Techniques for the Study of Infinite Products of Independent Random Functions (Random Multiplicative Multifractal Measures, Part III)

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ABSTRACT. This is the third of three papers devoted to a class of random measures generated by multiplicative processes. Part I surveys the main motivations which led B. Mandelbrot to introduce such statistically self-affine multifractal measures. These measures inspired Kahane's general theory of T-martingales. Part II completes this theory by exhibiting a class of T-martingales for which several fundamental problems, namely non-degeneracy, finiteness of moments, dimension of the carrier and multifractal analysis can be studied and solved. This class contains the already known examples of statistically self-similar T-martingales, and is also illustrated by new constructions. This Part III provides the proofs of the main results obtained in Part II.

1. Introduction

This paper is devoted to the proofs of the main results of Part II [**BM3**]. Techniques developed to study the "Canonical cascade measures" (CCM) [**M1, M2, KP, Bi, WaWi, Mol, B2, B3, Li1, Li2**] and their refinements for the study of "Multifractal products of cylindrical pulses" (MPCP) [**BM1**] are shown to also work in a larger class of measures, which is a subclass of T-martingales ([**K1**]). Apart from CCM and MPCP, this class includes in particular the "Multifractal products of pulses" introduced with MPCP in [**M3**], as well as the extension of MPCP performed in [**BaMu**], namely the "Log-infinitely divisible cascades".

2. Proof of Theorem 4.1

Theorem 2.2 follows from the computations performed in proving Theorem 4.1, and in particular (2.1) below. Other results of Section 2 and 3 are corollaries of Theorem 2.2 and we leave the verifications to the reader.

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Theorem 4.1 is a consequence of the following proposition and its corollary.

For $w \in A^*$, $\varepsilon \in (0,1]$ and $\gamma \in \Gamma$ define

$$Y_\varepsilon(w,\gamma) = \widetilde{\mu}^\gamma_{b^{-|w|}\varepsilon}(\mathcal{A}_w) = \int_{I_w} Q_{b^{-|w|}\varepsilon}(t,\gamma)\, d\sigma(t).$$

PROPOSITION 2.1. *Assume* **(A1)** *holds and* Γ *is an open set. For every* $w \in A^*$*, with probability one, the function* $Y_\varepsilon(w,\cdot)$ *converges uniformly on the compact subsets of* Γ*, as* $\varepsilon \to 0$*, to a nonnegative analytic function* $Y(w,\cdot)$*. Moreover, if* $\sigma(I_w) > 0$ *and* $(t,\gamma) \in I_w \times \Gamma \mapsto Q_\varepsilon(t,\gamma)$ *is positive almost surely for all* $\varepsilon \in (0,1]$*, then* $Y(w,\cdot)$ *is almost surely positive.*

COROLLARY 2.2. *Assume* **(A1)** *holds. With probability one, for all* $\gamma \in \Gamma$*, the measure* $\widetilde{\mu}^\gamma_\varepsilon$ *converges weakly, as* $\varepsilon \to 0$*, to a measure* $\widetilde{\mu}^\gamma$ *such that* $\widetilde{\mu}^\gamma(\mathcal{A}_w) = Y(w,\gamma)$ *for every* $w \in A^*$*. Consequently, the measure* μ^γ_ε *converges weakly, as* $\varepsilon \to 0$*, to* $\mu^\gamma = \widetilde{\mu}^\gamma \circ \pi^{-1}$*.*

Proof of Proposition 2.1. Fix K a compact subset of Γ. Let U_K and b be as in **(A1)**. For any $w \in A^*$ and any $m \geq 0$ consider the function $\widehat{Y}_m(w,z)$ of $z \in U_K$ defined by

$$\widehat{Y}_m(w,z) = \int_{I_w} \widehat{Q}_{b^{-|w|-m}}(t,z)\, d\sigma(t).$$

Now fix K' a compact subset of U_K (in $\mathbb{C}^d$), $w \in A^*$, and $p \in (1,2]$ as in **(A1)**(iii). Also fix β as in **(P'4)** and define $\varepsilon_m = b^{-|w|-m-1}$.

First step. We prove that there exists a constant $C = C(\beta,p,b)$ such that

(2.1)
$$\sup_{z\in K'} \mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p) \leq C \sup_{z\in K'} \sum_{v\in A^n} \sigma(I_{wv})^{p-1} \int_{I_{wv}} |\widehat{Q}_{\varepsilon_m}(t,z)|^p\, d\sigma(t).$$

In order to prove (2.1), we use **(P'2)** to write

$$\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z) = \int_{I_w} U(t)V(t)\, d\sigma(t)$$

with $U(t) = \widehat{Q}_{b\varepsilon_m}(t,z)$ and $V(t) = \widehat{Q}_{b\varepsilon_m,\varepsilon_m}(t,z) - 1$.

We divide I_w into b^m equal subintervals denoted J_k, $0 \leq k \leq b^m - 1$. Now let $N = N_\beta$ be the smallest integer larger than or equal to β and write

$$\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z) = \sum_{i=0}^{N-1} \sum_{0\leq Nk+i\leq b^m-1} \int_{J_{Nk+i}} U(t)V(t)\, d\sigma(t).$$

It is immediate that

$$|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p \leq N^{p-1} \sum_{i=0}^{N-1} \left| \sum_{0\leq Nk+i\leq b^m-1} \int_{J_{Nk+i}} U(t)V(t)\, d\sigma(t) \right|^p.$$

By construction, the functions $U(t)$ and $V(t)$ are independent (due to **(P'3)**). Moreover, it follows from assumptions **(P'1)** and **(P'4)** that for each $0 \leq i \leq N-1$, the restrictions of the function V to the J_{Nk+i}'s, $0 \leq Nk+i \leq b^m - 1$, are centered and mutually independent. So we are in a position to apply the following lemma.

LEMMA 2.3 ([**vBahrE**]). *Let $(V_i)_{i\geq 0}$ be a sequence of mutually independent complex random variables. Assume that $\sum_{i\geq 0} V_i$ is almost surely defined and that V_i is integrable with mean 0 for all $i \geq 0$. Then, for every $p \in [1,2]$*

$$\mathbb{E}\left|\sum_{i\geq 0} V_i\right|^p \leq 2^p \sum_{i\geq 1} \mathbb{E}|V_i|^p.$$

For each $0 \leq i \leq N-1$, conditionally on the σ-algebra $\mathcal{U}$ generated by the function $U(t)$, the random variables $V_{Nk+i} = \int_{J_{Nk+i}} U(t)V(t)\,d\sigma(t)$, $0 \leq Nk+i \leq b^m - 1$, satisfy the assumptions of Lemma 2.3. So

$$\mathbb{E}\left(\left|\sum_{0\leq Nk+i\leq b^m-1} V_{Nk+i}\right|^p \Big| \mathcal{U}\right) \leq 2^p \sum_{0\leq Nk+i\leq b^m-1} \mathbb{E}\left(|V_{Nk+i}|^p | \mathcal{U}\right).$$

By taking the unconditional expectation and summing over i we get

$$\mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p) \leq 2^p N^{p-1} \sum_{v\in A^m} \mathbb{E}\left(\left|\int_{I_{wv}} U(t)V(t)\,d\sigma(t)\right|^p\right).$$

For each $v \in A^m$ such that $\sigma(I_{wv}) > 0$, the Jensen inequality yields

$$\left|\int_{I_{wv}} U(t)V(t)\,d\sigma(t)\right|^p \leq \sigma(I_{wv})^{p-1} \int_{I_{wv}} |U(t)|^p|V(t)|^p\,d\sigma(t).$$

Therefore,

$$\mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p) \leq 2^p N^{p-1} \sum_{v\in A^m} \sigma(I_{wv})^{p-1} \int_{I_{wv}} \mathbb{E}|U(t)|^p \mathbb{E}|V(t)|^p\,d\sigma(t).$$

The conclusion comes from the factorization property and the fact that

$$\mathbb{E}|V(t)|^p \leq 2^{p-1}(1 + \mathbb{E}|\widehat{Q}_{b\varepsilon_m,\varepsilon_m}(t,z)|^p) \leq 2^p \mathbb{E}|\widehat{Q}_{b\varepsilon_m,\varepsilon_m}(t,z)|^p$$

since $\mathbb{E}\big(\widehat{Q}_{b\varepsilon_m,\varepsilon_m}(t,z)\big) = 1$ and $p \geq 1$.

Second step. We follow an idea of Biggins [**Bi**]: apply the Cauchy formula to get the uniform convergence, as $m \to \infty$, of $\widehat{Y}_m(w,\cdot)$ on the compact subsets of U_K.

Fix an arbitrary non-empty compact polydisc $D(z_0, 2\rho) \subset U_K$. For $z \in D(z_0,\rho)$ and $m \geq 0$ the Cauchy formula yields

$$\begin{aligned}&|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)| \\ &\leq \int_{[0,2\pi]^d} |\widehat{Y}_{m+1}\big(w, z_0{+}2\rho(e^{it_1},\dots,e^{it_d})\big) - \widehat{Y}_m(w, z_0{+}2\rho(e^{it_1},\dots,e^{it_d}))|\,\frac{dt_1}{\pi}\dots\frac{dt_d}{\pi}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E} \sup_{z\in D(z_0,\rho)} &|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)| \\ &\leq 2^d \sup_{z\in D(z_0,2\rho)} \mathbb{E}(|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|) \\ &\leq 2^d \sup_{z\in D(z_0,2\rho)} (\mathbb{E}|\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)|^p)^{1/p}.\end{aligned}$$

By the estimate (2.1) obtained in the *first step* and assumption **(A1)**(iii) for the compact $D(z_0, 2\rho)$, we get

$$\mathbb{E}\sum_{m=0}^{\infty} \sup_{z\in D(z_0,\rho)} |\widehat{Y}_{m+1}(w,z) - \widehat{Y}_m(w,z)| < \infty. \tag{2.2}$$

It follows that almost surely $\widehat{Y}_m(w,\cdot)$ converges uniformly on $D(z_0,\rho)$, and more generally on any compact subset of U_K. Due to the analyticity of the $\widehat{Y}_m(w,\cdot)$, the limit function $\widehat{Y}_{U_K}(w,\cdot)$ is almost surely analytic on U_K.

Now, fix an increasing sequence $(K_n)_{n\geq 1}$ of compact subsets of Γ such that each K_n is the closure of its interior and $\bigcup_{n\geq 1} K_n = \Gamma$. For every $n \geq 1$ denote by $Y_{K_n}(w,\cdot)$ the restriction of $\widehat{Y}_{U_{K_n}}(w,\cdot)$ to K_n. Each $Y_{K_n}(w,\cdot)$ is analytic in the interior of K_n and is the uniform limit of $Y_{b^{-m}}(w,\cdot)$ on K_n as $m \to \infty$. Consequently, with probability one, the family of functions $Y_{K_n}(w,\cdot)$ possesses an unique analytic extension to Γ, namely $Y(w,\cdot)$.

Third step. Now we prove that almost surely the function $Y_\varepsilon(w,\cdot)$ converges uniformly on any compact subset K of Γ to $Y(w,\cdot)$, as $\varepsilon \to 0$. Indeed what we proved in the *second step* is the convergence as $\varepsilon \to 0$ along the discrete sequence $(b^{-m})_{m\geq 0}$.

From (2.2) we learn that

$$\mathbb{E}\big(\sup_{\gamma\in K} |Y(w,\gamma) - Y_1(w,\gamma)|\big) < \infty. \tag{2.3}$$

For $t \geq 1$, denote by $\mathbb{F}_t$ the sub-σ-field of the Borel σ-field of $\big(C(K,\mathbb{R}), \|\ \|_\infty\big)$ generated by the random continuous functions

$$\gamma \in K \mapsto Y_{1/t'}(w,\gamma), \qquad 1 \leq t' \leq t.$$

Also denote respectively by M_t and M the random functions $Y_{1/t}(w,\cdot) - Y_1(w,\cdot)$ and $Y(w,\cdot) - Y_1(w,\cdot)$. Then the martingale $\big\{\mathbb{E}(M|\mathbb{F}_t), \mathbb{F}_t\big\}_{t\geq 1}$ is well defined due to (2.3). It follows from Proposition V-2-6 of [**N**] that any right-continuous modification of that martingale converges almost surely, as $t \to \infty$, uniformly to M. Consequently, the conclusion will follow from **(A1)**(i) if we show that for every $t \geq 1$, with probability one, $\mathbb{E}(M|\mathbb{F}_t) = M_t$. This indeed holds: By construction of the conditional expectation, for every $t \geq 1$ and $\gamma \in K$ one has, with probability one, $\mathbb{E}(M|\mathbb{F}_t)(\gamma) = \mathbb{E}(M(\gamma)|\mathbb{F}_t)$. Moreover, it follows from the second step that for every $m \geq 0$, with probability one, $\mathbb{E}(M(\gamma)|\mathbb{F}_{b^m}) = M_{b^m}(\gamma)$. By using the density of the countable set $K \cap \mathbb{Q}^d$ in K and the continuity in γ of the functions we deal with (we use **(A1)**(i)) we deduce that, with probability one, $\mathbb{E}(M|\mathbb{F}_{b^m}) = M_{b^m}$. Then the martingale properties of $\big\{\mathbb{E}(M|\mathbb{F}_t), \mathbb{F}_t\big\}_{t\geq 1}$ and $\big\{M_t, \mathbb{F}_t\big\}_{t\geq 1}$ yields the conclusion.

Fourth step. Assume that $\sigma(I_w) > 0$. We prove that if $t \in I_w \mapsto Q_\varepsilon(t,\gamma)$ is positive almost surely for all $\gamma \in \Gamma$ and $\varepsilon \in (0,1]$ then, with probability one, $Y(w,\gamma) > 0$ for every $\gamma \in \Gamma$.

It suffices to prove this property for any compact subset K of Γ instead of Γ. Fix such a set. We assume without loss of generality that K is the hypercube $[0,1]^d$. For any sub-hypercube C of K let

$$S_C^w = \{\omega \in \Omega : \ \exists \gamma \in C, \ Y(w,\gamma) = 0\}.$$

It is an event since $\gamma \mapsto Y(w,\gamma)$ is continuous. Moreover, due to the factorization property **(P'2)**, it is straightforward to verify that for every $m \geq 0$, S_C^w belongs to the σ-algebra generated by the random functions $(t,\gamma) \mapsto Q_{b^{-m},b^{-k}}(t,\gamma)$, $k > m$. So S_C^w is a tail event with respect to $\big(\sigma((t,\gamma) \in D \times K \mapsto Q_{b^{-k},b^{-k-1}}(t,\gamma) : \ k \geq n)\big)_{n\geq 0}$. Due to the property of independence **(P'3)** and the Kolmogorov zero-one law, its probability is 0 or 1. We claim that $\mathbb{P}(S_K^w) = 0$.

Otherwise, S_K^w has probability one. Then, there necessarily exists a closed dyadic sub-hypercube of K of the first generation, namely C_1, such that $\mathbb{P}(S_{C_1}^w) > 0$. By the above remark, this probability must be 1. This implies the existence of a closed sub-hypercube $C_2 \subset C_1$ of the second generation such that $\mathbb{P}(S_{C_2}^w) = 1$, and so on. Hence, there exists a decreasing sequence $(C_n)_{n\geq 1}$ of closed sub-hypercubes of $[0,1]^d$ such that $\mathbb{P}(C_n) = 1$ for all $n \geq 1$. Let γ_0 be the unique point in $\bigcap_{n\geq 1} C_n$. By the continuity of $Y(w,\cdot)$, we have $\mathbb{P}\big(Y(w,\gamma_0) = 0\big) = 1$. However, $Y(w,\gamma_0)$ is the limit in L^p norm of an L^p-bounded martingale with mean $\sigma(I_w) > 0$ by the second step. So $Y(w,\gamma_0)$ cannot be zero almost surely. This proves that $\mathbb{P}(S_K^w) = 0$.

Proof of Corollary 2.2. Since A^* is countable, the conclusions of Proposition 2.1 hold almost surely for all $w \in A^*$. It follows that, with probability one, for all $\gamma \in \Gamma$, the family of additive functions on cylinders $\big(\mathcal{A}_w \mapsto \widetilde{\mu}_\varepsilon^\gamma(\mathcal{A}_w)\big)_{\varepsilon\in(0,1]}$ converges, as $\varepsilon \to 0$, to the additive function $\mathcal{A}_w \mapsto Y(w,\gamma)$. Since ∂A^* is totally disconnected, each of these additive functions extends uniquely in a measure $\widetilde{\mu}^\gamma$ on $(\partial A^*, \mathcal{A}^*)$. Moreover, by construction, with probability one, for all $\gamma \in \Gamma$, $\widetilde{\mu}^\gamma$ is the weak limit of $\widetilde{\mu}_\varepsilon^\gamma$ as $\varepsilon \to 0$. This yields the almost sure weak convergence, for all $\gamma \in \Gamma$, of μ_ε^γ to $\mu^\gamma = \widetilde{\mu}^\gamma \circ \pi^{-1}$, since $\mu_\varepsilon^\gamma = \widetilde{\mu}_\varepsilon^\gamma \circ \pi^{-1}$.

3. Proof of Theorem 4.2

For $w \in A^*$, $\varepsilon \in (0,1]$ and $\gamma \in \Gamma$ define

$$Z_\varepsilon(w,\gamma) = \sigma(I_w)^{-1}\widetilde{\mu}_\varepsilon^{\gamma,\mathcal{A}_w}(\mathcal{A}_w) = \int_{I_w} Q_{b^{-|w|},b^{-|w|}\varepsilon}(t,\gamma)\,\frac{d\sigma(t)}{\sigma(I_w)}$$

if $\sigma(I_w) > 0$, and $Z_\varepsilon(w,\gamma) = 0$ otherwise.

Proposition 3.1. *Suppose Γ is an open set and assumptions* **(A2)**$(i)(ii)$ *hold.*

1. *With probability one, for every $w \in A^*$, the function $Z_\varepsilon(w,\cdot)$ converges uniformly on the compact subsets of Γ, as $\varepsilon \to 0$, to a nonnegative analytic function $Z(w,\cdot)$; moreover, if $\sigma(I_w) > 0$ and $(t,\gamma) \in I_w \times \Gamma \mapsto Q_\varepsilon(t,\gamma)$ is positive almost surely for all $\varepsilon \in (0,1]$, then $Z(w,\cdot)$ is positive.*
2. *Let K be a compact subset of Γ, and fix the associated $p \in (1,2]$. One has*

$$\sup_{w\in A^*,\gamma\in K} \mathbb{E}\big(Z(w,\gamma)^p\big) < \infty, \qquad \sup_{1\leq i\leq d}\ \sup_{w\in A^*,\gamma\in K} \mathbb{E}\left(\left|\frac{\partial Z}{\partial \gamma_i}(w,\gamma)\right|^p\right) < \infty.$$

Proof. We proceed as in the proof of Proposition 2.1. For $w \in A^*$ such that $\sigma(I_w) > 0$ and $m \geq 0$, consider the function $\widehat{Z}_m(w,\gamma) := Z_{b^{-m}}(w,\gamma)$. It possesses the following analytic extension

$$\widehat{Z}_m(w,z) = \int_{I_w} \widehat{Q}_{b^{-|w|},b^{-|w|-m}}(t,z)\,\frac{d\sigma(t)}{\sigma(I_w)}.$$

It follows from computations similar to those performed in the first step of the proof of Proposition 2.1 that for some constant $C = C(\beta,p,b)$

$$\begin{aligned} (3.1)\qquad & \mathbb{E}(|\widehat{Z}_{m+1}(w,z) - \widehat{Z}_m(w,z)|^p) \\ \leq\ & C\mathbb{E}\left(|\widehat{Q}_{b^{-|w|},b^{-|w|-m-1}}(t,z)|^p\right) \sum_{v\in A^m} (\sigma(I_{wv})/\sigma(I_w))^p. \end{aligned}$$

Define $\varepsilon_{K'} = -\widehat{\varphi}^{(b)}(p) - \sup_{z\in K'} \widehat{\theta}^{(b)}(z,p)$. Our assumption $\mathbf{(A2)}(ii)(\alpha)$ is $\varepsilon_{K'} > 0$. Moreover, by our assumption $\mathbf{(A2)}(ii)(\beta)$, there exist $C' > 0$ and $n_0 \geq 0$ such that for all $w \in A^*$ with $|w| \geq n_0$ and $z \in K'$, we have

$$(3.2)\qquad \mathbb{E}\left(|\widehat{Q}_{b^{-|w|},b^{-|w|-m-1}}(t,z)|^p\right) \sum_{v\in A^m} (\sigma(I_{wv})/\sigma(I_w))^p \leq C' b^{-(m+1)\varepsilon_{K'}/2}.$$

Then (1) follows from the same arguments as in the proof of Proposition 2.1.

