''_k(w)} \leq C.$$

Notice that $0 < \nu(D^{I_w}) \leq \nu(D_{b^{-m_k}})$. Applying the Lemma 3.3 to $a = \nu(D_{b^{-m_k}})$, $b = \beta$ and $\xi = \nu(D^{I_w})$, we get

$$\begin{aligned} &\mathbb{E}[\nu(D_{b^{-m_k}}) + N(D^{I_w})]^p \beta^{p(1+\eta)N(D^{I_w})} \\ &\qquad \leq C\nu(D_{b^{-m_k}})^2 \exp\left(\nu(D_{b^{-m_k}})\left[\beta^{(1+\eta)(1+\eta^2)} - 1\right]\right) \end{aligned} \tag{3.17}$$

Notice that $\mathbb{E}e^{rN(B^{I_w})}$ is bounded for any fixed $r > 0$ and for all w. In order to estimate the second expectation at the right hand side of (3.16), we apply the Hölder inequality and the Lemma 3.4. We get

$$\mathbb{E}M^{q(1+\eta)[N(B^{I_w}) + s'_k(w)]} \leq C. \tag{3.18}$$

Let

$$\begin{aligned} U_\beta(\eta) &= \frac{\gamma\eta^2}{1+\eta^2} - \eta(\beta - \epsilon)\log\beta, \\ V_\beta(\eta) &= 1 - \beta + \frac{\beta^{(1+\eta)(1+\eta^2)} - 1}{1+\eta^2}. \end{aligned}$$

It is clear that

$$\begin{aligned} U_\beta(\eta) &= -\eta(\beta-\epsilon)\log\beta + O(\eta^2), \\ V_\beta(\eta) &= \eta\beta\log\beta + O(\eta^2). \end{aligned}$$

In both above expressions, the constant involved in $O(\eta^2)$ is independent of β. Since $\nu(D_{b^{-m_k}}) = L'_{n_k}$, combining (3.15)-(3.18) gives rise to

$$\begin{aligned} \mathbb{E}|g'_{k,\eta}(\beta)| &\leq C\nu(D_{b^{-k}})^2 \cdot \exp\left(U_\beta(\eta)L'_{n_k} + V_\beta(\eta)\nu(D_{b^{-m_k}})\right) \\ &= CL'^2_{n_k}\exp\left(L'_{n_k}[U_\beta(\eta) + V_\beta(\eta)]\right) \\ &\leq CL'^2_{n_k}\exp\left(L'_{n_k}[\eta\epsilon\log\beta + O(\eta^2)]\right) \\ &\leq CL'^2_{n_k}\exp\left(L'_{n_k}\frac{\eta\epsilon}{2}\log\beta\right) \end{aligned}$$

if η is sufficiently small (recall that $\beta \in K \subset (0,1)$).

Similarly, we show that

$$\sup_{\beta\in K}\mathbb{E}g_{k,\eta}(\beta) \leq C\exp\left(-\frac{\eta\varepsilon\alpha}{2}L'_{n_k})\right).$$

□

Proof of Corollary 3.2. Notice that

$$N'_{n_k^{(b)}}(t) \leq N_{n_k^{(b)}}(t) \leq N''_{n_k^{(b)}}(t)$$

for any t. It follows from Proposition 3.1 that, with probability one, for all $\beta > 0$ such that $d_{\widehat{\alpha}}(\beta) > 0$, there exists an integer $b \geq 2$ such that for P^β-almost every $t \in [\delta, 1-\delta]$, we have

$$\lim_{k\to\infty} \frac{N_{n_k^{(b)}}(t)}{L'_{n_k^{(b)}}} = \beta.$$

Recall that $L_n \sim L'_n$ as $n \to \infty$. Moreover, by construction $L'_{n_k^{(b)}} \sim L'_{n_{k+1}^{(b)}}$. We deduce that with probability one, for all $\beta > 0$ such that $d_{\widehat{\alpha}}(\beta) > 0$, for P^β-almost every $t \in [\delta, 1-\delta]$,

$$\lim_{n\to\infty} \frac{N_n(t)}{L_n} = \beta.$$

That is to say, F_β carries the restriction of P^β to $[\delta, 1-\delta]$. So, $\dim F_\beta \geq \dim P^\beta$. However $\dim P^\beta$ is larger than or equal to $1 - \overline{\alpha}(1 - \beta + \beta \log \beta)$ (Theorems 2.1 and 2.2). □

3.3. Proofs of Lemmas.

Proof of Lemma 3.3 It is a consequence of $\mathbb{E} b^N = e^{\xi(b-1)}$. Differentiating it with respect to b leads to

$$\mathbb{E} N b^N = \xi b e^{\xi(b-1)}, \qquad \mathbb{E} N^2 b^N = \xi b(1+\xi b) e^{\xi(b-1)}.$$

It follows that

$$\mathbb{E}(a+N)^2 b^N = (a^2 + 2a\xi b + \xi b(1+\xi b)) e^{\xi(b-1)}.$$

□

Proof of Lemma 3.4. We first estimate $\mathbb{E} e^{r s'_k}$.

Fix $k \geq 1$ and $w \in A^{m_k}$. For any integer $m \geq 1$ such that $\lambda'_m \geq b^{-m_k}$, let

$$D_m^{I_w} = D^{I_w} \cap (\mathbb{R} \times \{\lambda'_m\}), \qquad B_m^{I_w} = B^{I_w} \cap (\mathbb{R} \times \{\lambda'_m\}).$$

In other word, $D_m^{I_w}$ and $B_m^{I_w}$ are respectively the intersections of D^{I_w} and B^{I_w} with the horizontal line in the plan of height λ'_m. We call $\{(\omega_n, \ell'_n)\}$ the Dvoretzky points in the plan (they are all located in the strip $[0,1] \times \mathbb{R}+$ or more precisely in the square $[0,1] \times [0,1]$). For any plan Borel set B, we denote by $\mathcal{D}(B)$ the number of Dvoretzky points contained in B. This definition is similar to that of $N(B)$ which is the number of Poisson points contained in B. Recall that the intensity of the Poisson process is $\lambda \otimes \mu$ with $\mu = \sum_{n=1}^\infty \delta_{\ell'_n}$.

According to the construction of the Poisson process, it is easy to geometrically check that

$$s'_k(w) \leq N(B^{I_w}) + \mathcal{D}(B^{I_w}) + |N(D^{I_w}) - \mathcal{D}(D^{I_w})|.$$

So, by using the Hölder inequality, we have only to show that for any $r > 0$ there exist constants $C = C(r) > 0$ such that

$$\mathbb{E} e^{rN(B^{I_w})} \leq C, \quad \mathbb{E} e^{r\mathcal{D}(B^{I_w})} \leq C, \quad \mathbb{E} e^{r|N(D^{I_w}) - \mathcal{D}(D^{I_w})|} \leq C. \tag{3.19}$$

The validity of the first inequality in (3.19) concerning a Poisson variable is due to the fact that $\nu(B^{I_w})$ is bounded for all w. For the second one, remark that

$$\mathcal{D}(B^{I_w}) = \sum_{m:\lambda'_m \geq b^{-m_k}} \sum_{n=\frac{m(m-1)}{2}+1}^{\frac{m(m+1)}{2}} \mathbf{1}_{B_m^{I_w}}(\omega_n, \lambda'_m).$$

This is a sum of n_k identically distributed independent random variables, each variable taking the value 1 with probability $2b^{-m_k}$ and the value 0 with probability $1-2b^{-m_k}$. So, it is a binomial variable. It follows that

$$\mathbb{E}e^{r\mathcal{D}(B^{I_w})} = (1-2b^{-m_k}+2b^{-m_k}e^r)^{n_k} \leq e^{2(e^r-1)b^{-m_k}n_k} = O(1)$$

because $b^{-m_k}n_k = b^{-m_k}\mathrm{Card}(\Lambda'_{b^{-m_k}}) = O(1)$, by the assumption (**H**).

