Applied Probability Trust (9 February 2007)

RENEWAL OF SINGULARITY SETS OF RANDOM SELF-SIMILAR MEASURES

JULIEN BARRAL,* *INRIA Rocquencourt*

STEPHANE SEURET,** *Université Paris-XII*

Abstract

This paper investigates new properties concerning the multifractal structure of a class of random self-similar measures. These measures include the well-known Mandelbrot multiplicative cascades, sometimes called independent random cascades. We evaluate the scale at which the multifractal structure of these measures becomes discernible. The value of this scale is obtained through what we call the growth speed in Hölder singularity sets of a Borel measure. This growth speed yields new information on the multifractal behavior of the rescaled copies involved in the structure of statistically self-similar measures. Our results are useful to understand the multifractal nature of various heterogeneous jump processes.

Keywords: Random measures—Large deviations—Hausdorff dimension—Self-similarity—Fractals

AMS 2000 Subject Classification: Primary 60G57—60F10—28A78

Secondary 28A80

1. Introduction

This paper investigates new properties concerning the multifractal structure of random self-similar measures. The class of measures to which our results apply includes the well-known Mandelbrot multiplicative cascades [39], sometimes called independent

* Postal address: Equipe Sosso2, Domaine de Voluceau Rocquencourt, 78153 Le Chesnay cedex, France. Email-address: julien.barral@inria.fr

** Postal address: Laboratoire d'Analyse et de Mathématiques Appliquées - Faculté de Sciences et Technologie, Bat. P3, 4ème étage, 61, avenue du Général de Gaulle, 94010 Créteil cedex, France. Email-address: seuret@univ-paris12.fr

random cascades. The case of another important class, the random Gibbs measures, is treated in [14]. Since these two important subclasses of random self-similar measures are extensively discussed in the sequel, in order to fix ideas, we recall now their definitions:

- **Random Gibbs measure**, as considered in [14], are obtained as follows.
  Let $b \geq 2$ be an integer. Let ϕ be a Hölder continuous function on $\mathbb{R}^d$. Assume that ϕ is 1-periodic in each variable. Let also $\omega = (\omega_n)_{n\geq 0}$ be a sequence of independent random phases uniformly distributed in $[0,1]^d$. Let T be the shift transformation on $[0,1)^d$: $T(t_1, \cdots, t_d) = (bt_1 \mod 1, \cdots, bt_d \mod 1)$. The Birkhoff sums associated with ϕ are defined by
  $$\forall\, t \in [0,1)^d,\ \forall\, n \geq 1,\ S_n(\phi,\omega)(t) = \sum_{k=0}^{n-1} \phi(T^k t + \omega_k),$$
  where T^k stands for the k^{th} iterate of T. It follows from the thermodynamic formalism [48, 32] that the random sequence of measures on $[0,1]^d$ defined by $\mu_n(dt) = \frac{\exp\left(S_n(\phi,\omega)(t)\right)}{\int_{[0,1]} \exp\left(S_n(\phi,\omega)(u)\right)\, du} dt$ converges almost surely, as $n \to \infty$, to a measure called random Gibbs measure.
- We also focus on **Independent random cascades,** (also referred to as canonical cascades [39]).
  Let X be a real valued random variable. Define $L:\ q \in \mathbb{R} \mapsto d\log(b) + \log \mathbb{E}(\exp(qX))$, and assume that $L(1) < \infty$. For every b-adic box B included in $[0,1]^d$, let X_B be a copy of X. Moreover, assume that the X_B's are mutually independent. A branching random walk S_n is then defined by
  $$\forall\, t \in [0,1)^d,\ \forall\, n \geq 1,\ S_n(t) = \sum_{1\leq j\leq n} \sum_{B\in\mathcal{I}_j} \mathbf{1}_B(t) X_B,$$
  where $\mathcal{I}_j$ stands for the set of b-adic cubes of generation j.
  The canonical cascade measure μ is then obtained as the almost sure weak limit of the sequence μ_n on $[0,1]^d$ given by $\mu_n(dt) = \left(\mathbb{E}(\exp(X))\right)^{-n} \exp(S_n(t))\, dt$. Let $\theta:\ q \in \mathbb{R} \mapsto \frac{qL(1)-L(q)}{\log(b)}$. In [39, 31], it is shown that $\theta'(1^-) > 0$ is a necessary and sufficient condition for μ to be almost surely a positive measure with support equal to $[0,1]^d$.

We now expose the purpose and the main results of this work, and their connection with multifractal analysis. Multifractal analysis is a field introduced by physicists in the context of fully developed turbulence [24]. It is now widely accepted as a pertinent tool in modeling other physical or social phenomena characterized by extreme spatial (or temporal) variability [40, 44, 35]. Given a positive measure μ defined on a compact subset of $\mathbb{R}^d$, performing the multifractal analysis of μ consists in computing (or estimating) the Hausdorff dimension $d_\mu(\alpha)$ of Hölder singularities sets E^μ_α. Let $B(t,r))$ stand for the closed ball of radius r centered at t. These sets E^μ_α are the level sets associated with the Hölder exponent

$$h_\mu(t) = \lim_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)}$$

(whenever it is defined at t). Thus

$$E^\mu_\alpha = \{t : h_\mu(t) = \alpha\}. \tag{1.1}$$

Of course, these limit behaviors are numerically incalculable, both when simulating model measures or when processing real data. Nevertheless, this difficulty can be circumvented since the Hausdorff dimension $d_\mu(\alpha)$ of E^μ_α can sometimes be numerically estimated by counting at scale 2^{-j} the number of dyadic boxes B such that $\mu(B) \approx 2^{-j\alpha}$ (this could be done with any regular fine grid). This number can formally be defined, for any scale $j \geq 0$, $\varepsilon > 0$ and $\alpha > 0$, by

$$N^\varepsilon_j(\alpha) = \#\left\{B \in \mathcal{I}_j : b^{-j(\alpha+\varepsilon)} \leq \mu(B) \leq b^{-j(\alpha-\varepsilon)}\right\}. \tag{1.2}$$

Then when some multifractal formalisms are fulfilled, it can be shown that, for some $\alpha > 0$,

$$d_\mu(\alpha) = \lim_{\varepsilon\to 0} \lim_{j\to+\infty} \frac{\log N^\varepsilon_j(\alpha)}{\log 2^j}. \tag{1.3}$$

This is the case for instance for the multifractal measures used as models [39] in the applications mentioned above. It is thus natural to seek theoretical results giving estimates of the first scale from which a substantial part of the singularity set E^μ_α is discernible when measuring the μ-mass of the elements B of the regular grid. In other words, we search for the first generation $J \geq 0$ such that for every $j \geq J$, $N^\varepsilon_j(\alpha) \approx 2^{jd_\mu(\alpha)}$. This is of course important for numerical applications and modeling.

The properties studied in this paper and in [14] rely on this question. We provide new accurate information on the fine structure of multiplicative cascades. They bring some answers to the above problems. As an interesting by-product of this work, we obtain that Mandelbrot measures and Gibbs measures have very different behaviors from the statistical self-similarity point of view, while it is known that they cannot be distinguished by the form of their multifractal spectra. Finally, our results are critical tools for the Hausdorff dimension estimate of a new class of limsup sets (see (1.8)) involved in multifractal analysis of recent jump processes [12, 11, 13].

1.1. A definition of random self-similar measures

We now specify what we mean by random self-similar measure. Our point of view takes into account a structure which often arises in the construction of random measures generated by multiplicative processes.

Let $\mathcal{I}$ be the set of closed b-adic sub-hypercubes of $[0,1]^d$. A random measure $\mu(\omega)$ on $[0,1]^d$ ($d \geq 1$) is said to be random self-similar if there exist an integer $b \geq 2$, a sequence $Q_n(t,\omega)$ of random non-negative functions, and a sequence of random measures $(\mu^I)_{I\in\mathcal{I}}$ on $[0,1]^d$ such that:

1. For every $I \in \mathcal{I}$ and $g \in C([0,1]^d,\mathbb{R})$, ($\stackrel{d}{\equiv}$ means equality in distribution)
$$\int_{[0,1]^d} g(u)\mu(\omega)(du) \stackrel{d}{\equiv} \int_{[0,1]^d} g(u)\mu^I(\omega)(du). \tag{1.4}$$
2. With probability one, for every $n \geq 1$, $I \in \mathcal{I}$ of generation n and every $g \in C(I,\mathbb{R})$,
$$\int_I g(v)\mu(\omega)(dv) = \ell(I)\int_I Q_n(v,\omega)\, g(v)\mu^I(\omega)\circ f_I^{-1}(dv), \tag{1.5}$$
where f_I stands for a similitude that maps $[0,1]^d$ onto I and ℓ is the Lebesgue measure.

