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Introduction to Infinite Products of Independent Random Functions

(Random Multiplicative Multifractal Measures, Part I)

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ABSTRACT. This is the first of three papers devoted to a class of random measures generated by multiplicative processes.

This Part I surveys the main motivations that led B. Mandelbrot to introduce such statistically self-affine multifractal measures, from the initial limit lognormal processes to multiplicative cascades of random weights, and finally the multifractal products of pulses. A discussion contrasts the recent class of multifractal products of cylindrical pulses with the well-known canonical cascade measures.

Part II will present these examples as particular elements of a class of random measures generated by multiplications of functions for which several fundamental problems, namely non-degeneracy, finiteness of moments, dimension of the carrier and multifractal analysis can be studied and solved. The results complete Kahane's general theory of T-martingales and will be applied to new examples.

Part III will provide the proofs of the results obtained in Part II.

1. Introduction

This paper describes the main motivations that led from the limit lognormal multifractals to the canonical cascade measures (CCM) and then, more recently, to the "multifractal products of pulses"(MPP). The main mathematical related problems are discussed for CCM and a subclass of MPP, namely the "multifractal products of cylindrical pulses" (MPCP) (an alternative account of the CCM and their extensions is found in [**P2**]).

The Kolmogorov lognormal hypothesis for intermittent turbulence [**Ko**] was inspiring but very sketchy. Implementing the underlying idea mathematically turned

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out to be difficult. To do so, [**M2**] introduced measures that are limits of suitable sequences of lognormal random processes. Such sequences form measure-valued positive martingales. They converge but may almost surely converge to zero. Therefore, the first fundamental problem raised in [**M2**] was to determine whether or not the limit measure is non-degenerate, that is, positive with a positive probability. A sufficient condition for degeneracy was provided, as well as a necessary and sufficient condition (NSC) of convergence in L^p (p integer ≥ 2), yielding sufficient conditions for non-degeneracy; a NSC was only conjectured. Another fundamental question was raised and partially answered as a corollary. When the limit measure is non-degenerate, which is the necessary and sufficient condition for a moment of given positive order to be finite ? This question was reformulated in [**M4**] as the one of divergence of high moments.

All those questions turned out to be too difficult for the conjectures to be tackled in complete rigor. In response, [**M4**] and [**M5**] introduced and discussed an alternative class of martingale limit measures, namely the canonical cascade measures (CCM). The presence of a cascade is a strongly restrictive but simplifying element. It allowed the previous conjectures about non-degeneracy and moments to be reformulated and again partially solved, and a third main conjecture about the dimension of the turbulence carrier was proposed. All these questions were answered in [**K1**], [**P1**] and [**KP**].

It is time to introduce the notion of multifractality. The first result was that, when a CCM μ is non-degenerate, it possesses μ-almost everywhere the same Hölder exponent D. This result yields the value $f_H(D) = D$ for the function f_H that is now commonly called multifractal spectrum of CCM. The D, conjectured in [**M4, M5**] and confirmed in [**KP**], solves the "dimension of the carrier" problem. Moreover, [**M5**] studied the probability distribution of the martingale defining CCM using the Cramer–Chernoff large deviation theorem. That theorem involves the Legendre transform and yields preasymptotic results that are formally identical to the expression now called large deviations spectrum of CCM.

The values of the Legendre transform $f(\alpha)$ were interpreted as fractal dimensions in [**FrPa**], which also introduced the term "multifractal". Then a number of deterministic multifractal measures were exhibited and studied; the heuristic interpretation of $f(\alpha)$ in [**FrPa**] was confirmed ([**HaJeKaPrSh**], [**CoLePo**], [**BoRa**], [**Ra**], [**BrMiP**]), giving birth to the term "multifractal formalism". In [**M7, M9**], CCM were shown to be complex enough to illustrate a new concept associated with randomness, that of (latent) negative values for the "dimension" $f(\alpha)$. Later, in the 90's, the technique used in [**KP**] became a basic tool in several studies of the multifractal spectrum of CCM ([**K5**], [**HoWa**], [**Mol**], [**B2**]). As multifractal analysis developed, controlling moments of negative orders of the total mass of martingale-limit measures also became a fundamental problem.

[**K2**] returned to the limit lognormal model of [**M2**], confirmed its conjectures, and developed the notion of "Gaussian multiplicative chaos". For certain limits of log-normal multiplicative cascades that can be related to some CCM, definitive results were obtained for the three problems mentioned above: non-degeneracy, high moments, and dimension. To study more general "positives martingales and random measures" and their applications, [**K2, K3, Fa1, K4, Fa2, K5, Fa3, Fa4**] developed a number of further tools.

CCM simplified the limit lognormal construction in [**M2**] by injecting a prescribed b-adic grid of intervals of $[0, 1]$. While this grid was necessary for technical reasons, it has no counterpart in nature. Moreover, in addition to the canonical cascades, [**M4**] had considered microcanonical cascades that are locally conservative (see also [**dW, Y**]). In every way, the move from microcanonical to canonical randomness brought far richer structure, hinting that the removal of the grid might also yield additional interesting results.