To get (2), let $\widehat{Z}(w,\cdot)$ be the limit of $\widehat{Z}_m(w,\cdot)$ on K', which is chosen to be a closed polydisc $D(z_0, 2\rho)$ as in the proof of Proposition 2.1.

It follows from (3.1), (3.2), the triangle inequality for the L^p norm, and the fact that $\widehat{Z}_0(w,\cdot) \equiv 1$ together, that

$$(3.3)\qquad \sup_{w\in A^*, z\in K'} \mathbb{E}\big(\widehat{Z}(w,z)^p\big) < \infty.$$

Moreover, applying the Cauchy formula to the partial derivatives $\frac{\partial \widehat{Z}_m}{\partial z_i}(w,z)$, $1 \leq i \leq d$, we get

$$\begin{aligned} \mathbb{E}\left(\sup_{z\in D(z_0,\rho)} \left|\frac{\partial \widehat{Z}_m}{\partial z_i}(w,z)\right|^p\right)^{1/p} &\leq \frac{2^d}{\rho^d} \sup_{z\in D(z_0,\rho)} \Big(\mathbb{E}(|\widehat{Z}_m(w,z)|^p)\Big)^{1/p} \\ \leq\ \frac{2^d}{\rho^d} \sup_{z\in D(z_0,\rho)} \Big(\mathbb{E}(|\widehat{Z}_0(w,z)|^p)\Big)^{1/p} & \\ +\frac{2^d}{\rho^d}\sum_{m=0}^{\infty} \sup_{z\in D(z_0,\rho)} \Big(\mathbb{E}(|\widehat{Z}_{m+1}(w,z) - \widehat{Z}_m(w,z)|^p)\Big)^{1/p}. & \end{aligned}$$

So we deduce from (3.1) and (3.2) that

$$(3.4)\qquad \sup_{1\leq i\leq d}\ \sup_{w\in A^*, z\in D(z_0,\rho)} \mathbb{E}\left(\left|\frac{\partial \widehat{Z}}{\partial z_i}(w,z)\right|^p\right) < \infty.$$

(2) is a consequence of (3.3) and (3.4).

Proof of Theorem 4.2. The fact that **(A1)** holds is a consequence of the estimate obtained in the proof of Proposition 3.1(1). The lower bound for the lower Hausdorff dimensions are consequences of $\mathcal{P}(\gamma)$, $\mathcal{P}'(\gamma)$ and a Billingsley Lemma ([**Bil**] pp 136–145).

Proof of (1). We treat the case where Γ is not a singleton. We can assume that $\mathcal{C}$ is a compact subset K of Γ, and that there exists a C^1 function $g : [0,1] \to K$ such that $K = g([0,1])$.

To prove the result on $\mathcal{P}(\gamma)$, it is enough to show that

$$(3.5)\qquad \mathbb{P}-a.s.\quad \forall\, \gamma\in K,\quad \widetilde{\mu}^\gamma\neq 0\quad \text{implies}\quad \widetilde{\mu}^\gamma\left(\limsup_{n\to\infty} E^c_{n,\varepsilon}(\gamma)\right)=0,$$

where

$$E_{n,\varepsilon}(\gamma)=\left\{\tilde{t}\in\partial A^*:\ \frac{\log\widetilde{\mu}^\gamma\big(\mathcal{A}_n(\tilde{t})\big)}{\log\widetilde{\sigma}\big(\mathcal{A}_n(\tilde{t})\big)}\geq \underline{D}(\gamma,\sigma)-\varepsilon\right\}.$$

In order to prove (3.5), by the Borel-Cantelli lemma, it suffices to show that for every $\varepsilon>0$

$$(3.6)\qquad \mathbb{P}-a.s.\quad \forall\, \gamma\in K,\quad \widetilde{\mu}^\gamma\neq 0\quad \text{implies}\quad \sum_{n\geq 1}\widetilde{\mu}^\gamma\left(\limsup_{n\to\infty} E^c_{n,\varepsilon}(\gamma)\right)<\infty.$$

Consider $X:\tilde{t}\mapsto\sigma\big(\mathcal{A}_n(\tilde{t})\big)^{-\underline{D}(\gamma,\sigma)+\varepsilon}\widetilde{\mu}^\gamma\big(\mathcal{A}_n(\tilde{t})\big)$ as a random variable with respect to the probability measure $\widetilde{\mu}^\gamma/\|\widetilde{\mu}^\gamma\|$ whenever $\|\widetilde{\mu}^\gamma\|\neq 0$. The definition of $E_{n,\varepsilon}(\gamma)^c$ means that $X(\tilde{t})>1$. For any positive number $\eta>0$, the Tchebitchev inequality leads to

$$\begin{aligned}\widetilde{\mu}^\gamma(E_{n,\varepsilon}(\gamma)^c) &\leq \int_{\partial A^*}\sigma\big(\mathcal{A}_n(\tilde{t})\big)^{\eta\left(-\underline{D}(\gamma,\sigma)+\varepsilon\right)}\big(\widetilde{\mu}^\gamma(\mathcal{A}_n(\tilde{t})\big)^\eta\widetilde{\mu}^\gamma(d\tilde{t})\\ (3.7)\qquad &= \sum_{w\in A^n}\sigma(\mathcal{A}_w)^{\eta\left(-\underline{D}(\gamma,\sigma)+\varepsilon\right)}(\widetilde{\mu}^\gamma(\mathcal{A}_w))^{1+\eta},\end{aligned}$$

where the last inequality is due to the fact that the random variable X is constant on each n-cylinder.

Now we use the construction of $\widetilde{\mu}^\gamma(\mathcal{A}_w)$ and **(A2)**(v) to get, for $w\in A^*$ such that $\widetilde{\sigma}(\mathcal{A}_w)>0$,

$$(3.8)\qquad \widetilde{\mu}^\gamma(\mathcal{A}_w)^{1+\eta}\leq M_w(\eta)\,(Q_{b^{-|w|}}(t_w,\gamma))^{1+\eta}\,\widetilde{\sigma}(\mathcal{A}_w)^{1+\eta}Z(w,\gamma)^{1+\eta}$$

where $Z(w,\gamma)$ was defined in Proposition 3.1. This, together with (3.7) yields

$$\widetilde{\mu}^\gamma(E_{n,\varepsilon}(\gamma)^c)\leq f_{n,\eta}(\gamma)$$

with

$$f_{n,\eta}(\gamma)=\sum_{w\in A^n}\widetilde{\sigma}(\mathcal{A}_w)^{1+\eta\left(-\underline{D}(\gamma,\sigma)+\varepsilon+1\right)}M_w(\eta)\,(Q_{b^{-|w|}}(t_w,\gamma))^{1+\eta}\,Z(w,\gamma)^{1+\eta}.$$

The positive number ε being fixed, the problem is reduced to find a positive number η such that

$$\mathbb{P}-a.s.\quad \forall\, x\in[0,1],\quad \sum_{n\geq 1} f_{n,\eta}\circ g(x)<\infty.$$

This will be done if we find $\eta>0$ such that

(1) There exists a constant $C=C(K,\eta)>0$ such that for all $n\geq 1$

$$(3.9)\qquad \sup_{1\leq i\leq d}\,\sup_{\gamma\in K}\mathbb{E}\left(\left|\frac{\partial f_{n,\eta}}{\partial\gamma_i}(\gamma)\right|\right)\leq Cb^{n\frac{\varphi_\sigma(1^+)\eta\varepsilon}{2}}.$$

(2) Let $\gamma_0=g(0)$. We have

$$(3.10)\qquad \mathbb{P}\text{-}a.s.\quad \sum_{n=1}^\infty f_{n,\eta}(\gamma_0)<\infty.$$

Indeed, if (1) holds, by using the Fubini Theorem we get

$$\mathbb{E}\int_0^1 \sum_{n=1}^{\infty} \left|\frac{d\, f_{n,\eta}\circ g}{dx}(x)\right|\, dx \le \int_0^1 \sum_{n=1}^{\infty}\sum_{i=1}^{d} \mathbb{E}\left|\frac{\partial f_{n,\eta}}{\partial \gamma_i}\big(g(x)\big)\right| |g_i'(x)|\, dx < \infty.$$

Therefore $\mathbb{P}$-almost surely $\displaystyle\int_0^1 \sum_{n=1}^{\infty} \left|\frac{d\, f_{n,\eta}\circ g}{dx}(x)\right|\, dx < \infty$. This yields $\mathbb{P}$-almost surely for all $\gamma \in K$

$$\sum_{n=1}^{\infty} |f_{n,\eta}(\gamma) - f_{n,\eta}(\gamma_0)| \le \int_0^1 \sum_{n=1}^{\infty} \left|\frac{d\, f_{n,\eta}\circ g}{dx}(x)\right|\, dx < \infty.$$

This, together with (2), allows us to conclude:

$$\mathbb{P}\text{-}a.s.\quad \sup_{\gamma\in K} \sum_{n\ge 1} f_{n,\eta}(\gamma) < \infty.$$

Proof of (3.9): for $w \in A^*$, $\eta > 0$ and $\gamma \in \Gamma$ define

$$\widehat{\sigma}_{w,\eta}(\gamma) = \widetilde{\sigma}(\mathcal{A}_w)^{1+\eta\left(-\underline{D}(\gamma,\sigma)+\varepsilon+1\right)}.$$

We have

$$\mathbb{E}\left(\left|\frac{\partial f_{n,\eta}}{\partial \gamma_i}(\gamma)\right|\right) \le F_{n,\eta}(\gamma) + (1+\eta)\big(G_{n,\eta}(\gamma) + H_{n,\eta}(\gamma)\big)$$

with

$$\begin{aligned}
F_{n,\eta}(\gamma) &= \sum_{w\in A^n} \left|\frac{\partial \widehat{\sigma}_{w,\eta}}{\partial \gamma_i}(\gamma)\right| \mathbb{E}\left(M_w(\eta) Q_{b^{-n}}(t_w,\gamma)^{1+\eta}\right) \mathbb{E}\left(Z(w,\gamma)^{1+\eta}\right),\\
G_{n,\eta}(\gamma) &= \sum_{w\in A^n} \widehat{\sigma}_{w,\eta}(\gamma)\\
&\qquad \times \mathbb{E}\left(M_w(\eta) Q_{b^{-n}}(t_w,\gamma)^{\eta} \left|\frac{\partial Q_{b^{-n}}}{\partial \gamma_i}(t_w,\gamma)\right|\right) \mathbb{E}\big(Z(w,\gamma)^{1+\eta}\big),\\
H_{n,\eta}(\gamma) &= \sum_{w\in A^n} \widehat{\sigma}_{w,\eta}(\gamma) \mathbb{E}\left(M_w(\eta) Q_{b^{-n}}(t_w,\gamma)^{1+\eta}\right) \mathbb{E}\left(Z(w,\gamma)^{\eta} \left|\frac{\partial Z}{\partial \gamma_i}(w,\gamma)\right|\right).
\end{aligned}$$

We now give estimates for the above quantities in the case $\mathbf{(A2)}(v)(\beta)$ (the other case $\mathbf{(A2)}(v)(\alpha)$ is simpler and left to the reader).

Let us make the following remarks:

(1) It follows from Proposition 3.1 that if η is small enough, $\mathbb{E}(Z(w,\gamma)^{1+\eta})$ and $\mathbb{E}\Big(Z(w,\gamma)^{\eta}\left|\frac{\partial Z}{\partial \gamma_i}(w,\gamma)\right|\Big)$ are uniformly bounded over $\gamma \in K$ and $w \in A^*$.

(2) We have

$$\left|\frac{\partial \widehat{\sigma}_{w,\eta}}{\partial \gamma_i}(\gamma)\right| = \frac{-1}{\varphi_\sigma'(1^+)} \left|\frac{\partial^2 \theta}{\partial \gamma_i \partial p}(\gamma, 1^+)\right| \eta\, |\log(\widetilde{\sigma}(\mathcal{A}_w))|\, \widetilde{\sigma}(\mathcal{A}_w)^{1+\eta\left(-\underline{D}(\gamma,\sigma)+\varepsilon+1\right)}.$$

Consequently, due to the assumption $\mathbf{(A2)}(iii)$ and the atomless of $\widetilde{\sigma}$ $(-\varphi_\sigma(1^+) > 0)$, there exists a constant $C = C(\eta, K)$ such that

$$\left|\frac{\partial \widehat{\sigma}_{w,\eta}}{\partial \gamma_i}(\gamma)\right| \le C\widetilde{\sigma}(\mathcal{A}_w)^{1+\eta\left(-\underline{D}(\gamma,\sigma)+\varepsilon+1\right)-\eta^2} \quad (\gamma\in K,\ w\in A^*,\ \widetilde{\sigma}(\mathcal{A}_w)>0).$$

(3) In order to control

$$A(w,\gamma)=\mathbb{E}\left(M_w(\eta)Q_{b^{-n}}(t_w,\gamma)^{\eta}\left|\frac{\partial Q_{b^{-n}}}{\partial\gamma_i}(t_w,\gamma)\right|\right)$$

we apply the Hölder inequality with a pair (h,h') of positive numbers, to be specified later, and such that $\frac{1}{h}+\frac{1}{h'}=1$. We get

$$A(w,\gamma)\le\left(\mathbb{E}\left(M_w(\eta)^h\right)\right)^{1/h}\left(\mathbb{E}\left(Q_{b^{-n}}(t_w,\gamma)^{\eta h'}\left|\frac{\partial Q_{b^{-n}}}{\partial\gamma_i}(t_w,\gamma)\right|^{h'}\right)\right)^{1/h'}.$$

By using the factorization property **(P'2)** and the differentiability property involved in **(A2)**$(v)(\beta)(2)$, we get (since K is included in the interior of some compact subset of Γ)

$$\begin{aligned}\left|\frac{\partial Q_{b^{-n}}}{\partial\gamma_i}(t_w,\gamma)\right|^{h'} &= \left|\sum_{k=0}^{n-1}\frac{\partial Q_{b^{-k},b^{-k-1}}}{\partial\gamma_i}(t_w,\gamma)\prod_{\substack{k'=0\\k'\neq k}}^{n-1}Q_{b^{-k'},b^{-k'-1}}(t_w,\gamma)\right|^{h'}\\ &\le n^{h'-1}\sum_{k=0}^{n-1}\left|\frac{\partial Q_{b^{-k},b^{-k-1}}}{\partial\gamma_i}(t_w,\gamma)\right|^{h'}\prod_{\substack{k'=0\\k'\neq k}}^{n-1}Q_{b^{-k'},b^{-k'-1}}(t_w,\gamma)^{h'}.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbb{E}\left(Q_{b^{-n}}(t_w,\gamma)^{\eta h'}\left|\frac{\partial Q_{b^{-n}}}{\partial\gamma_i}(t_w,\gamma)\right|^{h'}\right)\\ \le\ & n^{h'-1}\sum_{k=0}^{n-1}\mathbb{E}\left(Q_{b^{-k},b^{-k-1}}(t_w,\gamma)^{\eta h'}\left|\frac{\partial Q_{b^{-k},b^{-k-1}}}{\partial\gamma_i}(t_w,\gamma)\right|^{h'}\right)\\ &\times\prod_{\substack{k'=0\\k'\neq k}}^{n-1}\mathbb{E}\left(Q_{b^{-k'},b^{-k'-1}}(t_w,\gamma)^{(1+\eta)h'}\right).\end{aligned}$$

Now, we use the fact that $1\le\mathbb{E}\left(Q_{b^{-k},b^{-k-1}}(t_w,\gamma)^{(1+\eta)h'}\right)$ and **(A2)**$(v)(\beta)(1)(2)$ together to conclude that η, h and h' being chosen

$$A(w,\gamma)\le\exp\left(o(n)\right)\prod_{k=0}^{n-1}\mathbb{E}\left(Q_{b^{-k},b^{-k-1}}(t_w,\gamma)^{(1+\eta)h'}\right)^{1/h'}.$$

(4) h and h' being choosen as in (3), we have

$$\mathbb{E}\left(M_w(\eta)Q_{b^{-|w|}}(t_w,\gamma)^{1+\eta}\right)\le\exp\left(o(n)\right)\prod_{k=0}^{n-1}\mathbb{E}\left(Q_{b^{-k},b^{-k-1}}(t_w,\gamma)^{(1+\eta)h'}\right)^{1/h'}.$$

If follows from the above remarks that $\eta>0$ and $h'>1$ being fixed, uniformly over K

$$\mathbb{E}\left(\left|\frac{\partial f_{n,\eta}}{\partial \gamma_i}(\gamma)\right|\right) \le \exp\big(o(n)\big)\mathbb{E}\left(Q_{b^{-n}}(t,\gamma)^{(1+\eta)h'}\right)^{1/h'} \times \sum_{w\in A^n} \widetilde{\sigma}(\mathcal{A}_w)^{1+\eta\left(-\underline{D}(\gamma,\sigma)+\varepsilon+1\right)-\eta^2}.$$

We fix $h' = 1+\eta^2$, and use $\mathbf{(A2)}(iv)(\alpha)$ to get $n_0(\eta) \ge 1$ such that for all $n \ge n_0(\eta)$ and $\gamma \in K$,

$$\frac{1}{n}\log_b \mathbb{E}\left(Q_{b^{-n}}(t,\gamma)^{(1+\eta)(1+\eta^2)}\right) \le \theta\big(\gamma,(1+\eta)(1+\eta^2)\big)+\eta^2$$

and

$$\frac{1}{n}\log_b \sum_{w\in A^n} \widetilde{\sigma}(\mathcal{A}_w)^{1+\eta\left(\varphi'_\sigma(1^+)+\theta'_\gamma(1^+)+\varepsilon+1\right)-\eta^2} \le \varphi_\sigma\left(1+\eta\big(-\underline{D}(\gamma,\sigma)+\varepsilon+1\big)-\eta^2\right)+\eta^2.$$

(for the second estimate we used the fact that φ_σ is by definition the uniform limit of convex functions on the compact subsets of $\mathbb{R}_+$). Thus, for $n \ge n_0(\eta)$

$$\mathbb{E}\left(\left|\frac{\partial f_{n,\eta}}{\partial \gamma_i}(\gamma)\right|\right) \le \exp\Big(o(n)+n\log(b)B(\gamma,\eta)\Big) \tag{3.11}$$

with

$$B(\gamma,\eta)=\frac{\theta\big(\gamma,(1+\eta)(1+\eta^2)\big)+\eta^2}{1+\eta^2}+\varphi_\sigma\left(1+\eta\big(-\underline{D}(\gamma,\sigma)+\varepsilon+1\big)-\eta^2\right)+\eta^2.$$

Due to $\mathbf{(A2)}(iv)(\beta)$ we have

$$\frac{\theta\big(\gamma,(1+\eta)(1+\eta^2)\big)+\eta^2}{1+\eta^2}=\eta\theta'_\gamma(1^+)+o(\eta) \tag{3.12}$$

where $o(\eta)$ does not depend on $\gamma\in K$. Moreover,

$$\varphi_\sigma\left(1+\eta\big(-\underline{D}(\gamma,\sigma)+\varepsilon+1\big)-\eta^2\right)+\eta^2=\eta\varphi_\sigma(1^+)\big(-\underline{D}(\gamma,\sigma)+\varepsilon+1\big)+o(\eta) \tag{3.13}$$

where $o(\eta)$ does not depend on $\gamma \in K$. Now choose initially η small enough so that $o(\eta) \le |\varphi_\sigma(1^+)|\varepsilon\eta/8$. Then choose $n'_0 \ge n_0(\eta)$ such that in (3.11) $o(n) \le n\log(b)|\varphi_\sigma(1^+)|\varepsilon\eta/4$ if $n \ge n'_0$. It follows from (3.12) and (3.13) and the definition of $\underline{D}(\gamma,\sigma)$ that for $n \ge n'_0$

$$\mathbb{E}\left(\left|\frac{\partial f_{n,\eta}}{\partial \gamma_i}(\gamma)\right|\right) \le b^{n\varphi_\sigma(1^+)\varepsilon\eta/2}.$$

Proof of (3.10): computations similar to the previous ones show that

$$\sup_{\gamma\in K}\mathbb{E}(f_{n,\eta}(\gamma)) \le Cb^{n\varphi_\sigma(1^+)\varepsilon\eta/2}. \tag{3.14}$$

The result concerning $\mathcal{P}'(\gamma)$ is obtained similarly; the proof is left to the reader. *Proof of (2).* We only establish the result concerning $\mathcal{P}(\gamma)$. The case of $\mathcal{P}'(\gamma)$ is left to the reader, as in the proof of (1).