The main difficulty is the proof of the third one. To prove it, we use the following trivial estimate

$$|N(D^{I_w}) - \mathcal{D}(D^{I_w})| \leq \sum_{m:\lambda'_m \geq b^{-m_k}} v_m$$

where

$$v_m = |N(D_m^{I_w}) - \mathcal{D}(D_m^{I_w})|.$$

The number v_m is nothing but the (absolute value) of the difference between the number of Poisson points and the number of Dvoretzky points located on the segment $D_m^{I_w}$. Since the v_m are independent, we will finish the proof by showing that there is some constant $c = c(r) > 0$ such that

$$\mathbb{E}e^{rv_m} \leq e^{c\sqrt{m}\lambda_m} \tag{3.20}$$

because due to (**H**)

$$\mathbb{E}e^{rs'_k} \leq \prod_{m=1}^{m_k} \mathbb{E}e^{rv_m} \leq \exp\left(c\sum_{m=1}^{\infty}\sqrt{m}\lambda_m\right) < \infty. \tag{3.21}$$

When it is conditioned on the event $\{N_m = n\}$, v_m is a binomial variable. Actually, when $N_m = n \leq m$, v_m is equal to the number of those Dvoretzky points $(\omega_{m(m-1)/2+j}, \lambda'_m)$ with $N_m = n < j \leq m$ located on the segment $D_m^{I_w}$. Such a point is located on $D_m^{I_w}$ with probability equal to the length of the segment $D_m^{I_w}$, say J_m. All these points are mutually independent and independent of N_m. So we have

$$\begin{aligned}\mathbb{E}\big(\mathbf{1}_{N_m \leq m}e^{rv_m}\big) &= \sum_{n=0}^{m}\mathbb{P}(N_m = n)\sum_{i=0}^{m-n}\binom{m-n}{i}J_m^i(1-J_m)^{m-n-i}e^{ri} \\ &= \sum_{n=0}^{m}\mathbb{P}(N_m = n)\big(1+J_m(e^r-1)\big)^{m-n}.\end{aligned}$$

When $N_m = n > m$, v_m is the number of those $\eta_{0,n}^m$ with $m+1 \leq n \leq N_m$ (see the construction of the Poisson process the Section 3.1), which are located on $D_m^{I_w}$. It follows that

$$\begin{aligned}\mathbb{E}\big(\mathbf{1}_{N_m>m}e^{rv_m}\big) &= \sum_{n=m+1}^{\infty}\mathbb{P}(N_m=n)\sum_{l=0}^{n-m}\binom{n-m}{i}J_m^i(1-J_m)^{n-m-i}e^{ri}\\ &= \sum_{n=m+1}^{\infty}\mathbb{P}(N_m=n)\big(1+J_m(e^r-1)\big)^{n-m}.\end{aligned}$$

To go further, we will use the following special Taylor formula

$$\sum_{n=0}^{m}\frac{\alpha^n}{n!}=e^{\alpha}\left(1-\int_0^{\alpha}e^{-u}\frac{u^m}{m!}du\right)$$

which is equivalent to

$$\sum_{n=m+1}^{\infty}\frac{\beta^n}{n!}=e^{\beta}\int_0^{\beta}e^{-u}\frac{u^m}{m!}du.$$

Recall that $\mathbb{P}(N_m=n)=e^{-m}\frac{m^n}{n!}$. Let

$$x=1+J_m(e^r-1),\quad A_m=e^{-m}x^me^{m/x},\quad B_m=e^{-m}x^{-m}e^{mx}.$$

Using the above two identities ($\alpha=m/x,\beta=mx$), we get

$$\begin{aligned}\mathbb{E}e^{rv_m} &= e^{-m}x^m\sum_{n=0}^{m}\frac{(m/x)^n}{n!}+e^{-m}x^{-m}\sum_{n=m+1}^{\infty}\frac{(mx)^n}{n!}\\ &= A_m+(B_m-A_m)\int_0^{m/x}e^{-u}\frac{u^m}{m!}du\\ &\quad +B_m\int_{m/x}^{mx}e^{-u}\frac{u^m}{m!}du.\end{aligned}\tag{3.22}$$

Elementary calculations give

$$A_m=1+O(mJ_m^2),\qquad B_m=1+O(mJ_m^2).\tag{3.23}$$

It follows that $B_m-A_m=O(mJ_m^2)$. For the two integrals, we claim that

$$\int_0^{m/x}e^{-u}\frac{u^m}{m!}du=O(1)\tag{3.24}$$

$$\int_{m/x}^{mx}e^{-u}\frac{u^m}{m!}du=(e^r-1)\sqrt{\frac{2}{\pi}}\sqrt{m}J_m+o(\sqrt{m}J_m).\tag{3.25}$$

Combining (3.22)-(3.25) and the fact $J_m\le\lambda_m$, we will get (3.20).

We finish the proof by showing (3.25). The proof of (3.24) is simpler and actually the integral in (3.24) is equivalent to $1/2$ as $m\to\infty$.

Let $x=1+\delta_m$ with $\delta_m=J_m(e^r-1)\to 0$. Then the integral in (3.25) is equal to

$$\begin{aligned}\int_{m/(1+\delta_m)}^{m(1+\delta_m)}e^{-u}\frac{u^m}{m!}du &= \frac{m^{m+1}}{m!}\int_{1/(1+\delta_m)}^{1+\delta_m}e^{-mv}v^m\,dv\\ &= \frac{m^{m+1}}{m!}e^{-m}\int_{1/(1+\delta_m)-1}^{\delta_m}e^{-mt}(1+t)^m\,dt.\end{aligned}\tag{3.26}$$

By the Stirling formula, we have

$$\frac{m^{m+1}}{m!}e^{-m} = \sqrt{\frac{m}{2\pi}} + o(m^{\frac{1}{2}}). \tag{3.27}$$

On the other hand, we have $e^{-mt}(1+t)^m \leq 1$ for all $t \in [1/(1+\delta_m)-1, \delta_m]$. So,

$$\begin{aligned}
\int_{1/(1+\delta_m)-1}^{\delta_m} e^{-mt}\frac{(1+t)^m}{m!}dt &= \left(\delta_m - \frac{1}{1+\delta_m} + 1\right) \\
&= (2\delta_m + O(\delta_m^2)) \\
&= 2(e^r-1)J_m + O(J_m^2). \qquad (3.28)
\end{aligned}$$

Finally, (3.25) follows from (3.26), (3.27) and (3.28). □

Now estimate $\mathbb{E}e^{rs_k''}$. Fix $k \geq 1$ and $w \in A^{m_k}$. For any integer $m \geq 1$ such that $\lambda_m' \geq b^{-m_k}$, let

$$\widetilde{B}_m^{I_w} = B_m^{I_w} \cup ([\min(I_w) - \lambda_m'', \min(I_w) - \lambda_m'] \times \{\lambda_m'\}.$$

Denote $\widetilde{B}^{I_w} = \cup_{m:\lambda_m' \geq b^{-m_k}} \widetilde{B}_m^{I_w}$. We have

$$s_k''(w) \leq N(B^{I_w}) + \mathcal{D}(\widetilde{B}^{I_w}) + |N(D^{I_w}) - \mathcal{D}(D^{I_w})|.$$