Property 1. asserts that the measures μ^I and μ have the same probability distribution. Property 2. asserts that, up to the density Q_n, the behavior of the restriction of μ to I is ruled by the rescaled copy μ^I of μ.

Of course, the random density $Q_n(t,\cdot)$ plays a fundamental rôle, both in the construction of the measure μ, which is often equal to the almost sure weak limit of $Q_n(t,\cdot)\cdot\ell$, and in the local behavior of μ at scale b^{-n}.

We restrict ourselves to measures with support equal to $[0,1]^d$. Up to technical refinements, our point of view can easily be extended to measures which support is the limit set of more general iterated random similitudes systems ([27, 21, 45, 1, 5]).

The classes of measures described above illustrate conditions (1.4) and (1.5). Gibbs measures appear in random dynamical systems [32]. Independent multiplicative cascades [39, 31]) are in fact contained in a wider class of $[0,1]^d$-martingales (in the sense of [29, 30]), see [9] for details. This larger class also contains compound Poisson cascades [8] and their extensions [3, 9]. As claimed above, these two classes are quite identical regarding their multifractal structure in the sense that any measure in these classes is ruled by the so-called multifractal formalisms [17, 46]. However, the study of their self-similarity properties reveals the notable differences between them. These differences are consequences of their construction's schemes: For the class of random Gibbs measures, the copies μ^I used in (1.4) and (1.5) only depend on the generation of I, while they are all different for $[0,1]^d$-martingales (since they depend on the interval I) as illustrated by the construction of canonical cascades.

This difference is quantitatively measured thanks to a notion which is related with the multifractal structure, namely the *growth speed* in the μ^I Hölder singularities sets $E_\alpha^{\mu^I}$ (see Section 1.3 and Theorems A and B in Section 1.4 below). This quantity is precisely defined and studied in the rest of the paper for independent random cascades. It yields an estimate of the largest scale from which the observation of the μ^I's mass distribution accurately coincides with the prediction of the multifractal formalism.

1.2. New limsup-sets and conditioned ubiquity

The notion of growth speed in Hölder singularities sets defined in next Sections 1.3 naturally appears in the computation of the Hausdorff dimension of a new type of limsup-sets. These limsup sets are closely connected to the level sets of the pointwise Hölder exponent of some heterogeneous jump processes considered in [12, 13]. This is why these sets are so interesting.

Let μ be a finite Borel measure whose support is $[0,1]$. The heterogeneous jump processes we consider are either purely discontinuous measures which have the form

$$\sum_{j\geq 0}\ \sum_{0\leq k\leq b^j-1} j^{-2}\ \mu([kb^{-j},(k+1)b^{-j}])\ \delta_{kb^{-j}},$$

or the Lévy processes X in multifractal time μ defined as $\big(X \circ \mu([0,t])\big)_{0\le t\le 1}$. The fractal geometry of limsup sets already occupied an important rôle in performing the multifractal nature of homogeneous sums of Dirac masses [22] and Lévy processes [28]. In this homogeneous context, the measure μ is the Lebesgue measure, and computing the dimension of these sets relies on the notion of ubiquity (see [19] for instance). The study of $\big(X \circ \mu([0,t])\big)_{0\le t\le 1}$ necessitates the notion of heterogeneous (or conditioned) ubiquity introduced in [12, 10]. The work achieved in [13] is a fundamental step in the study of the fractal nature of the path of processes under multifractal subordination. The importance of this topic comes from the fact that such processes have been introduced as relevant model for the applications, especially in mathematical finance [40, 42, 41]. The reader will find an extensive study of processes in multifractal time in [50], which provides for instance the large deviation spectrum of Lévy processes in multifractal time, that is to say a statistical description of the variations of these processes rather than the geometric one given in [13].

We now describe the nature of the limsup sets discussed above. Let μ be a random self-similar measure whose support is $[0,1]$. Let $\widetilde{x} = (x_n)_{n\ge 1}$ denote a countable set of points in $[0,1]^d$ and $\widetilde{\lambda} = (\lambda_n)_{n\ge 1}$ be a positive sequence decreasing to 0 such that $\limsup_{n\to\infty} B(x_n,\lambda_n) = [0,1]$. For $h>0$, $\xi>1$ and $\widetilde{\varepsilon} = (\varepsilon_n)_{n\ge 1}$ a positive sequence converging to 0, let

$$K(\mu,h,\xi,\widetilde{x},\widetilde{\lambda},\widetilde{\varepsilon}) = \bigcap_{N\ge 1} \bigcup_{n\ge N:\, \lambda_n^{h+\varepsilon_n} \le \mu([x_n-\lambda_n,x_n+\lambda_n]) \le \lambda_n^{h-\varepsilon_n}} [x_n-\lambda_n^{\xi}, x_n+\lambda_n^{\xi}]. \tag{1.6}$$

Heuristically, $K(\mu,h,\xi,\widetilde{x},\widetilde{\lambda},\widetilde{\varepsilon})$ contains the points that infinitely often belong to an interval of the form $[x_n-\lambda_n^{\xi}, x_n+\lambda_n^{\xi}]$, upon the condition that $\mu([x_n-\lambda_n,x_n+\lambda_n]) \sim \lambda_n^h$. This condition implies that μ has roughly a Hölder exponent h at x_n at scale λ_n. One of the main results of [12, 10, 11] is the computation of the Hausdorff dimension of $K(\mu,h,\xi,\widetilde{x},\widetilde{\lambda},\widetilde{\varepsilon})$. The value of this dimension is related to the free energy function τ_μ considered in the multifractal formalism for measures in [26, 17]. For every $q\in\mathbb{R}$ and for every integer $n\ge 1$, introduce the quantities

$$\tau_{\mu,n}(q) = -\frac{1}{n}\log_b \sum_{I\in\mathcal{I}_n} \mu(I)^q \quad\text{and}\quad \tau_\mu(q) = \liminf_{n\to\infty} \tau_{\mu,n}(q). \tag{1.7}$$

The Legendre transform τ_μ^* of τ_μ at $h>0$ is defined as $\tau_\mu^*(h) := \inf_{q\in\mathbb{R}} hq - \tau_\mu(q)$.

Under suitable assumptions, we prove in [12, 10] that, for all h such that $\tau_\mu^*(h) > 0$ and all $\xi \geq 1$, there exists $\widetilde{\varepsilon}$ such that, with probability one, (dim stands for the Hausdorff dimension)

$$\dim K(\mu, h, \xi, \widetilde{x}, \widetilde{\lambda}, \widetilde{\varepsilon}) = \tau_\mu^*(h)/\xi. \tag{1.8}$$

This achievement is a non-trivial generalization of what is referred to as "ubiquity properties" (see [19] and references therein) of the resonant system ([2]) $\{(x_n, \lambda_n)\}_n$. The main difficulty here lies in the fact that μ may be a multifractal measure and not just the uniform Lebesgue measure. Results on growth speed in Hölder singularity set are determinant to obtain estimate (1.8).

Suppose that μ is an independent random cascade whose support is $[0,1]$ and X a stable Lévy process of index $\beta \in (0,1)$. Let ν stands for the derivative of $\big(X \circ \mu([0,t])\big)_{0 \leq t \leq 1}$. It is shown in [13] that there exists $\widetilde{x}$, $\widetilde{\lambda}$ and $\widetilde{\varepsilon}$ as above such that for every $h \in (0, \tau_\mu'(1)/\beta]$, the level set E_h^ν differs from $K\big(\tau_\mu'(1), \tau_\mu'(1)/\beta h, \widetilde{x}, \widetilde{\lambda}, \widetilde{\varepsilon}\big)$ by (roughly speaking) only a small set.