It follows from the previous computations that under ($\widetilde{\mathbf{A2}}$), for every compact subset K of Γ, (3.14) holds if η is small enough. Consequently, for such a pair (K,η),

$$\mathbb{E}\left(\int_K \sum_{n=1}^{\infty}(f_{n,\eta}(\gamma)\, d\ell_d(\gamma)\right) < \infty,$$

where ℓ_d is the Lebesgue measure on $\mathbb{R}^d$. This shows that with probability one, there exists a subset $K(\omega)$ of K of full ℓ_d-measure such that $\sum_{n=1}^{\infty} f_{n,\eta}(\gamma) < \infty$; hence $\mathcal{P}(\gamma)$ holds for every γ such that $\widetilde{\mu}^\gamma \neq 0$. The conclusion follows by writing Γ as a countable union of compact subsets.

4. Proofs of Theorems 5.3, 5.4, 5.5, and 5.6

We mimick the proofs in [**BM1**] for the first three results. We say once again that the approach consists in reductions to the CCM case. Theorem 5.6 is established in [**KP**] for CCM, and it is implicit in the multifractal analysis of MPCP in [**BM1**]. We do not use the so-called Peyrière probability to show this result, preferring here a method like the one used in the proof of Theorem 4.2.

For $n \geq 0$ define $Y_n = \|\mu_{b^{-n}}\|$ and notice that by construction $\mathbb{E}(Y_n) = 1$. Remember that N_β denotes the smallest integer larger than or equal to the constant β in **(P4)**.

Proof of Theorem 5.3(1). It follows from equation (4.1) in Part II ([**BM3**]) and Lemma C of [**KP**] that if $h < 1$ is large enough and $n > m \geq 1$

$$Y_n^h \geq \sum_{w\in A^m} \mu_{b^{-n}}(I_w)^h - (1-h) \sum_{w\neq v\in A^m} \mu_{b^{-n}}(I_w)^{\frac{h}{2}} \mu_{b^{-n}}(I_v)^{\frac{h}{2}}. \tag{4.1}$$

Moreover, the Jensen inequality yields $\mu_{b^{-n}}(I_w)^h \geq f_{w,n,m}(h)$, with

$$\begin{aligned} f_{w,n,m}(h) &= \|\mu^{I_w}_{b^{m-n}}\|^h \int_{I_w} Q_{b^{-m}}(t)^h \frac{\mu^{I_w}_{b^{m-n}}(dt)}{\|\mu^{I_w}_{b^{m-n}}\|} \\ &= \|\mu^{I_w}_{b^{m-n}}\|^h \int_{I_w} \prod_{k=0}^{m-1} Q_{b^{-k},b^{-k-1}}(t)^h \frac{\mu^{I_w}_{b^{m-n}}(dt)}{\|\mu^{I_w}_{b^{m-n}}\|} \end{aligned}$$

if $\|\mu^{I_w}_{b^{m-n}}\| > 0$ and 0 otherwise. This yields almost surely if $\|\mu^{I_w}_{b^{m-n}}\| > 0$

$$\begin{aligned} &f'_{w,n,m}(1^-) \\ =\ &\sum_{k=0}^{m-1} \int_{I_w} Q_{b^{-k},b^{-k-1}}(t) \log\left(Q_{b^{-k},b^{-k-1}}(t)\right) \prod_{\substack{k'=0\\k'\neq k}}^{m-1} Q_{b^{-k'},b^{-k'-1}}(t)\, \mu^{I_w}_{b^{m-n}}(dt) \\ &\quad + \log\left(\|\mu^{I_w}_{b^{m-n}}\|\right) \int_{I_w} Q_{b^{-m}}(t)\, \mu^{I_w}_{b^{m-n}}(dt). \end{aligned}$$

Now taking the expectation by using properties **(P1)**, **(P3)**, **(P5)**, **(P6)** and Proposition 5.1(1) in Part II gives

$$\begin{aligned} & \mathbb{E}\left(f'_{w,n,m}(1^-)\right) \\ = \ & mb^{-m}\mathbb{E}\left(Q_{b^{-1}}(t)\log Q_{b^{-1}}(t)\right) + b^{-m}\mathbb{E}(Y_{n-m}\log Y_{n-m}) - mb^{-m}\log(b)\mathbb{E}(Y_{n-m}) \\ = \ & -m\log(b)b^{-m}\tau'(1^-) + b^{-m}\mathbb{E}(Y_{n-m}\log Y_{n-m}). \end{aligned}$$

Returning to (4.1) we get

$$\frac{\mathbb{E}(Y_n^h) - \sum_{w\in A^m}\mathbb{E}(f_{w,n,m}(h))}{h-1} \le \sum(h) := \sum_{w\neq v\in A^m}\mathbb{E}\left(\mu_{b^{-n}}(I_w)^{\frac{h}{2}}\mu_{b^{-n}}(I_v)^{\frac{h}{2}}\right)$$

and letting h tend to 1 and using the value of $\mathbb{E}\left(f'_{w,n,m}(1^-)\right)$ we get

$$m\,\log(b)\,\tau'(1^-) + \mathbb{E}(Y_n\,\log Y_n) - \mathbb{E}(Y_{n-m}\log Y_{n-m}) \le \sum(1).$$

By the martingale nature of $(Y_n)_{n\ge 1}$, $\mathbb{E}(Y_n\,\log Y_n) - \mathbb{E}(Y_{n-m}\log Y_{n-m}) \ge 0$. Hence $m\,\log(b)\,\tau'(1^-) \le \sum(1)$.

In order to evaluate $\sum(1)$, we invoke assumption $(\mathbf{C_1})$. Then, for every $(w,v)\in(A^m)^2$, by using the independence and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} & \mathbb{E}\left(\mu_{b^{-n}}(I_w)^{\frac{1}{2}}\mu_{b^{-n}}(I_v)^{\frac{1}{2}}\right) \\ \le \ & \mathbb{E}\left(\left(\sup_{s\in I_w} Q_{b^{-m}}(s)\right)^{\frac{1}{2}}\left(\sup_{s\in I_v} Q_{b^{-m}}(s)\right)^{\frac{1}{2}}\|\mu^{I_w}_{b^{m-n}}\|^{\frac{1}{2}}\|\mu^{I_v}_{b^{m-n}}\|^{\frac{1}{2}}\right) \\ \le \ & \left[\mathbb{E}\left(\sup_{s\in I_w} Q_{b^{-m}}(s)\right)\mathbb{E}\left(\sup_{s\in I_v} Q_{b^{-m}}(s)\right)\right]^{\frac{1}{2}}\mathbb{E}\left(\|\mu^{I_w}_{b^{m-n}}\|^{\frac{1}{2}}\|\mu^{I_v}_{b^{m-n}}\|^{\frac{1}{2}}\right) \\ = \ & \varphi(m)\mathbb{E}\left(\|\mu^{I_w}_{b^{m-n}}\|^{\frac{1}{2}}\|\mu^{I_v}_{b^{m-n}}\|^{\frac{1}{2}}\right). \end{aligned}$$

By assumption **(P4)**, for each $w\in A^m$, there are at most $2N_\beta$ elements $v\in A^m$ distinct of w such that $\|\mu^{I_w}_{b^{m-n}}\|$ and $\|\mu^{I_v}_{b^{m-n}}\|$ are not independent. In this case, by the Cauchy-Schwarz inequality and Proposition 5.1, $\mathbb{E}\left(\|\mu^{I_w}_{b^{m-n}}\|^{\frac{1}{2}}\|\mu^{I_v}_{b^{m-n}}\|^{\frac{1}{2}}\right) \le b^{-m}$. Otherwise, $\mathbb{E}\left(\|\mu^{I_w}_{b^{m-n}}\|^{\frac{1}{2}}\|\mu^{I_v}_{b^{m-n}}\|^{\frac{1}{2}}\right) = b^{-m}\left(\mathbb{E}(Y^{\frac{1}{2}}_{n-m})\right)^2$. This yields

$$\sum(1) \le \varphi(m)\left(2N_\beta b^m \times b^{-m} + b^{2m}\times b^{-m}\left(\mathbb{E}(Y^{\frac{1}{2}}_{n-m})\right)^2\right).$$

Finally,

$$m\left[\log(b)\tau'(1^-) - 2N_\beta\varphi(m)/m\right] \le b^m\left(\mathbb{E}(Y^{\frac{1}{2}}_{n-m})\right)^2$$

and since $\tau'(1^-)>0$ and $\varphi(m)=o(m)$, taking m large enough so that $\log(b)\tau'(1^-) - 2N_\beta\varphi(m)/m > 0$ yields $\inf_{n\ge 1}\mathbb{E}(Y_n^{\frac{1}{2}}) > 0$. Following **[KP]** we remark that since the supermartingale $(Y_n^{\frac{1}{2}})_{n\ge 1}$ is bounded in L^2 norm by $\mathbb{E}(Y_n)=1$, it is uniformly integrable; so $\mathbb{E}(\lim_{n\to\infty} Y_n^{\frac{1}{2}}) > 0$. But $\lim_{n\to\infty} Y_n = \|\mu\|$ almost surely, so μ is non-degenerate.

The fact $\mathbb{E}(\mu)=1$ follows from Section 5.1. of Part II. The fact $\mathbb{P}(\|\mu\|>0)=1$ if the martingale (Q_ε) is positive follows from the fact that in this case $\{\|\mu\|>0\}$ is a tail event with respect to $\left(\sigma(Q_{b^{-k},b^{-k-1}}(\cdot):\ k\ge n)\right)_{n\ge 1}$.

Proof of Theorem 5.3(2). Fix h as in the statement. By using (4.3) in [**BM3**], the sub-additivity of $x \mapsto x^h$, **(P6)**, Proposition 5.1(2) in [**BM3**] and $\mathbf{C_2(h)}$ together, we get

$$\mathbb{E}(\|\mu\|^h) \le b^m e^{\varphi_h(m)} \mathbb{E}\left(Q_{b^{-m}}(t)^h\right) b^{-mh} \mathbb{E}(\|\mu\|^h).$$

If μ is non degenerate this yields

$$1 \le b^{-m\left[\tau(h) + \frac{\varphi_h(m)}{m\log(b)}\right]}.$$

Since $\varphi_h(m) = o(m)$ this forces $\tau(h) \le 0$. Since τ is a concave function and $\tau(1) = 0$, we get $\tau'(1^-) \ge 0$.

Proof of Theorem 5.4(1). It suffices to show that $(Y_n)_{n\ge 1}$ is bounded in L^h norm. The case $h \in (1, 2]$ is a consequence of Corollary 2.3 in [**BM3**].

Fix $n > m > \log_b(N_\beta)$. Number the intervals I_w, $w \in A^m$, as they follow one another from 0 on the real line, and write $\{I_w : \ w \in A^m\} = \{J_i;\, 0 \le i < b^m\}$. Then, for $i \in \{0, \dots, N_\beta - 1\}$ define

$$Z_{i,n} = \sum_{k:\ 0\le N_\beta k+i<b^m} \mu_{b^{-n}}(J_{N_\beta k+i})$$

and

$$N_i = \#\left\{k :\ 0 \le N_\beta k + i < b^m\right\} - 1.$$

We have

$$\mathbb{E}(Y_n^h) \le N_\beta^{h-1} \sum_{i=0}^{N_\beta-1} \mathbb{E}(Z_{i,n}^h). \tag{4.2}$$

Now we adapt the approach of [**KP**]. Let $\tilde h$ be the integer such that $\tilde h < h \le \tilde h + 1$ and use the sub-additivity of $x \mapsto x^{h/(\tilde h+1)}$ on $\mathbb{R}_+$ to write

$$Z_{i,n}^h \le \left[\sum_{k=0}^{N_i} \mu_{b^{-n}}(J_{N_\beta k+i})^{h/(\tilde h+1)}\right]^{\tilde h+1}.$$

It follows that

$$\mathbb{E}(Z_{i,n}^h) \le \sum_{k=0}^{N_i} \mathbb{E}\left(\mu_{b^{-n}}(J_{N_\beta k+i})^h\right) + \sum \alpha_{j_0\dots j_{N_i}} \mathbb{E}\left(\prod_{k=0}^{N_i} \mu_{b^{-n}}(J_{N_\beta k+i})^{j_k\frac{h}{\tilde h+1}}\right),$$

where in the last sum the j_l's are $\le \tilde h$, $j_0 + \cdots + j_{N_i} = \tilde h + 1$, $j_l \ge 0$ and $\sum \alpha_{j_0\dots j_{N_i}} = (N_i+1)^{(\tilde h+1)} - (N_i + 1)$.

On the one hand, given such a $j_0, \dots, j_{N_i}$ we have

$$\prod_{k=0}^{N_i} \mu_{b^{-n}}(J_{N_\beta k+i})^{j_k\frac{h}{\tilde h+1}} \le \prod_{k=0}^{N_i}\left(\sup_{s\in J_{N_\beta k+i}} Q_{b^{-m}}(s)\right)^{j_k\frac{h}{\tilde h+1}} \prod_{k=0}^{N_i} \|\mu_{b^{m-n}}^{J_{N_\beta k+i}}\|^{j_k\frac{h}{\tilde h+1}},$$

where the $\|\mu_{b^{m-n}}^{J_{N_\beta k+i}}\|$s are i.i.d. by **(P4)** and Proposition 5.1 in [**BM3**], and are also independent of $\prod_{k=0}^{N_i}\left(\sup_{s\in J_{N_\beta k+i}} Q_{b^{-m}}(s)\right)^{j_k\frac{h}{\tilde h+1}}$ by **(P3)**. Moreover, the random variables $\sup_{s\in J_{N_\beta k+i}} Q_{b^{-m}}(s)$ have the same probability distribution. Applying the generalized Hölder inequality, $\mathbf{C_2(h)}$ and the definition of τ successively, we get

$$\begin{aligned}\mathbb{E}\left(\prod_{k=0}^{N_i}\left(\sup_{s\in J_{N_\beta k+i}} Q_{b^{-m}}(s)\right)^{j_k\frac{h}{\tilde h+1}}\right) &\le \mathbb{E}\left(\sup_{s\in I_m} Q_{b^{-m}}(s)^h\right)\\ &\le e^{\varphi_h(m)}\mathbb{E}\left(Q_{b^{-m}}(t)^h\right)\\ &= e^{\varphi_h(m)} b^{-m(1-h+\tau(h))},\end{aligned}$$

where I_m is one of the I_w, $w\in A^m$, and $t\in(0,1)$.

Moreover, by using the independence, the Jensen inequality and Proposition 5.1 successively, we have

$$\begin{aligned}\mathbb{E}\left(\prod_{k=0}^{N_i}\|\mu_{b^{m-n}}^{J_{N_\beta k+i}}\|^{j_k\frac{h}{\tilde h+1}}\right) &= \prod_{k=0}^{N_i}\mathbb{E}\left(\|\mu_{b^{m-n}}^{J_{N_\beta k+i}}\|^{j_k\frac{h}{\tilde h+1}}\right)\\ &\le \prod_{k=0}^{N_i}\mathbb{E}\left(\|\mu_{b^{m-n}}^{J_{N_\beta k+i}}\|^{\tilde h}\right)^{\frac{j_k}{\tilde h}\frac{h}{\tilde h+1}}\\ &= b^{-mh}\left(\mathbb{E}(Y_{n-m}^{\tilde h})\right)^{h/\tilde h}.\end{aligned}$$

Thus, we obtained

$$\mathbb{E}\left(\prod_{k=0}^{N_i}\mu_{b^{-n}}(J_{N_\beta k+i})^{j_k\frac{h}{\tilde h+1}}\right)\le e^{\varphi_h(m)} b^{-m(1+\tau(h))}\left(\mathbb{E}(Y_{n-m}^{\tilde h})\right)^{h/\tilde h}.$$

On the other hand, for every $0\le k\le N_i$,

$$\mathbb{E}\left(\mu_{b^{-n}}(J_{N_\beta k+i})^h\right)\le e^{\varphi_h(m)}b^{-m(1+\tau(h))}\mathbb{E}(Y_{n-m}^h)\le e^{\varphi_h(m)}b^{-m(1+\tau(h))}\mathbb{E}(Y_n^h),$$

by the submartingale property of $(Y_n^h)_{n\ge1}$.

Returning to (4.2), we have now

$$\begin{aligned}\mathbb{E}(Y_n^h) &\le e^{\varphi_h(m)}N_\beta^{h-1}\sum_{i=0}^{N_\beta-1}(N_i+1)b^{-m(1+\tau(h))}\mathbb{E}(Y_n^h)\\ &\qquad +(N_i+1)^{(\tilde h+1)}b^{-m(1+\tau(h))}\left(\mathbb{E}(Y_{n-m}^{\tilde h})\right)^{h/\tilde h}\\ &= e^{\varphi_h(m)}N_\beta^{h-1}b^{-m\tau(h)}\mathbb{E}(Y_n^h)\\ &\qquad +\left(e^{\varphi_h(m)}b^{-m(1+\tau(h))}N_\beta^{h-1}\sum_{i=0}^{N_\beta-1}(N_i+1)^{(\tilde h+1)}\right)\left(\mathbb{E}(Y_{n-m}^{\tilde h})\right)^{h/\tilde h}.\end{aligned}$$

Since $\tau(h)>0$ and τ is concave with $\tau(1)=0$, we have $\tau(q)>0$ for all $q\in(1,h)$. Moreover, $\varphi_h(m)=o(m)$ so for m large enough $e^{\varphi_h(m)}N_\beta^{h-1}b^{-m\tau(h)}<1$; therefore

$$\mathbb{E}(Y_n^h)\le\frac{e^{\varphi_h(m)}b^{-m(1+\tau(h))}N_\beta^{h-1}\sum_{i=0}^{N_\beta-1}(N_i+1)^{(\tilde h+1)}}{1-e^{\varphi_h(m)}N_\beta^{h-1}b^{-m\tau(h)}}\left(\mathbb{E}(Y_{n-m}^{\tilde h})\right)^{h/\tilde h}.$$

It follows that $\sup_{n\geq 1}\mathbb{E}(Y_n^h)<\infty$ by induction on $\tilde h$ as in the proof of Theorem 2 in [**KP**].