Therefore, we have only to show that $\mathbb{E}e^{r\mathcal{D}(\widetilde{B}^{I_w})}$ is bounded by a constant independent of k. The reason for this boundedness is the following

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(r\mathcal{D}(\widetilde{B}^{I_w})\right)\right) \\
&= \prod_{m:\,\lambda_m' \geq b^{-m_k}} \prod_{n=\frac{m(m-1)}{2}+1}^{\frac{m(m+1)}{2}} \mathbb{E}\left(\exp\left(\mathbf{1}_{\widetilde{B}_m^{I_w}}(\omega_n, \lambda_m')r\right)\right) \\
&\leq \prod_{m:\,\lambda_m' \geq b^{-m_k}} \left(1 + e^r(\lambda_m'' - \lambda_m' + 2b^{-m_k})\right)^m \\
&\leq \exp\left(e^r \sum_{m:\,\lambda_m' \geq b^{-m_k}} (2b^{-m_k}m + (\lambda_m'' - \lambda_m')m\right) \\
&= \exp\left(O(b^{-m_k}n_k)\right) \times \exp\left(e^r \sum_{m=1}^{\infty} m(\lambda_m'' - \lambda_m')\right) \\
&= O(1).
\end{aligned}$$

The last sum is bounded because of the Lemma 2.7. □

4. Proofs of Theorems 1.1 and 1.2: upper bounds

4.1. **Upper bounds.** Assume $\overline{\alpha} > 0$ (there is nothing to prove when $\overline{\alpha} = 0$ since the lower bounds found in Section 3 are equal to 1).

For $k \geq 1$ define $m_k = m_k^{(2)}$ and $n_k = n_k^{(2)}$ as in Section 3 (see (3.1) and (3.2)). Since $L_{n_k} \sim L'_{n_k} \sim k\log 2 \sim L'_{n_{k+1}}$ and ℓ_n is decreasing, by the definition of $\overline{\alpha}$, we may find a strictly increasing sequence of integers $(k_j)_{j\geq 1}$ such that

$$\lim_{j\to\infty} \frac{L'_{n_{k_j}}}{-\log \ell_{n_{k_j}}} = \overline{\alpha}.$$

When the limsup defining $\overline{\alpha}$ is a limit, we can take $k_j = j$. For $\beta \geq 0$, define

$$\underline{F}_\beta^{\inf} = \left\{ t \in \mathbb{T} : \liminf_{j\to\infty} \frac{N_{n_{k_j}}(t)}{L'_{n_{k_j}}} \leq \beta \right\}$$

$$\underline{F}_\beta^{\sup} = \left\{ t \in \mathbb{T} : \liminf_{j\to\infty} \frac{N_{n_{k_j}}(t)}{L'_{n_{k_j}}} \geq \beta \right\}$$

$$\overline{F}_\beta^{\inf} = \left\{ t \in \mathbb{T} : \limsup_{j\to\infty} \frac{N_{n_{k_j}}(t)}{L'_{n_{k_j}}} \leq \beta \right\}$$

$$\overline{F}_\beta^{\sup} = \left\{ t \in \mathbb{T} : \limsup_{j\to\infty} \frac{N_{n_{k_j}}(t)}{L'_{n_{k_j}}} \geq \beta \right\}$$

Define

$$\beta_{\min} = \inf\{\beta \geq 0 : d_{\overline{\alpha}}(\beta) \geq 0\},$$
$$\beta_{\max} = \sup\{\beta \geq 0 : d_{\overline{\alpha}}(\beta) \geq 0\}$$

We put our estimates on the Hausdorff dimensions of the four sets defined above into two propositions. The second proposition may be proved as the first one with minor changes.

Proposition 4.1. *With probability one, we have*

1. $\dim(\overline{F}_\beta^{\inf}) \leq d_{\overline{\alpha}}(\beta)$ for all $\beta \in [0,1) \cap [\beta_{\min}, \beta_{\max}]$;
2. $\overline{F}_\beta^{\inf} = \varnothing$ for all $\beta \in [0,1) \setminus [\beta_{\min}, \beta_{\max}]$;
3. $\dim(\underline{F}_\beta^{\sup}) \leq d_{\overline{\alpha}}(\beta)$ for all $\beta \in (1,\infty) \cap [\beta_{\min}, \beta_{\max}]$;
4. $\underline{F}_\beta^{\sup} = \varnothing$ for all $\beta \in (1,\infty) \setminus [\beta_{\min}, \beta_{\max}]$.

Proposition 4.2. *With probability one we have*

1. $\dim \underline{F}_\beta^{\inf} \leq d_{\overline{\alpha}}(\beta)$ for all $\beta \in [0,1) \cap [\beta_{\min}, \beta_{\max}]$;
2. $\underline{F}_\beta^{\inf} = \varnothing$ for all $\beta \in [0,1) \setminus [\beta_{\min}, \beta_{\max}]$;
3. $\dim \overline{F}_\beta^{\sup} \leq d_{\overline{\alpha}}(\beta)$ for all $\beta \in \cap(1,\infty) \cap [\beta_{\min}, \beta_{\max}]$;
4. $\overline{F}_\beta^{\sup} = \varnothing$ for all $\beta \in (1,\infty) \setminus [\beta_{\min}, \beta_{\max}]$

In order to deduce from the above propositions the desired upper bounds on $\dim \overline{F}_\beta$ and $\dim \underline{F}_\beta$, we need the following proposition.

Proposition 4.3. *Assume* $(\mathbf{H})$ *and* $\overline{\alpha} = \lim_{n\to\infty} \frac{L_n}{-\log \ell_n}$, *or* $(\mathbf{H}_\infty)$. *With probability one,*

$$\sup_{t\in\mathbb{T}} \sum_{n=n_{k-1}+1}^{n_k} 1_{(0,\ell_n)}(t-\omega_n) = o(L_{n_k}).$$

Proof. For $k \geq 1$, denote by X_k the supremum in question. Let $c > 1$ be any constant larger than 1. It was proved in [FK1] (Lemma 1 with evident changes) that for $\alpha > 1$ and $\lambda > 0$ we have

$$\mathbb{E}(\exp(\lambda X_k)) \leq \frac{1}{(\alpha - 1)\ell_{n_k}} \exp\left(\alpha(e^\lambda - 1) \sum_{n=n_{k-1}+1}^{n_k} \ell_n\right).$$

This yields that for every $k \geq 1$, $\alpha > 1$ and $\epsilon, \lambda > 0$,

$$\mathbb{P}(X_k \geq \epsilon L_{n_k}) \leq \frac{1}{(\alpha - 1)\ell_{n_k}} \exp\left(\alpha(e^\lambda - 1)(L_{n_k} - L_{n_{k-1}}) - \lambda\epsilon L_{n_k}\right).$$

Now, distinguish the cases $(\mathbf{H})$ and $\overline{\alpha} = \lim_{n\to\infty} \frac{L_n}{-\log \ell_n}$, and $(\mathbf{H}_\infty)$.

