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Gibbs measures on self-affine Sierpinski carpets and their singularity spectrum

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Abstract. We consider a class of Gibbs measures on self-affine Sierpinski carpets and perform the multifractal analysis of its elements. These deterministic measures are Gibbs measures associated with bundle random dynamical systems defined on probability spaces whose geometrical structure plays a central rôle.

A special subclass of these measures is the class of multinomial measures on Sierpinski carpets. Our result improves the already known result concerning the multifractal nature of the elements of this subclass by considerably weakening and even eliminating in some cases a strong separation condition of geometrical nature.

1. *Introduction*

The singularity spectrum of a finite positive Borel measure on $\mathbb{R}^d$ is defined as the mapping

$$\alpha \geq 0 \mapsto \dim E_\mu(\alpha), \quad E_\mu(\alpha) = \left\{ t \in \operatorname{supp}(\mu) : \lim_{r \to 0^+} \frac{\log \mu\big(B(t,r)\big)}{\log r} = \alpha \right\},$$

where dim stands for the Hausdorff dimension. This function has been studied extensively for measures obtained as geometric realization of Gibbs measures defined on a symbolic space ([**5, 27, 4, 14, 22, 9, 24, 25, 12, 1**]). These measures possess a kind of self-similarity property. This paper deals with the case when the self-similarity is relaxed in self-affinity property and computes the singularity spectrum of a class of Gibbs measures on Sierpinski carpets.

Special elements of this class of measures are studied in [**17**] (and in [**22**] on Sierpinski sponges in $\mathbb{R}^d$). These measures are multinomial measures distributed

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on the (rectangular) cells of the carpet (see Section 1.3.1). Due to the self-affinity property of the carpet, the multifractal analysis of these measures meets the same difficulties as the computation of the Hausdorff dimension of the Sierpinski carpet ([**19, 2**]) and more general self-affine sets (see [**6, 7, 15, 10**] and references therein), and it is a delicate issue. Moreover, in [**17**] and [**22**] a rather strong separation condition is assumed in the construction of the carpet. Let us also mention that [**28**] studies the singularity spectrum for multinomial measures on more general self-affine sets: Fixing a probability vector $(p_i)_{1\le i\le m}$ and a family $(T_1,\dots,T_m)$ of linear contractions on $\mathbb{R}^n$ such that $\|T_i\|<\frac{1}{3}$, the authors obtain for almost all vectors $(a_1,\dots,a_m)\in\mathbb{R}^{nm}$ a part of the singularity spectrum of the unique probability measure μ on the attractor of the IFS $\{S_1=T_1+a_1,\dots,S_m=T_m+a_m\}$ satisfying $\mu=\sum_{i=1}^m p_i\mu\circ S_i^{-1}$. In [**8**], Gibbs measures including multinomial measures are considered on these attractors (under the weaker assumption $\|T_i\|<\frac{1}{2}$) and almost sure results are obtained for the generalized dimensions of these measures. It will be seen in Section 1.3.2 that when the attractor is a Sierpinski carpet, these measures form a subclass of the class studied in this paper.

Another special subclass of the set of Gibbs measures considered in [**8**] consists in self-affine generalized Riesz products on $[0,1]^2$: Let $W:\mathbb{R}^2\to\mathbb{R}_+^*$ be 1-periodic with respect to the first and second variable. Suppose that there exists $\alpha\in(0,1]$ and $C>0$ such that if $z,z'\in\mathbb{R}^2$ then $|W(z)-W(z')|\le C\|z-z'\|^\alpha$ ($\|\cdot\|$ being some norm). Let $2\le r_1\le r_2$ be two integers. Then, the Ruelle-Perron-Frobenius Theorem [**23**] applied for the dynamical system $\big((\mathbb{R}/\mathbb{Z})^2,\sigma\big)$, with $\sigma(x,y)=(\sigma_1(x)=r_1x \mod 1,\sigma_2(y)=r_2y \mod 1)$, and the potential $\log W$ ensures that the sequence of measures on $[0,1]^2$ defined by

$$\nu_n(dxdy)=\frac{\prod_{k=0}^{n-1}W(r_1^kx,r_2^ky)}{\int_{[0,1]^2}\prod_{k=0}^{n-1}W(r_1^kx',r_2^ky')\,dx'dy'}\,dxdy \tag{1.1}$$

converges weakly to a measure ν supported by $[0,1]^2$. The measure ν is a Gibbs measure associated with $\big((\mathbb{R}/\mathbb{Z})^2,\sigma\big)$ and the potential $\varphi(x,y)=\log W(x,y)$. We also use the terminology "generalized Riesz product" for ν by reference to the Riesz products in dimension 1, and also in order to underline the fact that these measures form a strict subclass of the objects we shall consider in this paper. If $r_1=r_2$, the measure ν possesses some self-similarity property and the singularity spectrum of such a measure is obtained by considering a family of auxiliary measures ν_q, $q\in\mathbb{R}$, obtained as follows ([**9, 25**]): ν_q is a Gibbs measure associated with the potential $q\varphi$ and the same dynamical system $\big((\mathbb{R}/\mathbb{Z})^2,\sigma\big)$, and it is supported by the singularity set $E_\nu(\alpha(q))$, where $\alpha(q)=(P(1)-P'(q))/\log(r_1)$, $P(q)$ being the topological pressure of $q\varphi$; moreover, the Hausdorff dimension of ν_q is that of the set $E_\nu(\alpha(q))$. In this case the auxiliary measures are exactly of the same nature as ν. When $r_1<r_2$, the measure ν possesses a self-affine rather than self-similar property, and the situation is subtler. This cannot be perceived immediately for self-affine multinomial measures supported by $[0,1]^2$ (and more generally by a Sierpinski carpet) because their multifractal analysis is performed by using a family of auxiliary measures ν_q constructed exactly in the same way. For self-affine

generalized Riesz products, it turns out that computing their singularity spectrum leads us to adopt the following point of view. Let ν be a self-affine generalized Riesz product as constructed above. The projection of ν on the first axis is equivalent to a ergodic measure $\mathbb{P}$ on $(\Omega = \mathbb{R}/\mathbb{Z}, \sigma_1)$, and ν is equivalent to a Gibbs measure associated with the dynamical system $(\mathbb{R}/\mathbb{Z}, \sigma_2)$, considered as a random bundle, on the probability space $(\Omega, \mathbb{P})$ and the random potential $\varphi(x, \cdot)$, in the sense of [**3, 16, 13**]. Then, for $q \in \mathbb{R}$, the auxiliary measure ν_q involved in the multifractal analysis of ν is a Gibbs measure associated with the potential $q\varphi(x, \cdot)$, but with a different random dynamical system, in the sense that the probability $\mathbb{P}$ is replaced by another one $\mathbb{P}_q$, in such a way that the random dynamical systems involved in the problem are all of the same nature. A remarkable fact is that a central rôle is played by the geometric properties of the measures $\mathbb{P}_q$. Indeed, they possess the so-called quasi-Bernoulli property (see (1.3) and Section 2.2). Thus, our study provides a frame in which quasi-Bernoulli measures are naturally involved and generated. A natural way to obtain such measures on (Ω, σ_1) is to consider Gibbs measures on (Ω, σ_1). But there is no obvious argument ensuring that conversely a quasi-Bernoulli measure like $\mathbb{P}$ or $\mathbb{P}_q$ is equivalent to a Gibbs measure on (Ω, σ_1). Consequently, since we have no way to prove that $\mathbb{P}_q$ is a Gibbs measure, contrarily to what happens when $r_1 = r_2$, the measure ν_q on $[0, 1]^2$ cannot be obtained as a generalized Riesz product (see Section 1.3.3 for more details) and belongs to an a priori larger class of Gibbs measures.

This class forms a subset of the Gibbs measures on general Sierpinski carpets rather than only on $[0, 1]^2$ considered in the sequel. Let μ be such a Gibbs measure. The singularity spectrum of μ will be obtained as the Legendre transform of some function β (which, up to an affine transformation, coincides with a topological pressure function when $r_1 = r_2$). While β is analytic for a self-affine multinomial measure, another delicate point in this paper is to establish the differentiability of this function in the general case. This uses some ideas from [**11**].

Also, while (as we already mentioned) in [**17, 22**] a rather restrictive separation hypothesis is assumed in the Sierpinski carpet construction for the computation of the singularity spectrum of self-affine multinomial measures, our results improve those established in [**17, 22**] by assuming a considerably weaker assumption (see Remark 1.4 and Theorem 1.2). More precisely, without assuming any restriction in the carpet construction we determine the whole decreasing part of the singularity spectrum as well as a lower bound for the increasing part which is sharp under our weak technical assumption.

For Gibbs measures associated with potentials satisfying some Dini and periodicity conditions, our results hold without any geometrical assumption (see Corollary 1.1).

Let us now introduce some definitions and notations. Then, in Sections 1.1 and 1.2 Gibbs measures on the product of two symbolic spaces will be defined as well as their natural projection on a Sierpinski carpet. Section 1.3 details the special examples mentioned above. Eventually, Section 1.4 provides our main results, and the rest of the paper is devoted to the proof of these results.

Definitions and notations.

Let $2 \le r_1 < r_2$ be two integers. For $i \in \{1,2\}$ let A_i denote the set $\{0,\dots,r_i-1\}$. Let $A_i^* = \bigcup_{k\ge 0} A_i^k$ ($A_i^0 = \{\emptyset\}$) and let $\mathbb{A}_i$ denote the symbolic space $A_i^{\mathbb{N}^*}$. The length of an element w of $A_i^* \cup \mathbb{A}_i$ is denoted $|w|$. The set $A_i^* \cup \mathbb{A}_i$ is endowed with the concatenation operation: if $w \in A_i^*$ and $w' \in A_i^* \cup \mathbb{A}_i$, then $w \cdot w'$ denotes the word obtained by juxtaposition of w and w'.

Then, for $w \in A_i^*$, $[w]$ stands for the cylinder $w \cdot \mathbb{A}_i = \{w \cdot w',\ w' \in \mathbb{A}_i\}$.

If $z = z_1 z_2 \cdots z_p \cdots \in \mathbb{A}_i$ and $n \in \mathbb{N}$ then $z|n$ stands for the prefix $z_1 \cdots z_n$ of z if $n \ge 1$ and the empty word otherwise.

For $z, z' \in A_i^* \cup \mathbb{A}_i$, let $z \wedge z'$ stands for the word u of maximal length in $A_i^* \cup \mathbb{A}_i$ such that u is a prefix of z and z'.

The set $\mathbb{A}_i$ is endowed with the ultrametric distance $d_i : (z,z') \in \mathbb{A}_i^2 \mapsto r_i^{-|z\wedge z'|}$.

Let σ_i stand for the shift transformation on $\mathbb{A}_i$ and denote by σ the transformation (σ_1, σ_2) on $\mathbb{A}_1 \times \mathbb{A}_2$.

The product $\mathbb{A}_1 \times \mathbb{A}_2$ is endowed with the ultrametric distance

$$d\big((x,y),(x',y')\big) = \max\big(d_1(x,x'), d_2(y,y')\big).$$

For every $n \ge 1$, let $\mathcal{F}_n$ be the set of balls of radius r_2^{-n} in $(\mathbb{A}_1 \times \mathbb{A}_2, d)$. Let $g(n)$ be the smallest integer m such that $r_1^{-m} \le r_2^{-n}$. It is easy to see that

$$\mathcal{F}_n = \Big\{[w_1 \cdot \widetilde{w}_1] \times [w_2] : (w_1, \widetilde{w}_1, w_2) \in A_1^n \times A_1^{g(n)-n} \times A_2^n\Big\}. \tag{1.2}$$

1.1. *Construction of Gibbs measures on $\mathbb{A}_1 \times \mathbb{A}_2$.*

Let A be a non-empty subset of $A_1 \times A_2$ and define $[A] = \{[i] \times [j] : (i,j) \in A\}$.

Then define the compact subset K of $\mathbb{A}_1 \times \mathbb{A}_2$ by $K = \bigcap_{n\ge 0} \sigma^{-n}([A])$.

Let $\varphi : \mathbb{A}_1 \times \mathbb{A}_2 \to \mathbb{R} \cup \{-\infty\}$ be a function such that $K = \{(x,y) \in \mathbb{A}_1 \times \mathbb{A}_2 : \varphi(x,y) > -\infty\}$. We assume that

(H1) φ satisfies the Dini condition

$$\int_{[0,1]} \sup_{\substack{z,z' \in K \\ d(z,z') \le r}} |\varphi(z) - \varphi(z')| \, \frac{dr}{r} < \infty.$$

Then, let $\widetilde{A}_1 = \{i \in A_1 :\ \exists\, j \in A_2,\ (i,j) \in A\}$ and $\widetilde{A}_2 = \{j \in A_2 :\ \exists\, i \in A_1,\ (i,j) \in A\}$.

In order to avoid trivial cases in the sequel, we assume

(H2) $\min(\#\, \widetilde{A}_1, \#\, \widetilde{A}_2) \ge 2$, where $\#S$ denotes the cardinality of the set S. We set $\widetilde{r}_1 = \#\widetilde{A}_1$.

For $i \in \{1,2\}$ let $\widetilde{A}_i^* = \bigcup_{k\ge 0} \widetilde{A}_i^k$ and let $\widetilde{\mathbb{A}}_i$ denote the symbolic space $\widetilde{A}_i^{\mathbb{N}^*}$.