For these reasons, a new class of random multifractal measures called "multifractal products of pulses", MPP, was introduced in [**M10**] together with a number of heuristic arguments and corresponding mathematical conjectures. The key virtues of the MPP are that – just as the limit lognormal multifractals – the MPP involve no b-adic grid, and that – contrary to the limit lognormal multifractals – the MPP are not bound to limit lognormality.

The MPP construction was inspired by the sums of pulses introduced in [**M8**] and developed in [**M14**]. Those processes do not involve a grid and are not bound to normality. Sums and products illuminate each other. Both were motivated directly by the modeling of situations in which power laws characterize both the tail of the marginal distribution and the long statistical dependence. These are two characteristics of what [**M11**] calls "wild variability" in natural and social phenomena, and their joint occurrences provided a strong challenge to the probabilists.

Let us outline the reasons, so-far unpublished, which suggested that those concrete requirements may be fulfilled by using pulses. The underlying basic structure to be used in Figure 1 had already been introduced when [**M3**] set up the problem of the covering of the real line by intervals. In terms of the present discussion, removal of an interval (or "trema") is equivalent to the multiplication of a density by a pulse equal to 0 in the interval and to 1 elsewhere.

In the specific case of sums of pulses, the point of departure was the classical series that defines the Weierstrass continuous non-differentiable function. In [**M6**], the original non random terms were multiplied by Gaussian prefactors and random phases were introduced. However, this procedure was restrictive, insofar as all the cycles of the sine function were randomized simultaneously and identically. The goal being to represent intermittent phenomena, it seemed better for each cycle's amplitude and phase to be randomized separately and independently. The resulting pulse was one half period of the sine function. The next step was to allow pulses that are not sinusoidal. The simplest are the cylindrical ones, but other shapes were examined and the behavior of sums of pulses was found to greatly depend on the pulse shape. In a last step, the Weierstrass discrete (lacunary) distribution of pulse lengths was made continuous, as in [**M3**].

In the case of canonical cascade measures, the definition already included cylindrical pulses of random amplitude. Therefore the first innovation consisted in freeing the pulse shape and injecting a random phase. This in turn allowed a second innovation consisting in eliminating the base b, in analogy to [**M3**] and what had been done for sums of pulses.

The subclass MPCP of MPP was studied in [**BM2**], which solved almost completely the three main problems and also obtained the multifractal spectrum of MPCP.

Sections 2 and 3 will define these MPCP and contrast them on several accounts with the familiar CCM. The absence of grid brings in a great increase of realism and versatility which is very valuable for the applications ([**M13**]). Those improvements are due to several novelties, essential to a varying degree, that Section 3 will discuss. It will be noted that an irreducible part of the common role of the basis will be played by a constant ρ, called "density", which is formally a generalized replacement for $1/\log b$. A third family, natural "semi-grid free" intermediate between CCM and MPCP will also be introduced, and denoted as Poisson canonical cascade measures, PCCM.

The theory in [**K3**] applies to CCM, PCCM and MPCP, but is too general to yield the finest results on non-degeneracy, finiteness of moments, and dimension of the carrier. On the other hand, it was observed in [**BM2**] that techniques developed in [**KP**] to study CCM can be adapted for MPCP. In fact, we shall see that the structure of MPCP turns out to be complex enough to allow the techniques used for MPCP to be also applicable to a larger subclass of the general construction in [**K3**]. This class will be studied in Part II [**BM3**] and illustrated with new examples, including MPP.

2. Sketches of the constructions of MPCP, CCM, and PCCM

Measures obtained by either process will be denoted by μ. We begin with MPCP. We continue by defining CCM in the same spirit as MPCP instead of the original construction in [**M4**] or [**KP**] involving successive generations of b-adic subintervals of $[0,1]$.

In the strip $\mathbb{R}\times(0,1]$ of the plane, denote by $S=\{(t_h,\lambda_h)\}$ a Poisson point process with the intensity

$$\Lambda_\rho(dtd\lambda)=\frac{\rho}{2}\frac{dtd\lambda}{\lambda^2}\qquad(\rho>0).$$

The "cylindrical pulses" investigated in [**BM2**] are a denumerable family of functions $P_h(t)$, each of which is constant and equal to 1 outside of an interval $[t_h-\lambda_h,t_h+\lambda_h]$ called "trema", and constant and equal to W_h within $[t_h-\lambda_h,t_h+\lambda_h]$. Here, the weights W_h are copies of a non-negative integrable random variable W, independent of one another and independent of S. We shall write $V=\mathbb{E}(W)$.