Suppose $(\mathbf{H})$ *and* $\overline{\alpha} = \lim_{n\to\infty} \frac{L_n}{-\log \ell_n}$ *holds.* It follows from Proposition 7.4 and the definitions of $\{\ell'_n\}$ and $\overline{\alpha}'$ that

$$\log((\ell'_n)^{-1}) = O(L'_n) = O(L_n).$$

So there exists $C > 0$ such that for k large enough

$$\frac{1}{\ell_{n_k}} \leq \frac{1}{\ell'_{n_k}} \leq \exp(CL_{n_k}).$$

On the other hand, there exists $C' > 0$ such that for k large enough we have

$$L_{n_k} - L_{n_{k-1}} \leq L''_{n_k} - L'_{n_k} + L'_{n_k} - L'_{n_{k-1}} \leq C',$$

since $L'_{n_k} \sim k \log 2$ by construction and $L''_n - L'_n = O(1)$ by Proposition 7.4 again.

The last estimates show that for fixed $\alpha > 1$, $\lambda > 0$ and $\epsilon > 0$, if k is large enough we have

$$\mathbb{P}(X_k \geq \epsilon L_{n_k}) \leq \frac{\exp\left(\alpha(e^\lambda - 1)C'\right)}{\alpha - 1} \exp\left((C - \lambda\epsilon)L_{n_k}\right).$$

Recall that $L_{n_k} \sim L'_{n_k} \sim k \log 2$. To conclude, take $\lambda > 2C/\epsilon$ and apply the Borel-Cantelli lemma.

Suppose $(\mathbf{H}_\infty)$ *holds.* We have $\frac{1}{n} = o(\ell'_n)$. Hence, we have (one can also use (7.4))

$$\log n_k \sim \sum_{n=1}^{n_k} \frac{1}{n} = o\left(\sum_{n=1}^{n_k} \ell'_n\right) = o(L'_{n_k}) \sim o(L_{n_k}),$$

where we used Proposition 7.4 for the last equivalence (actually we have $\log n = o(L_n)$). It follows that

$$1/\ell_{n_k} \leq 1/\ell'_{n_k} = o(n_k) = o\big(\exp(o(L_{n_k}))\big).$$

We also have

$$L_{n_k} - L_{n_{k-1}} \leq L''_{n_k} - L'_{n_k} + L'_{n_k} - L'_{n_{k-1}} = o(L_{n_k})$$

since $L''_n \sim L'_n$ by Proposition 7.4.

The last estimates show that for fixed $\alpha > 1$, $\lambda > 0$ and $\epsilon > 0$,

$$\mathbb{P}(X_k \geq \epsilon L_{n_k}) = o\left(\exp\left(\left((1 + \alpha(e^\lambda - 1)\right) o(L_{n_k}) - \lambda\epsilon L_{n_k}\right)\right).$$

Since for k large enough one has $o(L_{n_k}) \le \epsilon^2 L_{n_k}$, taking $\lambda = \sqrt{\epsilon}$ shows that for fixed $\alpha > 1$, for every $\epsilon > 0$ small enough, there exists $C > 0$ such that for k large enough,

$$\mathbb{P}(X_k \ge \epsilon L_{n_k}) \le C \exp(-\epsilon^{3/2} L_{n_k}).$$

The conclusion follows as in the previous case.

□

Since $L_{n_k} \sim L'_{n_k} \sim L'_{n_{k+1}}$, we have

$$F_\beta \subset \overline{F}_\beta^{\inf} \quad \text{for} \quad \beta < 1; \qquad F_\beta \subset \underline{F}_\beta^{\sup} \quad \text{for} \quad \beta > 1.$$

So, the upper bounds concerning $\dim F_\beta$ in Theorems 1.1 and 1.2 follow from Proposition 4.1. Of course, they also follow from Proposition 4.2.

If the limsup defining $\overline{\alpha}$ is a limit, by taking $k_j = j$ and applying Proposition 4.3 we get

$$\overline{F}_\beta \subset \overline{F}_\beta^{\inf} \cap \overline{F}_\beta^{\sup}, \qquad \underline{F}_\beta \subset \underline{F}_\beta^{\inf} \cap \underline{F}_\beta^{\sup}$$

(use the fact that $L_{n_k} \sim L'_{n_k} \sim L'_{n_{k+1}} \sim k \log 2$). Then we can get the upper bounds concerning $\dim \overline{F}_\beta$ and $\dim \underline{F}_\beta$ as stated in Theorem 1.2.

4.2. **Proof of Proposition 4.1.** Without loss of generality, we can only consider $\underline{F}_\beta \cap [\delta, 1-\delta]$ and $\overline{F}_\beta \cap [\delta, 1-\delta]$, where $\delta = 2^{-m_0}$. For sake of simplicity, we will still write them as $\underline{F}_\beta$ and $\overline{F}_\beta$.

We will use $H^\alpha(E)$ to denote the α-dimensional Hausdorff measure of a set E. We will estimate the Hausdorff measure of a set by using dyadic intervals. For this reason, we will consider the dyadic tree $A^* = \bigcup_{n=0} A^n$ with $A = \{0, 1\}$.

We have only to show that for every small enough $\epsilon > 0$, with probability one, we have

$$H^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}(\overline{F}_\beta^{\inf}) = 0, \qquad (\forall \beta \in [\beta_{\min}, \beta_{\max}] \cap [0,1)) \tag{4.1}$$

$$H^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}(\underline{F}_\beta^{\sup}) = 0, \qquad (\forall \beta \in [\beta_{\min}, \beta_{\max}] \cap [1,\infty)) \tag{4.2}$$

Given a closed interval $[a_1, a_2] \subset (0,1)$, et $K = [a_1, a_2] \cup \{0\}$. In order to prove (4.1), it is enough to show that for small $\epsilon > 0$, with probability one, we have

$$H^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}(\overline{F}_\beta^{\inf}) = 0, \qquad (\forall \beta \in K). \tag{4.3}$$

Fix $0 < \epsilon < 1 - a_2$ and $M \ge \frac{1}{\epsilon}$. Assume $\beta \in K$. For $t \in \overline{F}_\beta$, there exists $n \ge 1$ such that for every $k \ge n$

$$N'_{n_k}(t) \le (\beta + \epsilon) L'_{n_k} \tag{4.4}$$

(we used the facts $N'_n(t) \le N_n(t)$ and $L'_n \sim L_n$). It follows that

$$\overline{F}_\beta^{\inf} \subset \bigcup_{n=1}^{\infty} \overline{F}_\beta^{\inf}(n)$$

where

$$\overline{F}_\beta^{\inf}(n) = \left\{ t : N'_{n_k}(t) \le (\beta + \epsilon) L'_{n_k} \quad \forall\ k \ge n \right\}.$$

So, by the sub-additivity of the Hausdorff measure, (4.3) is reduced to the fact that for small $\epsilon > 0$ and for any $n \geq 1$, with probability one, we have

$$H^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}(\overline{F}_{\beta}^{\inf}(n)) = 0, \qquad (\forall \beta \in K). \tag{4.5}$$

Since $\beta + \epsilon \leq a_2 + \epsilon < 1$, the fact (4.4) implies

$$1 \leq (\beta+\epsilon)^{N'_{n_k}(t)} (\beta+\epsilon)^{-L'_{n_k}(\beta+\epsilon)}. \tag{4.6}$$

Let I_w with $w \in A^{m_k}$ be a dyadic interval containing a point t such that (4.6) holds (for such a I_w we have $I_w \cap (\delta, 1-\delta) \neq \varnothing$). Then

$$1 \leq M^{s'_k(w)} (\beta+\epsilon)^{N(D^{I_w})} (\beta+\epsilon)^{-L'_{n_k}(\beta+\epsilon)}. \tag{4.7}$$