We have

$$K \subset \widetilde{\mathbb{A}}_1 \times \widetilde{\mathbb{A}}_2,$$

and the sets $\widetilde{\mathbb{A}}_1$ and $\widetilde{\mathbb{A}}_2$ are the projections of K on $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively.

From now on, the space $(\widetilde{\mathbb{A}}_1, \sigma_1)$ plays a particular rôle, but we explain in Remark 1.3 that favoring $(\widetilde{\mathbb{A}}_2, \sigma_2)$ yields the same result.

Let $\mathbb{P}$ be a ergodic probability measure on $(\widetilde{\mathbb{A}}_1, \sigma_1)$ and suppose that $\mathbb{P}$ obeys the quasi-Bernoulli property ([**4**]):

(H3) There exists $C > 0$ such that for every $n, p \geq 1$, for every $(w_1, \widetilde{w}_1), \in \widetilde{A}_1^n \times \widetilde{A}_1^p$ one has

$$C^{-1} \leq \frac{\mathbb{P}\big([w_1 \cdot \widetilde{w}_1]\big)}{\mathbb{P}\big([w_1]\big)\mathbb{P}\big([\widetilde{w}_1]\big)} \leq C. \tag{1.3}$$

Let ℓ_i stand for the Haar measure on the compact set $\widetilde{\mathbb{A}}_i$ considered with its natural structure of additive group.

For $i \in \widetilde{A}_1$, let $\widetilde{A}_2(i)$ stand for $\{j \in A_2 : \varphi_{|[i]\times[j]} \not\equiv -\infty\}$. Then for $x = x_1 x_2 \cdots x_p \cdots \in \widetilde{\mathbb{A}}_1$ and $n \geq 1$ let

$$K_x = \{y \in \widetilde{\mathbb{A}}_2 : (x, y) \in K\} \quad \text{and} \quad \widetilde{A}_2(x|n) = \prod_{k=1}^{n} \widetilde{A}_2(x_k).$$

By construction

$$K_x = \bigcap_{n\geq 1} K_x^n, \quad \text{with } K_x^n = \bigcup_{w_2 \in \widetilde{A}_2(x|n)} [w_2].$$

In particular, K_x is a compact set. We then denote by $\ell_{2,x}$ the (unique) "branching" measure on K_x such that

$$\ell_{2,x}([w_2] \cap K_x) = \prod_{k=1}^{n} \big(\# \widetilde{A}_2(x_k)\big)^{-1}, \quad n \geq 1,\ w_2 \in \widetilde{A}_2^n(x|n).$$

REMARK 1.1. *If $K = \widetilde{\mathbb{A}}_1 \times \widetilde{\mathbb{A}}_2$ then $K_x = \widetilde{\mathbb{A}}_2$ and $\ell_{2,x} = \ell_2$ for all $x \in \widetilde{\mathbb{A}}_1$.*

DEFINITION 1.1. *For $n \geq 1$ and $(x, y) \in \mathbb{A}_1 \times \mathbb{A}_2$, let*

$$S_n\varphi(x, y) = \begin{cases} \sum_{k=0}^{n-1} \varphi\big(\sigma_1^k \cdot x, \sigma_2^k \cdot y\big) & \text{if } (x, y) \in K \\ -\infty & \text{otherwise} \end{cases}.$$

Also, for $n \geq 1$ and $x \in \widetilde{\mathbb{A}}_1$ define on K_x the measure

$$\mu_n^x(dy) = \frac{\exp\big(S_n\varphi(x, y)\big)}{\int_{K_x} \exp\big(S_n\varphi(x, u)\big)\, \ell_{2,x}(du)}\, \ell_{2,x}(dy).$$

Then define on K the measure

$$\mu_n(dx, dy) = \mathbb{P}(dx)\, \mu_n^x(dy) \tag{1.4}$$

and denote by $\mathcal{M}$ the set of weak limits of subsequences of $(\mu_n)_{n\geq 1}$.

We shall relate $\mathcal{M}$ to the concept of Gibbs measure, and then describe the multifractal nature of the elements of $\mathcal{M}$. The following proposition, which is a simple consequence of (2.4) in the proof of Lemma 2.2, shows that all the elements of $\mathcal{M}$ have the same multifractal nature.

PROPOSITION 1.1. *There exists $C > 0$ such that for every $\mu, \nu \in \mathcal{M}$ and Borel set E in K one has $C^{-1}\nu(E) \leq \mu(E) \leq C\nu(E)$.*

Before considering multifractal analysis, let us examine sufficient conditions on the set A for $\mathcal{M}$ to be a singleton (Proposition 1.2) and relate this property to the notion of Gibbs state. This uses the Ruelle-Perron-Fröbenius theorem established in [**3**] and requires to use double-ended infinite words on $\widetilde{A}_1$ to get an invertible shift operation.

Let $\overline{\mathbb{A}}_1 = \widetilde{A}_1^{\mathbb{Z}}$, let $\overline{\sigma}_1$ be the extension of σ_1 to $\overline{\mathbb{A}}_1$ and let $\overline{\sigma}$ be the transformation $(\overline{\sigma}_1, \sigma_2)$ on $\overline{\mathbb{A}}_1 \times \mathbb{A}_2$. If $\overline{x} = \cdots x_{-1}x_0x_1\cdots \in \overline{\mathbb{A}}_1$, we set $x_+ = x_1\cdots$, $\overline{K_{\overline{x}}} = K_{x_+}$ and $\overline{\ell}_{2,\overline{x}} = \ell_{2,x_+}$, and if $y \in \mathbb{A}_2$ we set $\overline{\varphi}(\overline{x}, y) = \varphi(x_+, y)$. Also, we set $\overline{K} = \{(\overline{x}, y) \in \overline{\mathbb{A}}_1 \times \mathbb{A}_2 : (x_+, y) \in K\}$.

If $i \in \widetilde{A}_1$, let $m(i) = \min_{j \in \widetilde{A}_2(i)} j$ and $M(i) = \max_{j \in \widetilde{A}_2(i)} j$. Then define the random transition matrix

$$B(\overline{x}) = \left(\mathbf{1}_{\widetilde{A}_2(x_1)}(j_1)\mathbf{1}_{\widetilde{A}_2(x_2)}(j_2)\right)_{\substack{j_1 \in \widetilde{A}_2 \cap [m(x_1), M(x_1)] \\ j_2 \in \widetilde{A}_2 \cap [m(x_2), M(x_2)]}} \quad (\overline{x} = \cdots x_{-1}x_0x_1x_2\cdots \in \overline{\mathbb{A}}_1)$$

By construction

$$\overline{K_{\overline{x}}} = \left\{y \in \widetilde{\mathbb{A}}_2 : B_{y_k, y_{k+1}}(\overline{\sigma}_1^{(k-1)}\overline{x}) = 1 \ \forall\ k \geq 1\right\}. \tag{1.5}$$

We denote by $\overline{\mathbb{P}}$ the ergodic extension of $\mathbb{P}$ to $\overline{\mathbb{A}}_1$, which we also denote by Ω.

In the setting of [**3**], the set $\overline{K}$ is a compact bundle over Ω with fibers (the sets $\overline{K_{\overline{x}}}$) in $\widetilde{\mathbb{A}}_2$. For $\overline{x} \in \Omega$ let $\phi(\overline{x})$ be the restriction to $\overline{K_{\overline{x}}}$ of the function σ_2. The map $\phi(\overline{x})$ is continuous from $\overline{K_{\overline{x}}}$ to $\overline{K_{\overline{\sigma}_1\overline{x}}}$. It follows that in the setting of [**3**] the functions $\phi(\overline{x})$ define a bundle random dynamical system on $\Omega \times \widetilde{\mathbb{A}}_2$. Moreover, due to (1.5), the maps $\phi(\overline{x})$ and the matrices $B(\overline{x})$ define a random subshift of finite type. Let us introduce the following assumption on the set A:

$$\forall i \in \widetilde{A}_1,\ \widetilde{A}_2(i) = \widetilde{A}_2 \cap [m(i), M(i)]. \tag{1.6}$$

REMARK 1.2. *Instead of considering random subshifts defined with random alphabets of the form $\{1, \ldots, l(\omega)\}$ inside $\mathbb{N}^*$ (that is made of the $l(\omega)$ first positive integers) as in [**3**] and [**13**], we work with the random alphabets $\widetilde{A}_2 \cap [m(x_1), M(x_1)]$ made of integers belonging to $\widetilde{A}_2$.*

*It is easily seen that (1.6) is the necessary and sufficient condition for $B(\overline{x})$ to satisfy the aperiodicity condition of [**13**] (which weakens that of [**3**]): For $\overline{\mathbb{P}}$-almost every $\overline{x} \in \overline{\mathbb{A}}$, there exists $N(\overline{x}) \geq 1$ such that all the entries of $B(\overline{x}) \cdots B(\overline{\sigma}_1^{N(\overline{x})-1}\overline{x})$ are positive. Moreover, under (1.6), we can take $N(\overline{x}) = 1$.*

Property (1.6) obviously holds if $K = \widetilde{\mathbb{A}}_1 \times \widetilde{\mathbb{A}}_2$. This is the case for generalized Riesz products considered in Section 1.

The following result is then a consequence of the random tranfer operator theorem obtained in [**3**] (Theorem 2.3 (iv)) for the random Perron-Fröbenius operator from $C(\overline{K}_{\overline{x}})$ (the space of continuous functions on $\overline{K}_{\overline{x}}$) to $C(\overline{K}_{\overline{\sigma}_1\overline{x}})$ defined for $\overline{x} \in \Omega$ by

$$\mathcal{L}_{\overline{\varphi}}^{\overline{x}}(g) = y \mapsto \sum_{y' \in \overline{K}_{\overline{x}}:\ \phi(\overline{x})(y')=y} e^{\overline{\varphi}(\overline{x},y')} g(y').$$

PROPOSITION 1.2. *Assume that* (1.6) *holds and that* φ *is a Hölder function, i.e. there exists* $\alpha \in (0,1]$ *and* $C > 0$ *such that if* $z, z' \in K$ *then* $|\varphi(z) - \varphi(z')| \leq Cd(z,z')^\alpha$. *Then, for* $\mathbb{P}$*-almost every* x, *the measures* μ_n^x *converge weakly to a probability measure* μ^x *on* K_x *as* n *tends to infinity. Consequently, the sequence of measures* μ_n *converges weakly to the measure* $\mathbb{P}(dx)\mu^x(dy)$ *as* n *goes to infinity and* $\mathcal{M}$ *is a singleton.*

Proof. For $\overline{x} \in \widetilde{\mathbb{A}}_1$ and $n \geq 1$ let $\mathcal{L}_{\overline{\varphi}}^{\overline{x},n} = \mathcal{L}_{\overline{\varphi}}^{\overline{\sigma}_1^{n-1}\cdot\overline{x}} \circ \cdots \circ \mathcal{L}_{\overline{\varphi}}^{\overline{\sigma}_1\cdot\overline{x}} \circ \mathcal{L}_{\overline{\varphi}}^{\overline{x}}$. An elementaty computation shows that if $f \in C(K_{x_+})$ then

$$\int_{K_{x_+}} f(y)\mu_n^x(dy) = \frac{\int_{\overline{K}_{\overline{x}}} \mathcal{L}_{\overline{\varphi}}^{\overline{x},n} f(u_+)\overline{\ell}_{2,\overline{x}}(d\overline{u})}{\int_{\overline{K}_{\overline{x}}} \mathcal{L}_{\overline{\varphi}}^{\overline{x},n} \mathbf{1}(\overline{u})\overline{\ell}_{2,\overline{x}}(d\overline{u})},$$

where $\mathbf{1}(\cdot)$ stands for the function identically equal to 1. The assumptions $C1$, $C2$ (slightly weakend here) and $C3$ of Section 2.5 in [**3**] are fullfilled by the random potential $\overline{\varphi}(\overline{x},\cdot)$ and the random matrix $S(\overline{x})$ (respectively denoted by $\phi(\omega)$ and $A(\omega)$ in [**3**]). Thus, due to [**3**] (Theorem 2.3 (iv)), for $\overline{\mathbb{P}}$-almost every $\overline{x} \in \overline{\mathbb{A}}_1$, for every $f \in C(K_{x_+})$, the sequence $\frac{\int_{\overline{K}_{\overline{x}}} \mathcal{L}_{\overline{\varphi}}^{\overline{x},n} f(u_+)\overline{\ell}_{2,\overline{x}}(d\overline{u})}{\int_{\overline{K}_{\overline{x}}} \mathcal{L}_{\overline{\varphi}}^{\overline{x},n} \mathbf{1}(\overline{u})\overline{\ell}_{2,\overline{x}}(d\overline{u})}$ converges. Since the limit depends only on x_+, if follows that μ_n^x converges weakly for $\mathbb{P}$-almost every x. The weak convergence of μ_n to $\mu_{\mathbb{P},\varphi}$ is then immediate. □

If (1.6) holds and φ is a Hölder function, it is $\overline{\mathbb{P}}$-almost sure that the measure $\mu_n^{x_+}$ converges weakly to a measure μ^{x_+} (see the proof below). Then, the measure $\mathbb{P}(d\overline{x})\mu^{x_+}(dy)$ is a Gibbs measure on $\overline{K}$ in the sense of [**3**]. By extension, we call $\mathbb{P}(dx)\mu^x(dy)$ a Gibbs measure on K, as well as any element of $\mathcal{M}$ even if (1.6) does not hold. Thus:

- We fix an element $\mu_{\mathbb{P},\varphi}$ of $\mathcal{M}$ and denote by μ the extention of $\mu_{\mathbb{P},\varphi}$ to the Borel subsets E of $\mathbb{A}_1 \times \mathbb{A}_2$ defined by $\mu(E) = \mu_{\mathbb{P},\varphi}(E \cap K)$.