One defines the approximating measures μ_ε, $0<\varepsilon\leq 1$, as having a density with respect to the Lebesgue measure ℓ given by

$$\frac{d\mu_\varepsilon}{d\ell}(t)=\varepsilon^{\rho(V-1)}\prod_{(t_h,\lambda_h)\in S,\,\lambda_h\geq\varepsilon}P_h(t).$$

The measure μ is defined on the whole real line as the weak limit (on compact subsets) of the approximating measures μ_ε. In the particular case where $W=0$, this construction is considered in [**K5**] in a lecture devoted to the random covering of the real line ([**M3**]) by the intervals $(t_h,t_h+\lambda_h)$.

The familiar CCM are also defined as products of cylindrical pulses, but on $[0,1]$ rather than $\mathbb{R}$ and with the deterministic rather than random set $S=\{(\frac{k+1/2}{b^n},\frac{1}{2b^n});$ integer n and k; $n\geq 1$, $0\leq k<b^n\}$ $(b\geq 2)$. In this case one must take $V>0$ and the countable family of approximating measures $(\mu_n)_{n\geq 1}$ is given by

$$\frac{d\mu_n}{d\ell}(t) = V^{-n} \prod_{(t_h,\lambda_h)\in S,\, \lambda_h \geq \frac{1}{2b^n}} P_h(t).$$

The intermediate PCCM are obtained by taking for S a Poisson point process in $\mathbb{R} \times (0,1]$ with intensity

$$\widetilde{\Lambda}_\rho = \ell \otimes \sum_{n\geq 1} \frac{\rho}{2} \log(b) b^n \delta_{b^{-n}} \qquad (\rho > 0).$$

The countable family of approximating measures $(\mu_n)_{n\geq 1}$ is given by

$$\frac{d\mu_n}{d\ell}(t) = b^{-n\rho(V-1)} \prod_{(t_h,\lambda_h)\in S,\, \lambda_h \geq b^{-n}} P_h(t).$$

In all cases, the normalizing factors were selected to insure that one deals with a measure-valued martingale. Writing $b^{-n} = \varepsilon$ and $1/\log(b) = \rho$ rephrases the normalizing factor for CCM as $\varepsilon^{\rho \log V}$. Moreover, the densities of approximating measures are formally the same for MPCP and PCCM.

It turns out from their construction that these measure share the following important property: taking $\varepsilon = b^{-n}$ and $\rho = 1/\log(b)$ for CCM, given a point $t \in [0,1]$, the expected number of (non-unit) factors in the previous products is $\rho \log(1/\varepsilon)$. For MPCP and PCCM the same holds for all $t \in \mathbb{R}$.

In terms of statistical self-similarity, each measure μ inherits the properties of the set S: roughtly speaking, the grid free MPCP are statistically self-similar in the sense that, up to a multiplicative random variable, the restriction of μ to any nontrivial subinterval of $\mathbb{R}$ of length smaller than 1 is a rescaled copy of μ. For the semi-grid free PCCM, the same holds for subintervals of $\mathbb{R}$ whose length is a negative integer power of b. For CCM, the same holds for b-adic subintervals of $[0,1]$.

3. Discussion with Examples.

A draft of this discussion was included in [**BM1**], an early unpublished version of [**BM2**].

Motivations. The cascades behind CCM are not part of physical reality, but an artificial device made up to simplify definitions and proofs. The same is true of the restriction of their self-similarity properties to reduction ratios of the form b^{-n}, with integers b and n.

The reason why [**M4**] and [**M5**] introduced the terms "microcanonical" (often replaced by "conservative") and "canonical" is worth mentioning. These terms call to two physical ensembles in the Gibbs statistical theory; canonical is less constrained statistically than microcanonical. The Gibbs theory then continues by introducing "grand canonical ensembles" which are made of a Poisson distributed number of canonical ensembles, therefore are infinitely divisible [**M1**].

The move from CCM to MPP loosens statistical constraints in the further spirit of grand canonical ensembles. Let us show how. The definition of the CCM approximating measures can be restated as follows. Let $W(t)$ be a function of positive t that is constant in the intervals between successive integers and whose values in different intervals are statistically independent and with the distribution of W. Then

$$\frac{d\mu_n}{d\ell}(t) = V^{-n} \prod_{0<m\leq n} W_m(b^m t),$$

where the functions $W_m(t)$ are statistically independent and distributed as $W(t)$. Similarly, the corresponding approximating measures of the limit lognormal measures of [**M2**] are products of statistically independent sinusoids. In his powerful advocacy of Fourier analysis, Norbert Wiener often pointed out to sinusoids as providing the proper base for the study of stationary phenomena. But, by design, multifractals are not stationary, either visually or in the usual mathematical sense. (They are conditionally stationary sporadic functions, as defined in chapter 10 in [**M12**]). One response is to replace sinusoids by wavelets. The response of [**M8**] and the present paper is to use "pulses".

Digression on a generalization. The product $\prod W_n(b^n t)$ remains meaningful if the base ceases to be an integer. It is made more elegant and extended from $[0,1]$ to $\mathbb{R}$ if random phases φ_n are introduced and the multipliers replaced by $W_n(b^n t + \varphi_n)$.