In fact, (4.7) is a direct consequence of (4.6) when $N(D^{I_w}) \leq N'_{n_k}(t)$. Assume now $N(D^{I_w}) > N'_{n_k}(t)$. Then (4.7) will follow from (4.6) and the fact $M^{-s'_k(w)} \leq (\beta+\epsilon)^{N(D^{I_w}) - N'_{n_k}(t)}$, which follows from $\epsilon^{s'_k(w)} \leq \epsilon^{N(D^{I_w}) - N'_{n_k}(t)}$ (notice that $M^{-1} \leq \epsilon$). Indeed, the last fact is true because

$$N'_{n_k}(t) \geq -s'_k(w) + N^P_{2^{-k}}(t) \geq -s'_k(w) + N(D^{I_w}),$$

a fortiori, $s'_k(w) \geq N(D^{I_w}) - N'_{n_k}(t)$. So, for such an interval I_w we have

$$|I_w|^{(d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon})} \leq 2^{-m_k(d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon})} \cdot M^{s'_k(w)} (\beta+\epsilon)^{N(D^{I_w})} (\beta+\epsilon)^{-L'_{n_k}(\beta+\epsilon)} \tag{4.8}$$

It follows from (4.8) that

$$H^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}(\overline{F}_{\beta}^{\inf}(n)) \leq \liminf_{k\to\infty} f_k(\beta)$$

where

$$f_k(\beta) = 2^{-m_k(d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon})} (\beta+\epsilon)^{-L'_{n_k}(\beta+\epsilon)} \sum_{\substack{w \in A^{m_k} \\ I_w \subset [\delta, 1-\delta]}} M^{s'_k(w)} (\beta+\epsilon)^{N(D^{I_w})}.$$

Let $(k_j)_{j\geq 1}$ be the sequence chosen at the beginning of the present section. We claim that for every small enough ϵ with probability one,

$$\sum_{j\geq 1} f_{k_j}(\beta) < \infty \qquad (\forall \beta \in K) \tag{4.9}$$

which implies (4.5). We will prove the claim by distinguishing $\beta = 0$ and $\beta \in [a_1, a_2]$.

Consider first $\beta = 0$. We have

$$f_k(0) = \sum_{\substack{w \in A^{m_k} \\ I_w \subset [\delta, 1-\delta]}} 2^{-m_k(d_{\overline{\alpha}}(0)+\sqrt{\epsilon})} \cdot M^{s'_k(w)} \cdot \epsilon^{N(D^{I_w})} \epsilon^{-\epsilon L'_{n_k}}$$

So,

$$\mathbb{E}(f_k(0)) \leq 2^{m_k} 2^{-m_k(d_{\overline{\alpha}}(0)+\sqrt{\epsilon})} \epsilon^{-\epsilon L'_{n_k}} \cdot \mathbb{E}(M^{s'_k(w)} \epsilon^{N(D^{I_w})}) \tag{4.10}$$

where $w \in A^{m_k}$ is generic such that $I_w \subset [\delta, 1-\delta]$. Since s'_k is exponentially integrable (Lemma 3.4), by the Hölder inequality, we get

$$\begin{aligned} \mathbb{E}(M^{s'_k(w)} \epsilon^{N(D^{I_w})}) &\leq C_\epsilon \left(\mathbb{E}\epsilon^{(1+\epsilon)N(D^{I_w})}\right)^{1/(1+\epsilon)} \\ &\leq C_\epsilon \exp\left(\frac{\epsilon^{1+\epsilon}-1}{1+\epsilon} L'_{n_k}\right) \end{aligned}$$

where in each inequality above C_ϵ is a constant depending on ϵ, but independent of k (we used the facts $L'_{n_k} = \nu(D_{2^{-m_k}})$ and $0 \leq \nu(D_{2^{-m_k}}) - \nu(D^{I_w}) = O(1)$). Thus, using the fact $d_{\overline{\alpha}}(0) = 1 - \overline{\alpha}$, we get

$$\mathbb{E}(f_k(0)) \leq C_\epsilon \exp\left[m_k(\overline{\alpha} - \sqrt{\epsilon})\log 2 - L'_{n_k}\left(\frac{1-\epsilon^{1+\epsilon}}{1+\epsilon} + \epsilon\log\epsilon\right)\right]$$

Notice that

$$\frac{1-\epsilon^{1+\epsilon}}{1+\epsilon} + \epsilon\log\epsilon = 1 + O(\epsilon|\log\epsilon|)$$

and that for large j we have

$$L'_{n_{k_j}} \geq -(\overline{\alpha} - \epsilon)\log\ell_{n_{k_j}} > (\overline{\alpha} - \epsilon)(m_{k_j} - 1)\log(2).$$

Then, for small $\epsilon > 0$, we have

$$\sum_j \mathbb{E}f_{k_j}(0) \leq C_\epsilon \sum_j 2^{-m_{k_j}(\sqrt{\epsilon} + O(\epsilon|\log\epsilon|))} < \infty. \tag{4.11}$$

Next suppose that $\beta \in [a_1, a_2]$. We have

$$\begin{aligned}\mathbb{E}f_k(\beta) \leq\ & 2^{m_k} 2^{-m_k(d_{\overline{\alpha}}(\beta) + \sqrt{\epsilon})}(\beta + \epsilon)^{-(\beta+\epsilon)L'_{n_k}} \\ & \times \mathbb{E}(M^{s'_k(w)}(\beta + \epsilon)^{N(D^{I_w})})\end{aligned} \tag{4.12}$$

In the same way, we apply the Hölder inequality to get

$$\mathbb{E}(M^{s'_k(w)}(\beta + \epsilon)^{N(D^{I_w})}) \leq C_\epsilon \exp\left(\frac{(\beta+\epsilon)^{1+\epsilon} - 1}{1+\epsilon} L'_{n_k}\right).$$

However

$$\frac{(\beta+\epsilon)^{1+\epsilon} - 1}{1+\epsilon} - (\beta+\epsilon)\log(\beta+\epsilon) = \beta - 1 - \beta\log(\beta) + O(\epsilon)$$

which is negative for small ϵ and again $L'_{n_{k_j}} \geq (\overline{\alpha} - \epsilon)(m_{k_j} - 1)\log(2)$ for large j. So, we can get

$$\sum_j \mathbb{E}f_{k_j}(\beta) \leq C_\epsilon \sum_j 2^{-m_{k_j}(\sqrt{\epsilon} + o(\sqrt{\epsilon}))} < \infty. \tag{4.13}$$

The same approach as the one used in proving Proposition 3.1 will show that

$$\sum_j \max_{\beta\in[a_1,a_2]} \mathbb{E}f'_{k_j}(\beta) \leq C_\epsilon \sum_j k_j^2 2^{-m_{k_j}(\sqrt{\epsilon} + o(\sqrt{\epsilon}))} < \infty, \tag{4.14}$$

where f'_{k_j} denotes the derivative of f_{k_j}. The combination of the last two estimates implies

$$\sum_j \mathbb{E} \max_{\beta\in[a_1,a_2]} f_{k_j}(\beta) < \infty \tag{4.15}$$

Finally, we get (4.9) from (4.11) and (4.15).

The part *1* of Proposition 4.1 is proved.

To prove part *2*, we make the following observations: even if a number d is negative, one can define formally by the usual way the d-dimensional Hausdorff measure of a set; this measure is equal to $+\infty$ for any non-empty set. Another observation is that the above estimations remain true even when $d_{\overline{\alpha}}(\beta) < 0$. These two observations allow us to conclude for part *2*.