Proof of Theorem 5.4(2). Assume $\mathbf{C_3(h)}$. By the super-additivity of $x\mapsto x^h$ on $\mathbb{R}_+$ and Proposition 5.1, for every $n>m\geq 1$ we have

$$\begin{aligned}\mathbb{E}(Y_n^h)\geq\sum_{w\in A^m}\mathbb{E}\big(\mu_{b^{-n}}(I_w)^h\big) &\geq \sum_{w\in A^m}\mathbb{E}\Big(\inf_{s\in I_w}Q_{b^{-m}}(s)^h\Big)\,\mathbb{E}\Big(\|\mu_{b^{m-n}}^{I_w}\|^h\Big)\\ &\geq b^m e^{-\varphi_h(m)}\mathbb{E}\left(Q_{b^{-m}}(t)^h\right)b^{-mh}\mathbb{E}(Y_{n-m}^h)\\ &= e^{-\varphi_h(m)}b^{-m\tau(q)}\mathbb{E}(Y_{n-m}^h).\end{aligned}$$

Since $0<\mathbb{E}(\|\mu\|^h)$, μ is non-degenerate, and we saw that $\mathbb{E}(\mu)=1$. Consequently the martingale $(Y_n)_{n\geq 1}$ is uniformly integrable and $\mathbb{E}(\|\mu\|^h)<\infty$ implies that Y_n converges in L^p norm to $\|\mu\|$ as $n\to\infty$. This yields $1\geq e^{-\varphi_h(m)}b^{-m\tau(q)}$ via the previous inequalities, and forces $\tau(q)\geq 0$ since $\varphi_h(m)=o(m)$.

Now assume $\mathbf{C_3'(h)}$. Denoting by I_m an interval among the I_ws, $w\in A^m$, one has

$$\mathbb{E}\big(\mu_{b^{-n}}(I_m)^h\big)=\mathbb{E}(Q_m^h)\mathbb{E}\left(\int_{I_m}\overline{Q}_n(t)\,\mu_{b^{m-n}}^{I_m}(dt)\right)^h.$$

By using the Jensen inequality for conditional expectations and the independence successively, we get

$$\begin{aligned}&\mathbb{E}\left(\left(\int_{I_m}\overline{Q}_m(t)\,\mu_{b^{m-n}}^{I_m}(dt)\right)^h\Big|\overline{\mathcal{F}}_{b^{-m}}\right)\\ \geq\ &\left(\mathbb{E}\left(\int_{I_m}\overline{Q}_m(t)\,\mu_{b^{m-n}}^{I_m}(dt)\Big|\overline{\mathcal{F}}_{b^{-m}}\right)\right)^h\\ =\ &\left(\int_{I_m}\mathbb{E}\big(\overline{Q}_m(t)\big)\,\mu_{b^{m-n}}^{I_m}(dt)\right)^h\\ =\ &\mathbb{E}\big(\overline{Q}_m(t)\big)^h\|\mu_{b^{m-n}}^{I_m}\|^h.\end{aligned}$$

It follows that here again

$$\mathbb{E}(Y_n^h)\geq e^{-\varphi_h(m)}b^{-m\tau(q)}\mathbb{E}(Y_{n-m}^h).$$

One concludes as under $\mathbf{C_3(h)}$.

Proof of Theorem 5.5. Fix $n\geq 1$ such that $\mathbb{E}\big((\inf_{s\in I_n}Q_{b^{-n}}(s))^q\big)<\infty$. Due to **(P2)**, **(P3)** and **(P6)**, the same property holds for the positive multiples of n. Fix such a number m such that moreover, $b^m>2N_\beta$. Then, let $J_0=[0,b^{-m}]$ and $J_1=[1-b^{-m},1]$. As in the proof of Theorem 5.4(2), we can get

$$Y_n\geq\inf_{s\in J_0}Q_{b^{-m}}(s)\,\|\mu_{b^{m-n}}^{J_0}\|+\inf_{s\in J_1}Q_{b^{-m}}(s)\,\|\mu_{b^{m-n}}^{J_1}\|.$$

Then, letting n tend to ∞ and using Proposition 5.1 yields

$$Y=\|\mu\|\geq\inf_{s\in J_0}Q_{b^{-m}}(s)\,b^{-m}Y_0+\inf_{s\in J_1}Q_{b^{-m}}(s)\,b^{-m}Y_1$$

where Y_0 and Y_1 are independent copies of Y (because of $d(J_0,J_1)\geq N_\beta b^{-m}$ and **(P4)**), and Y_0 and Y_1 are also independent of

$$(B_0,B_1):=\big(\inf_{s\in J_0}Q_{b^{-m}}(s),\inf_{s\in J_1}Q_{b^{-m}}(s)\big).$$

Since $B_0 \overset{d}{\equiv} B_1$ and $\mathbb{E}(B_0^q) < \infty$, the approach used in [**Mol**] for generalized CCM yields $\mathbb{E}(Y^q) < \infty$. Let us give the proof. From the relation

$$Y \geq B_0 Y_0 + B_1 Y_1 \geq 2\sqrt{B_0 Y_0 B_1 Y_1},$$

and the fact that $Y > 0$ almost surely (the martingale $(Q_\varepsilon)_\varepsilon$ is positive), we get for any $h > 0$

$$\mathbb{E}(Y^{-h}) \leq 2^{-h}\mathbb{E}\left(B_0^{-\frac{h}{2}} B_1^{-\frac{h}{2}}\right)\left(\mathbb{E}(Y^{-\frac{h}{2}})\right)^2 \leq 2^{-h}\mathbb{E}\left(B_0^{-h}\right)\left(\mathbb{E}(Y^{-\frac{h}{2}})\right)^2. \tag{4.3}$$

Assume we have shown that $\mathbb{E}(Y^{-\varepsilon}) < \infty$ for some $\varepsilon \in (0, -q/2)$. Using k times (4.3) successively with $h = 2^i\varepsilon$, $1 \leq i \leq k$ and $2^k\varepsilon > -q/2 \geq 2^{k-1}\varepsilon$ yields $\mathbb{E}(Y^{q/2}) < \infty$. A last application of (4.3) with $h = -q$ yields the conclusion. The iterations stop because we only know that $\mathbb{E}(B_0^q) < \infty$. If, for example, B_0 and B_1 are independent,

$$\mathbb{E}(Y^{-h}) \leq 2^{-h}\left(\mathbb{E}(B_0^{-\frac{h}{2}})\right)^2\left(\mathbb{E}(Y^{-\frac{h}{2}})\right)^2$$

and one gets $\mathbb{E}(Y^{2q}) < \infty$. This is what happens for CCM but not for MPCP (see Section 6.3 for more details).

To show the existence of an ε as above, one uses the Laplace transform ϕ of Y ([**K2, Mol, B1, B2, Li1, Li2**]) which satisfies

$$\phi(t) \leq \mathbb{E}\left(\phi(B_0 t)\phi(B_1 t)\right).$$

The most elegant approach is the one of [**Li1, Li2**]. Let $p \in (0,1)$ be a number small enough so that $p\mathbb{E}(B_0^q) < 1$. The Cauchy-Schwarz inequality gives

$$\phi(t)^2 \leq \left(\mathbb{E}\big(\phi(B_0 t)^2\big)\right)^2 = o\left(\mathbb{E}\big(\phi(B_0 t)^2\big)\right).$$

So there exists $t_0 > 0$ such that for all $t \geq t_0$

$$\phi(t)^2 \leq p\mathbb{E}\big(\phi(B_0 t)^2\big).$$

Let $\psi = \phi^2$. Let $(\widetilde{B}_i)_{i\geq 1}$ be a sequence of independent copies of B_0. Since $\psi \leq 1$, for $t \geq t_0$

$$\begin{aligned}
\psi(t) &\leq p\mathbb{P}(B_0 t < t_0) + p\mathbb{E}\left(\mathbf{1}_{\{B_0 t \geq t_0\}}\psi(B_0 t)\right)\\
&\leq p\mathbb{E}(B_0^q)(t/t_0)^q + p^2\mathbb{E}\left(\mathbf{1}_{\{B_0 t\geq t_0\}}\psi(B_0\widetilde{B}_1 t)\right)\\
&\leq p\mathbb{E}(B_0^q)(t/t_0)^q + p^2\mathbb{E}\left(\psi(B_0\widetilde{B}_1 t)\right)\\
&\leq p\mathbb{E}(B_0^q)(t/t_0)^q + \left(p\mathbb{E}(B_0^q)\right)^2(t/t_0)^q + p^2\mathbb{E}\left(\mathbf{1}_{\{B_0\widetilde{B}_1 t\geq t_0\}}\psi(B_0\widetilde{B}_1 t)\right)\\
&\leq (t/t_0)^q\sum_{k=1}^n\left(p\mathbb{E}(B_0^q)\right)^k + p^n\mathbb{E}\left(\mathbf{1}_{\{B_0\widetilde{B}_1\cdots\widetilde{B}_{n-1}t\geq t_0\}}\psi(B_0\widetilde{B}_1\cdots\widetilde{B}_{n-1}t)\right)
\end{aligned}$$

for every $n \geq 1$. Since $\psi \leq 1$ and p and $p\mathbb{E}(B_0^q)$ are in $(0,1)$, it follows that for $t \geq t_0$

$$\psi(t) \leq \frac{p\mathbb{E}(B_0^q)}{1 - p\mathbb{E}(B_0^q)}(t/t_0)^q,$$

so $\phi(t) = O(t^{q/2})$. Then it is standard that $\mathbb{E}(Y^{-\varepsilon}) < \infty$ for all $\varepsilon \in (0, -q/2)$.

Proof of Theorem 5.6. The approach used in the proof of Theorem 4.2 allows to reduce the problem to showing that for every $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sum_{n\geq 1}\sum_{w\in A^n}\mathbb{E}\left(\widetilde{\mu}(\mathcal{A}_w)^{1+\eta}\right)b^{n\eta\left(\tau'(1)-\varepsilon\right)}+\mathbb{E}\left(\widetilde{\mu}(\mathcal{A}_w)^{1-\eta}\right)b^{-n\eta\left(\tau'(1)+\varepsilon\right)}<\infty.$$

Fix $\eta>0$ and $n\geq 1$. On the one hand, for every $w\in A^n$ we have almost surely

$$\begin{aligned}\widetilde{\mu}(\mathcal{A}_w)^{1+\eta} &= \lim_{m\to\infty}\left(\int_{I_w} Q_{b^{-n}}(t)\,\mu^{I_w}_{b^{-m}}(dt)\right)^{1+\eta}\\ &\leq \lim_{m\to\infty}\|\mu^{I_w}_{b^{-m}}\|^{\eta}\int_{I_w} Q_{b^{-n}}(t)^{1+\eta}\,\mu^{I_w}_{b^{-m}}(dt),\end{aligned}$$

by the Jensen inequality. Consequently, by the Fatou Lemma used together with the independence and Proposition 5.1 of [**BM3**] we get

$$\begin{aligned}\mathbb{E}\left(\widetilde{\mu}(\mathcal{A}_w)^{1+\eta}\right) &\leq b^{-n(1+\eta)}\mathbb{E}\left(Q_{b^{-n}}(t)^{1+\eta}\right)\mathbb{E}(\|\mu\|^{1+\eta})\\ &= b^{-n\left(1+\tau(1+\eta)\right)}\mathbb{E}(\|\mu\|^{1+\eta}).\end{aligned}$$

On the other hand, if η is small enough, by using $\mathbf{C_2(1-\eta)}$ we get

$$\mathbb{E}\left(\widetilde{\mu}(\mathcal{A}_w)^{1-\eta}\right)\leq e^{\varphi_{1-\eta}(n)}b^{-n\left(1+\tau(1-\eta)\right)}\mathbb{E}(\|\mu\|^{1-\eta}).$$

Moreover, if h is as in the statement, it follows from Theorem 5.4 that

$$\sup_{h'\in[0,h]}\mathbb{E}(\|\mu\|^{h'})<\infty.$$

So we are led to show that

$$\sum(\eta):=\sum_{n\geq 1}b^{-n\tau(1+\eta)}b^{n\eta\left(\tau'(1)-\varepsilon\right)}+e^{\varphi_{1-\eta}(n)}b^{-n\tau(1-\eta)}b^{-n\eta\left(\tau'(1)+\varepsilon\right)}<\infty$$

for η small enough. We first fix η small enough so that $-\tau(1+\eta)+\eta\tau'(1)-\eta\varepsilon<-\eta\varepsilon/2$ and $-\tau(1-\eta)-\eta\tau'(1)-\eta\varepsilon<-3\eta\varepsilon/4$. Then, from $\varphi_{1-\eta}(n)=o(n)$ we get that for n large enough $e^{\varphi_{1-\eta}(n)}b^{-n\tau(1-\eta)}b^{-n\eta\left(\tau'(1)+\varepsilon\right)}\leq b^{-n\eta\varepsilon/2}$. Therefore $\sum(\eta)<\infty$.

Remark 4.1. Since the proofs of Theorems 5.3 and 5.4 are inspired from those of corresponding results for MPCP, for the convenience of the readers of [**BM1**], we mention three minor blemishes in [**BM1**]: the first blemish concerns the proof of Theorem 1(i), which corresponds to Theorem 5.3(1). Instead of writing "letting h tend to 1" as we do here, we wrote "letting h tend to 0".

The second one concerns the proof of Lemma 6 involved in adapting the size-biazing method of [**WaWi**] for CCM to get the converse of Theorem 1(i) under some additional conditions. A random variable X_k and a probability measure $\mathbb{P}_t$ are defined. The explanation of the fact $\mathbb{E}_{\mathbb{P}_t}(X_k^2)>0$ is confused. In fact, if $\mathbb{E}_{\mathbb{P}_t}(X_k^2)=0$ then, with the notations of part II, $\log Q_{b^{-1}}(t)=\log(b)$ almost surely. This contradicts $\mathbb{E}\big(Q_{b^{-1}}(t)\big)=1$ (see also the proof of Theorem 6.6 of [**BM3**] in this paper).

The third one concerns the proof of Theorem 2(ii) (it corresponds to the proof of Theorem 5.4 under $(\mathbf{C_3'(h)})$, which involves a Lemma 4(i)(c). The proof of Lemma 4(i)(c) uses the conditional expectation with respect to $\sigma(\mathcal{F}_\varepsilon:0<\varepsilon\leq b^{-m})$ ($\mathcal{F}_\varepsilon$ is defined as in [**BM3**]). The correct σ-field to consider is of course $\overline{\mathcal{F}}_{b^{-m}}$, as here.

5. Proofs of Theorems 5.11 and 5.12

Assertions (3) of these results are standard (adapt the proof of Lemma 4.4 in [**O**] for box-multifractal analysis and use this lemma for centered multifractal analysis).

Proof of Theorem 5.11 (1)(2). We begin by invoking a standard series of inequalities that may be found for example in [**BrMiP**], [**F**] and [**P**] (see also [**L-VVoj**]). We write these relations for $\widetilde{\mu}$ but they hold for any positive Borel measure on $(\partial A^*, \mathcal{A}^*)$. With probability one, for every $\alpha \geq 0$ and $S \in \{\widetilde{E}, \overline{\widetilde{E}}, \underline{\widetilde{E}}\}$ one has

$$\dim S_\alpha \leq \operatorname{Dim} S_\alpha \leq (-\widetilde{\varphi}_{\widetilde{\mu}})^*(\alpha),$$

and

$$\dim \widetilde{E}_\alpha \leq \widetilde{f}(\alpha) \leq (-\widetilde{\varphi}_{\widetilde{\mu}})^*(\alpha).$$

Now we show the following proposition.

Proposition 5.1. *Assume $Q_{b^{-1}} > 0$, the condition $(\mathbf{C_2(q)})$ is satisfied for every $q \in \mathcal{J} \cap \mathbb{R}_+ \setminus [1,2]$, and the condition $(\mathbf{C_4(q)})$ is satisfied for every $q \in \mathcal{J} \cap \mathbb{R}_-$. With probability one, $\widetilde{\varphi}_{\widetilde{\mu}}(q) \leq -\tau(q)$ for all $q \in \mathcal{J}$.*

Proof. We first notice that an alternative definition for $\varphi_{\widetilde{\mu}}(q)$ is

$$\varphi_{\widetilde{\mu}}(q) = \inf\{t : \limsup_{n\to\infty} C_n(q,t) = 0\},$$

where

$$C_n(q,t) = \sum_{w \in A^n} \widetilde{\mu}(\mathcal{A}_w)^q b^{-nt}.$$

The function $-\tau$ being convex and $\widetilde{\varphi}_{\widetilde{\mu}}$ almost surely convex, it is enough to show that $\widetilde{\varphi}_{\widetilde{\mu}}(q) \leq -\tau(q)$ for every $q \in \mathcal{J} \setminus \{0\}$ almost surely. For such a $q \notin (0,1)$, we have seen (Remark 5.9 of [**BM3**]) that $\mathbb{E}(\|\mu\|^q) < \infty$. Moreover, since the mapping $x \mapsto x^q$ is convex on $(0,\infty)$, we can use the computations done in the proof of Theorem 5.6. This yields for $t \in \mathbb{R}$ and $n \geq 1$

$$\mathbb{E}\left(\sum_{w \in A^n} \widetilde{\mu}(\mathcal{A}_w)^q b^{-nt}\right) \leq b^{-n(t+\tau(q))}\mathbb{E}(\|\mu\|^q).$$

It follows that $\mathbb{E}\sum_{n\geq 1} C_n(q, -\tau(q)+\varepsilon) < \infty$ for every $\varepsilon > 0$, hence $\varphi_{\widetilde{\mu}}(q) \leq -\tau(q)$ almost surely. For $q \in (0,1)$ use property $\mathbf{C_2(q)}$ as in the proof of Theorem 5.6 and proceed as above.

We continue the proof of Theorem 5.11. It follows from Proposition 5.1 and assumption $(\mathbf{C})(1)(2)$ that with probability one, for all $q \in \mathcal{J}$, $(-\varphi_{\widetilde{\mu}})^*(\tau'(q)) \leq \tau^*(\tau'(q))$. It remains to show that with probability one, for all $q \in \mathcal{J}$, $\dim \widetilde{E}_{\tau'(q)} \geq \tau'(q)q - \tau(q)$. According to [**BBeP**], it is enough to establish the following lemma.

Lemma 5.2. *For every $\varepsilon > 0$, with probability one, for every $q \in \mathcal{J}$ there exists a positive Borel measure $\widetilde{\mu}_q$ on ∂A^* such that*

$$\limsup_{n\to\infty} \frac{\widetilde{\mu}_q(\mathcal{A}_n(\tilde{t}))}{\widetilde{\mu}(\mathcal{A}_n(\tilde{t}))^q\, b^{n(\tau(q)+\varepsilon)}} < \infty \quad \widetilde{\mu}_q - \text{almost everywhere.} \tag{5.1}$$

Indeed, this lemma implies (see [**BBeP**]) that, with probability one, for every $q \in \mathcal{J}$, $\mu_q(\widetilde{E}_{\tau'(q)}) > 0$ and $\dim(\widetilde{\mu}_q) = \tau'(q)q - \tau(q)$; hence $\dim\, \widetilde{E}_{\tau'(q)} \geq \tau'(q)q - \tau(q)$. Then, the equality of $\widetilde{\varphi}_{\widetilde{\mu}}$ and $-\tau$ on $\mathcal{J}$ follows, and assertions (1) and (2) of Theorem 5.11 are established.

Proof of Lemma 5.2. *Construction of* $(\widetilde{\mu}_q)_{q\in\mathcal{J}}$. For every $\omega \in \Omega$, $t \in [0,1]$, $\varepsilon \in (0,1)$ and $q \in \mathcal{J}$, define

$$Q_\varepsilon(t,q,\omega) = \frac{Q_{b^{-n}}(t,\omega)^q}{\mathbb{E}\left(Q_{b^{-n}}(t)^q\right)} \quad (\text{if } \varepsilon \in (b^{-n-1}, b^{-n}],\ n \geq 0).$$

$Q_\varepsilon(t,q,\omega)$ is denoted by $Q_\varepsilon(t,q)$ in the sequel.