1.3. Growth speed in μ^I's Hölder singularity sets

Let μ be a random self-similar positive Borel measure as described in Section 1.1. As we said, multifractal analysis of μ [27, 34, 43, 23, 1, 7, 5] usually considers Hölder singularities sets of the form (1.1) and their Hausdorff dimension $d_\mu(\alpha)$, which is a measure of their size. The method used to compute $d_\mu(\alpha)$ is to find a random measure μ_α (of the same nature as μ) such that μ_α is concentrated on $E_\alpha^\mu \cap E^{\mu_\alpha}_{\tau_\mu^*(\alpha)}$. This measure μ_α is often referred to as an analyzing measure of μ at α. Then, by the Billingsley lemma ([16] pp 136–145), we get $d_\mu(\alpha) = \tau_\mu^*(\alpha)$, and the multifractal formalism for measures developed in [17] is said to hold for μ at α. Finally, the estimate (1.3) is a direct consequence of the multifractal formalism ([49]) for the large deviation spectrum. Thus the existence of μ_α has important consequences regarding the possibility of measuring the mass distribution of μ at large enough resolutions.

In this paper we refine the classical approach by considering, instead of the level sets E_α^μ, the finer level sets $\widetilde{E}_{\alpha,p}^\mu$ and $\widetilde{E}_\alpha^\mu$ defined for a sequence $(\varepsilon_n)_n$ going down to 0 by

$$\begin{aligned} \widetilde{E}^{\mu}_{\alpha,p} &= \left\{ t\in[0,1]^d : \forall\ n\geq p,\ b^{-n(\alpha+\varepsilon_n)} \leq \mu\big(I_n(t)\big)\big) \leq b^{-n(\alpha-\varepsilon_n)} \right\} && (1.9)\\ \text{and}\quad \widetilde{E}^{\mu}_{\alpha} &= \bigcup_{p\geq 1} E^{\mu}_{\alpha,p} && (1.10) \end{aligned}$$

($I_n(t)$ stands for the b-adic cube of generation n containing t). It is possible to choose $(\varepsilon_n)_{n\geq 1}$ so that with probability one, for all the exponents α such that $\tau_\mu^*(\alpha)>0$, we have $\mu_\alpha(\widetilde{E}^{\mu}_{\alpha}) = \|\mu_\alpha\|$ ($\|\mu_\alpha\|$ stands for the total mass of μ_α).

Since the sets sequence $(\widetilde{E}^{\mu}_{\alpha,p})_{p\geq 1}$ is non-decreasing and $\mu_\alpha(\widetilde{E}^{\mu}_{\alpha}) = \|\mu_\alpha\|$, we can define the growth speed of $(\widetilde{E}^{\mu}_{\alpha,p})$ as the smallest value of p for which the μ_α-measure of $\widetilde{E}^{\mu}_{\alpha,p}$ reaches a certain positive fraction $f\in(0,1)$ of the mass of μ_α, i.e.

$$GS(\mu,\alpha) = \inf\left\{ p : \mu_\alpha(\widetilde{E}^{\mu}_{\alpha,p}) \geq f\,\|\mu_\alpha\| \right\}.$$

For each copy μ^I of μ, the corresponding family of analyzing measures μ^I_α exists and are related with μ^I as μ_α is related with μ. The result we focus on in the following is the asymptotic behavior as the generation of I goes to ∞ of

$$GS(\mu^I,\alpha) = \inf\left\{ p : \mu^I_\alpha\big(\widetilde{E}^{\mu^I}_{\alpha,p}\big) \geq f\,\|\mu^I_\alpha\| \right\}. \tag{1.11}$$

This number yields an estimate of the number of generations needed to observe a substantial amount of the singularity set $E^{\mu^I}_{\alpha}$. Let

$$\mathcal{N}_n(\mu^I,\alpha) = \#\big\{ B\in\mathcal{I}_n : b^{-n(\alpha+\varepsilon_n)} \leq \mu^I(B) \leq b^{-n(\alpha-\varepsilon_n)} \big\}.$$

As a counterpart to controlling $GS(\mu^I,\alpha)$, we shall also control the smallest rank n from which $\mathcal{N}_n(\mu^I,\alpha)$ behaves like $b^{n\tau^*_\mu(\alpha)}$. This rank is defined by

$$GS'(\mu^I,\alpha) = \inf\big\{ p : \forall\ n\geq p, b^{n(\tau^*_\mu(\alpha)-\varepsilon_n)} \leq \mathcal{N}_n(\mu^I,\alpha) \leq b^{n(\tau^*_\mu(\alpha)+\varepsilon_n)} \big\} \tag{1.12}$$

and yields far more precise information than a result like (1.3). We should expect that (ε_n) and $GS(\mu^I,\alpha)$ are related through some constraints. This is indeed the case and this point is discussed in Theorem 3.2, Remarks 3.1 and 3.2.

1.4. A simplified version of the main results

In this paper, we focus on the one-dimensional case and we deal with independent random cascades, which are a slight extension of the first example of $[0,1]$-martingales introduced in [39] (see Section 3.1). We start with a recall of the theorem proved in [14], and then we give simplified versions of the main results detailed in Section 3.

Theorem A. *Let μ be a random Gibbs measure as defined above (in particular in (1.4) and (1.5) $\mu^I = \mu^{I'}$ if I and I' are of the same generation). Let $\beta > 0$. There exists a choice of $(\varepsilon_n)_{n\geq 1}$ such that, with probability one, for all $\alpha > 0$ such that $\tau_\mu^*(\alpha) > 0$, if I is of generation j large enough, then $GS(\mu^I, \alpha) \leq \exp\sqrt{\beta \log j}$.*

The fact that $GS(\mu^I, \alpha)$ behaves like $o(j)$ as $j \to \infty$ is a crucial property needed to establish (1.8) for random Gibbs measures.

Theorem **B** shall be compared with Theorem **A**. Under suitable assumptions, we have (see Theorem 3.2)

Theorem B. *Let μ be an independent random cascade. Let $\eta > 0$. There exists a choice of $(\varepsilon_n)_{n\geq 1}$ such that, with probability one, for all $\alpha > 0$ such that $\tau_\mu^*(\alpha) > 0$, if I is of generation j large enough, then $GS(\mu^I, \alpha) \leq j \log^\eta j$.*

Consequently the uniform upper bound for $GS(\mu^I, \alpha)$ by $o(j)$ when $I \in \mathcal{I}_j$ is lost. In fact this "worse" behavior is not surprising. Indeed, for an independent random cascade, at each resolution j, the behaviors at small scales of b^j distinct measures μ^I have to be controlled simultaneously. Nevertheless, this technical difficulty can be circumvented, by using a refinement of Theorem B (see Theorem 3.3 and 5.1), which is enough to get (1.8).

Theorem C. *Let μ be an independent random cascade. Let $\eta > 0$. There exists a choice of $(\varepsilon_n)_{n\geq 1}$ such that for every $\alpha > 0$ such that $\tau_\mu^*(\alpha) > 0$, with probability one, for μ-almost every t, for j large enough, $GS(\mu^{B_j(t)}, \alpha) \leq j \log^{-\eta} j$ (where $B_j(t)$ stands for the b-adic box of generation j containing t).*

The paper is organized as follows. Section 2 gives new definitions and establishes two general propositions useful for our main results. In Section 3 independent random cascades are defined in an abstract way. This makes it possible to consider Mandelbrot measures as well as their substitute in the critical case of degeneracy. Then the main results (Theorems 3.1, 3.2 and 3.3) are stated and proved. Theorem 3.4 is a counterpart of Theorem B in terms of $GS'(\mu^I, \alpha)$ (recall (1.12)). Theorem 3.5 deals with a problem connected with the estimate of the growth speed in singularities sets, namely the estimation of the speed of convergence of $\tau_{\mu,n}$ towards τ_μ. Section 4 provides the proofs of the results stated in Section 3. Eventually, Section 5 is devoted to the version of Theorem 3.3 needed to get (1.8).

The techniques presented in this paper can be applied to derive similar results for other random self-similar $[0,1]$-martingales described in [8, 3, 9].