The parts *3* and *4* may be proved in the same way with minor changes. Let us just point out what should be changed. Now we work with $[a_1, a_2] \subset (1, \infty)$. Instead of (4.4), we will have

$$N''_{n_k} \geq (\beta - \epsilon) L'_{n_k}.$$

The counterpart of $f_k(\beta)$ is

$$g_k(\beta) = 2^{-k(d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon})} \beta^{-L'_{n_k}(\beta-\epsilon)} \sum_{\substack{w \in A^k \\ I_w \subset [\delta, 1-\delta]}} M^{N(B^{I_w}) + s''_k(w)} (\beta + \epsilon)^{N(D^{I_w})}.$$

□

4.3. **Proof of Proposition 4.2.** We have only to make a small change of the proof of Proposition 4.1. The estimations obtained in the proof of Proposition 4.1 are still useful. Actually we have used the fact that the sequence $f_{k_j}(\beta)$ (as well as g_{k_j}) tends uniformly to zero but we have proved the uniform convergence of the series $\sum_j f_{k_j}(\beta)$ (as well as the series involving g_{k_j}). Now we really need the uniform convergence of the series.

For every $\epsilon > 0$ and $\beta \in [0, 1)$, we have

$$\underline{F}_\beta^{\inf} \subset \bigcap_{n \geq 1} \bigcup_{j \geq n} \underline{F}_\beta^{\inf}(j)$$

where

$$\underline{F}_\beta^{\inf}(j) = \left\{ t : N'_{n_{k_j}}(t) \leq (\beta + \epsilon) L'_{n_{k_j}} \right\}.$$

It follows that for $n \geq 1$

$$H_{2^{-m_{k_n}}}^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}(\underline{F}_\beta^{\inf}) \leq \sum_{j \geq n} \sum |I_w|^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}$$

where the second sum is taken over all $w \in A^{m_{k_j}}$ such that

$$I_w \subset [\delta, 1-\delta], \quad \exists\, t \in I_w \text{ such that } N'_{n_{k_j}}(t) \leq (\beta + \epsilon) L'_{n_{k_j}}.$$

By using the estimations obtained in the proof of Proposition 4.1, we get that with probability one

$$H_{2^{-m_{k_n}}}^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}(\underline{F}_\beta^{\inf}) \leq \sum_{j \geq n} f_{k_j}(\beta).$$

Thus we get the parts *1* and *2*.

For $\beta \in (1, \infty)$, we may prove in a similar way that with probability one,

$$H_{2^{-n}}^{d_{\overline{\alpha}}(\beta)+\sqrt{\epsilon}}(\overline{F}_\beta^{\sup}) \leq \sum_{j \geq n} g_{k_j}(\beta).$$

□

5. Proof of Theorem 1.3

Recall that ℓ'_n and ℓ''_n were defined in the subsection 2.5 and $m_k = m_k^{(2)}$ and $n_k = n_k^{(2)}$ were defined in the subsection 3.1.

Due to Proposition 4.3, it suffices to show that for every $\beta < 1$ close enough to 1, with probability one,

$$\underline{F}_\beta^{\inf} \cup \overline{F}_{1/\beta}^{\sup} = \varnothing$$

where $\{k_j\} = \{j\}$ is chosen for defining $\underline{F}_\beta^{\inf}$ and $\overline{F}_\beta^{\sup}$.

We proceed as in the proof of Proposition 4.2, but instead of cutting $[\delta, 1-\delta]$ into subintervals of length 2^{-m_k}, we divide it into subintervals of length $(1-2\delta)/n_k$. We denote this collection of intervals by $\mathcal{J}_k$.

We compare the Dvoretzky covering with the Poisson covering by defining, for each interval $I \in \mathcal{J}_k$, the quantities

$$\overline{s}'(I) = \sup_{t\in I} |N'_{n_k}(t) - N^P_{b^{-m_k}}(t)|,$$

$$\overline{s}''(I) = \sup_{t\in I} |N''_{n_k}(t) - N^P_{b^{-m_k}}(t)|.$$

(Analogous quantities were introduced and studied in the subsection 3.1.) In order to estimate the size of these variables, we introduce

$$\overline{D}^I = D_{b^{-m_k}}(\inf I) \bigcap D_{b^{-m_k}}(\sup I), \qquad \overline{B}^I = \bigcup_{t\in I} D_{b^{-m_k}}(t) \setminus \overline{D}^I.$$

For any integer $m \geq 1$ such that $\lambda'_m \geq b^{-m_k}$, denote

$$\overline{D}^I_m = \overline{D}^I \cap (\mathbb{R} \times \{\lambda'_m\}), \qquad \overline{B}^I_m = \overline{B}^I \cap (\mathbb{R} \times \{\lambda'_m\})$$

We also introduce

$$\widetilde{\overline{B}}^I = \bigcup_{m:\lambda'_m \geq b^{-m_k}} \widetilde{\overline{B}}^I_m$$

where

$$\widetilde{\overline{B}}^I_m = B^I_m \bigcup [\inf I - \lambda''_m, \inf I - \lambda'_m] \times \{\lambda'_m\}.$$

(Analogous sets were introduced in the subsection 2.4.)

We need the following intermediate result.

Lemma 5.1.

1. $L''_n \sim L'_n$.

2. For any $r > 0$, there exist constants $C_1 = C_1(r)$ and $C_2 = C_2(r)$ such that for any $I \in \mathcal{J}_k$ we have

$$\mathbb{E}e^{rN(\overline{B}^I)} + \mathbb{E}e^{r\mathcal{D}(\overline{B}^I)} \leq C_1$$

$$\mathbb{E}e^{r\overline{s}'(I)} + \mathbb{E}e^{r\overline{s}''(I)} \leq \exp(o(L'_{n_k})).$$

$$\mathbb{E}e^{r|N(\overline{D}^I) - \mathcal{D}(\overline{D}^I)|} \leq C_1 \exp\left(C_2 \sum_{m:\ \lambda'_m \geq 2^{-m_k}} \sqrt{m}\lambda'_m\right)$$

$$\mathbb{E}e^{r\mathcal{D}(\widetilde{\overline{B}}^I)} \leq C_1 \exp\left(C_2 \sum_{\lambda'_m \geq 2^{-m_k}} m(\lambda''_m - \lambda'_m)\right)$$

The first point $L'_n \sim L''_n$ is contained in Proposition 7.4. The other estimates follow the same lines as those proved in Section 3 for analogous quantities.

Now continue our proof. Fix $\beta \in (0,1)$ and $d < 0$. Let $M = 1/\beta$. For $\epsilon > 0$ and $k \geq 1$, define

$$\overline{f}_k(\beta) = n_k^{-d} \beta^{-L'_{n_k}(\beta+\epsilon)} \sum_{I \in \mathcal{J}_k} M^{\overline{s}'(I)} \beta^{N(\overline{D}^I)},$$

$$\overline{g}_k(\beta^{-1}) = n_k^{-d} \beta^{L'_{n_k}(\beta-\epsilon)} \sum_{I \in \mathcal{J}_k} M^{N(\overline{B}^I)+\overline{s}''(I)} \beta^{-N(\overline{D}^I)}.$$

Choose $\epsilon > 0$ such that $\beta + \epsilon < 1$ and $\beta^{-1} - \epsilon > 1$. The computations performed in the previous section yields

$$H^d_{n_k^{-1}}(\underline{F}^{\inf}_\beta) \leq (1-2\delta)^d \sum_{j \geq k} \overline{f}_j(\beta)$$

$$H^d_{n_k^{-1}}(\overline{F}^{\sup}_{1/\beta}) \leq (1-2\delta)^d \sum_{j \geq k} \overline{g}_j(\beta^{-1})$$