The family $\{Q_\varepsilon(\cdot,\cdot,\cdot)\}$ satisfies condition **(A1)** of Section 4.2 with $\Gamma = \mathcal{J}$: since the function τ takes finite values on $\mathcal{J}$, the analyticity of $z \in \mathbb{C} \mapsto Q_\varepsilon(t,\omega)^z$ at fixed (t,ω) and the dominated convergence Theorem imply that for every non-trivial compact subinterval K of $\mathcal{J}$, there exists a deterministic neighbourhood of K, namely U_K, such that the mapping $q \in K \mapsto \mathbb{E}\left(Q_{b^{-1}}(t)^q\right)$ possesses the analytic extension $z \in U_K \mapsto \mathbb{E}\left(Q_{b^{-1}}(t)^z\right)$. Morever, choosing U_K small enough, the modulus of this extension takes only positive values. Then, it is straightforward that properties **(A1)**(i) and (ii) hold with

$$\widehat{Q}_{b^{-n}}(t,z,\omega) = \frac{Q_{b^{-n}}(t,\omega)^z}{\left(\mathbb{E}\left(Q_{b^{-1}}(t)^z\right)\right)^n}.$$

Now we show that U_K can be chosen so that **(A1)**(iii) holds. Because of **(P2), (P3), (P5)** and **(P6)** and the fact that $\sigma = \ell$ here, property (4.4) in [**BM3**] means that for every compact subset K' of U_K, there exists $p \in (1,2]$ such that

$$\sup_{z\in K'} 1 - p + \log_b \mathbb{E}\left(\left|\widehat{Q}_{b^{-1}}(t,z)\right|^p\right) < 0,$$

For $z = q \in K$, using the definition of τ shows that $1-p+\log_b \mathbb{E}\left(\left|\widehat{Q}_{b^{-1}}(t,z)\right|^p\right) < 0$ is equivalent to $p\tau(q) - \tau(pq) < 0$. Since τ is twice continuously differentiable, we have

$$p\tau(q) - \tau(pq) = (1-p)\big(\tau'(q)q - \tau(q)\big) + O\big((p-1)^2\big) \quad (\forall\ q \in K),$$

where $O\big((p-1)^2\big)$ is uniform over K. It follows from the definition of $\mathcal{J}$ that if p is close enough to 1, we indeed have $\sup_{q\in K} p\tau(q) - \tau(pq) < 0$. This makes it possible to choose the neighborhood U_K such that $\sup_{z\in U_K} 1-p+\log_b \mathbb{E}\left(\left|\widehat{Q}_{b^{-1}}(t,z)\right|^p\right) < 0$; hence **(A1)**$(iii)$ is fulfilled.

It follows from the above remarks and Theorem 4.1 that, with probability one, for all $q \in \mathcal{J}$, the measure $\widetilde{\mu}^q_\varepsilon$ converges weakly, as $\varepsilon \to 0$, to a measure $\widetilde{\mu}^q$ whose support is ∂A^*. Due to the self-similarity property, properties **(A2)**$(i)(ii)$ hold, so Proposition 3.1 can be applied. We denote $\widetilde{\mu}^q$ by $\widetilde{\mu}_q$.

End of the proof. The approach is as in the proof of Theorem 4.2. Given $\varepsilon > 0$, applying almost surely for every $q \in \mathcal{J}$ the Tchebitchev inequality to the random variable $X_q : \tilde{t} \mapsto \widetilde{\mu}_q\big(\mathcal{A}_n(\tilde{t})\big)$ in order to bound $\widetilde{\mu}_q\big(\{\tilde{t} \in \partial A^* :\ \widetilde{\mu}_q\big(\mathcal{A}_n(\tilde{t})\big) > \widetilde{\mu}\big(\mathcal{A}_n(\tilde{t})\big)^q\, b^{n(\tau(q)+\varepsilon)}\}\big)$, we reduce the problem to proving the following fact: for every $\varepsilon > 0$ and every nontrivial compact subinterval K of $\mathcal{J}$, there exists $\eta > 0$

such that with probability one, for all $q \in \mathcal{J}$,

$$\sum_{n\geq 0} f_n(q) < \infty$$

where

$$f_n(q) = b^{-n\eta(\tau(q)+\varepsilon)} \sum_{w\in A^n} \widetilde{\mu}(\mathcal{A}_w)^{-\eta q}\widetilde{\mu}_q(\mathcal{A}_w)^{1+\eta}.$$

Fix such $\varepsilon > 0$ and K. We give the proof under **(C)**(3)(β). Under this assumption, with the notations of the proof of Proposition 3.1, for every $\eta > 0$

$$\begin{aligned} f_n(q) &\leq b^{-n\eta(\tau(q)+\varepsilon)} \sum_{w\in A^n} \left(\sup_{t\in I_w} Q_{b^{-n}}(t)^{-\eta q}\right) b^{n\eta q} Z(w,1)^{-\eta q} \\ &\qquad\qquad \times \frac{\sup_{t\in I_w} Q_{b^{-n}}(t)^{(1+\eta)q}}{\left(\mathbb{E}\left(Q_{b^{-n}}(t)^q\right)\right)^{1+\eta}} b^{-n(1+\eta)} Z(w,q)^{1+\eta} \\ &\leq b^{-n\eta(\tau(q)+\varepsilon)} \sum_{w\in A^n} M_w(\eta) \frac{Q_{b^{-n}}(t_w)^q b^{-n(1+\eta(1-q))}}{b^{-n(1+\eta)(1-q+\tau(q))}} Z(w,1)^{-\eta q} Z(w,q)^{1+\eta} \\ &= b^{n(\tau(q)-q-\eta\varepsilon)} \sum_{w\in A^n} M_w(\eta) Q_{b^{-n}}(t_w)^q Z(w,1)^{-\eta q} Z(w,q)^{1+\eta} \\ &=: b^{n(\tau(q)-q-\eta\varepsilon)} g_n(q). \end{aligned}$$

The same approach as in the proof of Theorem 4.2 shows that it suffices to prove that if η is small enough, there exists $C = C(K,\eta) > 0$ such that for every $n \geq 1$,

$$\sup_{q\in K} b^{n(\tau(q)-q-\eta\varepsilon)} \mathbb{E}(g_n'(q)) \leq C b^{-n\varepsilon\eta/2} \tag{5.2}$$

$$\sup_{q\in K} b^{n(\tau(q)-q-\eta\varepsilon)} \mathbb{E}(g_n(q)) \leq C b^{-n\varepsilon\eta/2}. \tag{5.3}$$

For every $w \in A^n$ and $q \in K$, the random variables $M_w(\eta)Q_{b^{-n}}(t_w)^q$ and $Z(w,1)^{-\eta q}Z(w,q)^{1+\eta}$ are independent by construction. Moreover, since K is bounded and $\|\mu\|$ possesses finite moments of negative orders (by assumption the hypothesis of Theorem 5.5 are fulfilled for every $q \in \mathcal{J} \cap \mathbb{R}_-$), it follows from Proposition 3.1 and Hölder inequalities that for η small enough,

$$\sup_{w\in A^*,\ q\in K} \mathbb{E}\left(Z(w,1)^{-\eta q} Z(w,q)^{1+\eta}\right) + \mathbb{E}\left(\left|\frac{dZ(w,1)^{-\eta q} Z(w,q)^{1+\eta}}{dq}\right|\right) < \infty.$$

So we are led to show (5.2) for

$$g_n(q) = \sum_{w\in A^n} M_w(\eta) Q_{b^{-n}}(t_w)^q.$$

The same computations as in remarks (3) and (4) in the proof of Theorem 4.2 together with properties **(P2)**,**(P3)**,**(P5)** and **(P6)** show that for η small enough, $h, h' > 1$ such that $1/h + 1/h' = 1$, and $q \in K$

$$\mathbb{E}\left(M_w(\eta)\left|\frac{dQ_{b^{-n}}(t_w)^q}{dq}\right|\right) \leq \left(\mathbb{E}\left(M_w(\eta)^h\right)\right)^{1/h} A(w,q)^{1/h'},$$

where

$$\begin{aligned} A(w,q) &= n^{h'-1}\sum_{k=0}^{n-1}\frac{\mathbb{E}\Big(Q_{b^{-k},b^{-k-1}}(t_w)^{qh'}\big|\log(Q_{b^{-k},b^{-k-1}}(t_w))\big|^{h'}\Big)}{\mathbb{E}\left(Q_{b^{-k},b^{-k-1}}(t_w)^{qh'}\right)} \\ &\qquad\qquad\times\prod_{k'=0}^{n-1}\mathbb{E}\Big(Q_{b^{-k'},b^{-k'-1}}(t_w)^{qh'}\Big) \\ &= n^{h'}\frac{\mathbb{E}\Big(Q_{b^{-1}}(t_w)^{qh'}\big|\log(Q_{b^{-1}}(t_w))\big|^{h'}\Big)}{\mathbb{E}(Q_{b^{-1}}(t_w)^{qh'})}\Big(\mathbb{E}(Q_{b^{-1}}(t_w)^{qh'})\Big)^n \\ &= n^{h'}\frac{\mathbb{E}\Big(Q_{b^{-1}}(t_w)^{qh'}\big|\log(Q_{b^{-1}}(t_w))\big|^{h'}\Big)}{\mathbb{E}(Q_{b^{-1}}(t_w)^{qh'})}b^{-n(1-qh'+\tau(qh'))}. \end{aligned}$$

On the one hand, $\left(\mathbb{E}\left(M_w(\eta)^h\right)\right)^{1/h}=\exp\big(o(n)\big)$ by **(C)**(3)(β) with $o(n)$ uniform over $w\in A^n$ (notice that here the probability distributions of the random variables we are dealing with do not depend on w). On the other hand, due to the fact that τ is finite in a neighborhood of 0, all the moments of $\big|\log(Q_{b^{-1}}(t_w))\big|$ are finite. Consequently, if h' is chosen close enough to 1, an application of the Hölder inequality yields

$$\sup_{q\in K}\frac{\mathbb{E}\Big(Q_{b^{-1}}(t_w)^{qh'}\big|\log(Q_{b^{-1}}(t_w))\big|^{h'}\Big)}{\mathbb{E}(Q_{b^{-1}}(t_w)^{qh'})}<\infty.$$

From now on take $h'=1+\eta^2$. Including $n^{h'}$ in $\exp\big(o(n)\big)$ we get a constant $C=C(K,\eta)>0$ such that for every $q\in K$,

$$\mathbb{E}(|g_n'(q)|)\le Cb^n\exp\big(o(n)\big)b^{-n\frac{1-q(1+\eta^2)+\tau(q(1+\eta^2))}{1+\eta^2}}=C\exp\big(o(n)\big)b^{-n\left(-q+\tau(q)+O(\eta^2)\right)}$$

where $O(\eta^2)$ is uniform over $q\in K$. Consequently, for $q\in K$ one has

$$b^{n(\tau(q)-q-\eta\varepsilon)}\mathbb{E}(|g_n'(q)|)\le C\exp\big(o(n)\big)b^{-n\left(\eta\varepsilon+O(\eta^2)\right)}.$$

To conclude, choose η small enough so that $O(\eta^2)\le\eta\varepsilon/4$. Finally, since for n large enough one has also $o(n)\le\log(b)\eta\varepsilon n/4$, (5.2) follows. (5.3) is obtained similarly.

Proof of Theorem 5.12(1)(2). It follows from the definitions of the functions b_μ and B_μ in [**O**] that $b_\mu(q)\le B_\mu(q)\le\varphi_\mu(q)$ (see Lemma 4.2 in [**BBeP**]). Moreover, Theorem 5.11 applied with $q=1$ shows that $\widetilde{\mu}$ is almost surely atomless. Since $\mu=\widetilde{\mu}\circ\pi^{-1}$, the analog of Proposition 5.1 for μ instead of $\widetilde{\mu}$ holds. This yields $\varphi_\mu\le-\tau$ on $\mathcal{J}$ almost surely by using **(C')**(1)(2).

Then, due to [**BBeP**], the conclusion follows from the following Lemma.

LEMMA 5.3. *For every $\varepsilon>0$, with probability one, for every $q\in\mathcal{J}$ there exists a Borel measure μ_q on $[0,1]$, such that*

$$\text{(5.4)}\qquad\begin{cases}\limsup\limits_{n\to\infty}\dfrac{\mu_q\big(I_n(t)\big)}{\mu\big(I_n(t)\big)^q\,b^{n(\tau(q)+\varepsilon)}}<\infty\quad\mu_q-\text{almost everywhere}\\[2ex] \limsup\limits_{r\to 0}\dfrac{\mu_q\big(I_r(t)\big))}{\mu\big(I_r(t)\big)^q\,r^{-(\tau(q)+\varepsilon)}}<\infty\quad\mu_q-\text{almost everywhere.}\end{cases}$$

Indeed, it then follows from Lemma 4.6 in [**BBeP**] that, with probability one, for every $q \in \mathcal{J}$, $b_\mu(q) \geq -\tau(q)$ for every $q \in \mathcal{J}$; so all the functions mentioned above coincide on $\mathcal{J}$ and are differentiable. Moreover, one gets the correct lower bound for the Hausdorff dimensions of the sets $S_{\tau'(q)}$ for $q \in \mathcal{J}$ and $S \in \{\overline{E}, \underline{E}, E, \overline{F}, \underline{F}, F\}$ (see the proofs of Lemma 4.7 and Theorem 4.8 in [**BBeP**]). The correct upper bounds follow from Theorem 1 in [**BrMiP**] and Propositions 2.5 and 2.6 in [**O**].

Proof of Lemma 5.3. A family of positive measures on $(\partial A^*, \mathcal{A}^*)$, $(\widetilde{\mu}_q)_{q\in\mathcal{J}}$, was constructed in the proof of Lemma 5.2. Let $(\mu_q)_{q\in\mathcal{J}}$ be the family of measures on $[0,1]$ obtained as $\mu_q = \widetilde{\mu}_q \circ \pi^{-1}$. We have seen that, with probability one, for all $q \in \mathcal{J}$, $\dim(\mu_q) = \tau'(q)q - \tau(q) > 0$. In particular the μ_q's are atomless and the useful (for computations) relation $\mu_q(I_w) = \widetilde{\mu}_q(\mathcal{A}_w)$ holds for every $w \in A^*$.

It follows from the proofs of Lemmas 4.4 and 4.6 in [**BBeP**] that we only have to prove that for every $\varepsilon > 0$ and every nontrivial compact subinterval K of $\mathcal{J} \cap \mathbb{R}_-^*$ or $\mathcal{J} \cap \mathbb{R}_+$, there exists $\eta > 0$ such that with probability one, for all $q \in K$,

$$\sum_{n\geq 0} f_n(q) < \infty$$

where

$$f_n(q) = b^{-n\eta(\tau(q)+\varepsilon)} \sum_{\substack{v,w\in A^n\\ \delta(v,w)\leq b'}} \mu(I_v)^{-\eta q}\mu_q(I_w)^{1+\eta}.$$

with $b' = 3$ if $q < 0$ and $4b + 2$ otherwise.

By using **(C')**$(3)(i)$ as **(C)**(3) in the proof of Lemma 5.2 the problem is reduced to showing that for η small enough there exists $C = C(K, \eta) > 0$ such that

$$\sup_{q\in K} b^{n(\tau(q)-q-\eta\varepsilon)}\mathbb{E}(h_n'(q)) \leq Cb^{-n\varepsilon\eta/2} \tag{5.5}$$

$$\sup_{q\in K} b^{n(\tau(q)-q-\eta\varepsilon)}\mathbb{E}(h_n(q)) \leq Cb^{-n\varepsilon\eta/2} \tag{5.6}$$

where

$$h_n(q) = \sum_{\substack{v,w\in A^n\\ \delta(v,w)\leq b'}} M_{v,w}(\eta) Q_{b^{-n}}(t_v)^{-\eta q} Q_{b^{-n}}(t_w)^{(1+\eta)q} Z(v,1)^{-\eta q} Z(w,q)^{1+\eta}.$$

The proof ends in the same way as that of Lemma 5.2 by using **(C')**$(3)(ii)$.

6. Proofs of Proposition 6.1 and Theorem 6.2

Proof of Proposition 6.1. The density of $\widetilde{\mu}_{b^{-n}}$ can be reformulated as follows: there exists a sequence of independent copies of W, $(W_w)_{w\in A^*}$, and a sequence of independent random phases $(\phi_w)_{w\in A^*}$, such that $\sigma(W_w : w \in A^*)$ and $\sigma(\phi_w : w \in A^*)$ are independent and for every $n \geq 1$ and $w = w_1\cdots w_n \in A^n$,

$$\widetilde{\widetilde{Q}}_{b^{-n}}(\tilde{t}) = \prod_{k=1}^{n} W_{w_1\cdots w_k}\, \widetilde{W}\big(b^k(\pi(\tilde{t}) + \phi_{w_1\cdots w_k})\big) \quad (\forall\ \tilde{t} \in \mathcal{A}_w). \tag{6.1}$$

This, together with the definition of ψ yield for $q, q' \in \mathbb{R}$, $n \geq 1$ and $v, w \in A^n$

$$\begin{aligned}&\sup_{t\in I_w}\widehat{Q}_{b^{-n}}(t)^q\sup_{t\in I_v}\widehat{Q}_{b^{-n}}(t)^{q'}\\ \leq\;& e^{(|q|+|q'|)\psi(n)}\prod_{k=1}^{n}W^{q}_{w_1\cdots w_k}W^{q'}_{v_1\cdots v_k}\widetilde{W}\big(b^k(t_w+\phi_{w_1\cdots w_k})\big)^q\widetilde{W}\big(b^k(t_v+\phi_{v_1\cdots v_k})\big)^{q'}\\ =\;& e^{(|q|+|q'|)\psi(n)}\widehat{Q}_{b^{-n}}(t_w)^q\widehat{Q}_{b^{-n}}(t_v)^{q'}.\end{aligned}$$

Moreover,

$$\inf_{t\in I_w}\widehat{Q}_{b^{-n}}(t)^q\geq e^{-|q|\psi(n)}\widehat{Q}_{b^{-n}}(t_w)^q.$$

This is enough to get assertions (1), (2), (3) and (4), as well as (5) for **(C)** and **(C')**(1)(2)(3)(i). Notice that we are in the cases **(C)**(3)(α) and **(C')**(3)(i)(α). Therefore, due to the proof of Theorem 5.12, we can assume that $h'=1$ in establishing **(C')**(3)(ii). To do this, we have to estimate

$$\sum_{\substack{v,w\in A^n\\ 0<\delta(v,w)\leq b'}}\mathbb{E}\left[\prod_{k=1}^{n}W^{(1+\eta)q}_{w|k}W^{-\eta q}_{v|k}\widetilde{W}(b^k(t_w+\phi_{w|k}))^{(1+\eta)q}\widetilde{W}(b^k(t_v+\phi_{v|k}))^{-\eta q}\right],$$

where for every $v\in A^*$ of length ≥ 1 and $1\leq k\leq|v|$, $v|k$ denotes the word $v_1\cdots v_k$.

As in [**BBeP**], we begin by a preliminary remark: if v and w are words of length n, and if $\dot{v}$ and $\dot{w}$ stand for their prefixes of length $n-1$, then $\delta(\dot{v},\dot{w})>k$ implies $\delta(v,w)>bk$. It results that, given two integers $n\geq m>0$ and two words v and w in A^n such that $b^{m-1}<\delta(v,w)\leq b^m$, there exist two prefixes $\bar{v}$ and $\bar{w}$ of v and w respectively of common length $n-m$ such that $\delta(\bar{v},\bar{w})\leq 1$.