The multifractal functions $\tau(q)$ and $f(\alpha)$. Among several possible choices for the function τ associated with a positive measure μ on $[0,1]$, the simplest uses a b-adic grid (see for example [**HaJeKaPrSh, BrMiP**]) and takes the form

$$\tau(q) = \liminf_{n\to\infty} -\frac{1}{n}\log_b \sum_{0\leq k<b^n} \mu\big([kb^{-n},(k+1)b^{-n})\big)^q.$$

Authors who also deal with thermodynamic formalism often prefer to eliminate the base b and use the integral $\displaystyle\int_{[0,1]} \mu\big(B(t,r)\big)^{q-1}\, d\mu(t)$ instead of the sum in the above expression when $r \approx b^{-n}$.

A third approach, adopted in [**O**], consists in defining multifractal functions as dimensions associated with multifractal generalizations of Hausdorff or packing measures. All these definitions coincide for the measures we deal with. For the equality of $\tau(q)$ defined above and the functions considered in [**O**], see [**BBeP**] and Part II Section 5.3 of [**BM3**].

Given the b-adic grid, the function $f(\alpha)$ is defined by

$$f(\alpha) = 1 + \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{\log_b \mathbb{P}_u\Big(\big\{2^{-n(\alpha+\varepsilon)} \leq \mu\big(I_n(x)\big) \leq 2^{-j(\alpha-\varepsilon)}\big\}\Big)}{n} \quad (\alpha\geq 0),$$

where $\mathbb{P}_u$ is the uniform probability on $[0,1)$ and $I_n(x)$ is the b-adic interval of the n^{th} generation, semi-open to the right, containing x. When μ is close to being statistically self-similar (as is the case for the measures we deal with), the "multifractal formalism" holds: if $f(\alpha) > 0$ then (A) $f(\alpha)$ is equal to the Legendre transform of τ at α, that is $\inf_{q\in\mathbb{R}} \alpha q - \tau(q)$, denoted $\tau^*(\alpha)$; (B) this infimum is also the Hausdorff dimension of the set $\Big\{x:\ \lim_{n\to\infty} \frac{\log\mu\big(I_n(x)\big)}{\log\ell\big(I_n(x)\big)} = \alpha\Big\}$.

A source of novelty is that $\tau(q)$ takes altogether different forms for CCM and (PCCM, MPCP). For the former, $\tau(q)$ was conjectured in [**M4**], [**M5**] and confirmed by numerous authors under various assumptions ([**KP**], [**K5**], [**HoWa**], [**Mol**], [**B2**], [**B3**]). The now classical expression is conveniently written

$$\tau_{\mathrm{CCM}}(q) = -1 + q[1 + \log_b V] - \log_b E(\mathbf{1}_{\{W>0\}} W^q).$$

For the sake of symmetry with τ_{MPCP}, it is best to inject $\rho = 1/\log b$ and write

$$\tau_{\mathrm{CCM}}(q) = -1 + q[1 + \rho \log V] - \rho \log \mathbb{E}(\mathbf{1}_{\{W>0\}} W^q).$$

On the contrary, [**BM2**], and Parts II and III ([**BM3, B4**]) examine the function τ for PCCM and MPCP and find when $W > 0$ that

$$\tau_{\mathrm{PCCM}}(q) = \tau_{\mathrm{MPCP}}(q) = -1 + q[1 + \rho(V-1)] - \rho\big(\mathbb{E}(W^q) - 1\big). \tag{3.1}$$

The role of $\tau'(1)$: condition of non-degeneracy and dimension of the non-degenerate "support". While the form of $\tau(q)$ changes, the condition for the non-degeneracy of μ remains $\tau'(1) > 0$ (see Theorems 5.3 and 6.6 of Part II ([**BM3**])). When it holds, $\tau'(1)$ is the Hausdorff–Besicovitch dimension of the "support" of the measure. For CCM, this is shown in [**M4**], [**P1**], [**KP**]. For MPCP, this is shown in [**BM2**] when $W > 0$. We shall see in Part II that this also holds for PCCM.