Notice that both $\underline{F}^{\inf}_\beta$ and $\overline{F}^{\sup}_\beta$ are increasing functions of $\beta \in (0,1)$. Since $d < 0$, we have to show that for any fixed β we have

$$\sum_j \mathbb{E}\overline{f}_j(\beta) < \infty, \qquad \sum_j \mathbb{E}\overline{g}_j(\beta) < \infty.$$

In fact, writing

$$d(x) = x - 1 - x\log x,$$

by Lemma 5.1, we get

$$\mathbb{E}\overline{f}_j(\beta) = n_k^{1-d} \exp\big(o(L'_{n_k}) + (d(\beta) + O(\epsilon))L'_{n_k}\big) \tag{5.1}$$

and

$$\mathbb{E}\overline{g}_j(\beta) = n_k^{1-d} \exp\big(o(L'_{n_k}) + (d(1/\beta) + O(\epsilon))L'_{n_k}\big) \tag{5.2}$$

Since $d(x) < 0$ for every $x \in (0,\infty) \setminus \{1\}$ and $L'_{n_k} \sim k\log 2$, in order to conclude, we have only to choose a small number $\epsilon > 0$ and to show that $\log n_k = o(L'_{n_k})$. This was done in the proof of Proposition 4.3.

6. Analogous results for Poisson coverings

We consider a Poisson point process as was constructed in Section 2. We assume that $\nu(D_\epsilon) \to \infty$ as $\epsilon \to 0$ and define

$$\underline{F}^P_\beta = \{t \in \mathbb{R}_+ : \liminf_{\epsilon \to 0} \frac{N^P_\epsilon(t)}{\nu(D_\epsilon)} = \beta\},$$

$$\overline{F}^P_\beta = \{t \in \mathbb{R}_+ : \limsup_{\epsilon \to 0} \frac{N^P_\epsilon(t)}{\nu(D_\epsilon)} = \beta\},$$

$$F^P_\beta = \underline{F}^P_\beta \bigcap \overline{F}^P_\beta.$$

Recall that the assumption (**HP**) was defined in Section 2. There is a counterpart of ($\mathbf{H}_\infty$), namely

$$(\mathbf{HP}_\infty) \qquad \lim_{\epsilon \to 0} \epsilon\, \mu([\epsilon, 1)) = +\infty.$$

We state the following results whose proofs are somehow easier.

Theorem 6.1 (Case $\overline{\alpha}^P = 0$)**.** *Assume* (**HP**)*, i.e.* $\limsup_{\epsilon\to 0} \epsilon\mu([\epsilon,1)) < \infty$*. Suppose* $\overline{\alpha}^P = 0$*. With probability one, for all* $\beta \geq 0$ *such that* $d_{\widehat{\alpha}^P}(\beta) > 0$*, we have*

$$\dim(F_\beta^P) = \dim(\underline{F}_\beta^P) = \dim(\overline{F}_\beta^P) = 1. \tag{6.1}$$

Theorem 6.2 (Case $\overline{\alpha}^P > 0$)**.** *Assume that* (**HP**)*, i.e.* $\limsup_{\epsilon\to 0} \epsilon\mu([\epsilon,1)) < \infty$*. Suppose* $0 < \overline{\alpha}^P < \infty$*. With probability one, for all* $\beta \geq 0$ *such that* $d_{\widehat{\alpha}^P}(\beta) > 0$*, we have*

$$\dim(F_\beta^P) = d_{\overline{\alpha}}(\beta); \tag{6.2}$$

and for all $\beta \geq 0$ *such that* $d_{\overline{\alpha}^P}(\beta) < 0$ *we have*

$$F_\beta^P = \varnothing. \tag{6.3}$$

If, moreover, $\overline{\alpha}^P$ *is defined by a limit (not just a limsup), (6.2) and (6.3) hold for* $\underline{F}_\beta^P$ *and* $\overline{F}_\beta^P$ *instead of* F_β^P*.*

Theorem 6.3 (Case $\overline{\alpha}^P = +\infty$)**.** *Assume* (**HP**$_\infty$)*, i.e.* $\lim_{\epsilon\to 0} \epsilon\, \mu([\epsilon,1)) = +\infty$*. Then almost surely we have*

$$\lim_{\epsilon\to 0} \frac{N_\epsilon^P(t)}{\nu(D_\epsilon)} = 1 \quad (\forall\ t \in \mathbb{R}).$$

As in case of the Dvoretzky covering, we make the following remark on F_0^P. By Theorem 6.2, $\overline{\alpha}^P > 1$ implies $F_0^P = \varnothing$; $\overline{\alpha}^P < 1$ then $\dim(F_0^P) = 1 - \overline{\alpha}^P > 0$. When $\overline{\alpha}^P = 1$, $\dim(F_0^P) = 0$ and $F_0^P \neq \varnothing$ if

$$\int_{(0,1)} \exp\left\{ \int_{(t,1)} \mu(s,1)\, ds \right\} dt < \infty.$$

7. Appendix

Here we get together some properties of the sequence $\{\ell_n\}$ under different conditions.

Proposition 7.1. *We have the following equivalences.*

(i) *The assumption* (**H**) *is equivalent to* $\limsup_{\epsilon\to 0}\ \epsilon \operatorname{Card}\Lambda_\epsilon < \infty$*.*

(ii) *The assumption* (**H**$_\infty$) *is equivalent to* $\lim_{\epsilon\to 0} \epsilon \operatorname{Card}\Lambda_\epsilon = +\infty$*.*

Proof. (i) Recall that $\operatorname{Card}\Lambda_\epsilon = \sum_{n:\ell_n \geq \epsilon} 1$. Suppose $\operatorname{Card}\Lambda_\epsilon \leq C\epsilon^{-1}$. Fix $N \geq 1$. There exists a unique k such that $2^{-k} \leq \ell_N < 2^{-k+1}$. Then

$$N \leq \operatorname{Card}\Lambda_{2^{-k}} = \big(2^{-k}\operatorname{Card}\Lambda_{2^{-k}}\big)2^k \leq \frac{2C}{\ell_N}.$$

That is to say $\ell_N \leq 2C/N$. Suppose $\ell_n \leq D/n$ for all n. Fix $\epsilon \in (0,\ell_1)$ and $k \geq 1$ such that $2^{-k} \leq \epsilon < 2^{-k+1}$. We have

$$\operatorname{Card}\Lambda_\epsilon \leq \sum_{n:D/n \geq 2^{-k}} 1 \leq 2D\epsilon^{-1}.$$

(ii) Fix $N \geq 1$. Choose k such that $2^{-k} \leq \ell_N < 2^{-k+1}$. We have

$$N \geq \operatorname{Card}\Lambda_{2^{-k+1}} = (2^{-k+1}\operatorname{Card}\Lambda_{2^{-k+1}})2^{k-1} \geq (2^{-k+1}\Lambda_{2^{-k+1}})\frac{1}{2\ell_N},$$

that is

$$N\ell_N \geq \frac{1}{2}2^{-k+1}\Lambda_{2^{-k+1}}.$$

Consequently, $\lim_{\epsilon\to 0} \epsilon \mathrm{Card}\,\Lambda_\epsilon = +\infty$ implies $(\mathbf{H}_\infty)$.