2. General estimates for the growth speed in singularity sets

2.1. Measure of fine level sets: a neighboring boxes condition

Let $(\Omega, \mathcal{B}, \mathbb{P})$ stand for the probability space on which the random variables in this paper are defined. Fix an integer $b \geq 2$.

Let $\mathcal{A} = \{0, \dots, b-1\}$. For every $w \in \mathcal{A}^* = \bigcup_{n\geq 0} \mathcal{A}^n$ ($\mathcal{A}^0 := \{\emptyset\}$), let I_w be the closed b-adic subinterval of $[0,1]$ naturally encoded by w. If $w \in \mathcal{A}^n$, we set $|w| = n$.

For $n \geq 1$ and $0 \leq k \leq b^n - 1$, $I_{n,k}$ denotes the interval $[kb^{-n}, (k+1)b^{-n})$. If $t \in [0,1)$, $k_{n,t}$ is the unique integer such that $t \in [k_{n,t}b^{-n}, (k_{n,t}+1)b^{-n})$. We denote by $w^{(n)}(t)$ the unique element w of $\mathcal{A}^n$ such that $I_w = [k_{n,t}b^{-n}, (k_{n,t}+1)b^{-n}]$.

With $w \in \mathcal{A}^n$ can be associated a unique number $i(w) \in \{0, 1, \dots, b^n - 1\}$ such that $I_w = [i(w)b^{-n}, (i(w)+1)b^{-n}]$. Then, if $(v,w) \in \mathcal{A}^n$, $\delta(v,w)$ stands for $|i(v) - i(w)|$.

Let μ and m be two positive Borel measures with supports equal to $[0,1]$.

Let $\widetilde{\varepsilon} = (\varepsilon_n)_{n\geq 0}$ be a positive sequence, $N \geq 1$, and $\alpha \geq 0$.

We consider a slight refinement of the sets introduced in (1.10): For $p \geq 1$, define

$$E^\mu_{\alpha,p}(N, \widetilde{\varepsilon}) = \left\{ t \in [0,1] : \begin{cases} \forall\, n \geq p,\ \forall w \in \mathcal{A}^n \text{ such that } \delta(w, w^{(n)}(t)) \leq N, \\ \forall\ \gamma \in \{-1, 1\}, \quad b^{\gamma n(\alpha - \gamma\varepsilon_n)} \mu(I_w)^\gamma \leq 1 \end{cases} \right\} \tag{2.1}$$

$$\text{and } E^\mu_{\alpha}(N, \widetilde{\varepsilon}) = \bigcup_{p\geq 1} E^\mu_{\alpha,p}(N, \widetilde{\varepsilon}). \tag{2.2}$$

This set contains the points t for which, for every n large enough, the μ-measure of the $2N+1$ neighbors of I_{n,k_t} belongs to $[b^{-n(\beta+\varepsilon_n)}, b^{-n(\beta-\varepsilon_n)}]$. The information on neighboring intervals is involved in the proof of (1.8).

For $n \geq 1$ and $\varepsilon, \eta > 0$, consider the quantity

$$S_n^{N,\varepsilon,\eta}(m, \mu, \alpha) = \sum_{\gamma \in \{-1,1\}} b^{n(\alpha - \gamma\varepsilon)\gamma\eta} \sum_{v,w \in \mathcal{A}^n:\ \delta(v,w) \leq N} m(I_v)\mu(I_w)^{\gamma\eta}. \tag{2.3}$$

The following result is established in [14], but the proof is given for completeness.

Proposition 2.1. *Let μ and m be two positive Borel measures with supports equal to $[0,1]$. Let $\widetilde{\varepsilon}=(\varepsilon_n)_{n\geq 0}$ be a positive sequence, $N\geq 1$, and $\alpha\geq 0$. Let $(\eta_n)_{n\geq 1}$ be a positive sequence.*

If $\sum_{n\geq 1} S_n^{N,\varepsilon_n,\eta_n}(m,\mu,\alpha)<+\infty$, then $E^\mu_\alpha(N,\widetilde{\varepsilon})$ is of full m-measure.

Remark 2.1. Similar conditions were used in [6] to obtain a comparison between the multifractal formalisms of [17] and [46].

Proof. For $\gamma\in\{-1,1\}$ and $n\geq 1$, define

$$E^\mu_\alpha(N,\varepsilon_n,\gamma)=\left\{ t\in[0,1]:\begin{cases}\forall\ w\in\mathcal{A}^n \text{ such that } \delta(w,w^{(n)}(t))\leq N,\\ b^{\gamma n(\alpha-\gamma\varepsilon_n)}\mu(I_w)^\gamma\leq 1\end{cases}\right\}. \tag{2.4}$$

For $t\in[0,1]$ and $k\in\{-N,....,N\}$, if there exists (a necessarily unique) $w\in\mathcal{A}^n$ such that $i(w)-i(w^{(n)}(t))=k$, this word w is denoted $w_k^{(n)}(t)$.

For $\gamma\in\{-1,1\}$, let $S_{n,\gamma}=\sum_{-N\leq k\leq N} m_{k,n}$ with

$$m_{k,n}=m\left(\left\{t\in[0,1]: i(w)-i(w^{(n)}(t))=k\Rightarrow b^{\gamma n(\alpha-\gamma\varepsilon_n)}\mu(I_w)^\gamma>1\right\}\right).$$

We clearly have

$$m\Big((E^\mu_\alpha(N,\varepsilon_n,-1))^c\bigcup E^\mu_\alpha(N,\varepsilon_n,1))^c\Big)\leq S_{n,-1}+S_{n,1}. \tag{2.5}$$

Fix $\eta_n>0$ and $-N\leq k\leq N$. Let $Y(t)$ be the random variable defined to be equal to $b^{\gamma n(\alpha-\gamma\varepsilon_n)\eta_n}\mu\big(I_{w_k^{(n)}(t)}\big)^{\gamma\eta_n}$ if $w_k^{(n)}(t)$ exists or 0 otherwise. The Markov inequality applied to $Y(t)$ with respect to m yields $m_{k,n}\leq\int Y(t)dm(t)$. Since Y is constant over each b-adic interval I_v of generation n, we get

$$m_{k,n}\leq\sum_{v,w\in\mathcal{A}^n:\, i(w)-i(v)=k} b^{n(\alpha-\gamma\varepsilon_n)\gamma\eta_n}m(I_v)\mu(I_w)^{\gamma\eta_n}.$$

Summing over $|k|\leq N$ yields $S_{n,-1}+S_{n,1}\leq S_n^{N,\varepsilon_n,\eta_n}(m,\mu,\alpha)$. The conclusion follows from (2.5) and from the Borel-Cantelli Lemma.

2.2. Uniform growth speed in singularity sets

Let Λ be a set of indexes, and Ω^* a measurable subset of Ω of probability 1. Some notations and technical assumptions are needed to state the general result that we shall apply in Section 3. These assumptions describe a common situation in multifractal analysis. In particular the measures in the following sections satisfy these requirements.

• For every $\omega \in \Omega^*$, we consider two sequences of families of measures $\Big(\{\mu_\lambda^w\}_{\lambda\in\Lambda}\Big)_{w\in\mathcal{A}^*}$ and $\Big(\{m_\lambda^w\}_{\lambda\in\Lambda}\Big)_{w\in\mathcal{A}^*}$ (indexed by $\mathcal{A}^*$) such that for every $w \in \mathcal{A}^*$, the elements of the families $\{\mu_\lambda^w\}_{\lambda\in\Lambda}$ and $\{m_\lambda^w\}_{\lambda\in\Lambda}$ are positive finite Borel measures whose support is $[0,1]$. $\{\mu_\lambda^\emptyset\}_{\lambda\in\Lambda}$ (resp. $\{m_\lambda^\emptyset\}_{\lambda\in\Lambda}$) is written $\{\mu_\lambda\}_{\lambda\in\Lambda}$ (resp. $\{m_\lambda\}_{\lambda\in\Lambda}$).