- We denote also by $\mathbb{P}$ the extension of $\mathbb{P}$ to the Borel subsets B of $\mathbb{A}_1$ defined by $\mathbb{P}(B) = \mathbb{P}(B \cap \widetilde{\mathbb{A}}_1)$.

REMARK 1.3. *By analogy with the construction of the previous bundle random dynamical system, for* $j \in \widetilde{A}_2$ *let* $\widetilde{A}_1(j)$ *stand for* $\{i \in A_1 :\ \varphi_{|[i]\times[j]} \not\equiv -\infty\}$. *Then for* $y = y_1y_2\cdots y_p\cdots \in \widetilde{\mathbb{A}}_2$ *and* $n \geq 1$ *let* $K_y = \{x \in \widetilde{\mathbb{A}}_1 : (x,y) \in K\}$ *and* $\widetilde{A}_1(y|n) = \prod_{k=1}^n \widetilde{A}_1(y_k)$. *Also let* $\ell_{1,y}$ *be the unique measure on* K_y *such that* $\ell_{1,y}([w_1] \cap K_y) = \prod_{k=1}^n \big(\#\widetilde{A}_1(y_k)\big)^{-1}$ *for all* $n \geq 1$ *and* $w_1 \in \widetilde{A}_1^n(y|n)$. *Then for*

$n \geq 1$ and $y \in \widetilde{\mathbb{A}}_2$ define on K_y the measure

$$\mu_n^y(dx) = \frac{\exp\big(S_n\varphi(x,y)\big)}{\int_{K_y} \exp\big(S_n\varphi(u,y)\big)\,\ell_{1,y}(du)}\,\ell_{1,y}(dx),$$

Then, considering the measure $\mathbb{P}$ on $(\widetilde{\mathbb{A}}_1, \sigma_1)$ is equivalent to chosing it on $(\widetilde{\mathbb{A}}_2, \sigma_2)$ if (1.3) holds. Indeed, computations similar to those done in Section 2.1 show that:

*(i) the projection $\mathbb{P}_2$ of μ on $\widetilde{\mathbb{A}}_2$ is quasi-Bernoulli (and thus equivalent to a ergodic quasi-Bernoulli measure (see [**11**] for instance));*

(ii) Any weak limit of the sequence $\mathbb{P}_2(dy)\mu_n^y(dx)$ is equivalent to μ.

(iii) A sufficient condition for $\mathbb{P}_2(dy)\mu_n^y(dx)$ to weakly converge is that $\widetilde{A}_1(j) = \widetilde{A}_1 \cap \Big[\min_{i\in\widetilde{A}_1(j)} i,\ \max_{i\in\widetilde{A}_1(j)} i\Big]$ for all $j \in \widetilde{A}_2$.

1.2. *Gibbs measures on the Sierpinski carpet.*

Let μ be the extension to $\mathbb{A}_1 \times \mathbb{A}_2$ of the Gibbs measure $\mu_{\mathbb{P},\varphi}$ considered after the statement of Proposition 1.2. Let

$$\pi_i : z \in \mathbb{A}_i \mapsto \sum_{k\geq 1} z_k r_i^{-k} \quad (i \in \{1,2\}) \text{ and } \pi = (\pi_1, \pi_2).$$

The measure $\widetilde{\mu} = \mu \circ \pi^{-1}$ is the natural projection of μ on $[0,1]^2$ and its support is the Sierpinski carpet $\pi(K)$. The set $\pi(K)$ is also the attractor of the iterated function system composed by the affine transformations

$$f_{i,j} : (x,y) \mapsto (ir_1^{-1} + r_1^{-1}x, jr_2^{-1} + r_2^{-1}y), \quad (i,j) \in A.$$

The measure $\widetilde{\mu}$ is called a Gibbs measure on the Sierpinski carpet $\pi(K)$.

1.3. *Basic examples.*

1.3.1. *Self-affine multinomial measures.* This corresponds to the measures considered in [**19, 2, 17, 22**] which are obtained by taking φ constant equal to a value $\varphi_{i,j}$ over each product $K \cap [i] \times [j]$ ($(i,j) \in A$) and $\mathbb{P}$ the Bernoulli measure such that $\mathbb{P}([i]) = \dfrac{\sum_{j\in\widetilde{A}_2(i)} \exp\varphi_{i,j}}{\sum_{i'\in\widetilde{A}_1}\sum_{j\in\widetilde{A}_2(i')} \exp\varphi_{i',j}}$ for $i \in \widetilde{A}_1$. In this example, $\mathcal{M}$ is reduced to one point even when (1.6) does not hold.

1.3.2. *Self-affine generalized Riesz products – Gibbs measures of [**8**].* Let ν be a generalized Riesz product as constructed in the introduction. Computations similar to those performed in the proof of Proposition 2.1 show that the projection ρ of ν on the first axis is equivalent to the image by π_1 of an ergodic quasi-Bernoulli measure $\mathbb{P}$. As a result the measure ν is equivalent to the projection of the measure $\mu_{\mathbb{P},\varphi}$ defined in the previous section, where $\varphi = \log W \circ (\pi_1, \pi_2)$. Recall that in this case the support of ν is $[0,1]^2$ and that of $\mu_{\mathbb{P},\varphi}$ is $\mathbb{A}_1 \times \mathbb{A}_2$ (in particular (1.6) holds and $\mathcal{M}$ is reduced to one point).

In [**8**], the more general following construction is considered, which is also a special case of our setting.

Let φ as in Section 1.1. Let $m = \#A$. There is a natural homeomorphism h between the symbolic space $I_\infty = \{1,\dots,m\}^{\mathbb{N}^*}$ endowed with the shift operation s and the set $(K, \sigma_{|K})$, such that $h \circ s = \sigma \circ h$. A Gibbs measure ν on (I_∞, s) can be associated with the potential $\varphi \circ h$, and in our setting the measure considered in [**8**] on the carpet $\pi(K)$ is the measure $\widetilde{\mu} = \mu \circ \pi^{-1}$, where $\mu := \nu \circ h^{-1}$. Here again, it is not difficult to see, using computations similar to those used for the proof of Proposition 2.1, that the projection ρ of μ on $\widetilde{A}_1$ is equivalent to an ergodic quasi-Bernoulli measure $\mathbb{P}$ and that μ is equivalent to the measure $\mu_{\mathbb{P},\varphi}$.

1.3.3. *Comment.* In each example, the multifractal analysis of the measure μ requires us to consider a family $\{\mu_q\}_{q\in\mathbb{R}}$ of auxiliary measures. For multinomial measures μ, each μ_q is itself multinomial. If ν is a generalized Riesz product associated with the function W then ν_q takes the form $\widetilde{\mu}_q$, where $\mu_q = \mu_{\mathbb{P}_q,\varphi_q}$ for some quasi-Bernoulli and ergodic measure $\mathbb{P}_q$ and the potential $\varphi_q = q \log W \circ (\pi_1, \pi_2)$ (see Section 2.2). If we knew that any quasi-Bernoulli measure is equivalent to a Gibbs measure, μ_q could be obtained as a generalized Riesz product. Indeed, since the quantity $I_{q,n}$ introduced below in Section 1.4 also possesses a quasi-Bernoulli structure (Lemma 2.1), we see on the Definitions 1.1 and 2.1 that there would exist a 1-periodic potential ψ_q on $[0,1]$ such that $\mathbb{P}_q([x|n])/I_{q,n}(x|n) \approx \exp(\sum_{k=0}^{n-1} \psi_q(\pi_1(\sigma_1^k x))$. Then, $\nu_q = \widetilde{\mu}_q$ would be the generalized Riesz product associated with W_q such that $W_q(x,y) = \exp(\psi_q(x))\, W(x,y)^q$. This expression strongly differs from the case $r_1 = r_2$ [**4, 9, 25**] for which the term ψ_q vanishes. The same remarks hold for the Gibbs measures considered in [**8**].

1.4. *Main results.*

The measure μ and its projection $\widetilde{\mu}$ are respectively defined as at the end of Section 1.1 and as in Section 1.2.

Let $s = \log(r_1)/\log(r_2)$. For $n \geq 1$, $w_1 \in \widetilde{A}_1^n$ and $q \in \mathbb{R}$, let

$$I_{q,n}(w_1) = \sum_{w_2 \in \widetilde{A}_2^n(w_1)} \sup_{(x,y)\in[w_1]\times[w_2]\cap K} \exp\big(qS_n\varphi(x,y)\big). \tag{1.7}$$

Then define

$$\rho_{q,n}([w_1]) = \mathbb{P}([w_1])^q \left(\frac{I_{q,n}(w_1)}{I_{1,n}(w_1)^q}\right)^s, \tag{1.8}$$

$$\beta_{\mu,n}(q) = -\frac{1}{n}\log_{r_1} \sum_{w_1 \in \widetilde{A}_1^n} \rho_{q,n}([w_1])$$

and

$$\beta_\mu(q) = \liminf_{n\to\infty} \beta_{\mu,n}(q).$$

For $f:\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$, the Legendre transform of f is defined by $f^*:\alpha\geq 0\mapsto \inf_{q\in\mathbb{R}}\alpha q-f(q)$.

Theorem 1.1. *(Singularity spectrum of μ)*

(i) The concave function β_μ is differentiable and non decreasing.

(ii) For every $\alpha\in\mathbb{R}_+$, one has $\dim E_\mu(\alpha)=\beta_\mu^*(\alpha)$ *if $\beta_\mu^*(\alpha)>0$ and $E_\mu(\alpha)=\emptyset$ if $\beta_\mu^*(\alpha)<0$.*

Let us introduce on the set A three types of properties, respectively denoted **(G1)**, **(G2)** and **(G3)**. Property **(G1)** is a strong separation condition of geometrical nature which weakens that of [**17, 22**] (see Remark 1.4 below). Property **(G2)** is a weak separation condition of geometrical nature, but neither **(G1)** implies **(G2)** nor **(G2)** implies **(G1)**. Property **(G3)** is a king of weak periodicity condition on the potential φ and the Sierpinski carpet, and it excludes **(G2)**.

(G1) $|i-i'|\geq 2$ for every pair (i,i') of distinct elements of $\widetilde{A}_1$.

(G2) $\{0,r_1-1\}\cap(A_1\setminus\widetilde{A}_1)\neq\emptyset$.

(G3) $\{0,r_1-1\}\subset\widetilde{A}_1$ and for all $q>0$,

$$\lim_{n\to\infty}\frac{1}{n}\log I_{q,n}(0^{\cdot n})=\lim_{n\to\infty}\frac{1}{n}\log I_{q,n}((r_1-1)^{\cdot n}),\tag{1.9}$$

where for $j\in A_i$ and $n\geq 1$, $j^{\cdot n}$ stands for the word of length n whose letters are all equal to j (the limit exist in (1.9) thanks to Lemma 2.1).

Theorem 1.2. *(Singularity spectrum of $\widetilde{\mu}$)*

(i) (Lower bound) For every $\alpha\in\mathbb{R}_+$ such that $\beta_\mu^*(\alpha)>0$, one has $\dim E_{\widetilde{\mu}}(\alpha)\geq\beta_\mu^*(\alpha)$.

(ii) (Upper bound: Case $\alpha\geq\beta_\mu'(0)$). If $\alpha\geq\beta_\mu'(0)$ then $\dim E_{\widetilde{\mu}}(\alpha)\leq\beta_\mu^*(\alpha)$ *and $E_{\widetilde{\mu}}(\alpha)=\emptyset$ if $\beta_\mu^*(\alpha)<0$.*

(iii) (Upper bound: Case $0\leq\alpha<\beta_\mu'(0)$). Suppose that one of the properties **(G1)**, **(G2)** *or* **(G3)** *holds.*

If $0\leq\alpha<\beta_\mu'(0)$ then $\dim E_{\widetilde{\mu}}(\alpha)\leq\beta_\mu^*(\alpha)$ *and $E_{\widetilde{\mu}}(\alpha)=\emptyset$ if $\beta_\mu^*(\alpha)<0$.*

The following corollary shows that our work yields the singularity spectrum of Gibbs measures directly constructed on a Sierpinski carpet in the torus $(\mathbb{R}/\mathbb{Z})^2$ (like generalized Riesz products) without particular geometrical assumptions.

Corollary 1.1. *Let $W:\mathbb{R}^2\to\mathbb{R}_+$. Suppose that $\pi(K)=\{z\in[0,1]^2:W(z)>0\}$ and W is 1-periodic with respect to the first and second variables. Suppose also that*

$$\int_{[0,1]}\sup_{\substack{z,z'\in\pi(K)\cup(\pi(K)+(-1,0))\\ \|z'-z\|\leq r}}\left|\log\frac{W(z')}{W(z)}\right|\frac{dr}{r}<\infty.$$

If φ is taken to be equal to $\widetilde{\varphi}\circ\pi$ over K and $-\infty$ elsewhere, then for every $\alpha\in\mathbb{R}_+$, one has $\dim E_{\widetilde{\mu}}(\alpha)=\beta_\mu^*(\alpha)$ *if $\beta_\mu^*(\alpha)>0$ and $E_{\widetilde{\mu}}(\alpha)=\emptyset$ if $\beta_\mu^*(\alpha)<0$.*

We leave the reader verify that this result follows from Theorem 1.2 and the fact that either **(G2)** or **(G3)** holds due to the periodicity and Dini assumptions.