Now suppose $(\mathbf{H}_\infty)$ hold. For any large number $M > 0$, there exists $n_0 \geq 1$ such that $n\ell_n \geq M$ for all $n \geq n_0$. Fix $\epsilon \in (0, \ell_1)$ and $k \geq 1$ such that $2^{-k} \leq \epsilon < 2^{-k+1}$. We have

$$\begin{aligned}
\mathrm{Card}\,\Lambda_\epsilon &\geq \sum_{n:\ \ell_n \geq 2^{-k+1}} 1 = \sum_{n:\ n\leq (n\ell_n)2^{k-1}} 1 \\
&\geq \sum_{n_0\leq n\leq M2^{k-1}} 1 = M2^{k-1} - n_0 \geq \frac{M}{2\epsilon} - n_0.
\end{aligned}$$

It follows that

$$\liminf_{\epsilon\to 0} \epsilon\,\mathrm{Card}\,\Lambda_\epsilon \geq M.$$

This finishes the proof since M may be arbitrary large. □

Recall that

$$\overline{\alpha} = \limsup_{N\to\infty} \frac{\sum_{n=1}^N \ell_n}{-\log \ell_N},$$

$$\overline{\alpha}_b = \limsup_{k\to\infty} \frac{\sum_{n:\ell_n\geq b^{-k}} \ell_n}{\log b^k}$$

where $b \geq 2$ is an integer.

Proposition 7.2. *We have $\overline{\alpha}_b = \overline{\alpha}$ for all $b \geq 2$.*

Proof. Fix $k \geq 1$. For sake of convenience, let $N_k = \mathrm{Card}\,\Lambda_{b^{-k}}$. Then $\ell_{N_k+1} < b^{-k} \leq \ell_{N_k}$. For any $\epsilon > 0$ and large k, we have

$$\sum_{n:\ell_n\geq b^{-k}} \ell_n = \sum_{n=1}^{N_k} \ell_n \leq (\overline{\alpha}+\epsilon)(-\log \ell_{N_k}) \leq (\overline{\alpha}+\epsilon)\log b^k.$$

It follows that $\overline{\alpha}_b \leq \overline{\alpha}$.

Fix $N \geq 1$. There exists a unique k such that $b^{-k} \leq \ell_N < b^{-k+1}$. Then for any $\epsilon > 0$ and large N

$$\sum_{n=1}^N \ell_n \leq \sum_{n:\ell_n\geq b^{-k}} \ell_n \leq (\overline{\alpha}_b+\epsilon)\log b^k \leq (\overline{\alpha}_b+\epsilon)\log(b\ell_N^{-1}).$$

It follows that $\overline{\alpha} \leq \overline{\alpha}_b$. □

As is pointed out in [K1] (p. 161), there is another formula for $\overline{\alpha}$:

$$\overline{\alpha} = \limsup_{n\to\infty} \frac{\sum_{j=1}^n \ell_j}{\log n}. \tag{7.4}$$

Proposition 7.3. *We have*

(i) $\overline{\alpha} \leq \widehat{\alpha}$;

(ii) The assumption $(\mathbf{H})$ *implies* $\widehat{\alpha} < \infty$.

Proof. (i) Fix $n \geq 1$ and $b \geq 2$. For every $N \geq 1$ denote by k_N the integer such that $b^{-k_N-1} < \ell_N \leq b^{-k_N}$. Since for N large enough

$$\frac{\sum_{j=1}^N \ell_j}{-\log \ell_N} \leq \frac{\sum_{\ell_j \in [b^{-k_N-1}, b^{-n}]} \ell_j + \sum_{\ell_j \in (b^{-n}, 1]} \ell_j}{\log b^{k_N}}$$

Write $k_N + 1 = n + m$ with $m = k_N + 1 - n$. Then $\log b^{k_N} = \log b^m + \log b^{n-1}$ and we see that

$$\begin{aligned} \overline{\alpha} &\leq \limsup_{m\to\infty} \frac{\sum_{\ell_j \in [b^{-m-n}, b^{-n}]} \ell_j + \sum_{\ell_j \in (b^{-n}, 1]} \ell_j}{\log b^m} \\ &\leq \sup_{m \geq 1} \frac{\sum_{\ell_j \in [b^{-m-n}, b^{-n}]} \ell_j}{\log b^m}. \end{aligned}$$

We conclude by using the definition of $\widehat{\alpha}$.

(ii) Let $\mu = \sum_{j=1}^\infty \delta_{\ell_j}$. For $n, m \geq 1$ and $b \geq 2$, one has

$$\sum_{\ell_j \in [b^{-m-n}, b^{-n}]} \ell_j = \int_{b^{-m-n}}^{b^{-n}} y d\mu(y) = \int_{b^{-m-n}}^{b^{-n}} \int_0^y dx d\mu(y).$$

Using Fubini Theorem yields

$$\begin{aligned} \sum_{\ell_j \in [b^{-m-n}, b^{-n}]} \ell_j &= b^{-n-m}\mu([b^{-n-m}, b^{-n}]) + \int_{b^{-m-n}}^{b^{-n}} \mu([x, b^{-n}]) dx \\ &\leq b^{-n-m}\mu([b^{-n-m}, 1]) + \sum_{k=n+1}^{n+m} \mu([b^{-k}, 1])(b^{-k+1} - b^{-k}). \end{aligned}$$

However $\mu([b^{-k}, 1])$ is nothing but $\mathrm{Card}\,\Lambda_{b^{-k}}$. By the assumption $(\mathbf{H})$, $\mu([b^{-k}, 1])b^{-k} = O(1)$ and then the last sum is $O(m)$. So, (ii) is proved. $\square$

Proposition 7.4. *Under the assumption* $(\mathbf{H})$, *i.e.* $\limsup_n n\ell_n < \infty$, *we have*

$$L''_n - L'_n = O(1).$$

Under the assumption $(\mathbf{H}_\infty)$, *i.e.* $\lim_n n\ell_n = \infty$, *we have*

$$L''_n \sim L'_n.$$

Proof. For any integer $n \geq 1$ there exists an integer m such that $\frac{m(m+1)}{2} \leq n < \frac{(m+1)(m+2)}{2}$. We have

$$L''_n - L'_n = L''_{\frac{m(m+1)}{2}} - L'_{\frac{m(m+1)}{2}} + O\left(m\ell_{\frac{m(m+1)}{2}}\right).$$

The the assumption $(\mathbf{H})$ implies that $m\ell_{\frac{m(m+1)}{2}}$ is bounded. So, we have only to show that $L''_{\frac{m(m+1)}{2}} - L'_{\frac{m(m+1)}{2}}$ is bounded. Indeed,

$$L''_{\frac{m(m+1)}{2}} - L'_{\frac{m(m+1)}{2}} \leq \sum_{j=1}^m j\left(\ell_{\frac{j(j-1)}{2}+1} - \ell_{\frac{j(j+1)}{2}+1}\right) = \sum_{j=1}^m \ell_{\frac{j(j-1)}{2}+1}.$$

The last sum is bounded, up to a multiplicative constant, by $\sum_{j=1}^\infty j^{-2}$. Thus the first assertion is proved.

To prove the second assertion, we apply the assumption ($\mathbf{H}_\infty$) instead of the assumption ($\mathbf{H}$). It suffices to remark that

$$m\ell_{\frac{m(m+1)}{2}} = o\Big(\sum_{k=1}^{m} k\ell_{\frac{k(k+1)}{2}}\Big).$$

This follows from

$$\sum_{k=1}^{m} k\ell_{\frac{k(k+1)}{2}} \geq \frac{m}{2}\sum_{k=m/2}^{m} \ell_{\frac{k(k+1)}{2}} \geq \frac{m^2}{2}\ell_{\frac{m(m+1)}{2}}.$$

□

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