Consequently, due to the independence and assumptions on moments of W and $\widetilde{W}$, in the above sum we can assume that $b'=1$. The pairs (v,w) such that $\delta(v,w)=1$ will be represented as follows. Define ρ_k to be the word consisting of k consecutive zeros and λ_k to be the word consisting of k consecutive $b-1$ (considered as a letter from the alphabet $\{0,1,2,\dots,b-1\}$). A representation of the set of pairs (v,w) in $\mathcal{A}^n$ such that $\delta(v,w)=1$ is:

$$\bigcup_{k=0}^{n-1}\bigcup_{u\in A^{n-1-k}}\{(u.j.\lambda_k,u.(j+1).\rho_k)\,:\,0\leq j\leq b-2\}.\tag{6.2}$$

For every $q\in\mathbb{R}$, we denote by $\mathbb{E}(\widetilde{W}^q)$ the moment $\int_{[0,1]}\widetilde{W}(t)^q\,dt$.

Denote

$$\sum_{\substack{v,w\in A^n\\ \delta(v,w)=1}}\mathbb{E}\left[\prod_{k=1}^{n}W^{(1+\eta)q}_{w|k}W^{-\eta q}_{v|k}\widetilde{W}(b^k(t_w+\phi_{w|k}))^{(1+\eta)q}\widetilde{W}(b^k(t_v+\phi_{v|k}))^{-\eta q}\right]$$

by $\sum(n,q,\eta)$. Using (6.2) and taking into account the independence we get

$$\sum(n,q,\eta)=2\sum_{k=0}^{n-1}B_{n,k}\sum_{j=0}^{b-2}\sum_{u\in A^{n-1-k}}C_{n,k,j}$$

with

$$\begin{cases} B_{n,k} = (\mathbb{E}(W^q))^{n-1-k} \left(\mathbb{E}(W^{(1+\eta)q})\mathbb{E}(W^{-\eta q})\right)^{k+1} \left(\mathbb{E}(\widetilde{W}^{(1+\eta)q})\mathbb{E}(\widetilde{W}^{-\eta q})\right)^{k+1} \\ C_{n,k,j} = \prod_{i=1}^{n-k-1} \mathbb{E}\left(\widetilde{W}\big(b^i(t_{u.j.\lambda_k} + \phi_{u|i})\big)^{(1+\eta)q}\widetilde{W}\big(b^i(t_{u.(j+1).\rho_k} + \phi_{u|i})\big)^{-\eta q}\right). \end{cases}$$

For every $u \in A^{n-1-k}$ and $0 \leq j \leq b-2$, since $|(t_{u.j.\lambda_k}+\phi_{u|i})-(t_{u.(j+1).\lambda_k}+\phi_{u|i})| \leq b^{k+1-n}$, by definition of ψ we have

$$\begin{aligned} &\prod_{i=1}^{n-k-1} \widetilde{W}\big(b^i(t_{u.j.\lambda_k} + \phi_{u|i})\big)^{(1+\eta)q}\widetilde{W}\big(b^i(t_{u.(j+1).\rho_k} + \phi_{u|i})\big)^{-\eta q} \\ \leq\ & e^{\eta|q|\psi(n-k-1)} \prod_{i=1}^{n-k-1} \widetilde{W}\big(b^i(t_{u.j.\lambda_k} + \phi_{u|i})\big)^q; \end{aligned}$$

hence

$$C_{n,k,j} \leq e^{\eta|q|\psi(n-k-1)} \left(\mathbb{E}(\widetilde{W}^q)\right)^{n-k-1}.$$

It follows from previous computations and the definition of τ that

$$\begin{aligned} &\sum(n,q,\eta) \\ \leq\ & 2(b-2)\sum_{k=0}^{n-1} b^{n-1-k}e^{\eta|q|\psi(n-k-1)} \\ & \qquad \times b^{-(n-k-1)\left(1-q+\tau(q)\right)}b^{-(k+1)\left[\left(1-(1+\eta)q+\tau((1+\eta)q)\right)+\left(1+\eta q+\tau(-\eta q)\right)\right]} \\ =\ & 2(b-2)b^{-n(\tau(q)-q)}\sum_{k=0}^{n-1} e^{\eta|q|\psi(n-k-1)}b^{(k+1)(-2-\tau(0)+O(\eta))} \end{aligned}$$

with $O(\eta)$ uniform over $q \in K$. Notice that $\tau(0) = -1$. Finally, applying the Cauchy-Schwarz inequality to the last above sum yields the desired control since it is straightforward that $\sum_{k=0}^{n-1} e^{2\eta|q|\psi(n-k-1)} = \exp\left(2\eta|q|o(n)\right)$ and $\sum_{k=0}^{n-1} b^{2(k+1)(-2-\tau(0)+O(\eta))}$ is bounded for η small enough.

It remains to verify property (5.8) of [**BM3**]. This is elementary and left to the reader.

Proof of Theorem 6.2. We use the size biasing method involved in [**WaWi**] for CCM.

For every $n \geq 1$, define $\overline{\mathbb{P}}_n$ the probability measure on $(\Omega, \mathcal{F}_{b^{-n}})$ with density with respect to $\mathbb{P}$ equal to $Y_n = \mu_{b^{-n}}([0,1])$. Since $(Y_n, \mathcal{F}_{b^{-n}})_{n\geq 1}$ is a 1-mean martingale, $\{\overline{\mathbb{P}}_n\}$ is a consistent family of probability measures. Let $\overline{\mathbb{P}}$ be the Kolmogorov extension of the $\overline{\mathbb{P}}_n$'s to $\left(\Omega, \mathcal{F}_\infty = \sigma(\mathcal{F}_{b^{-n}}\ :\ n \geq 1)\right)$. By Theorem 2.5.20 of [**D-CDu**], $Y_n = \frac{d\overline{\mathbb{P}}_n}{d\mathbb{P}}$ converges $\frac{1}{2}(\mathbb{P}+\overline{\mathbb{P}})$-almost surely to a random variable Y_∞ in $\mathbb{R}_+ \cup \{\infty\}$ and if $\overline{\mathbb{P}}(Y_\infty = \infty) = 1$ then $\mathbb{P}(Y_\infty = 0) = 1$. Since $\|\mu\| = Y_\infty$ $\mathbb{P}$-almost surely, it is enough to show that $\overline{\mathbb{P}}(\limsup_{n\to\infty} Y_n = \infty) = 1$ to get the conclusion.

For every $t \in [0,1]$ and $n \geq 1$, define the measure $\mathbb{P}_{t,n}$ on $\mathcal{F}_{b^{-n}}$ by

$$\frac{d\mathbb{P}_{t,n}}{d\mathbb{P}}(\omega) = \widehat{Q}_{b^{-n}}(t,\omega).$$

Since $(\widehat{Q}_{b^{-n}}(t), \mathcal{F}_{b^{-n}})_{n\geq 1}$ is a 1-mean martingale, $\{\mathbb{P}_{t,n}\}$ is a consistent family of probability measures. Let $\mathbb{P}_t$ denote the Kolmogorov extension of the $\mathbb{P}_{t,n}$ to $\mathcal{F}_\infty$. Then for every $n \geq 1$ define on $(\Omega \times [0,1], \mathcal{F}_{b^{-n}} \otimes \mathcal{B}([0,1]))$ the probability measure $\mathcal{Q}_n(d\omega \times dt) = \mathbb{P}_{t,n}(d\omega)\ell(dt)$, and define $\mathcal{Q}$ on $(\Omega \times [0,1], \mathcal{F}_\infty \otimes \mathcal{B}([0,1]))$, the Kolmogorov extension of $(\mathcal{Q}_n)_{n\geq 1}$.

Let π_Ω be the first coordinate projection map on $\Omega \times [0,1]$. By construction, for every $n \geq 1$, $\overline{\mathbb{P}}_n = \mathcal{Q}_n \circ \pi_\Omega^{-1}$ and so $\overline{\mathbb{P}} = \mathcal{Q} \circ \pi_\Omega^{-1}$. Moreover $\mathcal{Q}(d\omega \times dt) = \mathbb{P}_t(d\omega)\ell(dt)$. Consequently, $\overline{\mathbb{P}}(\limsup_{n\to\infty} Y_n = \infty) = 1$ will follow after showing that $\mathbb{P}_t(\limsup_{n\to\infty} Y_n = \infty) = 1$ for ℓ-almost every every $t \in [0,1]$.

Fix $t = \sum_{k=0}^\infty t_k b^{-k} \in (0,1)$ $(t_k \in A)$. We only have to show that

$$\mathbb{P}_t(\limsup_{n\to\infty} \mu_{b^{-n}}(I_n(t)) = \infty) = 1.$$

It follows from (6.1) and the definition of ψ that for $n \geq 1$

$$\log\left(\mu_{b^{-n}}(I_n(t))\right) \geq -\psi(n) + \sum_{k=0}^{n-1} \log\left(W_{t_1\cdots t_k}\widetilde{W}\left(b^k(t+\phi_{t_1\cdots t_k})\right)\right) - \log(b). \tag{6.3}$$

Moreover, the random variables $X_k = \log\left(W_{t_1\cdots t_k}\widetilde{W}\left(b^k(t+\phi_{t_1\cdots t_k})\right)\right) - \log(b)$ are i.i.d. with respect to $\mathbb{P}_t$ with mean $\tau'(1^-)\log(b) = 0$ and positive variance. Indeed, if the variance of X_1 vanishes then $X_1 = 0$ $\mathbb{P}_t$-almost surely and so $\mathbb{P}$-almost surely by construction of $\mathbb{P}_t$. This implies $Q_{b^{-1}}(t) = b$ almost surely and contradicts $\mathbb{E}\big(Q_{b^{-1}}(t)\big) = 1$. Finally, the assumption on ψ, the law of the iterated logarithm applied to $(X_k)_{k\geq 0}$ with respect to $\mathbb{P}_t$ and (6.3) together yield the conclusion.

7. Proofs of Proposition 6.4, and Theorems 6.6 and 6.7

Proof of Proposition 6.4. We begin by preliminary definitions and remarks, as well as a lemma.

We will work under assumption **(H1)** or **(H2)** so we assume that $\widetilde{W}$ is continuous and fix $\underline{w}$ and $\overline{w}$ two numbers such that $0 < \underline{w} \leq \widetilde{W} \leq \overline{w} < \infty$ and $\underline{w}\overline{w} = 1$.

For every $w \in A^*$, recall that

$$T^{I_w} = \bigcap_{t\in I_w} \mathcal{C}_{b^{-|w|}}(t),$$

and

$$B^{I_w} = \left(\bigcup_{t\in I_w} \mathcal{C}_{b^{-|w|}}(t)\right) \setminus T^{I_w}$$

(see Figure 1 of Part II).

For every $n \geq 1$ and $w \in A^n$ we have

$$\begin{aligned} \widehat{Q}_{b^{-n}}(t) \quad &= \quad b^{-n\rho(V-1)} \prod_{M\in S\cap T^{I_w}} W_M\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right) \\ &\qquad \times \prod_{M\in S\cap B^{I_w}\cap \mathcal{C}_{b^{-n}}(t)} W_M\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right) \quad (\forall\ t\in I_w), \end{aligned}$$

where $V = \mathbb{E}(W)\mathbb{E}(\widetilde{W})$, and $t \in I_w \mapsto \prod_{M\in S\cap T^{I_w}} W_M \widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)$ and $t \in I_w \mapsto \prod_{M\in S\cap B^{I_w}\cap\mathcal{C}_{b^{-n}}(t)} W_M \widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)$ are independent.

Now we write

$$\prod_{M\in S\cap T^{I_w}} W_M\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right) = \prod_{k=0}^{n-1}\prod_{M\in S\cap T_k^{I_w}} W_M\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)$$

where $T_k^{I_w} = T^{I_w}\cap\{(t,\lambda):\ b^{-k-1}\le\lambda<b^{-k}\}$.

Also notice that for $t\in I_w$, $0\le k\le n-1$ and $M\in S\cap T_k^{I_w}$, one has

$$\left|\log\left(\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)\right)-\log\left(\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right)\right)\right|\le h(n-k-1)$$

where

$$h(k) = \sup_{\substack{u,v\in[0,1]\\|u-v|\le b^{-k}}}\left|\log\left(\widetilde{W}(u)\right)-\log\left(\widetilde{W}(v)\right)\right|.$$

It follows that for every $t\in I_w$

$$e^{-H(w)}\widehat{Q}(w)\le\prod_{M\in S\cap T^{I_w}} W_M\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)\le\ e^{H(w)}\widehat{Q}(w)$$

with

$$\begin{cases} H(w)=\sum_{k=0}^{|w|-1} h(|w|-k-1)\#\,S\cap T_k^{I_w}\\ \widehat{Q}(w)=\prod_{M\in S\cap T^{I_w}} W_M\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right).\end{cases}$$

If K_0 is a bounded subset of $\mathbb{R}$, define

$$\begin{cases} M(K_0,w)=\sup_{q\in K_0,t\in I_w}\prod_{M\in S\cap B^{I_w}\cap\mathcal{C}_{b^{-|w|}}(t)} W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q\\ M'(K_0,w)=\sup_{q\in K_0,t\in I_w}\dfrac{\prod_{M\in S\cap B^{I_w}\cap\mathcal{C}_{b^{-|w|}}(t)} W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q}{\prod_{M\in S\cap B^{I_w}\cap\mathcal{C}_{b^{-|w|}}(t_w)} W_M^q\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right)^q}.\end{cases}$$

For $k\ge 0$ and $q\in\mathbb{R}$ define

$$\widetilde{h}(k,q)=\max(0,1-\overline{w}^{2|q|}b^{-k}/2).$$

For $k\ge0$ and $t\in[0,1]$ define $\widehat{\mathcal{C}}_k(t)=\mathcal{C}_{b^{-k-1}}(t)\setminus\mathcal{C}_{b^{-k}}(t)$. For $\Lambda\in\{\Lambda_\rho,\widetilde{\Lambda}_\rho\}$, the Λ-measure of $\widehat{\mathcal{C}}_k(t)$ does not depend on t; so we will write $\Lambda(\widehat{\mathcal{C}}_k)$.

Finally, if $(q,q')\in\mathbb{R}^2$ is so that $\mathbb{E}(W^{q'})<\infty$, define

$$\begin{cases}\widehat{M}(q,q',w)=\exp\left(\sum_{k=0}^{|w|-1}\Lambda(\widehat{\mathcal{C}}_{|w|-1-k})\left(e^{qh(k)}-1\right)\mathbb{E}(W^{q'})\mathbb{E}(\widetilde{W}^{q'})\right)\\ \widehat{m}(q,q',w)=\exp\left(\sum_{k=0}^{|w|-1}\Lambda(\widehat{\mathcal{C}}_{|w|-1-k})\left(\widetilde{h}(k,q')e^{qh(k)}-1\right)\mathbb{E}(W^{q'})\mathbb{E}(\widetilde{W}^{q'})\right).\end{cases}$$

LEMMA 7.1. *Let* $\Lambda\in\{\Lambda_\rho,\widetilde{\Lambda}_\rho\}$.

(1) *For every* $q\in\mathbb{R}$, $n\geq 1$ *and* $t\in[0,1]$,

$$\begin{aligned}&\mathbb{E}\left(\prod_{M\in S\cap\mathcal{C}_{b^{-n}}(t)} W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q\right)\\ =\ & \exp\Big[\Lambda\big(\mathcal{C}_{b^{-n}}(t)\big)\Big(\mathbb{E}(W^q)\mathbb{E}(\widetilde{W}^q)-1\Big)\Big]=b^{n\rho\left(\mathbb{E}(W^q)\mathbb{E}(\widetilde{W}^q)-1\right)}.\end{aligned}$$

(2) *For every* $(q,q')\in\mathbb{R}^2$,

$$b^{-2\rho}\widehat{m}(q,q',w)\leq\frac{\mathbb{E}\left(e^{qH(w)}\widehat{Q}(w)^{q'}\right)}{\exp\Big(\Lambda\big(\mathcal{C}_{b^{-|w|}}(t_w)\big)\Big(\mathbb{E}(W^{q'})\mathbb{E}(\widetilde{W}^{q'})-1\Big)\Big)}\leq b^{2\rho}\widehat{M}(q,q',w).$$

(3) *For every* $q\geq 0$

$$\mathbb{E}\left(\inf_{t\in I_w}\prod_{M\in S\cap B^{I_w}\cap\mathcal{C}_{b^{-|w|}}(t)} W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q\right)\geq\exp\left(\Lambda(B^{I_w})\left(\mathbb{E}(W_q)-1\right)\right),$$

where $W_q=\min(1,\underline{w}^qW^q)$.

(4) *Let* K_0 *be a bounded subset of* $\mathbb{R}$ *and* $w_{K_0}=1+\underline{w}^{\inf K_0}+\overline{w}^{\sup K_0}$.

(*i*)

$$\mathbb{E}\left(M(K_0,w)\right)\leq\exp\Big(\Lambda(B^{I_w})\Big(\mathbb{E}(\widehat{W}_{K_0})-1\Big)\Big),$$

where $\widehat{W}_{K_0}=w_{K_0}(1+W^{\inf K_0}+W^{\sup K_0})$.

(*ii*) *For every* $h\geq 1$,

$$\mathbb{E}\left(M'(K_0,w)^h\right)\leq\exp\Big(\Lambda(B^{I_w})\Big(\mathbb{E}(\widehat{W}'_{K_0})-1\Big)\Big),$$

where $\widehat{W}'_{K_0}=w_{K_0}^{2h}(1+W^{h\inf K_0}+W^{h\sup K_0}+W^{-h\inf K_0}+W^{-h\sup K_0})$.

REMARK 7.2. The following lines show that the equality in Lemma 7.1(1) is valid for any locally bounded Borel intensity Λ invariant by horizontal translations.

Proof. (1) We will write $\Lambda(\mathcal{C}_{b^{-n}})$ for $\Lambda(\mathcal{C}_{b^{-n}}(t))$. We have

$$\begin{aligned}&\mathbb{E}\left(\prod_{M\in S\cap\mathcal{C}_{b^{-n}}(t)} W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q\right)\\ =\ &\sum_{k=0}^{\infty}\mathbb{E}\left(\mathbf{1}_{\{\#S\cap\mathcal{C}_{b^{-n}}(t)=k\}}W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q\right).\end{aligned}$$

By construction, conditionally on $\#S\cap\mathcal{C}_{b^{-n}}(t)=k\geq 1$, $S\cap\mathcal{C}_{b^{-n}}(t)$ is a set of k i.i.d. random variables M_k whose probability distribution is the restriction of Λ to $\mathcal{B}(\mathcal{C}_{b^{-n}}(t))$ normalized by $\Lambda(\mathcal{C}_{b^{-n}})$. Moreover, the random variables W_{M_i} are i.i.d. with W and independent of the M_js. So writing $\Lambda=\ell\otimes\nu$ and defining $p_k=\mathbb{P}(\#S\cap\mathcal{C}_{b^{-n}}(t)=k)$ we have

$$
\begin{aligned}
&\mathbb{E}\left(\mathbf{1}_{\{\#S\cap\mathcal{C}_{b^{-n}}(t)=k\}}W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q\right)\\
&= p_k\mathbb{E}(W^q)^k\left[\mathbb{E}\left(\widetilde{W}\left(\frac{t-t_{M_1}+\lambda_{M_1}}{2\lambda_{M_1}}\right)^q\right)\right]^k\\
&= p_k\mathbb{E}(W^q)^k\left[\frac{1}{\Lambda(\mathcal{C}_{b^{-n}})}\int_{\mathcal{C}_{b^{-n}}(t)}\widetilde{W}\left(\frac{t-s+\lambda}{2\lambda}\right)^q\ell(ds)\nu(d\lambda)\right]^k.
\end{aligned}
$$

Since $\mathcal{C}_{b^{-n}}(t)=\{(s,\lambda)\in\mathbb{R}\times(0,\infty):\ b^{-n}\le\lambda<1,\ t-\lambda<s\le t+\lambda\}$, the change of variable $t'=\frac{t-s+\lambda}{2\lambda}$ yields

$$
\begin{aligned}
\int_{\mathcal{C}_{b^{-n}}(t)}\widetilde{W}\left(\frac{t-s+\lambda}{2\lambda}\right)^q\ell(ds)\nu(d\lambda) &= \int_{[b^{-n},1)}(2\lambda)\int_{[0,1]}\widetilde{W}(t')^q\,dt'\,\nu(d\lambda)\\
&= \mathbb{E}(\widetilde{W}^q)\int_{[b^{-n},1)}2\lambda\nu(d\lambda)\\
&= \mathbb{E}(\widetilde{W}^q)\Lambda(\mathcal{C}_{b^{-n}}).
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\left(\mathbf{1}_{\{\#S\cap\mathcal{C}_{b^{-n}}(t)=k\}}W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q\right)=p_k\mathbb{E}(W^q)^k\mathbb{E}(\widetilde{W}^q)^k.
$$

Since $p_k=e^{-\Lambda(\mathcal{C}_{b^{-n}})}\frac{\Lambda(\mathcal{C}_{b^{-n}})^k}{k!}$, we get the conclusion.