REMARK 1.4. *In the case of Example 1.3.1, i.e. self-affine multinomial measures, the function β_μ takes the simple analytic form*

$$\beta_\mu(q) = -\log_{r_1} \sum_{i\in\widetilde{A}_1} \left[\frac{\sum_{j\in\widetilde{A}_2(i)} \exp\varphi_{i,j}}{\sum_{i'\in\widetilde{A}_1}\sum_{j\in\widetilde{A}_2(i')} \exp\varphi_{i',j}}\right]^q \left[\frac{\sum_{j\in\widetilde{A}_2(i)} \exp q\varphi_{i,j}}{\left(\sum_{j\in\widetilde{A}_2(i)} \exp\varphi_{i,j}\right)^q}\right]^s$$

*obtained in [**17**].*

*The strong separation assumption considered in [**17, 22**] requires that for every pair (i,i') of distinct elements of $\widetilde{A}_1$ one has $|i-i'|\geq 2$, and if $i\in\widetilde{A}_1$ then if j and j' are two distinct elements of $\widetilde{A}_2(i)$) one has $|j-j'|\geq 2$. Thus property* **(G1)** *weakens this assumption and property* **(G2)** *describes a large class of configurations which are completed by property* **(G3)**, *which holds if $\widetilde{A}_2(0)=\widetilde{A}_2(r_1-1)\neq\emptyset$ and $\varphi(0,j)=\varphi(r_1-1,j)$ for all $j\in\widetilde{A}_2(0)$.*

REMARK 1.5. *Condition* **(G3)** *is also illustrated by potentials φ for which (1) $\widetilde{A}_2(0)=\widetilde{A}(r_1-1)\neq\emptyset$; (2) there exist an integer $p\geq 1$ such that for every $w_2\in\widetilde{A}_2(0^{\cdot p})=\widetilde{A}_2((r_1-1)^{\cdot p})$, the restrictions of φ to $[0^{\cdot p}]\times[w_2]\cap K$ and $[(r_1-1)^{\cdot p}]\times[w_2]\cap K$ are constant and take the same value.*

In the sequel, in order to simplify the computations, we slightly modify the definition of the quantity $I_{q,n}(w_1)$.

If $n\geq 1$ and $(w_1,w_2)\in\widetilde{A}_1^n\times\widetilde{A}_2^n$ is such that $[w_1]\times[w_2]\cap K\neq\emptyset$ (i.e. $w_2\in\widetilde{A}_2(w_1)$) let $(\overline{w_1},\overline{w_2})$ be an element of $[w_1]\times[w_2]\cap K$. Then, due to **(H1)**, a bounded distortion principle yields $C>0$ depending on φ only such that

$$C^{-|q|}\leq I_{q,n}(w_1)^{-1}\sum_{w_2\in\widetilde{A}_2^n(w_1)}\exp\left(qS_n\varphi(\overline{w_1},\overline{w_2})\right)\leq C^{|q|}.$$

Consequently, from now on we set

$$I_{q,n}(w_1)=\sum_{w_2\in\widetilde{A}_2^n(w_1)}\exp\left(qS_n\varphi(\overline{w_1},\overline{w_2})\right) \tag{1.10}$$

without affecting our results.

2. *Auxiliary functions and measures*

2.1. *Four basic properties.*

LEMMA 2.1. *Let L be a compact subset of $\mathbb{R}$. There exists a constant $C>0$ such that for every $n,p\geq 1$, $q\in L$, and $(w_1,\widetilde{w}_1)\in\widetilde{A}_1^n\times\widetilde{A}_1^p$ one has*

$$C^{-1}\leq\frac{I_{q,n+p}([w_1\cdot\widetilde{w}_1])}{I_{q,n}([w_1])I_{q,p}([\widetilde{w}_1])}\leq C. \tag{2.1}$$

The following lemma provides an extension to μ of (1.3) (which holds for $\mathbb{P}$).

LEMMA 2.2. *There exists $C > 0$ such that for every $n, p \geq 1$, for every $(w_1, \widetilde{w}_1, w_2, \widetilde{w}_2) \in A_1^n \times A_1^p \times A_2^n \times A_2^p$ such that $[w_1 \cdot \widetilde{w}_1] \times [w_2 \cdot \widetilde{w}_2] \cap K \neq \emptyset$ one has*

$$C^{-1} \leq \frac{\mu\big([w_1 \cdot \widetilde{w}_1] \times [w_2 \cdot \widetilde{w}_2]\big)}{\mu\big([w_1] \times [w_2]\big)\mu\big([\widetilde{w}_1] \times [\widetilde{w}_2]\big)} \leq C. \tag{2.2}$$

LEMMA 2.3. *There exists $C > 0$ such that for every $n \geq 1$, for every $[w_1 \cdot \widetilde{w}_1] \times [w_2] \in \mathcal{F}_n$ such that $[w_1 \cdot \widetilde{w}_1] \times [w_2] \cap K \neq \emptyset$ one has*

$$C^{-1} \leq \frac{\mu\big([w_1 \cdot \widetilde{w}_1] \times [w_2]\big)}{\mu\big([w_1] \times [w_2]\big)\mathbb{P}([\widetilde{w}_1])} \leq C.$$

LEMMA 2.4. *There exists $C > 0$ such that for every $n \geq 1$, for every $(w_1, w_2) \in A_1^n \times A_2^n$ such that $[w_1] \times [w_2] \cap K \neq \emptyset$ and every $(x, y) \in [w_1] \times [w_2] \cap K$ one has*

$$C^{-1}\mu([w_1] \times [w_2]) \leq \mathbb{P}([w_1])\frac{\exp\big(S_n\varphi(x, y)\big)}{I_{1,n}(w_1)} \leq C\mu([w_1] \times [w_2]).$$

Proof of Lemma 2.1. Let $q \in L$, $n, p \geq 1$ and $(w_1, \widetilde{w}_1) \in \widetilde{A}_1^n \times \widetilde{A}_1^p$. Recall (1.10).

$$\begin{aligned}
I_{q,n+p}([w_1 \cdot \widetilde{w}_1]) &= \sum_{(w_2,\widetilde{w}_2)\in\widetilde{A}_2(w_1)\times\widetilde{A}_2(\widetilde{w}_1)} \exp\big(qS_{n+p}(\overline{w_1 \cdot \widetilde{w}_1}, \overline{w_2 \cdot \widetilde{w}_2})\big) \\
= \sum_{(w_2,\widetilde{w}_2)\in\widetilde{A}_2(w_1)\times\widetilde{A}_2(\widetilde{w}_1)} &\exp\left(\sum_{k=0}^{n-1} q\varphi(\sigma_1^k \cdot \overline{w_1 \cdot \widetilde{w}_1}, \sigma_2^k \cdot \overline{w_2 \cdot \widetilde{w}_2}\right) \\
&\times \exp\left(\sum_{k=n}^{n+p-1} q\varphi(\sigma_1^k \cdot \overline{w_1 \cdot \widetilde{w}_1}, \sigma_2^k \cdot \overline{w_2 \cdot \widetilde{w}_2}\right).
\end{aligned}$$

Since φ satisfies the Dini property **(H1)**, a standard bounded distortion principle implies that there exists a constant c such that for all $q \in L$ one has

$$\left|\sum_{k=0}^{n-1} q\varphi(\sigma_1^k \cdot \overline{w_1 \cdot \widetilde{w}_1}, \sigma_2^k \cdot \overline{w_2 \cdot \widetilde{w}_2}) - \sum_{k=0}^{n-1} q\varphi(\sigma_1^k \cdot \overline{w_1}, \sigma_2^k \cdot \overline{w_2})\right| \leq c$$

$$\text{and } \left|\sum_{k=n}^{n+p-1} q\varphi(\sigma_1^k \cdot \overline{w_1 \cdot \widetilde{w}_1}, \sigma_2^k \cdot \overline{w_2 \cdot \widetilde{w}_2}) - \sum_{k=0}^{p-1} q\varphi(\sigma_1^k \cdot \overline{\widetilde{w}_1}, \sigma_2^k \cdot \overline{\widetilde{w}_2})\right| \leq c.$$

This yields the conclusion. □

Proof of Lemma 2.2. For $k \geq 1$ we denote by f_k the function $\exp(S_k\varphi)$. Let $n \geq 1$, $m \geq 0$ and $(w_1, w_2) \in A_1^n \times A_2^n$ such that $[w_1] \times [w_2] \cap K \neq \emptyset$ (notice that $w_1 \in \widetilde{A}_1^n$ and $w_2 \in \widetilde{A}_2(w_1)$).

$$\mu_{n+m}\big([w_1] \times [w_2]\big) = \int_{[w_1]\times[w_2]} \frac{f_{n+m}(x, y)}{\int_{K_x} f_{n+m}(x, u)\, \ell_{2,x}(du)}\, \ell_{2,x}(dy)\mathbb{P}(dx). \tag{2.3}$$

Again because of φ satisfies **(H1)**, a standard bounded distortion principle implies that there exists $C > 0$, independent of n, m and $(w_1', w_2') \in A_1^n \times A_2^n$, such that for all $(x, u) \in [w_1'] \times [w_2'] \cap K$,

$$C^{-1} \leq \frac{f_{n+m}(x,u)}{f_n\big(\overline{w_1'}, \overline{w_2'}\big) f_m(\sigma_1^n x, \sigma_2^n u)} \leq C.$$

Consequently,

$$C^{-1} \leq \frac{\int_{K_x} f_{n+m}(x,u)\, \ell_{2,x}(du)}{\sum_{w_2' \in \widetilde{A}_2(w_1')} f_n\big(\overline{w_1'}, \overline{w_2'}\big) \int_{[w_2'] \cap K_x} f_m(\sigma_1^n x, \sigma_2^n u) \ell_{2,x}(du)} \leq C.$$

On the other hand, one has

$$\int_{[w_2'] \cap K_x} f_m(\sigma_1^n x, \sigma_2^n u) \ell_{2,x}(du) = \ell_{2,x}([w_2'] \cap K_x) \int_{K_{\sigma_1^n x}} f_m(\sigma_1^n x, v) \ell_{2,\sigma_1^n x}(dv),$$

and we see on the right hand side that due to the definition of $\ell_{2,x}$ this quantity does not depend on w_2'. Incorporating the above estimates in (2.3) yields

$$C^{-1} \leq \mu_{n+m}\big([w_1] \times [w_2]\big) \frac{\sum_{w_2' \in \widetilde{A}_2(w_1)} f_n(\overline{w_1}, \overline{w_2'})}{f_n(\overline{w_1}, \overline{w_2}) \mathbb{P}([w_1])} \leq C.$$

In other words (recall (1.10))

$$C^{-1} \leq \mu_{n+m}\big([w_1] \times [w_2]\big) \frac{I_{1,n}([w_1])}{f_n(\overline{w_1}, \overline{w_2}) \mathbb{P}([w_1])} \leq C \tag{2.4}$$

Moreover, it follows from the proof of Lemma 2.1 and the quasi-Bernoulli property of $\mathbb{P}$ that there exists a constant $C_1 > 0$ independent of n, p and $(w_1, \widetilde{w}_1, w_2, \widetilde{w}_2) \in A_1^n \times A_1^p \times A_2^n \times A_2^p$ such that if $[w_1 \cdot \widetilde{w}_1] \times [w_2 \cdot \widetilde{w}_2] \cap K \neq \emptyset$ then

$$C_1^{-1} \leq \frac{I_{1,n+p}([w_1 \cdot \widetilde{w}_1]) / f_{n+p}(\overline{w_1 \cdot \widetilde{w}_1}, \overline{w_2 \cdot \widetilde{w}_2}) \mathbb{P}([w_1 \cdot \widetilde{w}_1])}{\Big(I_{1,n}([w_1]) / f_n(\overline{w_1}, \overline{w_2}) \mathbb{P}([w_1])\Big) \Big(I_{1,p}([\widetilde{w}_1]) / f_p(\overline{\widetilde{w}_1}, \overline{\widetilde{w}_2}) \mathbb{P}([\widetilde{w}_1])\Big)} \leq C_1. \tag{2.5}$$

Let now $n, p \geq 1$ and $(w_1, \widetilde{w}_1, w_2, \widetilde{w}_2) \in A_1^n \times A_1^p \times A_2^n \times A_2^p$ such that $[w_1 \cdot \widetilde{w}_1] \times [w_2 \cdot \widetilde{w}_2] \cap K \neq \emptyset$. Due to (2.4) and (2.5), for all $m \geq 0$ one has

$$(C^3 C_1)^{-1} \leq \frac{\mu_{n+p+m}\big([w_1 \cdot \widetilde{w}_1] \times [w_2 \cdot \widetilde{w}_2]\big)}{\mu_{n+m}\big([w_1] \times [w_2]\big) \mu_{p+m}\big([\widetilde{w}_1] \times [\widetilde{w}_2]\big)} \leq C^3 C_1. \tag{2.6}$$

Since the indicator function of any cylinder of the form $[w_1] \times [w_2]$ is continuous, letting m tend to ∞ in (2.6) yields the result. $\square$

Proof of Lemma 2.3. Let us write

$$\mu\big([w_1 \cdot \widetilde{w}_1] \times [w_2]\big) = \sum_{\widetilde{w}_2 \in A_2^{g(n)-n}} \mu\big([w_1 \cdot \widetilde{w}_1] \times [w_2 \cdot \widetilde{w}_2]\big).$$

The result is then a simple consequence of Lemma 2.2 and the fact that $\mathbb{P}$ is the projection of μ on $\mathbb{A}_1$. $\square$

Proof of Lemma 2.4. Let m tend to ∞ in (2.4).