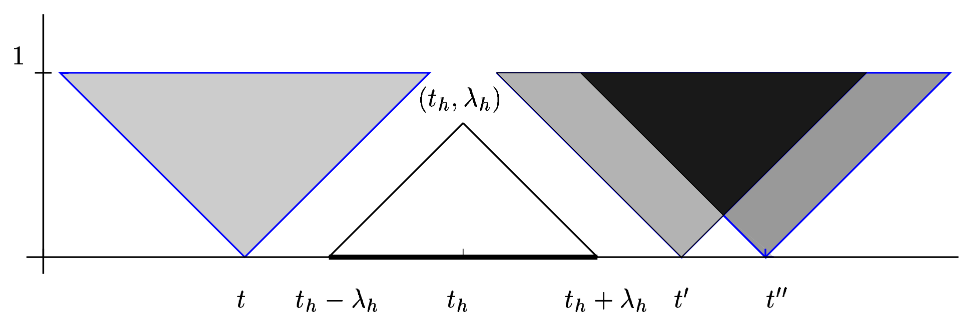

FIGURE 1. Each pulse is represented by an address point in the "address space" $\mathbb{R}\times(0,1]$. To the left: set in $\mathbb{R}\times(0,1]$ containing the addresses of those pulses that affect a given t in $\mathbb{R}$. Middle: set on the time axis $\mathbb{R}$ containing the instants t affected by a pulse (t_h, λ_h). To the right: sets in $\mathbb{R}\times(0,1]$ containing the addresses of those pulses that affect, respectively, a given t' but not a given t'', both t' and t'', and t'' but not t'.

Covariance and the role of $\tau(2)$. Given $\varepsilon > 0$, denote by μ'_ε the density of the approximating measure μ_ε and consider two points t' and t'' with $r = |t' - t''| > 2\varepsilon$. If μ is a non-degenerate MPCP, the dependence between μ_ε at t' and t'' is measured by

$$\mathbb{E}[\mu'_\varepsilon(t')\mu'_\varepsilon(t'')] = \varepsilon^{2\rho(V-1)}\mathbb{E}\{[\Pi'_\varepsilon P_h(t')][\Pi''_\varepsilon P_h(t'')]\},$$

where $\mathbb{E}$ denotes the expected value and Π'_ε and Π''_ε are products of the pulses that rule t' and t''.

Denote by N_L and N_R the numbers of pulses associated with points in $\mathbb{R}\times(\varepsilon, 1]$ that only affect t' and t''. The pulses that affect only t' or t'', but not both, contribute $V^{N_L+N_R}$ in the product $\Pi'_\varepsilon \Pi''_\varepsilon$. The N_0 pulses associated with points in $\mathbb{R} \times (\varepsilon, 1]$ that affect both t' and t'' contribute $[\mathbb{E}(W^2)]^{N_0}$.

Since $\varepsilon < r/2$, the Λ_ρ-measure of the subset $S(t', t'')$ of $\mathbb{R} \times (0, 1]$ whose pulses rule both t' and t'' does not depend on ε (see Figure 1). Moreover, elementary computations based on the construction (and helped by Figure 1) yield

$$\mathbb{E}(V^{N_L+N_R})\mathbb{E}(V^{N_0})^2 = \varepsilon^{-2\rho(V-1)},$$
$$\mathbb{E}(V^{N_0}) = e^{\Lambda(S(t',t''))(V-1)},$$
$$\mathbb{E}([\mathbb{E}(W^2)]^{N_0}) = e^{\Lambda(S(t',t''))(\mathbb{E}(W^2)-1)}.$$

Thus

$$\mathbb{E}[\mu'_\varepsilon(t')\mu'_\varepsilon(t'')] = e^{\Lambda(S(t',t''))\{[\mathbb{E}(W^2)-1]-2(V-1)\}},$$

which does not depend on ε. The correlation of μ at t' and t'' is the limit as $\varepsilon \to 0$ of

$$\frac{\mathbb{E}[\mu'_\varepsilon(t')\mu'_\varepsilon(t'')]}{[\mathbb{E}(\mu'_\varepsilon(t'))]^2} - 1.$$

Since $[\mathbb{E}(\mu'_\varepsilon(t'))]^2 = 1$ and $\Lambda(S(t', t''))$ behaves like $-\rho \log(r/2)$ as $r \ll 2$, this correlation behaves like $r^{\tau(2)-1}$ as $r \ll 2$ ($\tau(2) - 1 < 0$ if W is not the constant 1). For $r > 2$, the correlation vanishes.

The same holds for PCCM. For CCM, a formally identical expression of the correlation holds but with the physical Euclidean distance between t' and t'' replaced by the artificial ultrametric distance.

Upper critical power $q_{\text{crit.pos}}$ and conditions under which it is finite. For non-degenerate CCM, PCCM or MPCP, $\mathbb{E}[\mu([0, 1])] < \infty$ and if $q > 1$, the condition of finiteness of $\mathbb{E}[\mu([0, 1])^q]$ is $\tau(q) > 0$.

The critical power $q_{\text{crit.pos}}$ was introduced in [**M4, M5**] for a CCM as the supremum of $\{q \geq 0 : \mathbb{E}(\mu([0, 1])^q) < \infty\}$. It is also defined for MPCP, and when the equation $\tau(q) = 0$ has a solution > 1, that solution is $q_{\text{crit.pos}}$.

Conditions for finiteness of $q_{\text{crit.pos}}$. These conditions bring out a third difference between (PCCM,MPCP) and CCM, and a third source of novelty.