(2) We have

$$
\mathbb{E}\left(e^{qH(w)}\widehat{Q}(w)^{q'}\right)=\prod_{k=0}^{|w|-1}\mathbb{E}\left(\prod_{M\in S\cap T_k^{I_w}}e^{h(|w|-1-k)q}W_M^{q'}\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right)^{q'}\right).
$$

Since $T_k^{I_w}=\{(s,\lambda)\in\mathbb{R}\times(0,\infty):\ b^{-k-1}\le\lambda<b^{-k},\ t_w-\lambda+b^{-|w|}/2<s\le t_w+\lambda-b^{-|w|}/2$, computations similar to those done to get (1) yield for every $0\le k\le|w|-1$

$$
\begin{aligned}
&\mathbb{E}\left(\prod_{M\in S\cap T_k^{I_w}}e^{h(|w|-1-k)q}W_M^{q'}\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right)^{q'}\right)\\
&= e^{-\Lambda(T_k^{I_w})}\sum_{j=0}^{\infty}\frac{\Lambda(T_k^{I_w})^j}{j!}\left(e^{h(|w|-1-k)q}\mathbb{E}(W^{q'})\right)^j(I(k,q',w))^j
\end{aligned}
$$

with

$$
I(k,q',w)=\frac{1}{\Lambda(T_k^{I_w})}\int_{[b^{-k-1},b^{-k})}(2\lambda)\int_{[b^{-|w|}/4\lambda,1-b^{-|w|}/4\lambda]}\widetilde{W}(t')^{q'}\,dt'\nu(d\lambda).
$$

Since

$$
\max(0,1-\overline{w}^{2|q'|}b^{-|w|+k+1}/2)\mathbb{E}(\widetilde{W}^{q'})\le\int_{[b^{-|w|}/4\lambda,1-b^{-|w|}/4\lambda]}\widetilde{W}(t')^{q'}\,dt'\le\mathbb{E}(\widetilde{W}^{q'})
$$

we get

$$\begin{aligned} & e^{-\Lambda(T_k^{I_w})} \exp\left(\widetilde{h}(|w|-k-1,q')e^{h(|w|-1-k)q}\mathbb{E}(W^{q'})\mathbb{E}(\widetilde{W}^{q'})\Lambda(\widehat{\mathcal{C}}_k)\right) \\ \leq\ & \mathbb{E}\left(\prod_{M\in S\cap T_k^{I_w}} e^{h(|w|-1-k)q} W_M^{q'}\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right)^{q'}\right) \\ \leq\ & e^{-\Lambda(T_k^{I_w})}\exp\left(e^{h(|w|-1-k)q}\mathbb{E}(W^{q'})\mathbb{E}(\widetilde{W}^{q'})\Lambda(\widehat{\mathcal{C}}_k)\right). \end{aligned}$$

Returning to $\mathbb{E}\left(e^{qH(w)}\widehat{Q}(w)^{q'}\right)$ we get the conclusion since $\prod_{k=0}^{|w|-1} e^{-\Lambda(T_k^{I_w})} = e^{-\Lambda(T^{I_w})}$ and $0\leq \Lambda(\mathcal{C}_{b^{-n}})-\Lambda(T^{I_w})\leq \Lambda(B^{I_w})\leq 2\rho\log(b)$.

(3) This is due to the inequality

$$\prod_{M\in S\cap B^{I_w}\cap \mathcal{C}_{b^{-|w|}}(t)} W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q \geq \prod_{M\in S\cap B^{I_w}} (\min(1,\underline{w}W_M))^q$$

for all $t\in I_w$.

(4)(i) This is due to the inequality

$$\prod_{M\in S\cap B^{I_w}\cap \mathcal{C}_{b^{-|w|}}(t)} W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q \leq \prod_{M\in S\cap B^{I_w}} w_{K_0}(1+W_M^{\inf K_0}+W_M^{\sup K_0})$$

for all $t\in I_w$ and $q\in K_0$.

(4)(ii) Proceed as above after writing

$$\begin{aligned} & \frac{\prod_{M\in S\cap B^{I_w}\cap \mathcal{C}_{b^{-|w|}}(t)} W_M^q\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)^q}{\prod_{M\in S\cap B^{I_w}\cap \mathcal{C}_{b^{-|w|}}(t_w)} W_M^q\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right)^q} \\ \leq\ & \left(\prod_{M\in S\cap B^{I_w}} w_{K_0}^2\right)\frac{\prod_{M\in S\cap B^{I_w}\cap\left(\mathcal{C}_{b^{-|w|}}(t)\setminus \mathcal{C}_{b^{-|w|}}(t_w)\right)} W_M^q}{\prod_{M\in S\cap B^{I_w}\cap\left(\mathcal{C}_{b^{-|w|}}(t_w)\setminus \mathcal{C}_{b^{-|w|}}(t)\right)} W_M^q} \end{aligned}$$

for all $t\in I_w$ and $q\in K_0$.

We now prove the assertions of Proposition 6.4.

Proof of (1) and (2). Fix $q\geq 0$, $n\geq 1$ and $w\in A^n$. We can assume that $\mathbb{E}(W^q)<\infty$. Indeed if this moment is infinite, $\mathbf{C_2(q)}$ holds automatically since $\mathbb{E}(Q_{b^{-n}}(t)^q)=\infty$ by Lemma 7.1(1), and the same holds for $\mathbf{C_1}$ if $q=1$. From the inequality

$$\sup_{s\in I_w}\widehat{Q}_{b^{-n}}(s)^q\leq b^{-n\rho(V-1)q}e^{qH(w)}\widehat{Q}(w)^q M(\{q\},w)$$

together with the independences between random variables and Lemma 7.1(2)(4)(i), we deduce that

$$\begin{aligned} & \mathbb{E}\left(\sup_{s\in I_w} \widehat{Q}_{b^{-n}}(s)^q\right) \\ \leq\ & b^{2\rho}\widehat{M}(q,q,w)\mathbb{E}(M(\{q\},w))b^{-n\rho(V-1)q}\exp\left(\Lambda(\mathcal{C}_{b^{-n}})\left(\mathbb{E}(W^q)\mathbb{E}(\widetilde{W}^q)-1\right)\right) \\ =\ & b^{2\rho}\widehat{M}(q,q,w)\mathbb{E}(M(\{q\},w))\mathbb{E}\left(\widehat{Q}_{b^{-n}}(t_w)^q\right). \end{aligned}$$

Now, on the one hand, $\Lambda(B^{I_w})$ being uniformly bounded (by $2\rho\log(b)$) over $w \in A^*$, so is $\mathbb{E}(M(\{q\},w))$. On the other hand, $\Lambda(\widehat{\mathcal{C}}_k)$ being uniformly bounded (it is equal to $\rho\log(b)$) over $k\in\mathbb{N}$, due to the fact that $h(j)\to 0$, as $j\to\infty$, we have

$$\log\widehat{M}(q,q,w) = O(\psi(n))$$

uniformly over $w\in A^n$. This is enough to conclude.

Proof of (3). Proceed as previously and use Lemma 7.1(3) to get

$$\mathbb{E}\left(\inf_{s\in I_w}\widehat{Q}_{b^{-n}}(s)^q\right) \geq C(q,w)\widehat{m}(-q,q,w)\mathbb{E}\left(\widehat{Q}_{b^{-n}}(t_w)^q\right),$$

with

$$C(q,w) = b^{-2\rho}\exp\left(\Lambda(B^{I_w})\left(\mathbb{E}(W_q)-1\right)\right).$$

The conclusion follows the same lines as in proving (1) and (2) since $\widetilde{h}(k,q)\to 1$ as $k\to\infty$.

Proof of (4). Fix $q<0$. Writing $\left(\inf_{s\in I_w}\widehat{Q}_{b^{-n}}(s)\right)^q = \left(\sup_{s\in I_w}\widehat{Q}_{b^{-n}}(s)^{-1}\right)^{-q}$ makes it possible to use computations similar to those used in proving (2) with W^{-1} instead of W and $-q$ instead of q.

Proof of (5). We assume that all the moments of W are finite, and we begin by proving that **(C)** holds. In fact, due to (2) and (4), we only have to prove **(C)**(3).

The inequality (5.5) of [**BM3**] is satisfied with the random variable

$$M_w(\eta) = M'(K_{-\eta},w)M'(K_{1+\eta},w)e^{C_K(\eta)H(w)},$$

where $K_\beta = \{\beta q:\ q\in K\}$ if $\beta\in\mathbb{R}$ and $C_K(\eta) = \sup\{q\in K:\ (1+2\eta)|q|\}$. Lemma 7.1(2) and (4)($ii$) together with any Hölder inequality guarantee that property **(C)**(3)(β) holds (all the moments of W are finite).

Now we prove **(C')**(3). The inequality (5.6) of [**BM3**] is satisfied with the random variable

$$M_{v,w}(\eta) = M'(K_{-\eta},v)M'(K_{1+\eta},w)e^{C_K(\eta)H(v)}e^{C'_K(\eta)H(w)},$$

where $C_K(\eta) = \sup\{q\in K:\ \eta|q|\}$ and $C'_K(\eta) = \sup\{q\in K:\ (1+\eta)|q|\}$. Here again Lemma 7.1(2) and (4)(ii) together with any Hölder inequality insure property **(C')**(3)(i)(β) holds, yielding **(C')**(3)(i).

Now we establish **(C')**(3)(ii). Here we only need $\mathbb{E}(W^r)<\infty$ for r in a neighborhood of $[0,1]$.

For $n\geq 1$, $(v,w)\in A^n$ such that $0<\delta(v,w)\leq b'$, $h'>1$ and $q\in K$ define $t_{v,w}$ as the middle of $[t_v,t_w]$ and

$$\begin{cases} H(v,w) = \delta(v,w) \displaystyle\sum_{\substack{k=0 \\ b^{n-k} \geq \frac{1+\delta(v,w)}{2}}}^{n-1} h(n-k-1) \# S \cap T_k^{I_v} \cap T_k^{I_w}, \\ \widehat{Q}(v,w) = \displaystyle\prod_{M \in S \cap T^{I_v} \cap T^{I_w}} W_M \widetilde{W}\left(\frac{t_{v,w} - t_M + \lambda_M}{2\lambda_M}\right). \end{cases}$$

Also define $\widetilde{W}_{M,t} = \widetilde{W}\Big(\dfrac{t - t_M + \lambda_M}{2\lambda_M}\Big)$ and

$$\begin{cases} M_1(\eta, h', v, w, K) = \displaystyle\sup_{q \in K} \sup_{t \in I_v} \prod_{M \in S \cap (B^{I_v} \setminus B^{I_w}) \cap \mathcal{C}_{b^{-|w|}}(t)} W_M^{-\eta h' q} \widetilde{W}_{M,t}^{-\eta h' q} \\ \qquad \cdot \displaystyle\sup_{t \in I_w} \prod_{M \in S \cap B^{I_v} \cap T^{I_w} \cap \mathcal{C}_{b^{-|w|}}(t)} W_M^{(1+\eta) h' q} \widetilde{W}_{M,t}^{(1+\eta) h' q} \\ M_2(\eta, h', v, w, K) = \displaystyle\sup_{q \in K} \sup_{t \in I_w} \prod_{M \in S \cap (B^{I_w} \setminus B^{I_v}) \cap \mathcal{C}_{b^{-|w|}}(t)} W_M^{(1+\eta) h' q} \widetilde{W}_{M,t}^{(1+\eta) h' q} \\ \qquad \cdot \displaystyle\sup_{t \in I_v} \prod_{M \in S \cap B^{I_w} \cap T^{I_v} \cap \mathcal{C}_{b^{-|w|}}(t)} W_M^{-\eta h' q} \widetilde{W}_{M,t}^{-\eta h' q} \\ M_3(\eta, h', v, w, K) = \displaystyle\sup_{q \in K} \sup_{t \in I_v} \prod_{M \in S \cap B^{I_v} \cap B^{I_w} \cap \mathcal{C}_{b^{-|w|}}(t)} W_M^{-\eta h' q} \widetilde{W}_{M,t}^{-\eta h' q} \\ \qquad \cdot \displaystyle\sup_{t \in I_w} \prod_{M \in S \cap B^{I_w} \cap B^{I_v} \cap \mathcal{C}_{b^{-|w|}}(t)} W_M^{(1+\eta) h' q} \widetilde{W}_{M,t}^{(1+\eta) h' q} \\ M_4(\eta, h', v, w, K) = \displaystyle\sup_{q \in K} \prod_{M \in S \cap \left(T^{I_v} \setminus (T^{I_w} \cup B^{I_w})\right)} W_M^{-\eta h' q} \overline{w}^{\eta h' |q|} \\ M_5(\eta, h', v, w, K) = \displaystyle\sup_{q \in K} \prod_{M \in S \cap \left(T^{I_w} \setminus (T^{I_v} \cup B^{I_v})\right)} W_M^{(1+\eta) h' q} \overline{w}^{(1+\eta) h' |q|}. \end{cases}$$

The random variables $M_i(\eta, h', v, w, K)$ are mutually independent. Define their product

$$C_{v,w}(\eta, h') = \prod_{i=1}^{5} M_i(\eta, h', v, w, K).$$

The random variable $C_{v,w}(\eta, h')$ is itself independent of $e^{(1+2\eta)|q| h' H(v,w)} \widehat{Q}(v,w)^{q h'}$. Moreover, for every $q \in K$ one has

$$\begin{aligned} (7.1) \qquad & \widehat{Q}_{b^{-n}}\left(t_w\right)^{(1+\eta) q h'} \widehat{Q}_{b^{-n}}\left(t_v\right)^{-\eta q h'} \\ \leq \quad & C_{v,w}(\eta, h') e^{(1+2\eta)|q| h' H(v,w)} b^{-n\rho(V-1) q h'} \widehat{Q}(v,w)^{q h'}. \end{aligned}$$

This is obtained after writing for $q' \in \mathbb{R}$ and $\epsilon \in \{-1, 1\}$

$$\widehat{Q}_{b^{-n}}\left(t_v\right)^{\epsilon q'} = b^{-n\rho(V-1)\epsilon q'} \Pi_1(v) \Pi_2(v) \Pi_3(v) \Pi_4(v)$$

where

$$\Pi_1(v) = \prod_{M \in S \cap (B^{I_v} \setminus B^{I_w}) \cap \mathcal{C}_{b^{-n}}(t_v)} W_M^{\epsilon q'} \widetilde{W}\left(\frac{t_v - t_M + \lambda_M}{2\lambda_M}\right)^{\epsilon q'},$$

$$\Pi_2(v) = \prod_{M\in S\cap B^{I_v}\cap B^{I_w}\cap \mathcal{C}_{b^{-n}}(t_v)} W_M^{\epsilon q'}\widetilde{W}\left(\frac{t_v - t_M + \lambda_M}{2\lambda_M}\right)^{\epsilon q'},$$

$$\Pi_3(v) = \prod_{M\in S\cap\left(T^{I_v}\setminus T^{I_w}\right)} W_M^{\epsilon q'}\widetilde{W}\left(\frac{t_v - t_M + \lambda_M}{2\lambda_M}\right)^{\epsilon q'},$$

and

$$\begin{aligned}\Pi_4(v) &= \prod_{M\in S\cap T^{I_v}\cap T^{I_w}} W_M^{\epsilon q'}\widetilde{W}\left(\frac{t_v - t_M + \lambda_M}{2\lambda_M}\right)^{\epsilon q'}\\ &= \widehat{Q}(v,w)^{\epsilon q'}\frac{\prod_{M\in S\cap T^{I_v}\cap T^{I_w}}\widetilde{W}\left(\frac{t_v-t_M+\lambda_M}{2\lambda_M}\right)^{\epsilon q'}}{\prod_{M\in S\cap T^{I_v}\cap T^{I_w}}\widetilde{W}\left(\frac{t_{v,w}-t_M+\lambda_M}{2\lambda_M}\right)^{\epsilon q'}}\\ &\leq \widehat{Q}(v,w)^{\epsilon q'}e^{|q'|H(v,w)};\end{aligned}$$

the last inequality is due, on the one hand, to the fact that $T_k^{I_v}\cap T_k^{I_w}\neq\emptyset$ only for $b^{-k}\geq \frac{1+\delta(v,w)}{2}b^{-n}$ (see Figure 1), and on the other hand to the definition of $h(k)$ and the fact that if $M\in S\cap T_k^{I_v}\cap T_k^{I_w}\neq\emptyset$ one has $|\frac{t_v-t_{v,w}}{2\lambda_M}|\leq\frac{\delta(v,w)}{4}b^{k+1-n}\leq \delta(v,w)b^{k+1-n}$.

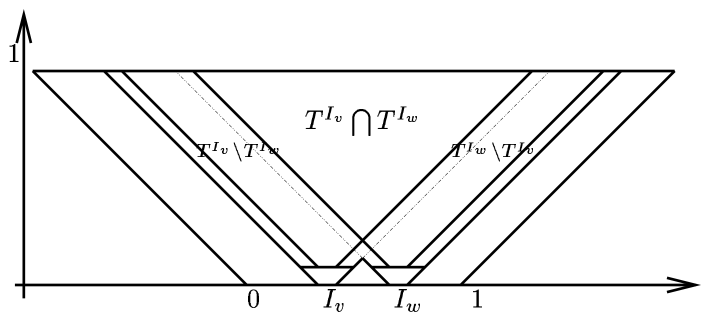

FIGURE 1. Illustration (to complete with Figure 1 of Part II) of some sets involved in the definitions of $H(v,w)$, $\widehat{Q}(v,w)$ and $M_i(\eta,h',v,w,K)$.