2.2. *Auxiliary measures.*

PROPOSITION 2.1. *Let $q \in \mathbb{R}$. There exists a quasi-Bernoulli ergodic measure $\mathbb{P}_q$ and a constant $C > 0$ such that for all $n \geq 1$ and $w_1 \in \widetilde{A}_1^n$*

$$C^{-1} r_1^{n\beta_\mu(q)} \rho_{q,n}([w_1]) \leq \mathbb{P}_q([w_1]) \leq C\, r_1^{n\beta_\mu(q)} \rho_{q,n}([w_1]) \tag{2.7}$$

DEFINITION 2.1. *Let $q \in \mathbb{R}$. Let $\mathbb{P}_q$ be the measure obtained in Proposition 2.1. We set $\varphi_q(x,y) = q\varphi(x,y)$ if $(x,y) \in K$ and $\varphi_q(x,y) = -\infty$ otherwise. Then, μ_q stands for the measure constructed from $(\mathbb{P}_q, \varphi_q)$ as μ is constructed from $(\mathbb{P}, \varphi)$. By construction μ and μ_q have the same support K.*

The proof of Proposition 2.1 requires the following lemmas whose proofs are postponed to after that of Proposition 2.1.

LEMMA 2.5. *For every compact subset L of $\mathbb{R}$, there exists $C > 0$ such that for all $q \in L$ and $n, p \geq 1$*

$$C^{-1} \leq \frac{\exp\big((n+p)\beta_{\mu,n+p}(q)\big)}{\exp\big(n\beta_{\mu,n}(q) + p\beta_{\mu,p}(q)\big)} \leq C. \tag{2.8}$$

Consequently, $\beta_{\mu,n}$ converges uniformly to β_μ on L and $\|n(\beta_{\mu,n} - \beta_\mu)\|_{\infty,L} \leq \log C$.

DEFINITION 2.2. *For $q \in \mathbb{R}$ and $n \geq 1$, let $\widetilde{\rho}_{q,n}$ be the probability measure defined on $\widetilde{\mathbb{A}}_1$ by*

$$\widetilde{\rho}_{q,n} = r_1^{n\beta_{\mu,n}(q)} \sum_{w_1 \in \widetilde{A}_1^n} \rho_{q,n}([w_1]) \widetilde{r}_1^{\,n} \ell_{1|[w_1]},$$

where $\ell_{1|[w_1]}$ stands for the restriction of ℓ_1 to $[w_1]$ and $\widetilde{r}_1 = \#\widetilde{A}_1$.

LEMMA 2.6. *For every compact subset L of $\mathbb{R}$, there exists $C > 0$ such that for all $n, p \geq 1$, $q \in L$ and $w_1 \in \widetilde{A}_1^n$*

$$C^{-1} \leq \frac{\widetilde{\rho}_{q,n+p}([w_1])}{\widetilde{\rho}_{q,n}([w_1])} \leq C.$$

Proof of Proposition 2.1. Let $\widetilde{\rho}_q$ be the weak limit of a subsequence of $(\widetilde{\rho}_{q,n})$. It is immediate from the definition of $\rho_{q,n}$ and Lemmas 2.1, 2.5 and 2.6 that $\widetilde{\rho}_q$ is a quasi-Bernoulli measure and that there exists $C > 0$ such that for all $n \geq 1$ and $w_1 \in \widetilde{A}_1^n$ we have

$$C^{-1} r_1^{n\beta_\mu(q)} \rho_{q,n}([w_1]) \leq \widetilde{\rho}_q([w_1]) \leq C\, r_1^{n\beta_\mu(q)} \rho_{q,n}([w_1]).$$

Now, since $\widetilde{\rho}_q$ is quasi-Bernoulli, it follows from [**11**] that it is equivalent to a quasi-Bernoulli ergodic measure $\mathbb{P}_q$. □

Proof of Lemma 2.5. Property (2.8) is a consequence of the definition of $\beta_{\mu,n}$, Lemma 2.1 and the quasi-Bernoulli property of $\mathbb{P}$. Then, the uniform control of $n(\beta_{\mu,n}(q) - \beta_\mu(q))$ over L follows from the standard fact that u_n/n converges to its infimum if the sequence $(u_n)_{n\geq 1}$ is subadditive. □

Proof of Lemma 2.6. Let $q \in L$, $n, p \geq 1$ and $w_1 \in \widetilde{A}_1^n$. By definition we have

$$\widetilde{\rho}_{q,n+p}([w_1]) = \sum_{\widetilde{w}_1 \in \widetilde{A}_1^p} \widetilde{\rho}_{q,n+p}([w_1 \cdot \widetilde{w}_1]) = \sum_{\widetilde{w}_1 \in \widetilde{A}_1^p} r_1^{(n+p)\beta_{\mu,n+p}(q)} \rho_{q,n+p}([w_1 \cdot \widetilde{w}_1]).$$

It follows from Lemma 2.1, Lemma 2.5 and the quasi-Bernoulli property of $\mathbb{P}$ that there exists $C > 0$ independent of q, n and p such that for all $\widetilde{w}_1 \in \widetilde{A}_1^p$ we have

$$C^{-1} \leq \frac{r_1^{(n+p)\beta_{\mu,n+p}(q)} \rho_{q,n+p}([w_1 \cdot \widetilde{w}_1])}{\widetilde{\rho}_{q,n}([w_1])\widetilde{\rho}_{q,p}([\widetilde{w}_1])} \leq C.$$

We conclude by using the identity $\sum_{\widetilde{w}_1 \in \widetilde{A}_1^p} \widetilde{\rho}_{q,p}([\widetilde{w}_1]) = 1$. □

2.3. *Comparing β_μ with the L^q-spectrum of μ. Differentiability of β_μ.*

DEFINITION 2.3. *Let ν be a positive finite Borel measure on $\mathbb{A}_1 \times \mathbb{A}_2$. The L^q-spectrum τ_ν of ν is defined by*

$$\tau_\nu(q) = \liminf_{n \to \infty} \tau_{\nu,n}(q), \quad \textit{where } \tau_{\nu,n}(q) = -\frac{1}{n} \log_{r_2} \sum_{\mathcal{C} \in \mathcal{F}_n} \nu(\mathcal{C})^q$$

with the convention $0^q = 0$ (recall that $\mathcal{F}_n$ is defined in (1.2)).

Let ρ be a positive finite Borel measure on $\mathbb{A}_1$. The L^q-spectrum τ_ρ of ρ is defined by

$$\tau_\rho(q) = \liminf_{n \to \infty} \tau_{\rho,n}(q), \quad \textit{where } \tau_{\rho,n}(q) = -\frac{1}{n} \log_{r_1} \sum_{w \in A_1^n} \rho([w])^q.$$

PROPOSITION 2.2. *One has $\beta_\mu \geq \tau_\mu$. Moreover, if $\mathbb{P}$ is the Lebesgue measure and φ does not depend on the first variable, then $\beta_\mu = \tau_\mu$.*

REMARK 2.1. *The case when $\mathbb{P}$ is the Lebesgue measure and φ does not depend on the first variable is the extension to our general setting of the case when the column vector $(\varphi_{ij})_{0 \leq j < r_2}$ does not depend on i in [**17, 22**] (see Section 1.3.1).*

PROPOSITION 2.3. *The function τ_μ is differentiable at 1, and so is β_μ, with $\beta'_\mu(1) = \tau'_\mu(1)$.*

PROPOSITION 2.4. *For all $q, r \in \mathbb{R}$ one has*

$$\beta_{\mu_q}(r) = \beta_\mu(qr) - r\beta_\mu(q). \tag{2.9}$$

COROLLARY 2.1. *The function β_μ is differentiable.*

Proof of Proposition 2.2. Let $n \geq 1$. Due to Lemmas 2.3 and 2.4 we have

$$\begin{aligned}
\left[\sum_{\mathcal{C}\in\mathcal{F}_n} \mu(\mathcal{C})^q\right]^s &= \left[\sum_{\substack{(w_1,\widetilde{w}_1,w_2)\in A_1^n\times A_1^{g(n)-n}\times A_2^n,\\ [w_1\cdot\widetilde{w}_1]\times[w_2]\cap K\neq\emptyset}} \mu([w_1\cdot\widetilde{w}_1]\times[w_2])^q\right]^s \\
&\geq C^{-s}\left[\sum_{\substack{(w_1,\widetilde{w}_1,w_2)\in A_1^n\times A_1^{g(n)-n}\times A_2^n,\\ [w_1\cdot\widetilde{w}_1]\times[w_2]\cap K\neq\emptyset}} \mathbb{P}([w_1])^q\mathbb{P}([\widetilde{w}_1])^q\frac{\exp\left(qS_n\varphi(\overline{w_1},\overline{w_2})\right)}{I_{1,n}(w_1)^q}\right]^s \\
&= C^{-s}\left[\sum_{\widetilde{w}_1\in\widetilde{A}_1^{g(n)-n}} \mathbb{P}([\widetilde{w}_1])^q\right]^s\left[\sum_{w_1\in\widetilde{A}_1^n}\mathbb{P}([w_1])^q\frac{I_{q,n}(w_1)}{I_{1,n}(w_1)^q}\right]^s
\end{aligned}$$

for some positive constant C. The concavity of the function $x\mapsto x^s$ on $\mathbb{R}_+$ implies (via Jensen's inequality applied with the probability measure P_n on $\widetilde{A}_1^n$ defined by $P_n(\{w_1\}) = \mathbb{P}([w_1])^q/\sum_{w_1\in\widetilde{A}_1^n}\mathbb{P}([w_1])^q$)

$$\left[\sum_{w_1\in\widetilde{A}_1^n}\mathbb{P}([w_1])^q\frac{I_{q,n}(w_1)}{I_{1,n}(w_1)^q}\right]^s \geq \left[\sum_{w_1\in\widetilde{A}_1^n}\mathbb{P}([w_1])^q\right]^{s-1}\sum_{w_1\in\widetilde{A}_1^n}\mathbb{P}([w_1])^q\left(\frac{I_{q,n}(w_1)}{I_{1,n}(w_1)^q}\right)^s.$$

Now recall that due to the quasi-Bernoulli property of $\mathbb{P}$ there exists $C>0$ such that $C^{-1}\leq r_1^{n\tau_{\mathbb{P}}(q)}\sum_{w_1\in\widetilde{A}_1^n}\mathbb{P}([w_1])^q\leq C$ for all $n\geq 1$ (this is due to the subaddivity property of $n\tau_{\mathbb{P},n}(q)$) so there exists $C'>0$ such that for all $n\geq 1$ $\left[\sum_{\widetilde{w}_1\in\widetilde{A}_1^{g(n)-n}}\mathbb{P}([\widetilde{w}_1])^q\right]^s\left[\sum_{w_1\in\widetilde{A}_1^n}\mathbb{P}([w_1])^q\right]^{s-1}\geq C'$. The previous estimates yield

$$\limsup_{n\to\infty}\frac{1}{n}\log_{r_2}\sum_{\mathcal{C}\in\mathcal{F}_n}\mu(\mathcal{C})^q \geq \limsup_{n\to\infty}\frac{1}{n}\log_{r_1}\sum_{w_1\in\widetilde{A}_1^n}\mathbb{P}([w_1])^q\left(\frac{I_{q,n}(w_1)}{I_{1,n}(w_1)^q}\right)^s.$$

□

Proof of Proposition 2.3. τ_μ is a linear combination of $\tau_{\mathbb{P}}$ and the function

$$\widetilde{\tau}_\mu(q) = \liminf_{n\to\infty} -\frac{1}{n}\log_{r_2}\sum_{\substack{(w_1,w_2)\in A_1^n\times A_2^n,\\ [w_1]\times[w_2]\cap K\neq\emptyset}}\mu([w_1]\times[w_2])^q.$$

Indeed, it follows from Lemma 2.3 that

$$\tau_\mu = s\tau_{\mathbb{P}} + \widetilde{\tau}_\mu. \tag{2.10}$$

We know that $\tau_{\mathbb{P}}$ is differentiable at 1 because $\mathbb{P}$ is quasi-Bernoulli (Theorem 3.1 in [**11**]). Moreover, since the measure μ is quasi-Bernoulli on products of cylinders by Lemma 2.2, the same arguments as those used in proving Theorem 3.1 in [**11**] show that $\widetilde{\tau}_\mu$ is also differentiable at 1. This yields the existence of $\tau_\mu'(1)$.