• We consider an integer $N \geq 1$, a positive sequence $\widetilde{\varepsilon} = (\varepsilon_n)_{n\geq 1}$, and a family of positive numbers $(\alpha_\lambda)_{\lambda\in\Lambda}$. Then, remembering (2.4), consider for every $j \geq 0$, $w \in \mathcal{A}^j$ and $p \geq 1$ the sets

$$E^{\mu_\lambda^w}_{\alpha_\lambda,p}(N,\widetilde{\varepsilon}) = \bigcap_{n\geq p} E^{\mu_\lambda^w}_{\alpha_\lambda}(N,\varepsilon_n,-1) \cap E^{\mu_\lambda^w}_{\alpha_\lambda}(N,\varepsilon_n,1). \tag{2.6}$$

The sets $\{E^{\mu_\lambda^w}_{\alpha_\lambda,n}(N,\widetilde{\varepsilon})\}_n$ form a non-decreasing sequence. We assume that m_λ^w *is concentrated on* $\lim_{p\to+\infty} E^{\mu_\lambda^w}_{\alpha_\lambda,p}(N,\widetilde{\varepsilon})$. We define the growth speed of $E^{\mu_\lambda^w}_{\alpha_\lambda,p}(N,\widetilde{\varepsilon})$ as

$$GS(m_\lambda^w,\mu_\lambda^w,\alpha_\lambda,N,\widetilde{\varepsilon}) = \inf\Big\{p\geq 1 : m_\lambda^w\big(E^{\mu_\lambda^w}_{\alpha_\lambda,p}(N,\widetilde{\varepsilon})\big) \geq \|m_\lambda^w\|/2\Big\}. \tag{2.7}$$

Since $\mu_\lambda^w(\lim_{p\to+\infty} E^{\mu_\lambda^w}_{\alpha_\lambda,p}(N,\widetilde{\varepsilon})) = \|m_\lambda^w\|$, $GS(m_\lambda^w,\mu_\lambda^w,\alpha_\lambda,N,\widetilde{\varepsilon}) < \infty$. This number is a measure of the number p of generations needed for $E^{\mu_\lambda^w}_{\alpha_\lambda,p}(N,\widetilde{\varepsilon})$ to recover a certain given fraction (here chosen equal to 1/2) of the measure m_λ^w.

• We assume that for every positive sequence $\widetilde{\eta} = (\eta_j)_{j\geq 0}$, there exist a random vector $\big(U(\widetilde{\eta}),V(\widetilde{\eta})\big) \in \mathbb{R}_+ \times \mathbb{R}_+^{\mathbb{N}}$ and a sequence $(U^w, V^w = (V_n^w)_{n\in\mathbb{N}})_{w\in\mathcal{A}^*}$ of copies of $\big(U(\widetilde{\eta}),V(\widetilde{\eta})\big)$ and finally a sequence $(\psi_j(\widetilde{\eta}))_{j\geq 0}$, such that for $\mathbb{P}$-almost every $\omega\in\Omega^*$,

$$\forall\, w\in\mathcal{A}^*,\ \forall n \geq \psi_{|w|}(\widetilde{\eta}),\ \begin{cases} U^w \leq \inf_{\lambda\in\Lambda} \|m_\lambda^w\|, \\ V_n^w \geq \sup_{\lambda\in\Lambda} S_n^{N,\varepsilon_n,\eta_n}(m_\lambda^w,\mu_\lambda^w,\alpha_\lambda), \end{cases} \tag{2.8}$$

where $S_n^{N,\varepsilon_n,\eta_n}(m_\lambda^w,\mu_\lambda^w,\alpha_\lambda)$ is defined in (2.3) (if $w\in\mathcal{A}_j$, remember that $|w| = j$). This provides us with a uniform control over $\lambda\in\Lambda$ of the families of measures $\{(m_\lambda^w,\mu_\lambda^w)\}_{w\in\mathcal{A}^*}$.

Proposition 2.2. (**Uniform growth speed in singularity sets.**) *Assume that two sequences of positive numbers $\widetilde{\eta} = (\eta_j)_{j\geq 0}$ and $(\rho_j)_{j\geq 0}$ are fixed.*
Let $(\mathcal{S}_j)_{j\geq 0}$ be a sequence of integers such that $\mathcal{S}_j \geq \psi_j(\widetilde{\eta})$. If

$$\sum_{j\geq 0} b^j \mathbb{P}\Big(U(\widetilde{\eta}) \leq b^{-\rho_j}\Big) < \infty \ \textit{and} \ \sum_{j\geq 0} b^j b^{\rho_j} \sum_{n\geq \mathcal{S}_j} \mathbb{E}\Big(V_n(\widetilde{\eta})\Big) < \infty, \tag{2.9}$$

then, with probability one, for every j large enough, for every $w \in \mathcal{A}^j$ and $\lambda \in \Lambda$, $GS(m_\lambda^w, \mu_\lambda^w, \alpha_\lambda, N, \widetilde{\varepsilon}) \leq \mathcal{S}_j$.

Proof. Fix $j \geq 1$ and $w \in \mathcal{A}^j$. As shown in the proof of Proposition 2.1, for every $n \geq 1$ and every $\lambda \in \Lambda$, we can write

$$m_\lambda^w\Big(\big(E_{\alpha_\lambda}^{\mu_\lambda^w}(N, \varepsilon_n, -1)\big)^c \cup \big(E_{\alpha_\lambda}^{\mu_\lambda^w}(N, \varepsilon_n, 1)\big)^c\Big) \leq S_n^{N, \varepsilon_n, \eta_n}(m_\lambda^w, \mu_\lambda^w, \alpha_\lambda).$$

Thus, using (2.8), we get

$$m_\lambda^w\Big(\bigcup_{n \geq \mathcal{S}_j} \big(E_{\alpha_\lambda}^{\mu_\lambda^w}(N, \varepsilon_n, -1)\big)^c \cup \big(E_{\alpha_\lambda}^{\mu_\lambda^w}(N, \varepsilon_n, 1)\big)^c\Big) \leq \sum_{n \geq \mathcal{S}_j} V_n^w. \tag{2.10}$$

We apply the "random self-similar control" (2.9) combined with the Borel-Cantelli lemma. On the one hand, the left part of (2.9) yields $\sum_{j \geq 1} \mathbb{P}\Big(\exists\ w \in \mathcal{A}^j,\ U^w \leq b^{-\rho_j}\Big) < \infty$. Hence, with probability one, for j large enough and for all $w \in \mathcal{A}^j$,

$$\sup_{\lambda \in \Lambda} \|m_\lambda^w\|^{-1} \leq (U^w)^{-1} \leq b^{\rho_j}. \tag{2.11}$$

On the other hand, the right part of (2.9) yields

$$\sum_{j \geq 1} \mathbb{P}\Big(\exists\ w \in \mathcal{A}^j,\ b^{\rho_j} \sum_{n \geq \mathcal{S}_j} V_n^w \geq 1/2\Big) \leq 2 \sum_{j \geq 1} b^j b^{\rho_j} \mathbb{E}\Big(\sum_{n \geq \mathcal{S}_j} V_n^w\Big) < \infty.$$

This implies that with probability one, $b^{\rho_j} \sum_{n \geq \mathcal{S}_j} V_n^w < 1/2$ for every j large enough and for all $w \in \mathcal{A}^j$.

Thus, by (2.11), $\sup_{\lambda \in \Lambda} \frac{\sum_{n \geq \mathcal{S}_j} V_n^w}{\|m_\lambda^w\|} < 1/2$. Combining this with (2.10) and (2.7), we get that for every $\lambda \in \Lambda$, $GS(m_\lambda^w, \mu_\lambda^w, \alpha_\lambda, N, \widetilde{\varepsilon}) \leq \mathcal{S}_j$.