For a non-degenerate CCM, $q_{\text{crit.pos}} = \infty$ holds for the elementary examples (binomial and multinomial), and in all the cases when $W < b\mathbb{E}(W)$. The latter condition necessarily holds for the conservative – as opposed to canonical – cascades. In fact, one has $q_{\text{crit.pos}} < \infty$ if and only if $\mathbb{P}(\{W > b\mathbb{E}(W)\}) > 0$ or $\mathbb{P}(W = b) \geq 1/b$. That is, the finiteness of $q_{\text{crit.pos}}$ depends on b and the tail of W. A finite $q_{\text{crit.pos}}$ is widely perceived as an anomaly. In terms of $f(\alpha)$ it is associated with the complication of negative Hölder-like components and negative dimensions described in [**M9**]. Indeed, the condition $\tau(q) = 0$ expresses that the tangent of $\tau^*(\alpha)$ whose slope is q crosses the vertical axis of abscissa $\alpha = 0$ at the point of ordinate 0. This is well-known to be the case for $q = 1$. But for $q > 1$, this cannot be the case unless the graph of $\tau^*(\alpha)$ crosses into the lower left quadrant where $\alpha < 0$ and $\tau^* < 0$.

This behavior of $\tau^*(\alpha)$ and the fact that $q_{\text{crit.pos}} < \infty$ occur in the limit lognormal multifractals introduced in [**M2**]. But those multifractals are not as widely known as they deserve to be. In any event, the deep importance of the case $q_{\text{crit.pos}} < \infty$ is not sufficiently widely appreciated and its frequent occurrences in applications continue to be a source of surprise.

For PCCM and MPCP, to the contrary, a simple sufficient condition for $q_{\text{crit.pos}} < \infty$ is that max $W > 1$. If so, the term $\mathbb{E}(W^q)$ in $\tau_{\text{MPCP}}(q)$ does not vanish at ∞, implying $\lim_{q\to\infty} \tau_{\text{MPCP}}(q) = -\infty$.

A guess. Consider the following sequence of multifractal processes: non-random cascades, conservative cascades, "effectively conservative" cascades defined as having the same $\tau(q)$ as a non-random or conservative cascade, canonical cascades and (PCCM,MPCP). For the binomial and other "effectively conservative" cascades, those constraints were natural. But otherwise, each of the above steps eliminates some constraint on randomness that simplified the theory but was arbitrary.

As a result, the following tentative conclusion deserves careful attention. It may be that in further evolution of the models, the cases where $q_{\text{crit}} = \infty$ will increasingly become "anomalous" and the cases where $q_{\text{crit}} < \infty$ will increasingly become the norm.

The concrete importance of $q_{\text{crit.pos}} < \infty$ and more generally of $f(\alpha)$ that is negative for some α (see [**M7, M9**]*).* In that case, a single sample of the process can only yield $f(\alpha)$ where it is positive. The negative $f(\alpha)$ can only be obtained by "supersampling" and characterize the level of randomness of the process. Therefore, if the above guess proves correct, random multifractals will prove to be typically highly random.

Lower critical power. The exponent $q_{\text{crit.neg}}$ and conditions under which it is finite. All non-degenerate CCM, PCCM and MPCP also involve a second critical power $q_{\text{crit.neg}} = \inf\{q : \mathbb{E}[\mathbf{1}_{\{\mu([0,1])>0\}}\mu([0,1])^q] < \infty\}$.

For CCM, [**B1, B2**] obtained

$$q_{\text{crit.neg}} = b \inf\{q : \mathbb{E}(W^q) < \infty\}$$

when $W > 0$, and [**Li1, Li2**] obtained

$$q_{\text{crit.neg}} = \inf\{q : b^{1-q}V^{-q}\big(\mathbb{P}(\mu = 0)\big)^{\frac{b-1}{b}}\mathbb{E}(\mathbf{1}_{\{W>0\}}W^q) < 1\}$$

when $\mathbb{P}(W = 0) > 0$. In both cases $q_{\text{crit.neg}}$ depends on W and also the artificial base b for CCM. To the contrary, providing a fourth source of novelty,

$$q_{\text{crit.neg}} = \inf\{q : \tau_{\text{MPCP}}(q) > -\infty\} = \inf\{q : \mathbb{E}(W^q) < \infty\}$$

for PCCM and MPCP when $W > 0$. So $q_{\text{crit.neg}}$ only depends on W and not on the counterpart of b provided by ρ.

Comment. Despite the symmetry between the definitions, the two critical power are extremely different in nature.