We have $\beta_\mu \geq \tau_\mu$, β_μ and τ_μ coincide at 1, and the both are concave. So the differentiability of τ_μ at 1 implies that of β_μ as well as the equality $\beta_\mu'(1) = \tau_\mu'(1)$. $\square$

Proof of Proposition 2.4. This is a simple consequence of the definitions of β_{μ_q} and Proposition 2.1. $\square$

Proof of Corollary 2.1. Fix $q \in \mathbb{R} \setminus \{0\}$. We can apply Proposition 2.3 to μ_q. Differentiating β_{μ_q} at 1 and using (2.9) at $r = 1$ yields the existence of $\beta_\mu'(q)$ and the relation

$$\beta_{\mu_q}'(1) = q\beta_\mu'(q) - \beta_\mu(q). \tag{2.11}$$

The differentiability of β_μ at $q = 0$ follows from the relation (2.10) and an argument very similar to that of [**11**] using the second part of Lemma 2.5, the concavity of the functions $\tau_{\mathbb{P},n}$, $\tau_{\mathbb{P}}$, $\widetilde{\tau}_{\mu,n}$ and $\widetilde{\tau}_\mu$ and the fact that $\tau_{\mathbb{P},n}(0) = \tau_{\mathbb{P}}(0)$ and $\widetilde{\tau}_{\mu,n}(0) = \widetilde{\tau}_\mu(0)$ for all $n \geq 1$. $\square$

3. *The singularity spectrum of the measure* μ

If $(x, y) \in \mathbb{A}_1 \times \mathbb{A}_2$ and $n \geq 0$, let $\mathcal{C}_n(x, y)$ stand for the unique element of $\mathcal{F}_n$ containing (x, y). We have $\mathcal{C}_n(x, y) = B((x, y), r_2^{-n})$. Also, for every $\alpha \geq 0$, $E_\mu(\alpha) = \Big\{z \in \mathrm{supp}(\mu) : \lim_{n\to\infty} \frac{\log \mu(B(z, r_2^{-n}))}{\log r_2^{-n}} = \alpha\Big\}$.

3.1. *Upper bound for* $\dim E_\mu(\alpha)$.

Our approach is similar to that used in [**17, 22**].

PROPOSITION 3.1. *Let $q \in \mathbb{R}$. For all $(x, y) \in K$, one has*

$$\limsup_{n\to\infty} \left(\frac{\mu_q(\mathcal{C}_n(x, y))}{\mu(\mathcal{C}_n(x, y))^q \big|\mathcal{C}_n(x, y)\big|^{-\beta_\mu(q)}}\right)^{1/n} \geq 1.$$

Then, using the same approach as in [**22**] one gets

COROLLARY 3.1. *For every $\alpha \geq 0$ one has* $\dim E_\mu(\alpha) \leq \beta_\mu^*(\alpha)$.

Proof of Proposition 3.1. Let $(x, y) \in K$. The set $\mathcal{C}_n(x, y)$ takes the form $[w_1 \cdot \widetilde{w}_1] \times [w_2]$, where $(w_1, \widetilde{w}_1) \in \widetilde{A}_1^n \times \widetilde{A}_1^{g(n)-n}$ and $w_2 \in \widetilde{A}_2(w_1)$. In the sequel, the symbol $\approx$ means that the quantities in the both sides of $\approx$ differ from a constant which depends only on φ, $\mathbb{P}$ and q. By construction, due to Lemma 2.3 and Proposition 2.1 we have

$$\begin{aligned}
\mu_q(\mathcal{C}_n(x, y)) &\approx \mathbb{P}_q([\widetilde{w}_1])\mu_q([w_1] \times [w_2]) \\
&\approx \mathbb{P}_q([\widetilde{w}_1])\mathbb{P}_q([w_1])\frac{\exp\big(qS_n\varphi(x, y)\big)}{I_{q,n}(w_1)} \\
&\approx r_1^{g(n)\beta_\mu(q)}\mathbb{P}([\widetilde{w}_1])^q\mathbb{P}([w_1])^q\mathcal{I}_{q,n}\frac{\exp\big(qS_n\varphi(x, y)\big)}{I_{q,n}(w_1)}
\end{aligned}$$

where
$$\mathcal{I}_{q,n} = \frac{I_{q,g(n)-n}(\widetilde{w}_1)^s}{I_{1,g(n)-n}(\widetilde{w}_1)^{sq}} \frac{I_{q,n}(w_1)^s}{I_{1,n}(w_1)^{sq}} \approx \frac{I_{q,g(n)}(w_1 \cdot \widetilde{w}_1)^s}{I_{1,g(n)}(w_1 \cdot \widetilde{w}_1)^{sq}}$$
(due to Lemma 2.1). Thus
$$\mu_q(C_n(x,y)) \approx r_1^{g(n)\beta_\mu(q)} \mathbb{P}([\widetilde{w}_1])^q \mathbb{P}([w_1])^q \frac{I_{q,g(n)}(w_1 \cdot \widetilde{w}_1)^s}{I_{1,g(n)}(w_1 \cdot \widetilde{w}_1)^{sq}} \frac{\exp\big(qS_n\varphi(x,y)\big)}{I_{q,n}(w_1)}.$$

On the other hand
$$\mu(\mathcal{C}_n(x,y))^q \big|\mathcal{C}_n(x,y)\big|^{-\beta_\mu(q)} \approx r_2^{n\beta_\mu(q)} \mathbb{P}([\widetilde{w}_1])^q \mathbb{P}([w_1])^q \frac{\exp\big(qS_n\varphi(x,y)\big)}{I_{1,n}(w_1)^q}.$$
This yields
$$\begin{aligned} \frac{\mu_q(\mathcal{C}_n(x,y))}{\mu(\mathcal{C}_n(x,y))^q |\mathcal{C}_n(x,y)|^{-\beta_\mu(q)}} &\approx \frac{I_{1,n}(w_1)^q}{I_{q,n}(w_1)} \frac{I_{q,g(n)}(w_1 \cdot \widetilde{w}_1)^s}{I_{1,g(n)}(w_1 \cdot \widetilde{w}_1)^{qs}} &(3.1)\\ &\approx \frac{I_{1,n}(x|n)^q}{I_{q,n}(x|n)} \frac{I_{q,g(n)}(x|g(n))^s}{I_{1,g(n)}(x|g(n))^{qs}}. &(3.2)\end{aligned}$$
Thus, denoting $u_n = \dfrac{I_{1,n}(x|n)^q}{I_{q,n}(x|n)}$, there exists $c = c(q, \varphi, \mathbb{P})$ such that
$$\frac{\mu_q(\mathcal{C}_n(x,y))}{\mu(\mathcal{C}_n(x,y))^q \big|\mathcal{C}_n(x,y)\big|^{-\beta_\mu(q)}} \geq c \frac{u_n}{u_{g(n)}^{n/g(n)}} u_{g(n)}^{n/g(n)-s}. \tag{3.3}$$
Now, since by construction there exists $0 < a < b < \infty$ independent of x such that $a^n \leq u_n \leq b^n$ for all $n \geq 1$, and $|n/g(n) - s| = O(1/n)$, there exists $c' > 0$ such that $u_{g(n)}^{n/g(n)-s} \geq c'$. The conclusion then comes from the fact that since $s = \lim_{n\to\infty} n/g(n) < 1$, we have that $\limsup_{n\to\infty} \frac{u_n^{1/n}}{u_{g(n)}^{1/g(n)}} \geq 1$ for any positive sequence $(u_n)_{n\geq 1}$ such that $u_n^{1/n}$ is bounded away from 0. □

3.2. *Lower bound for* dim $E_\mu(\alpha)$.

PROPOSITION 3.2. *Let $q \in \mathbb{R}^*$. The set $E_\mu(\beta'_\mu(q))$ is of full μ_q-measure. Consequently,* $\dim E_\mu(\beta'_\mu(q)) \geq q\beta'_\mu(q) - \beta_\mu(q)$.

Proof Let us begin with the case $q \neq 0$. Proposition 2.3 claims that $\tau'_{\mu_q}(1)$ and $\beta'_{\mu_q}(1)$ exist and are equal. The differentiability of τ_{μ_q} at 1 implies that $\lim_{r\to 0} \frac{\log \mu_q(B(z,r))}{\log r} = \tau'_{\mu_q}(1)$ μ_q-almost everywhere (by [**20**]). So μ_q is carried by sets of Hausdorff dimension at least $\tau'_{\mu_q}(1) = q\beta'_\mu(q) - \beta_\mu(q)$ (by (2.11)).

To conclude, it is enough to show that $\lim_{n\to\infty} n^{-1} \log \frac{\mu_q(\mathcal{C}_n(x,y))}{\mu(\mathcal{C}_n(x,y))^q |\mathcal{C}_n(x,y)|^{-\beta_\mu(q)}} = 0$ μ_q-almost everywhere.

Due to (3.2), this amounts to showing that

$$\lim_{n\to\infty} n^{-1}\log \frac{I_{1,n}(x|n)^q}{I_{q,n}(x|n)}\frac{I_{q,g(n)}(x|g(n))^s}{I_{1,g(n)}(x|g(n))^{qs}}=0 \quad \mathbb{P}_q\text{- } a.e. \tag{3.4}$$

Due to the submultiplicative property established in Lemma 2.1, the ergodicity of $\mathbb{P}_q$, and the fact that $\lim_{n\to\infty} n/g(n)=s$, the result follows from Kingman's subadditive ergodic theorem ([**18**]).

If $q=0$, let us suppose for a while that there exists $\alpha\geq 0$ such that the set $E_\mu(\alpha)$ is of full μ_0-measure. The measure μ_0 is generated by the potential φ_0 which is equal to 0 on K and equal to $-\infty$ elsewhere, as well as the measure $\widetilde{\rho}_0$. This measure belongs to the class of self-affine multinomial measures and it follows from [**19, 17**] that μ_0 is supported by $E_{\mu_0}(\beta_\mu(0))$ (in particular, the value of $\beta_\mu(0)$, which only depends on the structure of A as well as r_1 and r_2, is equal to the Hausdorff dimension of K). So dim $E_\mu(\alpha)\geq\beta_\mu(0)$. Moreover, it follows from Corollary 3.1 that if $\alpha\neq\beta'_\mu(0)$ then dim $E_\mu(\alpha)<\beta_\mu(0)$. So $\alpha=\beta'_\mu(0)$. Since $\beta^*_\mu(\beta'_\mu(0))=\beta_\mu(0)$ we get the desired lower bound.

The existence of α comes from Lemma 2.4 and the subadditive ergodic theorem applied with the ergodic measure μ_0. □

4. *The singularity spectrum of the measure* $\widetilde{\mu}$

4.1. *Intermediate results.*

This section provides the versions of Propositions 3.1 (Proposition 4.3) and Proposition 3.2 (Corollary 4.2) needed to establish Theorem 1.2 in Section 4.2.

We need the next proposition and its corollary which, for every $q\in\mathbb{R}$, provides precious information on the relationship between the measure μ_q and its projection on the Sierpinski carpet.

PROPOSITION 4.1. *Let $q\in\mathbb{R}$. For all $(w_1,w_2)\in A_1^*\times A_2^*$ one has $\mu_q(\{\underline{w_1}\}\times[w_2])=\mu_q([w_1]\times\{\underline{w_2}\})=0$.*

COROLLARY 4.1. *Let $q\in\mathbb{R}$. For all $(w_1,w_2)\in A_1^*\times A_2^*$ one has*

$$\widetilde{\mu}_q\big(\pi([w_1]\times[w_2])\big)=\mu_q([w_1]\times[w_2]).$$

If $i\in\{1,2\}$ and $w\in A_i^*$ then $\underline{w}$ stands for $w\cdot\underline{0}$, where $\underline{0}$ is the element of $\mathbb{A}_i$ whose letters are all equal to 0. Also, recall that if $j\in A_i$ and $n\geq 1$, $j^{\cdot n}$ stands for the word of length n whose letters are all equal to j.

Proof of Proposition 4.1. Let $(w_1,w_2)\in A_1^*\times A_2^*$ and $q\in\mathbb{R}$. We have $\mu_q(\{\underline{w_1}\}\times[w_2])\leq\mathbb{P}_q(\{\underline{w_1}\})$. Thus $\mu_q(\{\underline{w_1}\}\times[w_2])=0$ follows from the fact that the measure $\mathbb{P}_q$ is atomless because by construction it is supported by the full set $\widetilde{\mathbb{A}}_1$ and we assumed that $\#\widetilde{A}_1\geq 2$. Indeed, if $\mathbb{P}_q$ had an atom at $x=x_1\cdot x_2\cdots x_n\cdots$, the sequence $\mathbb{P}_q([x|n])/\mathbb{P}_q([x|n+1])$ would converge to 1 as n goes to ∞, so that $\mathbb{P}_q([x|n])/\mathbb{P}_q([x|n\cdot y_{n+1}])$, $y_{n+1}\in\widetilde{A}_1\setminus\{x_{n+1}\}$, should converge to ∞. This would be in contradiction with the property (1.3) satisfied by $\mathbb{P}_q$.

Let us show that $\mu_q([w_1]\times\{\underline{w_2}\})=0$. We could use the fact, claimed in use Remark 1.3, that the projection of μ_q over $\mathbb{A}_2$ is quasi-Bernoulli. Since this fact is not established explicitely in this paper, we provide another instructive approach. We leave the reader verify that we can assume without loss of generality that w_1 and w_2 are of the same generation. Then, due to the factorization provided by Lemma 2.2, to show that $\mu_q([w_1]\times\{\underline{w_2}\})=0$ it is enough to show that $\mu_q(\mathbb{A}_1\times\{\underline{0}\})=0$. Due to Lemma 2.4, this amounts to showing that

$$\lim_{n\to\infty}\sum_{\substack{w_1\in\widetilde{A}_1^n,\\ [w_1]\times[0^{\cdot n}]\cap K\neq\emptyset}} Q_n(w_1)\mathbb{P}_q([w_1])=0, \tag{4.1}$$

where

$$Q_n(w_1)=\frac{\exp qS_n\varphi(\overline{w_1},\overline{0^{\cdot n}})}{\sum_{w_2\in\widetilde{A}_2(w_1)}\exp qS_n\varphi(\overline{w_1},\overline{w_2})}.$$

Two cases must be distinguished.

Case 1: The following property $(\mathcal{P})$ holds.