3. Main results for independent random cascades

3.1. Definition

Let $v = (v_1, \ldots, v_{|v|}) \in \mathcal{A}^*$. For every $k \in \{1, \ldots, |v|\}$, $v|k$ is the truncated word $(v_1, \ldots, v_k) \in \mathcal{A}^k$, and by convention $v|0$ is the empty word $\emptyset$. If v and w belong to $\mathcal{A}^*$, the word obtained by concatenation of v and w is denoted either by vw or by $v \cdot w$. We focus in this paper on the measures introduced in [39] and more recently in [5]. A measure $\mu(\omega)$ is said to be an independent random cascade if it has the following property: There exist a sequence of random positive vectors

$\big(W(w) = (W_0(w), \dots, W_{b-1}(w))\big)_{w\in\mathcal{A}^*}$ and a sequence of random measures $(\mu^w)_{w\in\mathcal{A}^*}$ such that

- **(P1)** for all $v, w \in \mathcal{A}^*$, $\mu(I_{vw}) = \mu^v(I_w) \prod_{k=0}^{|v|-1} W_{v_{k+1}}(v|k)$ $(\mu^\emptyset = \mu)$,
- **(P2)** the random vectors $W(w)$, for $w \in \mathcal{A}^*$, are i.i.d. with a vector $W = (W_0, \dots, W_{b-1})$ such that $\sum_{k=0}^{b-1} \mathbb{E}(W_k) = 1$,
- **(P3)** for all $v \in \mathcal{A}^*$, $\big(\mu^v(I_w)\big)_{w\in\mathcal{A}^*} \equiv \big(\mu(I_w)\big)_{w\in\mathcal{A}^*}$. Moreover, for every $j \geq 1$, the measures μ^v, $v \in \mathcal{A}^j$, are mutually independent,
- **(P4)** for every $j \geq 1$, the σ-algebras $\sigma\big(W(w) :\ w \in \cup_{0\leq k\leq j-1}\mathcal{A}^k\big)$ and $\sigma(\mu^v(I_w) :$ $v \in \mathcal{A}^j,\ w \in \mathcal{A}^*)$ are independent.

Let $\big(W(w)\big)_{w\in\mathcal{A}^*}$ be as above. For $q \in \mathbb{R}$ define the function

$$\widetilde{\tau}_\mu(q) = -\log_b \mathbb{E}\Big(\sum_{k=0}^{b-1} W_k^q\Big) \in \mathbb{R} \cup \{-\infty\}. \tag{3.1}$$

We deal in the following with two classes of measures.

- **Non-degenerate multiplicative martingales when $\widetilde{\tau}_\mu'(1^-) > 0$.**
  Let $\big(W(w) = (W_0(w), \dots, W_{b-1}(w))\big)_{w\in\mathcal{A}^*}$ be a sequence of random positive vectors satisfying **(P2)**. With probability one, $\forall\, v \in \mathcal{A}^*$, the sequence of measures $\{\mu_j^v\}_{j\geq 0}$ defined on $[0,1]$ by

  $$\frac{d\mu_j^v}{d\ell}(t) = b^j \prod_{k=0}^{j-1} W_{t_{k+1}}(v\cdot t|k) \tag{3.2}$$

  converges weakly, as $n \to \infty$ to a measure μ^v.

  1. For $\mu = \mu^\emptyset$, properties **(P1)** to **(P4)** are satisfied;
  2. For $\mu = \mu^\emptyset$, if $\widetilde{\tau}_\mu'(1^-) > 0$, the total masses $\|\mu^v\|$ are almost surely positive, and their expectation is equal to 1 ([31, 20]).

- **The modified construction in the critical case $\widetilde{\tau}_\mu'(1^-) = 0$.** Suppose that $\widetilde{\tau}_\mu'(1^-) = 0$ and $\widetilde{\tau}_\mu(h) > -\infty$ for some $h > 1$. Then, with probability one, for all $v \in \mathcal{A}^*$, $\mu^v(I_w) = \lim_{j\to\infty} -H_j^v(w)$, where

  $$H_j^v(w) = \sum_{u\in\mathcal{A}^j} \Big(\prod_{k=0}^{|w|+j-1} W_{(w\cdot u)_{k+1}}\big(v\cdot(w\cdot u|k)\big)\Big) \log\Big(\prod_{k=0}^{|w|+j-1} W_{(w\cdot u)_{k+1}}\big(v\cdot(w\cdot u|k)\big)\Big),$$

defines a function on the b-adic intervals I_w which is a positive Borel measure supported by $[0,1]$ ([5, 37]).

1. For $\mu = \mu^{\emptyset}$, properties **(P1)** to **(P4)** are satisfied;
2. For $\mu = \mu^{\emptyset}$, $\mathbb{E}(\|\mu\|^h) < \infty$ for $h \in [0,1)$ but $\mathbb{E}(\|\mu\|) = \infty$.

3.2. Analyzing measures

Let $\mathcal{O}$ be the interior of the interval $\{q \in \mathbb{R} : \widetilde{\tau}'_\mu(q)q - \widetilde{\tau}_\mu(q) > 0\}$. We always have $(0,1) \subset \mathcal{O}$, and $\mathcal{O} \subset (-\infty, 1)$ if $\widetilde{\tau}'_\mu(1^-) = 0$. We assume that:

- If $\widetilde{\tau}'_\mu(1^-) > 0$, then $\mathcal{O}$ contains the closed interval $[0,1]$,
- If $\widetilde{\tau}'_\mu(1^-) = 0$, then $0 \in \mathcal{O}$.

For $q \in \mathcal{O}$, $v \in \mathcal{A}^*$, $j \geq 1$, let $\mu^v_{q,j}$ be the measure defined as μ^v_j in (3.2) but with the sequence $\big(W_q(v \cdot w) = (b^{\widetilde{\tau}_\mu(q)} W_0(v \cdot w)^q, \dots, b^{\widetilde{\tau}_\mu(q)} W_{b-1}(v \cdot w)^q)\big)_w$ instead of $\big(W(v \cdot w)\big)_w$. It is proved in [5] that on a set $\Omega^* \subset \Omega$ of probability 1, $\forall\, \omega \in \Omega^*$, $\forall\, v \in \mathcal{A}^*$ and $\forall\, q \in \mathcal{O}$, the sequence $(\mu^v_{q,j})$ converges weakly to a positive measure μ^v_q.

It is proved in [15, 5] that with probability one,

- the mappings $q \in \mathcal{O} \mapsto Y_q(v)$ are analytic and positive, where

$$\mu^{\emptyset}_q = \mu_q, \quad Y_q = \|\mu^{\emptyset}_q\| \;\text{ and }\; Y_q(v) = \|\mu^v_q\| \;\text{ for } v \in \mathcal{A}^*, \tag{3.3}$$

- $\tau_\mu \equiv \widetilde{\tau}_\mu$ on $\mathcal{O}$.

Eventually, we remark that $\mathcal{O} \supset \mathbb{R}_+$ (resp. $\mathbb{R}_-$) if and only if $\widetilde{\tau}_\mu(hq) - h\widetilde{\tau}_\mu(q) > 0$ for all $q \in \mathbb{R}_+$ (resp. $\mathbb{R}_-$) and $h > 1$, which amounts to saying that $\forall\, q \in \mathbb{R}_+$ (resp. $\mathbb{R}_-$), $\mathbb{E}(Y_q^h) < \infty$ and $h > 1$ (see the proof of Lemma 3).

3.3. Main results

For an independent random cascade μ, we assume that if $\widetilde{\tau}'_\mu(1^-) > 0$, then $\mathcal{O}$ contains $[0,1]$, and if $\widetilde{\tau}'_\mu(1^-) = 0$, then $\mathcal{O} \subset (-\infty, 1)$. We also suppose that $0 \in \mathcal{O}$.

Theorem 3.1. *Let μ be an independent random cascade. Let N be an integer ≥ 1 and $\widetilde{\varepsilon} = (\varepsilon_n)_{n\geq 1}$ a sequence of positive numbers going to 0. Assume that $\forall \chi > 0$, the series $\sum_{n\geq 1} n b^{-n\chi\varepsilon_n^2}$ converges.*

Then, with probability one, for every $q \in \mathcal{O}$, $\tau_\mu(q) = \widetilde{\tau}_\mu(q)$, and the two level sets $E^{\mu_q}_{\tau'_\mu(q)q - \tau_\mu(q)}(N, \widetilde{\varepsilon})$ and $E^{\mu}_{\tau'_\mu(q)}(N, \widetilde{\varepsilon})$ are both of full μ_q-measure.