Now, since $\mathbb{E}(W^r)<\infty$ for $r\in\mathcal{J}$ and $\Lambda\left(T^{I_w}\setminus T^{I_v}\right)$ and $|\Lambda(\mathcal{C}_{b^{-n}})-\Lambda(T^{I_v}\cap T^{I_w})|$ are uniformly bounded over $n\geq 1$ and those $(u,v)\in A^n$ such that $0<\delta(u,v)\leq b'$, computations very similar to those done in proving Lemma 7.1 below show that if η is small enough and h' close enough to 1, then $\mathbb{E}\big(M_i(\eta,h',v,w,K)\big)$ is uniformly bounded over $n\geq 1$ and those $(u,v)\in A^n$ such that $0<\delta(u,v)\leq b'$; consequently, the same holds for $\mathbb{E}\big(C_{v,w}(\eta,h')\big)$. On the other hand,

$$\mathbb{E}\left(e^{(1+2\eta)|q|h'H(v,w)}b^{-n\rho(V-1)qh'}\widehat{Q}(v,w)^{qh'}\right)\leq b^{o(n)-n\rho\left((V-1)qh'-(\mathbb{E}(W^{qh'})\mathbb{E}(\widetilde{W}^{qh'})-1)\right)},$$

where $o(n)$ is uniform over $q\in K$ and those $(u,v)\in A^n$ such that $0<\delta(u,v)\leq b'$. Notice that

$$\rho\big((V-1)qh'-(\mathbb{E}(W^{qh'})\mathbb{E}(\widetilde{W}^{qh'})-1)\big)=\tau(qh')-qh'+1.$$

Moreover, by taking $h' \le 1+\eta^2$, we can fix η small enough so that

$$\sup_{1\le h'\le 1+\eta^2, q\in K} |\tau(qh') - qh' - (\tau(q)-q)| \le \varepsilon\eta/8$$

(such a choice is possible because τ is of class C^2). Then, for n large enough the function $o(n)$ above is also less than $\varepsilon\eta n/8$. So, if η is small enough, there exists a constant $C = C(K,\eta)$ such that for all $1 \le h' \le 1+\eta^2$,

$$\mathbb{E}\left(e^{(1+2\eta)|q|h'H(v,w)} b^{-n\rho(V-1)qh'} \widehat{Q}(v,w)^{qh'}\right) \le Cb^{-n(\tau(q)-q+1-\eta\varepsilon/4)}.$$

Finally, returning to (7.1), taking the expectation and summing over the right pairs (u,v) whose quantity is less than $(2b'+2)b^n$, we get the first part of **(C')**$(3)(ii)$.

To get the second part of **(C')**$(3)(ii)$, i.e. (5.8), write

$$\widehat{Q}_{b^{-n},b^{-n-1}}(t_w)^{(1+\eta)q}\widehat{Q}_{b^{-n},b^{-n-1}}(t_v)^{-\eta q} = b^{-n\rho(V-1)q}\Pi_1(q)\Pi_2(q)\Pi_3(q)$$

with

$$\begin{cases} \Pi_1(q) = \prod_{M\in S\cap\widehat{\mathcal{C}}_n(t_w)\setminus\widehat{\mathcal{C}}_n(t_v)} W_M^{(1+\eta)q}\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right)^{(1+\eta)q}, \\ \Pi_2(q) = \prod_{M\in S\cap\widehat{\mathcal{C}}_n(t_w)\cap\widehat{\mathcal{C}}_n(t_v)} W_M^{q}\widetilde{W}\left(\frac{t_w-t_M+\lambda_M}{2\lambda_M}\right)^{(1+\eta)q}\widetilde{W}\left(\frac{t_v-t_M+\lambda_M}{2\lambda_M}\right)^{-\eta q}, \\ \Pi_3(q) = \prod_{M\in S\cap\widehat{\mathcal{C}}_n(t_v)\setminus\widehat{\mathcal{C}}_n(t_w)} W_M^{-\eta q}\widetilde{W}\left(\frac{t_v-t_M+\lambda_M}{2\lambda_M}\right)^{-\eta q}. \end{cases}$$

The denominator (resp. numerator) of (5.8) has to be lower (resp. upper) bounded uniformly over K if η is small enough and h' close enough to 1. For the denominator, the proof uses the independence of Π_1, Π_2 and Π_3 as well as the same approach as in the proof of Lemma 7.1(3), which gives a uniform lower bound for the expectation of $\Pi_i(q)^{h'}$ (one uses also the fact that $\Lambda(\widehat{\mathcal{C}}_n)$ is uniformly bounded over n). For the numerator, one needs an upper bound of the expectations of $\Pi_i(q)^{h'}$ and $\left|\frac{d\Pi_i}{dq}(q)\right|^{h'}$. The first expectation is controlled via the same computations as in the proof of Lemma 7.1(4)(i). Let us show how to control the expectation of $\left|\frac{d\Pi_i}{dq}(q)\right|^{h'}$ for $i=1$ (cases $i=2$ and $i=3$ are treated similarly).

Conditionally on $\#S\cap\widehat{\mathcal{C}}_n(t_w)\setminus\widehat{\mathcal{C}}_n(t_v) = k \ge 1$, let $M_1,\dots,M_k$ be the points of $S\cap\widehat{\mathcal{C}}_n(t_w)\setminus\widehat{\mathcal{C}}_n(t_v)$. One has

$$\frac{d\Pi_1}{dq}(q) = \sum_{i=1}^k (1+\eta) y_i^{(1+\eta)q}\log(y_i)\prod_{\substack{j=1\\ j\neq i}}^k y_j^{(1+\eta)q}$$

with

$$y_i = W_{M_i}\widetilde{W}\left(\frac{t_w - t_{M_i} + \lambda_{M_i}}{2\lambda_{M_i}}\right).$$

Then, using successively the convex inequality $|\sum_{i=1}^k x_i|^{h'} \le k^{h'-1}\sum_{i=1}^k |x_i|^{h'}$, the upper bound for $\widetilde{W}$, and the fact that the W_{M_i}'s are i.i.d. we get

$$\begin{aligned} &\mathbb{E}\left(\left|\frac{d\Pi_1}{dq}(q)\right|^{h'} \,\Big|\, \#S\cap\widehat{\mathcal{C}}_n(t_w)\setminus\widehat{\mathcal{C}}_n(t_v) = k\right) \\ \le\ & (1+\eta)^{h'}k^{h'}(\overline{w}^{(1+\eta)|q|h'})^k\left(\mathbb{E}(W^{(1+\eta)qh'})\right)^{k-1} \\ & \times\mathbb{E}\left(W^{(1+\eta)qh'}\left(|\log(W)| + \log(\overline{w})\right)^{h'}\right). \end{aligned}$$

Define $\lambda = \Lambda(\widehat{\mathcal{C}}_n(t_w) \setminus \widehat{\mathcal{C}}_n(t_v))$, $x = \mathbb{E}\left(W^{(1+\eta)qh'}\left(|\log(W)| + \log(\overline{w})\right)^{h'}\right)$, $y = \overline{w}^{(1+\eta)|q|h'}$ and $z = \mathbb{E}(W^{(1+\eta)qh'})$. We obtained

$$\begin{aligned}\mathbb{E}\left(\left|\frac{d\Pi_1}{dq}(q)\right|^{h'}\right) &= \sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\mathbb{E}\left(|\Pi_1'(q)|^{h'}|\#S \cap \widehat{\mathcal{C}}_n(t_w) \setminus \widehat{\mathcal{C}}_n(t_v) = k\right)\\ &\leq (1+\eta)^{h'} xy\lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{k^{h'}}{k!}(\lambda yz)^{k-1}.\end{aligned}$$

It is standard that there exists a constant $C_{h'}$ such that $\sum_{k=1}^{\infty}\frac{k^{h'}}{k!}(\lambda yz)^{k-1} \leq C_{h'}(1+\lambda yz)^{h'+2}\exp(\lambda yz)$. Our previous remark on the uniform boundedness of $\lambda = \Lambda(\widehat{\mathcal{C}}_n(t_w) \setminus \widehat{\mathcal{C}}_n(t_v))$ as well as the finiteness of $\mathbb{E}(W^r)$ for all $r \in \mathcal{J}$ yield the conclusion.

Proof of Theorem 6.6. It suffices to show that for any $t \in (0,1)$ one has $\mathbb{P}_t(\limsup_{n\to\infty}\mu_{b^{-n}}(I_n(t)) = \infty) = 1$, where the probability $\mathbb{P}_t$ is constructed as in the proof of Theorem 6.2 but with $\widehat{Q}$ as defined in this Section 7. Fix $t \in (0,1)$. We have

$$\mu_{b^{-n}}(I_n(t)) \geq b^{-n}\widehat{Q}_{b^{-n}}(t)\inf_{s\in I_n(t)}\frac{\widehat{Q}_{b^{-n}}(s)}{\widehat{Q}_{b^{-n}}(t)} \geq b^{-n}\widehat{Q}_{b^{-n}}(t)U_n(t)V_n(t),$$

with

$$U_n(t) = \inf_{s\in I_n(t)}\prod_{M\in\mathcal{C}_{b^{-n}}(t)\cap\mathcal{C}_{b^{-n}}(s)}\frac{\widetilde{W}\left(\frac{s-t_M+\lambda_M}{2\lambda_M}\right)}{\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)}$$

and

$$V_n(t) = \inf_{s\in I_n(t)}\frac{\prod_{M\in\mathcal{C}_{b^{-n}}(s)\setminus\mathcal{C}_{b^{-n}}(t)} W_M\widetilde{W}\left(\frac{s-t_M+\lambda_M}{2\lambda_M}\right)}{\prod_{M\in\mathcal{C}_{b^{-n}}(t)\setminus\mathcal{C}_{b^{-n}}(s)} W_M\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)}.$$

The same approach as in the proof of Theorem 6.2 shows that it remains to show that

$$|\log(U_n(t))| + |\log(V_n(t))| = o(\sqrt{n\log\log(n)}) \quad \mathbb{P}_t - \text{almost surely.} \tag{7.2}$$

With the notations introduced for the proof of Proposition 6.4, we have

$$\begin{aligned}|\log(U_n(t))| &\leq \sum_{k=0}^{n-1} h(n-k-1)\#\, S\cap\widehat{\mathcal{C}}_k(t)\cap\widehat{\mathcal{C}}_k(s)\\ &\leq \sum_{k=0}^{n-1} h(n-k-1)\#\, S\cap\widehat{\mathcal{C}}_k(t)\\ &\leq \left(\sum_{k=0}^{n-1} h(k)^p\right)^{1/p}\left(\sum_{k=0}^{n-1} c_k^q\right)^{1/q},\end{aligned}$$

where $c_k = \#\, S\cap\widehat{\mathcal{C}}_k(t)$ and (p,q) is any pair of positive numbers such that $\frac{1}{p}+\frac{1}{q} = 1$. Fix such a pair (p,q). Due to the choice of $\Lambda \in \{\Lambda_\rho, \widetilde{\Lambda}_\rho\}$, the random variables c_k^q

are i.i.d and integrable with respect to $\mathbb{P}_t$. So, by virtue of the law of large numbers,

$$\left(\sum_{k=0}^{n-1} c_k^q\right)^{1/q} = O\big(n^{1/q}\big) \quad \mathbb{P}_t - \text{almost surely.}$$

Moreover,

$$\left(\sum_{k=0}^{n-1} h(k)^p\right)^{1/p} = O(\psi(n)^{1/p}) = O(n^{\frac{1}{2p}-\frac{\varepsilon}{p}})$$

by the assumption made on $\widetilde{W}$. Therefore, choosing q large enough yields

$$|\log(U_n(t))| = O(\sqrt{n}) \quad \mathbb{P}_t - \text{almost surely.} \tag{7.3}$$

We claim that

$$\sup_{n\geq 1} \mathbb{E}_{\mathbb{P}_t}(|\log(V_n(t))|^{2+\gamma}) < \infty. \tag{7.4}$$

Then, $\mathbb{P}_t(|\log(V_n(t))| \geq \frac{\sqrt{n\log\log(n)}}{n^{\gamma/(8+4\gamma)}}) = O(n^{-(1+\gamma/4)})$ and due to the Borel-Cantelli Lemma, (7.3) and (7.4) together imply (7.2).

Proof of (7.4): it is easily seen that

$$|\log(V_n(t))| \leq A_n(t) + B_n(t)$$

with

$$\begin{cases} A_n(t) = \sum_{M\in S\cap(B^{I_n(t)}\setminus \mathcal{C}_{b^{-n}}(t))} |\log W_M| + \log(\overline{w}) \\ B_n(t) = \sum_{M\in S\cap(\mathcal{C}_{b^{-n}}(t)\setminus T^{I_n(t)})} |\log W_M| + \log(\overline{w}). \end{cases}$$

The random variable $A_n(t)$ is independent of $\widehat{Q}_{b^{-n}}(t)$ because $B^{I_n(t)}\setminus \mathcal{C}_{b^{-n}}(t)$ and $\mathcal{C}_{b^{-n}}(t)$ are disjoint. Hence, defining $p_k = \mathbb{P}(\#S\cap(B^{I_n(t)}\setminus \mathcal{C}_{b^{-n}}(t)) = k)$ and $M_1,\dots,M_k$ the elements of $S\cap(B^{I_n(t)}\setminus \mathcal{C}_{b^{-n}}(t))$ conditionally on $\#S\cap(B^{I_n(t)}\setminus \mathcal{C}_{b^{-n}}(t)) = k$, we get

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_t}\big(A_n(t)^{2+\gamma}\big) &= \mathbb{E}\big(A_n(t)^{2+\gamma}\big) \\ &\leq \sum_{k=0}^{\infty} p_k(2k)^{1+\gamma}\sum_{i=1}^{k}\mathbb{E}(|\log W_{M_i}|^{2+\gamma}|) + (\log(\overline{w}))^{2+\gamma} \\ &\leq \sum_{k=0}^{\infty} p_k(2k)^{2+\gamma}\max\big(\mathbb{E}(|\log W|^{2+\gamma}|), (\log(\overline{w}))^{2+\gamma}\big). \end{aligned}$$

$\Lambda(B^{I_n(t)}\setminus \mathcal{C}_{b^{-n}}(t))$ being bounded independently of $n\geq 1$ and $\mathbb{E}(|\log W|^{2+\gamma}|)$ finite by assumption, it follows from the value of p_k that $\sup_{n\geq 1}\mathbb{E}_{\mathbb{P}_t}\big(A_n(t)^{2+\gamma}\big) < \infty$. It remains to show that $\sup_{n\geq 1}\mathbb{E}_{\mathbb{P}_t}\big(B_n(t)^{2+\gamma}\big) < \infty$. We have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_t}(B_n(t)^{2+\gamma}) &= \mathbb{E}(\widehat{Q}_{b^{-n}}(t)B_n(t)^{2+\gamma}) \\ &= C_n(t)D_n(t) \end{aligned}$$

where

$$C_n(t) = b^{-n\rho(V-1)}\mathbb{E}\left(\prod_{M\in S\cap T^{I_n(t)}} W_M\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)\right)$$

is bounded independently of n by the computations performed in the proof of Lemma 7.1(2), and

$$D_n(t) = \mathbb{E}\left(B_n(t)^{2+\gamma} \prod_{M\in S\cap(\mathcal{C}_{b^{-n}}(t)\setminus T^{I_n(t)})} W_M \widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)\right).$$

Let $p_k = \mathbb{P}(\#S\cap(\mathcal{C}_{b^{-n}}(t)\setminus T^{I_n(t)}) = k)$ and let $M_1,\dots,M_k$ be the elements of $S\cap(\mathcal{C}_{b^{-n}}(t)\setminus T^{I_n(t)})$ conditionally on $\#S\cap(\mathcal{C}_{b^{-n}}(t)\setminus T^{I_n(t)}) = k$. We have

$$\begin{aligned} &D_n(t)\\ \le\ & \sum_{k=0}^{\infty} p_k(2k)^{1+\gamma}\sum_{i=1}^{k}\mathbb{E}\Big(W_{M_i}\overline{w}\big[|\log W_{M_i}|^{2+\gamma}+(\log(\overline{w}))^{2+\gamma}\big]\Big)\prod_{\substack{j=1\\ j\neq i}}^{k}\mathbb{E}(W_{M_j})\overline{w}\\ \le\ & \sum_{k=0}^{\infty} p_k(2k)^{2+\gamma}\big(\mathbb{E}(W)\overline{w}\big)^{k-1}\max\big(\overline{w}\mathbb{E}(W|\log W|^{2+\gamma}),\overline{w}(\log(\overline{w}))^{2+\gamma}\mathbb{E}(W)\big).\end{aligned}$$

Since $\Lambda(\mathcal{C}_{b^{-n}}(t)\setminus T^{I_n(t)})$ is bounded independently of $n\ge 1$ and $\mathbb{E}(W|\log W|^{2+\gamma}) < \infty$, it follows that $\sup_{n\ge1} D_n(t)<\infty$. This yields $\sup_{n\ge 1}\mathbb{E}_{\mathbb{P}_t}\big(B_n(t)^{2+\gamma}\big)<\infty$.

Proof of Theorem 6.7. We already saw that it suffices to show that for every $\varepsilon>0$ and every nontrivial compact subinterval K of $\mathcal{J}$, there exists $\eta>0$ such that with probability one, for all $q\in\mathcal{J}$,

$$\sum_{n\ge 0} f_n(q)+\widehat{f}_n(q)<\infty$$

where

$$f_n(q) = b^{-n\eta(\tau(q)+\varepsilon)}\sum_{w\in A^n}\widetilde{\mu}(\mathcal{A}_w)^{-\eta q}\widetilde{\mu}_q(\mathcal{A}_w)^{1+\eta}.$$

and

$$\widehat{f}_n(q) = b^{-n\eta(\tau(q)+\varepsilon)}\sum_{\substack{v,w\in A^n\\ \delta(v,w)\le b'}}\mu(I_v)^{-\eta q}\mu_q(I_w)^{1+\eta}$$

with $b'=3$ if $q<0$ and $4b+2$ otherwise.

The approach here consists in directly taking into account the specificities of the particular construction we are dealing with. With the notations introduced in establishing Proposition 6.4, if $q_K=\max\{|q|:\ q\in K\}$, we have for every $q\in K$

$$f_n(q)\le b^{n(\tau(q)-q-\eta\varepsilon)}g_n(q)$$

with

$$g_n(q) = b^{-n\rho q(V-1)}\sum_{w\in A^n} M(K_{-\eta},w)M(K_{1+\eta},w)e^{q_K H(w)}\widehat{Q}(w)^q Z(w,1)^{-\eta q}Z(w,q)^{1+\eta}$$

and

$$\widehat{f}_n(q)\le b^{n(\tau(q)-q-\eta\varepsilon)}\widehat{g}_n(q)$$

with

$$\widehat{g}_n(q) = b^{-n\rho q(V-1)}\sum_{\substack{v,w\in A^n\\ \delta(v,w)\le b'}} C_{v,w}(\eta,1)e^{(1+2\eta)q_K H(v,w)}\widehat{Q}(v,w)^q Z(v,1)^{-\eta q}Z(w,q)^{1+\eta}.$$

Now, recall that $\mathbb{E}(W^r)<\infty$ for all $r\in\mathcal{J}$. Moreover, $\Lambda(B^{I_w})$ and $\Lambda(T^{I_w}\setminus(T^{I_w}\cap T^{I_v}))$ are uniformly bounded over $n\ge 1$ and those pairs $(v,w)\in(A^n)^2$

such that $\delta(v,w) \le b'$. Consequently, for η small enough, the expectations of $M(K_{-\eta},w)M(K_{1+\eta},w)$ and $C_{v,w}(\eta,1)$ are uniformly bounded over these pairs. Then, by taking into account the independence, the problem is reduced as in the proofs of Theorems 5.11 and 5.12 to showing that for η small enough there exists $C = C(K,\eta) > 0$ such that

$$\begin{cases} \sup_{q\in K} b^{n(\tau(q)-q-\eta\varepsilon)}\mathbb{E}(|h_n'(q)|) \le Cb^{-n\varepsilon\eta/2} \\ \sup_{q\in K} b^{n(\tau(q)-q-\eta\varepsilon)}\mathbb{E}(h_n(q)) \le Cb^{-n\varepsilon\eta/2} \end{cases}$$

for $h \in \{h_1, h_2\}$, where

$$h_1 : \ q \mapsto b^{-n\rho q(V-1)} \sum_{w\in A^n} e^{q_K H(w)} \widehat{Q}(w)^q$$

and

$$h_2 : \ q \mapsto b^{-n\rho q(V-1)} \sum_{\substack{v,w\in A^n \\ \delta(v,w)\le b'}} e^{(1+2\eta)q_K H(v,w)} \widehat{Q}(v,w)^q.$$

The computations being very similar to those already done above, they are left to the reader.

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