$(\mathcal{P})$: For every $w_1\in\widetilde{A}_1$, if $[w_1]\times[0]\cap K\neq\emptyset$ then $[w_1]\times[w_2]\cap K=\emptyset$ for all $w_2\in\widetilde{A}_2\setminus\{0\}$.

Then (4.1) simplifies to be

$$\lim_{n\to\infty}\mathbb{P}_q\big(\{w_1\in\widetilde{A}_1^n:\ [w_1]\times[0^{\cdot n}]\cap K\neq\emptyset\}\big)=0 \tag{4.2}$$

Let $\widehat{A}_1=\{w_1\in\widetilde{A}_1:[w_1]\times[0]\cap K\neq\emptyset\}$. Since $\#\widetilde{A}_2\geq 2$, $(\mathcal{P})$ implies that $\widehat{A}_1$ is strictly included in $\widetilde{A}_1$. Moreover,

$$\bigcap_{n\geq 1}\ \bigcup_{\substack{w_1\in\widetilde{A}_1^n,\\ [w_1]\times[0^{\cdot n}]\cap K\neq\emptyset}}[w_1]=\widehat{\mathbb{A}}_1:=\widehat{A}_1^{\mathbb{N}^*}. \tag{4.3}$$

On the other hand, if $j\in\widetilde{A}_1\setminus\widehat{A}_1$, the expectation of the random variable $x\in\widetilde{\mathbb{A}}_1\mapsto\mathbf{1}_{\{j\}}(x_1)$ with respect to $\mathbb{P}_q$ is positive. Moreover, $\mathbb{P}_q$ is ergodic. Consequently, $\mathbb{P}_q(\widehat{\mathbb{A}}_1)=0$. We conclude by using (4.3) and (4.2).

Case 2: There exists $j\in\widetilde{A}_1$ such that $[j]\times[0]\cap K\neq\emptyset$ and there exists $l\in\widetilde{A}_2\setminus\{0\}$ such that $[j]\times[l]\cap K\neq\emptyset$. Fix such a pair (j,l). The expectation of the random variable $x\in\widetilde{\mathbb{A}}_1\mapsto\mathbf{1}_{\{j\}}(x_1)$ with respect to $\mathbb{P}_q$ is positive. Let us denote its value by c_j. We now that for $\mathbb{P}_q$-almost every $x=x_1\cdots x_k\cdots$, $\lim_{n\to\infty}\frac{1}{n}\#\{1\leq k\leq n: x_k=j\}=c_j$.

Let $\varepsilon\in(0,1)$. By the Egoroff lemma, there exists a Borel set $B\subset\widetilde{\mathbb{A}}_1$ and an integer $N\geq 1$ such that for all $x\in B$ and $n\geq N$, $\#\{1\leq k\leq n:x_k=j\}\geq c_jn/2$.

Now, for $n\geq N$ let $w_1\in\widetilde{A}_1^n$ such that $[w_1]\times[0^{\cdot n}]\cap K\neq\emptyset$ and $[w_1]\cap B\neq\emptyset$. Let $x\in[w_1]\cap B\neq\emptyset$. For each value of k between 1 and n such that $x_k=j$, the word $0^{\cdot n}(k)$ obtained by changing the k^{th} 0 in $0^{\cdot n}$ by l is such that $[w_1]\times[0^{\cdot n}(k)]\cap K\neq\emptyset$. Moreover, since φ is bounded over its support, there exists a constant $c>0$

depending only on φ such that $\frac{\exp qS_n\varphi(\overline{w_1},\overline{0^{\cdot n}})}{\exp qS_n\varphi(\overline{w_1},\overline{0^{\cdot n}(k)})} \leq c$. It follows from the previous remarks that

$$Q_n(w_1) \leq \frac{\exp qS_n\varphi(\overline{w_1},\overline{0^{\cdot n}})}{\sum_{\substack{1\leq k\leq n,\\ x_k=j}} \exp qS_n\varphi(\overline{w_1},\overline{0^{\cdot n}(k)})} \leq \frac{2c}{c_j n}.$$

Then for $n \geq N$

$$\sum_{\substack{w_1\in\widetilde{A}_1^n,\\ [w_1]\times[0^{\cdot n}]\cap K\neq\emptyset}} Q_n(w_1)\mathbb{P}_q([w_1]) \leq \frac{2c}{c_j n}\mathbb{P}_q(B) + \mathbb{P}_q(B^c) \leq \frac{2c}{c_j n} + \varepsilon.$$

This yields (4.1). □

Proof of Corollary 4.1. Let

$$\mathbb{D} = \bigcup_{(w_1,w_2)\in A_1^*\times A_2^*} \{\overline{w_1}\}\times[w_2]\cup[w_1]\times\{\overline{w_2}\}.$$

For $(w_1, w_2) \in A_1^* \times A_2^*$, the set $\pi([w_1]\times[w_2])$ is the rectangle $R(w_1, w_2)$ obtained as the product of the closed intervals $\pi_1([w_1])$ and $\pi_2([w_2])$. It is easily seen that $\pi^{-1}\big(R(w_1,w_2)\setminus\partial R(w_1,w_2)\big) \subset [w_1]\times[w_2]$ and $\pi^{-1}\big(\partial R(w_1,w_2)\big) \subset \mathbb{D}\cup[w_1]\times[w_2]$. Thus, the result follows from Proposition 4.1. □

If $(w, w') \in A_i^n$, $\pi_i([w]) = [kr_i^{-n}, (k+1)r_i^{-n}]$ and $\pi_i([w']) = [k'r_i^{-n}, (k'+1)r_i^{-n}]$ for some pair of integers $(k, k') \in \{0, \dots, r_i^n - 1\}^2$; let $\delta_i(w', w) = k' - k$. Conversely, given $w \in A_i^n$ and k such that $\pi_i([w]) = [kr_i^{-n}, (k+1)r_i^{-n}]$, fixing $k' \in \{0, \dots, r_i^n - 1\}$, there exists a unique $w' \in A_i^n$ such that $\delta_i(w', w) = k' - k$.

Definition 4.1. *If $(x, y) \in \mathbb{A}_1 \times \mathbb{A}_2$, $n \geq 1$, $\mathcal{C}_n(x, y) = [w_1 \cdot \widetilde{w}_1] \times [w_2]$ and $\epsilon = (\epsilon_1, \epsilon_2) \in \{-1, 0, 1\}^2$, we set $\mathcal{C}_n^\epsilon(x, y) = [w_1'] \times [w_2']$, where $(w_1', w_2') \in A_1^{g(n)} \times A_2^n$ is the only pair such that $\delta_1(w_1', w_1 \cdot \widetilde{w}_1) = \epsilon_1$ and $\delta_2(w_2', w_2) = \epsilon_2$.*

In the sequel, by convention if ν is a positive Borel measure on $\mathbb{A}_1 \times \mathbb{A}_2$ and $\mathcal{C}_n^\epsilon(x, y) \cap \mathrm{supp}(\nu) = \emptyset$, we set $\nu(\mathcal{C}_n^\epsilon(x, y)) = \nu(\mathcal{C}_n(x, y))$.

Proposition 4.2. *Suppose that one of the properties* **(G1)**, **(G2)** *or* **(G3)** *holds. For all $q > 0$, for all $(x, y) \in \mathrm{supp}(\mu)$,*

$$\limsup_{n\to\infty} \min_{\epsilon\in\{-1,0,1\}^2} \left(\frac{\mu_q(\mathcal{C}_n^\epsilon(x,y))}{\mu(\mathcal{C}_n^\epsilon(x,y))^q\big|\mathcal{C}_n^\epsilon(x,y)\big|^{-\beta_\mu(q)}}\right)^{1/n} \geq 1.$$

Remark 4.1. *The conclusion of Proposition 4.2 also holds for all $q \leq 0$ if one of the properties* **(G1)**, **(G2)** *or* **(G3)** *holds. We did not state this result to underline the fact that the next Proposition 4.3 does not involve such a property when $q \leq 0$.*

Proposition 4.3. *Let $C = 4r_1r_2$. Let $q \in \mathbb{R}$ and assume that one of the properties* **(G1)**, **(G2)** *or* **(G3)** *holds if $q > 0$.*

For all $z \in \mathrm{supp}(\widetilde{\mu})$,

$$\limsup_{r\to 0^+} \left(\frac{\widetilde{\mu}_q(B(z, Cr))}{\widetilde{\mu}(B(z,r))^q r^{-\beta_\mu(q)}}\right)^{1/\log r^{-1}} \geq 1.$$

PROPOSITION 4.4. *Let $q \in \mathbb{R}$. For all $\epsilon \in \{-1,0,1\}^2$ and $\nu \in \{\mu,\mu_q\}$ one has* $\lim_{n\to\infty} \dfrac{\log \nu(\mathcal{C}_n^\epsilon(x,y))}{\log \big|\mathcal{C}_n^\epsilon(x,y)\big|} = \lim_{n\to\infty} \dfrac{\log \nu(\mathcal{C}_n(x,y))}{\log \big|\mathcal{C}_n(x,y)\big|}$ *μ_q-almost everywhere.*

COROLLARY 4.2. *For all $q \in \mathbb{R}$ one has $\widetilde{\mu}_q\big(E_{\widetilde{\mu}}(\beta'_\mu(q))\big) = \widetilde{\mu}_q\big(E_{\widetilde{\mu}_q}(q\beta'_\mu(q) - \beta_\mu(q))\big) = 1$.*

Proof of Proposition 4.2. Let $q > 0$. Let $(x,y) \in K$. Let $n \geq 2$ and let $[w_1 \cdot \widetilde{w}_1] \times [w_2]$ be the element of $\mathcal{F}_n$ equal to $\mathcal{C}_n(x,y)$. Let $\epsilon \in \{-1,0,1\}^2$. We can suppose that $\mathcal{C}^\epsilon(x,y) \cap K \neq \emptyset$. The set $\mathcal{C}^\epsilon(x,y)$ takes the form $[v_1 \cdot \widetilde{v}_1] \times [v_2]$.

It follows from (3.1) that if we denote $\dfrac{I_{1,n}(w_1)^q}{I_{q,n}(w_1)} \dfrac{I_{q,g(n)}(w_1 \cdot \widetilde{w}_1)^s}{I_{1,g(n)}(w_1 \cdot \widetilde{w}_1)^{qs}}$ by $\mathcal{U}_n(w_1,\widetilde{w}_1)$ we have

$$\begin{cases} \dfrac{\mu_q(\mathcal{C}_n(x,y))}{\mu(\mathcal{C}_n(x,y))^q |\mathcal{C}_n(x,y)|^{-\beta_\mu(q)}} \approx \mathcal{U}_n(w_1,\widetilde{w}_1) \\ \\ \dfrac{\mu_q(\mathcal{C}_n^\epsilon(x,y))}{\mu(\mathcal{C}_n^\epsilon(x,y))^q |\mathcal{C}_n^\epsilon(x,y)|^{-\beta_\mu(q)}} \approx \mathcal{U}_n(v_1,\widetilde{v}_1) \end{cases}. \tag{4.4}$$

We have $|\delta_1(w_1 \cdot \widetilde{w}_1, v_1 \cdot \widetilde{v}_1)| \leq 1$. Let us distinguish several cases.

Case 1: $|\delta_1(w_1 \cdot \widetilde{w}_1, v_1 \cdot \widetilde{v}_1)| = 0$. In this case we have $\frac{\mathcal{U}_n(v_1,\widetilde{v}_1)}{\mathcal{U}_n(w_1,\widetilde{w}_1)} = 1$ and this holds in particular if **(G1)** holds (straightforward from the definition of **(G1)**).

Case 2: $|\delta_1(w_1 \cdot \widetilde{w}_1, v_1 \cdot \widetilde{v}_1)| = 1$ and **(G1)** does not hold.

If $|w_1 \cdot \widetilde{w}_1 \wedge v_1 \cdot \widetilde{v}_1| < g(n) - 1$, there exists a word u of length less than or equal to $g(n) - 2$ in $\widetilde{A}_1^*$ and an element $e \in \{0, \dots, r_1 - 2\}$ such that $(w_1 \cdot \widetilde{w}_1, v_1 \cdot \widetilde{v}_1) \in \big\{(u \cdot e \cdot (r_1-1)^{\cdot g(n)-|u|-1}, u \cdot (e+1) \cdot 0^{\cdot g(n)-|u|-1}), (u \cdot (e+1) \cdot 0^{\cdot g(n)-|u|-1}, u \cdot e \cdot (r_1-1)^{\cdot g(n)-|u|-1})\big\}$. If **(G2)** holds this implies that $\mathcal{C}^\epsilon(x,y) \cap K = \emptyset$, a contradiction with our initial assumption on ϵ. Thus if **(G2)** holds we have $|w_1 \cdot \widetilde{w}_1 \wedge v_1 \cdot \widetilde{v}_1| \geq g(n) - 1$, and the words $w_1 \cdot \widetilde{w}_1$ and $v_1 \cdot \widetilde{v}_1$ differ at most by their last letter, and it it is clear from the submultiplicative property of $I_{r,n}$ (see Lemma 2.1) that there exists a constant $c > 0$ which depends on q only such that $c^{-1} \leq \frac{\mathcal{U}_n(v_1,\widetilde{v}_1)}{\mathcal{U}_n(w_1,\widetilde{w}_1)} \leq c$.