The role of $\mathbb{E}(W)$; CCM only depend on $W/\mathbb{E}(W)$, while PCCM and MPCP also depend on $\mathbb{E}(W)$; this dependence is a major source of versatility. The PCCM and MPCP exhibit a major fifth source of novelty that is clarified by writing $W = W_1V$, where $W_1 = W/\mathbb{E}(W)$, therefore $\mathbb{E}(W_1) = 1$. For CCM, the normalization needed to define μ yields

$$\tau_{\text{CCM}}(q) = -1 + q - \rho \log \mathbb{E}(\mathbf{1}_{\{W_1>0\}}W_1^q).$$

That is, V drops out and τ is independent of V. To the contrary,

$$\tau_{\text{MPCP}}(q) = -1 + q[1 + \rho(V - 1)] - \rho[V^q\mathbb{E}(W_1^q) - 1]$$

involves both W_1 and V explicitly and inseparably. So does the dimension

$$\tau'_{\mathrm{MPCP}}(1) = 1 + \rho[(V-1) - V\log V - V\mathbb{E}(W_1 \log W_1)].$$

So do $\tau(2)$ and $q_{\mathrm{crit.pos}}$. To the contrary, $q_{\mathrm{crit.neg}}$ only involves W_1.

Special case 1: pulses of non random height V. They correspond to $W_1 \equiv 1$. For MPP, this case suffices to generate an interesting random multifractal measure with a single parameter V. This measure has no counterpart in CCM.

To pinpoint the origin of this novelty, recall the approximating measures μ_ε obtained by pulses of width $\geq \varepsilon$. For CCM, the number of pulses that affect μ_ε at a fixed t is non random and independent of t. Therefore, when W is non random, it degenerates to a constant that is eventually renormalized to 1. For PCCM and MPCP, this number is a Poisson random variable and its randomness suffices to create a non-degenerate process. It may, but need not, be useful in modeling.

Remark on a class of multidimensional Poisson random variables. Contrary to the Gaussian, the Poisson distribution has no intrinsic multivariable version. The logarithm of μ'_ε provides a "natural" candidate that is, insofar as we can tell, new: in the case of two instants t' and t'', the values of $\log \mu'_\varepsilon$ are of the form $\log \mu'_\varepsilon(t') = P_L + P_0$ and $\log \mu'_\varepsilon(t'') = P_0 + P_R$, where P_L , P_0, and P_R are independent Poisson variables that correspond to the three areas to the right of Figure 1. The same expressions (with Poisson replaced by Gaussian) hold for positively correlated Gaussian variables.

Special case 2: W uniformly distributed between 0 and $2V$. Fix $V > 0$ and assume that $W \in [0, 2V]$ and W is uniformly distributed. Since $V = \mathbb{E}(W)$

$$P_W(dx) = \mathbf{1}_{\{0 \leq W \leq 2V\}} \frac{dx}{2V}.$$

Then for every $q > -1$,

$$\mathbb{E}(W^q) = \frac{(2V)^q}{q+1}.$$

Both τ_{CCM} and $\tau_{\mathrm{PCCM}} = \tau_{\mathrm{MPCP}}$ are elementary functions and the degeneracy of μ and the finiteness of the critical values of q can be discussed explicitely.

• **The CCM case.** This case was studied in [**M7**] in the base $b = 2$. For every $b \geq 2$, we have $P(W < b\mathbb{E}(W)) = 1$ and, independently of V,

$$\tau_{\mathrm{CCM}}(q) = \begin{cases} -\infty & \text{if} \quad q \leq -1 \\ -1 + q(1 - \log_b(2)) + \log_b(q+1) & \text{if} \quad q > -1. \end{cases}$$

Thus $\lim_{q\to\infty} \tau_{\mathrm{CCM}}(q) > 0$; so $\tau'_{\mathrm{CCM}}(1) > 0$ hence μ is non-degenerate, and $q_{\mathrm{crit.pos}} = \infty$. Moreover, $q_{\mathrm{crit.neg}} = -b$.

• **The PCCM and MPCP cases.** For a general W, when $\max W \leq 1$, $q_{\mathrm{crit.pos}}$ is either $< \infty$ or $= \infty$ according to the value of ρ. $\mathbb{E}(W^q)$ vanishes at ∞ and

$$\tau'_{\mathrm{MPCP}}(1) = 1 + \rho(V-1) - \rho\mathbb{E}(W \log W)$$

with $\mathbb{E}(W \log W) \leq 0$. There are two subcases:

(1) $1/\rho \geq 1 - V$, that is $1 + \rho(V-1) \geq 0$, and $\lim_{q\to\infty} \tau_{\mathrm{MPCP}}(q) > 0$. Then $q_{\mathrm{crit.pos}} = \infty$. Moreover, such a ρ yields always $\tau'_{\mathrm{MPCP}}(1) > 0$ as can be seen on the expression of $\tau'_{\mathrm{MPCP}}(1)$.