Remark 3.1. The conclusions of Theorem 3.1 hold as soon as

$$\exists\ \eta > 0 \text{ such that for every } n,\ \varepsilon_n \geq n^{-1/2} \log(n)^{1/2+\eta}. \tag{3.4}$$

This condition on $(\varepsilon_n)_{n\geq 1}$ must be compared with the sharper one which holds if we consider only one measure μ_q generated by a vector $W_q = W_q(\emptyset)$ which satisfies the additional conservative condition $\sum_{k=0}^{b-1} W_{q,k} = 1$. Indeed, in this case, Y_q is equal to 1 almost surely, and as a function of (t,ω), the logarithmic density $\frac{\log \mu_q(I_n(t))}{n}$ is a sum of i.i.d. random variables with respect to the probability measure $\mathcal{Q}_q$ introduces later in Section 4.4. If $0 < \sigma_q^2 = \sum_{k=0}^{b-1} W_{q,k} \log^2 W_{q,k} < \infty$, then the law of the iterated logarithm yields that with probability one, μ_q-almost everywhere,

$$\limsup_{n\to\infty} \left| \frac{\big(\log \mu_q(I_n(t))\big) - n\big(\tau_\mu'(q)q - \tau_\mu(q)\big)}{\sigma_q \sqrt{n \log\log(n)}} \right| = 1;$$

In this case ε_n can be chosen equal to $\sigma' \sqrt{n \log\log(n)}$ (for $n \geq 3$) for any $\sigma' > \sigma_q$.

The estimate (3.4) comes from our will to control simultaneously the asymptotic behavior of an uncountable family of measures, and from the technique we use. Moreover, when W_q does not satisfy the conservative condition described above, it seems difficult to control the asymptotic behavior by a term of the form $\sigma' \sqrt{|v| \log\log(|v|)}$.

Theorem 3.2. (**Growth speed in Hölder singularity sets.**) *Under the assumptions of Theorem 3.1, assume that $(\varepsilon_n)_n$ satisfies (3.4) and that there exists $A > 1$ such that, with probability one, $A^{-1} \leq W_i$ (resp. $W_i \leq A$) for all $0 \leq i \leq b-1$. Let K be a compact subinterval of $\mathcal{O} \cap \mathbb{R}_+$ (resp. $\mathcal{O} \cap \mathbb{R}_-$).*

Then, with probability one, for j large enough, for all $q \in K$ and $w \in \mathcal{A}^j$,

$$\max\big(GS\big(\mu_q^w, \mu^w, \tau_\mu'(q), N, \widetilde{\varepsilon}\big), GS\big(\mu_q^w, \mu_q^w, \tau_\mu'(q)q - \tau_\mu(q), N, \widetilde{\varepsilon}\big)\big) \leq \mathcal{S}_j, \tag{3.5}$$

with $\mathcal{S}_j = \left[\exp\left(\left(j \log(j)^\eta\right)^{\frac{1}{1+2\eta}}\right)\right]$ ($[x]$ stands for the integer part of x).

If there exists $\eta > 0$ such that for every n, $\varepsilon_n \geq \log(n)^{-\eta}$, the above conclusion holds with $\mathcal{S}_j = \left[j \log(j)^{\eta'}\right]$, for any $\eta' > 2\eta$.

Remark 3.2. Our computations show that the faster (ε_n) decays with n, the faster the growth speed increases (in the sense that $\mathcal{S}_j$ increases faster than j). In Theorem 3.2, the first choice of ε_n corresponds to the fastest choice for the convergence speed of (ε_n) allowed by our technique (see Remark 3.1). As a counterpart, $\mathcal{S}_j$ is very large compared to j. To the contrary, the second choice for (ε_n) is the slowest one, but as a counterpart $\mathcal{S}_j$ corresponds to the "best" choice allowed to minimize $\mathcal{S}_j/j$.

Using the proof of Theorem 3.2, we can find other pairs $\big((\mathcal{S}_j)_{j\geq 1}, (\varepsilon_n)_{n\geq 1}\big)$ for which (3.5) holds and whose asymptotic behaviors are intermediate between those of the "extremal" pairs presented in the statement.

Remark 3.3. We assume that the number of neighbors N is fixed. In fact, it is not difficult to consider a sequence of neighbors N_n simultaneously with the speed of convergence ε_n. This number N_n can then go to ∞ under the condition that $\log N_n = o(n\varepsilon_n^2)$. Another modification would consist in replacing the fixed fraction f in (1.11) by a fraction f_j going to 1 as j goes to ∞. The choice $f_j = 1 - b^{-s_j}$ with $s_j = j$ is convenient. These two improvements yield technical complications, but comparable results are easily derived from the proofs we propose.

The growth speed obtained in Theorem 3.2 can be improved by considering results valid only almost surely, for almost every q, μ_q almost-everywhere. Recall that if $t \in [0,1)$ and $j \geq 1$, $w^{(j)}(t)$ is the unique w of $\mathcal{A}^j$ such that $t \in [i(w)b^{-j}, (i(w)+1)b^{-j})$.

Theorem 3.3. (**Improved growth speed.**) *Under the assumptions of Theorem 3.1, fix $\kappa > 0$ and assume that (3.4) holds. For $j \geq 2$, let $\mathcal{S}_j = \big[j\log(j)^{-\kappa}\big]$.*

1. *For every $q \in \mathcal{O}$, with probability one, the property $\mathcal{P}(q)$ holds, where $\mathcal{P}(q)$ is: For μ_q-almost every $t \in [0,1)$, if j is large enough, for $w = w^{(j)}(t)$,*

$$\max\big(GS\big(\mu_q^w, \mu^w, \tau_\mu'(q), N, \widetilde{\varepsilon}\big), GS\big(\mu_q^w, \mu_q^w, \tau_\mu'(q)q - \tau_\mu(q), N, \widetilde{\varepsilon}\big)\big) \leq \mathcal{S}_j.$$

2. *With probability one, for almost every $q \in \mathcal{O}$, $\mathcal{P}(q)$ holds.*

For $w \in \mathcal{A}^*$, $n \geq 1$ and $q \in \mathcal{O}$, let

$$\mathcal{N}_n(\mu^w, \alpha, \varepsilon_n) = \#\big\{b\text{-adic box } I \text{ of scale } n : |I|^{\alpha+\varepsilon_n} \leq \mu^w(I) \leq |I|^{\alpha-\varepsilon_n}\big\}, \qquad (3.6)$$

where $|I|$ stands for the diameter of I. Remember that $\tau_\mu = \widetilde{\tau}_\mu$ on $\mathcal{O}$.

Theorem 3.4. (**Renewal speed of large deviations spectrum.**) *Under the assumptions of Theorem 3.1, we also assume that (3.4) holds, $\mathcal{O} = \mathbb{R}$ (in particular $\widetilde{\tau}_\mu'(1) > 0$) and there exists $A > 1$ such that with probability one $A^{-1} \leq W_i \leq A$ for all*

$0 \le i \le b-1$. *Let* $\mathcal{S}_j$ *be defined as in Theorem 3.2. Let* K *be a compact subinterval of* $\mathbb{R}$ *and* $\beta = 1 + \max_{q\in K} |q|$.

With probability one, for j *large enough, for all* $q \in K$ *and* $w \in \mathcal{A}^j$*, we have*

$$\forall\, n \ge \mathcal{S}_j,\ Y_q(w) b^{n(\widetilde{\tau}'_\mu(q)q - \widetilde{\tau}_\mu(q) - \beta\varepsilon_n)} \le \mathcal{N}_n\big(\mu^w, \widetilde{\tau}'_\mu(q), \varepsilon_n\big) \le Y_q(w) b^{n(\widetilde{\tau}'_\mu(q)q - \widetilde{\tau}_\mu(q) + \beta\varepsilon_n)} \tag{3.7}$$

Heuristically, Theorem 3.4 asserts that for large j's, if $|w| = j$, the number of b-adic intervals of scale b^{-n} such that $\mu^w(I) \sim |I|^{\tau'_\mu(q)}$ is approximately equal to $b^{n(\widetilde{\tau}'_\mu(q)q - \widetilde{\tau}_\mu(q))}$ as soon as $n \ge \mathcal{S}_j$. Hence, the equality (3.7) carries precise information on the renewal of the large deviations spectrum $\alpha \mapsto \lim_{\varepsilon\to 0} \lim_{j\to+\infty} j^{-1} \log_b N_j^\varepsilon(\alpha)$ (see (1.2)).