Now, suppose that **(G3)** holds. For $r \in \{1,q\}$ there exists a positive number λ_r such that $\big|\lambda_r - \frac{1}{n}\log I_{r,n}(j^{\cdot n})\big| = O(1/n)$ for $j \in \{0, r_1 - 1\}$ (consequence of the submultiplicativity property established in Lemma 2.1). Consequently, due to the form taken by $(w_1 \cdot \widetilde{w}_1, v_1 \cdot \widetilde{v}_1)$, there exists a constant c' depending only on q such that $c'^{-1} \leq \frac{\mathcal{U}_n(v_1,\widetilde{v}_1)}{\mathcal{U}_n(w_1,\widetilde{w}_1)} \leq c'$.

In summary, if one of the properties **(G1)**, **(G2)** or **(G3)** holds, there exists $c" > 0$ depending on q only such that for all $(x,y) \in \operatorname{supp}(\mu)$ and $\epsilon \in \{-1,0,1\}^2$ such that $\mathcal{C}_n^\epsilon(x,y) \cap K \neq \emptyset$,

$$\frac{\mu_q(\mathcal{C}_n^\epsilon(x,y))}{\mu(\mathcal{C}_n^\epsilon(x,y))^q |\mathcal{C}_n^\epsilon(x,y)|^{-\beta_\mu(q)}} \geq c" \frac{\mu_q(\mathcal{C}_n(x,y))}{\mu(\mathcal{C}_n(x,y))^q |\mathcal{C}_n(x,y)|^{-\beta_\mu(q)}}.$$

Finally, the conclusion follows by using Proposition 3.1.

□

Proof of Proposition 4.3. Let $z \in \pi(K)$ and $(x, y) \in K$ such that $z = \pi(x, y)$ and $r \in (0, 1)$. Let $n_r \geq 1$ be the smallest integer n such that $\pi(\mathcal{C}_n(x, y)) \subset B(z, r)$.

Let us first suppose that $q \leq 0$. We have

$$\frac{\widetilde{\mu}_q(B(z, r))}{\widetilde{\mu}(B(z, r))^q} \geq \frac{\mu_q\big(\mathcal{C}_{n_r}(x, y)\big)}{\mu\big(\mathcal{C}_{n_r}(x, y)\big)^q}.$$

Suppose now that $q > 0$. Let $C_1 = 2r_2$ and $c_1 = (2r_1)^{-1}$. A verification shows that $B(z, c_1 r) \subset \bigcup_{\epsilon \in \{-1,0,1\}^2} \pi\big(\mathcal{C}^{\epsilon}_{n_r-1}(x, y)\big) \subset B(z, C_1 r)$. Let ϵ_r be such that $\mu\big(\mathcal{C}^{\epsilon_r}_{n_r-1}(x, y)\big) = \max_{\epsilon \in \{-1,0,1\}^2} \mu\big(\mathcal{C}^{\epsilon}_{n_r-1}(x, y)\big)$. We have

$$\frac{\widetilde{\mu}_q(B(z, C_1 r))}{\widetilde{\mu}(B(z, c_1 r))^q} \geq \frac{\mu_q\big(\mathcal{C}^{\epsilon_r}_{n_r-1}(x, y)\big)}{9^q \mu\big(\mathcal{C}^{\epsilon_r}_{n_r-1}(x, y)\big)^q}.$$

Since $\lim_{r \to 0^+} \log_{r_2}(r)/n_r = -1$, the conclusion follows from Proposition 3.1 when $q \leq 0$ and Proposition 4.2 when $q > 0$.

□

Proof of Proposition 4.4. Let $\epsilon = (\epsilon_1, \epsilon_2) \in \{-1, 0, 1\}^2$. If $(x, y) \in K$ and $n \geq 2$, write $C_n(x, y) = [w_1 \cdot \widetilde{w}_1(x, y)] \times [w_2(x, y)]$ and $\mathcal{C}^{\epsilon}_n(x, y) = [v_1 \cdot \widetilde{v}_1(x, y)] \times [v_2(x, y)]$.

By using the same approach as in the proof of Proposition 4.2, we get

$$(w_1 \cdot \widetilde{w}_1(\cdot), v_1 \cdot \widetilde{v}_1(\cdot)) \in \Big\{W^{(1)}_n(u, e) : u \in \bigcup_{k=0}^{g(n)-1} \widetilde{A}_1^k,\ e \in \widetilde{A}_1 \setminus \{r_1 - 1\}\Big\} \bigcup \Delta_1^{g(n)}$$

and

$$(w_2(\cdot), v_2(\cdot)) \in \Big\{W^{(2)}_n(u, e) : u \in \bigcup_{k=0}^{n-1} \widetilde{A}_2^k,\ e \in \widetilde{A}_2 \setminus \{r_2 - 1\}\Big\} \bigcup \Delta_2^n,$$

where $W^{(1)}_n(u, e) = \big\{(a, b), (b, a)\big\}$ with $a = u \cdot e \cdot (r_1 - 1)^{\cdot g(n)-|u|-1}$ and $b = u \cdot (e + 1) \cdot 0^{\cdot g(n)-|u|-1}$, $W^{(2)}_n(u, e) = \big\{(c, d), (d, c)\big\}$ with $c = u \cdot e \cdot (r_2 - 1)^{\cdot n-|u|-1}$ and $d = u \cdot (e + 1) \cdot 0^{\cdot n-|u|-1}$, and $\Delta_i^k = \{(u, u) : u \in \widetilde{A}_1^k\}$ for $i \in \{1, 2\}$ and $k \geq 1$.

We need the following lemma whose interpretation is that for μ_q- almost every (x, y), $|w_1 \cdot \widetilde{w}_1(x, y) \wedge v_1 \cdot \widetilde{v}_1(x, y)|$ and $|w_2(x, y) \wedge v_2(x, y)|$ are respectively asymptotically equivalent to $g(n)$ and n, that is to say the words $(w_1 \cdot \widetilde{w}_1(x, y), w_2(x, y))$ and $((v_1 \cdot \widetilde{v}_1(x, y), v_2(x, y))$ are almost the same.

LEMMA 4.1. *For every $\alpha \in (0, 1)$, for μ_q-almost every $(x, y) \in K$, for n large enough one has*

$$(w_1 \cdot \widetilde{w}_1(x), v_1 \cdot \widetilde{v}_1(x)) \in \Big\{W^{(1)}_n(u, e) : u \in \bigcup_{k=[(1-\alpha)g(n)]}^{g(n)-1} \widetilde{A}_1^k,\ 0 \leq e \leq r_1 - 2\Big\} \bigcup \Delta_1^{g(n)}$$

and

$$(w_2(y), v_2(y)) \in \Big\{W^{(2)}_n(u, e) : u \in \bigcup_{k=[(1-\alpha)n]}^{n-1} \widetilde{A}_2^k,\ 0 \leq e \leq r_2 - 2\Big\} \bigcup \Delta_2^n.$$

Let $\alpha \in (0,1)$. Let K_α be a subset of K of full μ_q-measure such that the conclusion of Lemma 4.1 holds. If $(x,y) \in K_\alpha$ and $\mathcal{C}_n^\epsilon(x,y) \cap K \neq \emptyset$, it follows from Lemmas 2.3 and 2.4 applied to μ and μ_q that there exists a constant depending on q only such that for $\nu \in \{\mu, \mu_q\}$

$$C^{-\alpha g(n)} \leq \frac{\nu\big(\mathcal{C}_n^\epsilon(x,y)\big)}{\nu\big(\mathcal{C}_n(x,y)\big)} \leq C^{\alpha g(n)} \quad \text{for } n \text{ large enough.} \tag{4.5}$$

Since (4.5) holds for all $(x,y) \in \bigcap_{p\geq 2} K_{1/p}$ which is of full μ_q-measure, we obtain the desired conclusion. □

Proof of Lemma 4.1. Let $\alpha \in (0,1)$. For $n \geq 2$ let $n(\alpha)$ and $n'(\alpha)$ stand for the integer part of $(1-\alpha)g(n)$ and $(1-\alpha)n$ respectively. Due to the Borel-Cantelli lemma, it is enough to show that

$$\sum_{n\geq 2} \mu_q\Big(\Big\{(w_1\cdot\widetilde{w}_1(\cdot), v_1\cdot\widetilde{v}_1(\cdot)) \in \Big\{W_n^{(1)}(u,e) : u \in \bigcup_{k=0}^{n(\alpha)} \widetilde{A}_1^k,\ 0\leq e\leq r_1-2\Big\}\Big\}\Big) < \infty$$

and

$$\sum_{n\geq 2} \mu_q\Big(\Big\{(w_2(\cdot), v_2(\cdot)) \in \Big\{W_n^{(2)}(u,e) : u \in \bigcup_{k=0}^{n'(\alpha)} \widetilde{A}_2^k,\ 0\leq e\leq r_2-2\Big\}\Big\}\Big) < \infty.$$

The first inequality is equivalent to

$$\sum_{n\geq 2} \sum_{k=g(n)-n(\alpha)-1}^{g(n)-1} \sum_{u\in\widetilde{A}_1^{g(n)-k-1}} \sum_{e=0}^{r_1-2} \big[\mathbb{P}_q([u\cdot e\cdot (r_1-1)^{\cdot k}]) + \mathbb{P}_q([u\cdot(e+1)\cdot 0^{\cdot k}])\big] < \infty$$

(we made the change of variable $k' = g(n)-k-1$) and because of the quasi-Bernoulli property of $\mathbb{P}_q$ this is also equivalent to

$$\sum_{n\geq 2} \sum_{k=g(n)-n(\alpha)-1}^{g(n)-1} \big[\mathbb{P}_q([(r_1-1)^{\cdot k}]) + \mathbb{P}_q([0^{\cdot k}])\big] < \infty.$$

Moreover, again due to the submultiplicativity properties of $\mathbb{P}_q$ and the fact that $\mathbb{P}_q$ is atomless (see the proof of Proposition 4.1), both $\mathbb{P}_q([(r_1-1)^{\cdot k}])$ and $\mathbb{P}_q([0^{\cdot k}])$ tend to 0 exponentially fast as k goes to ∞. Thus, there exists $C>0$ and $\lambda\in(0,1)$ such that for all $n\geq 2$, $\sum_{k=g(n)-n(\alpha)-1}^{g(n)-1} \mathbb{P}_q([(r_1-1)^{\cdot k}]) + \mathbb{P}_q([0^{\cdot k}]) \leq Cg(n)\lambda^{g(n)\alpha}$.

The second inequality is equivalent to

$$\sum_{n\geq 2} \sum_{w_1\in\widetilde{A}_1^n} \sum_{k=n-n'(\alpha)-1}^{n-1} \sum_{u\in\widetilde{A}_2^{n-k-1}} \sum_{e=0}^{r_2-2} \big[\mu_q([w_1]\times[c]) + \mu_q([w_1]\times[d])\big] < \infty,$$

with $c = u\cdot e\cdot (r_2-1)^{\cdot k}$ and $d = u\cdot(e+1)\cdot 0^{\cdot k}$. Due to Lemma 2.2, this is also equivalent to

$$\sum_{n\geq 2} \sum_{k=n-n'(\alpha)-1}^{n-1} \sum_{w\in\widetilde{A}_1^k} \big[\mu_q([w]\times[(r_2-1)^{\cdot k}]) + \mu_q([w]\times[0^{\cdot k}])\big] < \infty,$$

that is to say

$$\sum_{n\geq 2}\sum_{k=n-n'(\alpha)-1}^{n-1}\big[\mu_q(\mathbb{A}_1\times[(r_2-1)^{\cdot k}])+\mu_q(\mathbb{A}_1\times[0^{\cdot k}])\big]<\infty.$$

It is easily seen by using Lemma 2.2 again that for $e\in\{0,r_2-1\}$, the sequence $(\mu_q(\mathbb{A}_1\times[e^{\cdot k}]))_{k\geq 1}$ is submultiplicative. Moreover, we saw in the proof of Proposition 4.1 that $\mu_q(\mathbb{A}_1\times[0^{\cdot k}])$ goes to 0 as k tends to ∞. The same arguments show that it is also the case for $\mu_q(\mathbb{A}_1\times[(r_2-1)^{\cdot k}])$.

Consequently there exists $C'>0$ and $\lambda'\in(0,1)$ such that for all $k\geq 1$,

$$\sum_{k=n-n'(\alpha)-1}^{n-1}\big[\mu_q(\mathbb{A}_1\times[(r_2-1)^{\cdot k}])+\mu_q(\mathbb{A}_1\times[0^{\cdot k}])\big]\leq C'n\lambda'^{n\alpha},$$

and the conclusion follows. □

Proof of Corollary 4.2. Use the same relation

$$B(z,c_1r)\subset\bigcup_{\epsilon\in\{-1,0,1\}^2}\pi\big(\mathcal{C}^{\epsilon}_{n_r-1}(x,y)\big)\subset B(z,C_1r)$$

as in the proof of Corollary 4.3, as well as Proposition 4.4 and Proposition 3.2. □

4.2. *Proof of Theorem 1.2.*

The upper bound for the dimensions of the sets $E_\mu(\beta'_\mu(q))$ is a consequence of Proposition 4.3. Indeed, by using standard techniques one shows that under the assumptions of Proposition 4.3, the generalized Hausdorff dimension $b_{\widetilde{\mu}}(q)$ introduced in [**21**] is less than or equal to $-\beta_\mu(q)$. Moreover, Proposition 2.5 in [**21**] yields $\dim E_{\widetilde{\mu}}(\alpha)\leq(-b_{\widetilde{\mu}})^*(\alpha)$ for all $\alpha\geq 0$.

The lower bound follows immediately from Corollary 4.2 and the mass distribution principle (see for instance p. 43 in [**24**]).

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