(2) $1/\rho < 1-V$, that is $1+\rho(V-1) < 0$, and $\lim_{q\to\infty} \tau_{\mathrm{MPCP}}(q) = -\infty$. Hence $q_{\mathrm{crit.pos}} < \infty$. Non-degeneracy holds if and only if $1+\rho(V-1)-\rho\mathbb{E}(W \log W) > 0$. This yields the following condition to be satisfied by ρ:

$$1 - V + \mathbb{E}(W \log W) < 1/\rho < 1 - V.$$

Furthermore, in the special case 2

$$\tau_{\mathrm{MPCP}}(q) = \begin{cases} -\infty & \text{if} \quad q \leq -1 \\ -1 + q(1+\rho(V-1)) - \rho(\frac{(2V)^q}{q+1} - 1) & \text{if} \quad q > -1 \end{cases}$$

and

$$\tau'_{\mathrm{MPCP}}(1) = 1 + \rho(V-1) - \rho\theta(V)$$

with

$$\theta(V) = V\,[\log(2V) - 1/2].$$

Then, one has to distinguish the following cases:

• $V > 1/2$: in this case $\max W > 1$ and $q_{\mathrm{crit.pos}} < \infty$ as long as μ is non-degenerate. Moreover, the condition $\tau'_{\mathrm{MPCP}}(1) > 0$ is equivalent to $1/\rho > 1 - V + \theta(V)$.

• $0 < V \leq 1/2$: in this case $\max W \leq 1$, $\theta(V) < 0$ and the following three situations arise:

(1) If $1/\rho \geq 1 - V$ then $\tau'_{\mathrm{MPCP}}(1) > 0$ and $q_{\mathrm{crit.pos}} = \infty$.

(2) If $1 - V + \theta(V) < 1/\rho < 1 - V$ then $\tau'_{\mathrm{MPCP}}(1) > 0$ and $q_{\mathrm{crit.pos}} < \infty$. One can check that $1 - V + \theta(V)$ describes $[1/4, 1)$, so in this case ρ must be in $(1, 4)$.

(3) In all other cases, μ is degenerate.

Moreover, in all cases of non-degeneracy, $q_{\mathrm{crit.neg}} = -1$, a special case of the general rule.

Special case 3: pulses with $V = 1$. For them, $\tau'_{\mathrm{MPCP}}(1)$ takes a form familiar from the CCM case.

The general case $W = W_1 V$, with $P(\{W_1 = 1\}) < 1$ and $V > 0$. Observe that in the formula for the codimension $1 - \tau'_{\mathrm{MPCP}}(1)$, every term contains V. Therefore the codimension corresponding to $W_1 V$ is not the sum of the codimensions corresponding to W_1 and V taken separately. That is, the "typical behavior" of the intersection of "independent" sets is not applicable.

Marginal distribution of density for the approximating measures. Assume $W > 0$ almost surely. Then, up to the constant $\rho(V-1)\log\varepsilon$, the quantity $\log(d\mu_\varepsilon/d\ell)$ is the sum of N independent random variables of the form $\log W$, where N is a Poisson random variable of expectation $-\rho\log\varepsilon$, independent of the Ws. When $W \equiv V$, $\log(d\mu_\varepsilon/d\ell)$ is a Poisson random variable. In all other cases, $\log(d\mu_\varepsilon/d\ell)$ is a very special infinitely divisible random variable. The early de Finetti theory, later generalized by Lévy and Khinchine (see [**GKo**] p 68) involved the sum of this very special variable and of a Gaussian.

The Gaussian term alone is the foundation of the "limit lognormal" multifractals (LLNM) introduced in [**M2**]. In the Gaussian context, the whole process is determined by its covariance. In the context of of MPCP and more generally MPP (see Part II), it is not the case.

Scientific models often compromise between the numbers of parameters, the ease of calculation and the quality of fit. LLNM, PCCM and MPCP with W equal to a constant involve a single parameter. PCCM and MPCP are far easier to calculate.

Critical density. When W is fixed, the condition $\tau'(1) = 0$ defines a critical density $\rho_{\text{crit}}(1)$ beyond which $\mu = 0$. For CCM, there is also a critical ρ, but a critical base is only defined when $\exp(1/\rho_{\text{crit}}(1))$ is an integer. There is also for each q a critical density $\rho_{\text{crit}}(q)$ beyond which $\mathbb{E}[\mu([0,1])^q] = \infty$. For $W_1 \equiv 1$ and $V < e$, the function $\rho_{\text{crit}}(1)$ is two-to-one, that is, the same criticality $\mu \equiv 0$ can be achieved by a small V and a V close to e.

Final remarks. (1) The criteria of non-degeneracy and finiteness of moments obtained for statistically self-similar examples are not exhaustive. Generalized Riesz products with i.i.d. uniform random phases provide additional special examples of statistically self-similar measures obtained as limit of multiplicative martingales. For them [**BCM**] shows that the CNS for non-degeneracy differs completely from $\tau'(1) > 0$. Moreover, the non-degeneracy of the limit measure μ implies that $||\mu|| = 1$ almost surely. Hence the problem of the finiteness of moments is empty.

(2) [**BaMu**] generalizes the MPCP construction: a larger class of infinitely divisible laws is allowed for the logarithm of $\mu'_\varepsilon(t)$, including the Gaussian case. Moreover, the limit measures in [**BaMu**] possess an exact scaling property.

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