For $w \in \mathcal{A}^*$, $n \ge 1$ and $q \in \mathbb{R}$, introduce the functions ($\tau_n^\emptyset$ associated with $\mu^\emptyset = \mu$ is simply denoted by τ_n)

$$\tau_n^w(q) = -\frac{1}{n} \log_b \sum_{v\in\mathcal{A}^n} \mu^w(I_v)^q.$$

Theorem 3.5. (**Convergence speed of** τ_n^w **toward** $\widetilde{\tau}_\mu$**.**) *Under the assumptions of Theorem 3.4, let* K *be a compact subinterval of* $\mathbb{R}$*. There exists* $\theta > 0$ *and* $\delta \in (0,1)$ *such that, with probability one,*

1. *for* j *large enough,* $\big|\widetilde{\tau}_\mu(q) - \tau_j(q)\big| \le \big|\log_b Y_q\big| j^{-1} + \theta \log(j) j^{-1}$;
2. *for* j *large enough, for every* $n \ge j^\delta$*, for every* $w \in \mathcal{A}^j$*,* $\big|\widetilde{\tau}_\mu(q) - \tau_n^w(q)\big| \le \big|\log_b Y_q(w)\big| n^{-1} + \theta \log(n) n^{-1}$*, with* $\big|\log Y_q(w)\big| \le \theta \log(j)$.

The convergence speed obtained in last Theorem 3.5 provides precisions on the estimator of the function τ_μ discussed in [18, 47].

4. Proofs of the main results

4.1. Proof of Theorem 3.1

Fix K a compact subinterval of $\mathcal{O}$ and $\widetilde{\eta} = (\eta_n)_{n\ge 1}$ a bounded positive sequence to be specified later. For $\omega \in \Omega^*$ and $q \in K$, introduce (recall (2.3))

$$F_n(q) = S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q, \mu, \tau'_\mu(q)\big) \text{ and } G_n(q) = S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q, \mu_q, \tau'_\mu(q)q - \tau_\mu(q)\big). \tag{4.1}$$

We begin by giving estimates for $\mathbb{E}\big(H_n(q)\big)$ and $\mathbb{E}\big(H'_n(q)\big)$ for $H \in \{F, G\}$.

Lemma 1. *Under the assumptions of Theorem 3.1, if $\|\widetilde{\eta}\|_\infty$ is small enough, there exists $C_K > 0$ such that if $H \in \{F, G\}$,*

$$\forall\, q \in K,\ \max\big(\mathbb{E}\big(H_n(q)\big), \mathbb{E}\big(H_n'(q)\big)\big) \le C_K n b^{-n(\varepsilon_n \eta_n + O(\eta_n^2))}, \tag{4.2}$$

where $O(\eta_n^2)$ is uniform over $q \in K$.

The proof of this lemma is postponed to the next subsection.

Let q_0 be the left end point of K. Since $\sup_{q\in K} H_n(q) \le H_n(q_0) + \int_K |H_n'(q)|\, dq$, we have

$$\mathbb{E}\big(\sup_{q\in K} H_n(q)\big) \le C_K(1+|K|) n b^{-n(\varepsilon_n \eta_n + O(\eta_n^2))}. \tag{4.3}$$

Choosing $\eta_n = \varepsilon_n / A$ with A large enough yields $\varepsilon_n \eta_n + O(\eta_n^2) \ge \varepsilon_n^2/2A$. Using the assumptions of Theorem 3.1, we get the almost sure convergence of $\sum_{n\ge 1} \sup_{q\in K} H_n(q)$ for $H \in \{F, G\}$. We conclude with Proposition 2.1.

4.2. Proof of Lemma 1

• **The case $H = F$:** For $v, w \in \mathcal{A}^n$, $q \in \mathcal{O}$ and $\gamma \in \{-1, 1\}$, we write (recall that $Y_q(v)$ was defined in (3.3))

$$\mu_q(I_v)\mu(I_w)^{\gamma\eta_n} = Y_q(v) Y_1(w)^{\gamma\eta_n} b^{n\tau_\mu(q)} \prod_{k=0}^{n-1} W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}.$$

Moreover, it follows from estimates of Lemma 6 in [5] that for $\|\widetilde{\eta}\|_\infty$ small enough, the quantities

$$\begin{aligned} C_K'(\widetilde{\eta}) &= \sup_{\substack{q\in K,\ \gamma\in\{-1,1\}\\ n\ge 1,\ v,w\in\mathcal{A}^n}} \Big(\mathbb{E}\Big(\Big|\frac{d}{dq} Y_q(v) Y_1(w)^{\gamma\eta_n}\Big|\Big) + \mathbb{E}\big(Y_q(v) Y_1(w)^{\gamma\eta_n}\big)\Big) \\ \text{and}\quad C_K''(\widetilde{\eta}) &= \sup_{\substack{q\in K,\ \gamma\in\{-1,1\}\\ n\ge 1,\ v,w\in\mathcal{A}^n,\ 0\le k\le n-1}} \frac{\mathbb{E}\big(\big|\frac{d}{dq} W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big|\big)}{\mathbb{E}\left(W_{v_{k+1}}(v)^q W_{w_{k+1}}(w)^{\gamma\eta_n}\right)} \end{aligned}$$

are finite. Hence, due to the definition of $F_n(q)$ and the fact that $\widetilde{\tau}$ is continuously differentiable on $\mathcal{O}$, there exists a constant $C_K(\widetilde{\eta})$ such that for every $q \in K$, $\max\big(\mathbb{E}\big(F_n(q)\big), \mathbb{E}\big(F_n'(q)\big)\big) \le C_K(\widetilde{\eta}) T_n(q)$, where

$$T_n(q) = n\, b_n(q) \sum_{\substack{\gamma\in\{-1,1\},\\ v,w\in\mathcal{A}^n,\ \delta(v,w)\le N}} \prod_{k=0}^{n-1} \mathbb{E}\big(W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big),$$

where $b_n(q) = b^{n(\tau_\mu(q) + \gamma\eta_n(\tau_\mu'(q) - \gamma\varepsilon_n))}$. Let us make the following important remark.

Remark 4.1. If v and w are words of length n, and if $\bar v$ and $\bar w$ stand for their prefixes of length $n-1$, then $\delta(\bar v,\bar w) > k$ implies $\delta(v,w) > bk$. It implies that, given two integers $n\ge m>0$ and two words v and w in $\mathcal{A}^n$ such that $b^{m-1} < \delta(v,w)\le b^m$, there are two prefixes $\bar v$ and $\bar w$ of respectively v and w of common length $n-m$ such that $\delta(\bar v,\bar w)\le 1$. Moreover, for these words $\bar v$ and $\bar w$, there are at most b^{2m} pairs (v,w) of words in $\mathcal{A}^n$ such that $\bar v$ and $\bar w$ are respectively the prefixes of v and w.

Due to Remark 4.1 and the form of $T_n(q)$, there exists a constant C'''_K such that

$$\forall\, q\in K,\ \forall\, n\ge 1,\ T_n(q)\le C'''_K n b_n(q)\sum_{\substack{\gamma\in\{-1,1\},\\ v,w\in\mathcal{A}^n,\ \delta(v,w)\le 1}}\prod_{k=0}^{n-1}\mathbb{E}\big(W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big).$$

The situation is thus reducible to the case $N=1$. Now $T_n(q)\le n\big(T_{n,1}(q)+T_{n,2}(q)\big)$,

$$\text{where}\ \begin{cases} T_{n,1}(q)=b_n(q)\displaystyle\sum_{\gamma\in\{-1,1\},\,v\in\mathcal{A}^n}\ \prod_{k=0}^{n-1}\mathbb{E}\big(W_{v_{k+1}}(v|k)^{q+\gamma\eta_n}\big),\\ T_{n,2}(q)=b_n(q)\displaystyle\sum_{\gamma\in\{-1,1\},\,v,w\in\mathcal{A}^n,\,\delta(v,w)=1}\ \prod_{k=0}^{n-1}\mathbb{E}\big(W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big).\end{cases}$$

By using the twice continuous differentiability of τ_μ ($\tau_\mu=\widetilde\tau_\mu$), we immediately get that

$$T_{n,1}(q)=\sum_{\gamma\in\{-1,1\}} b^{n\big(\tau_\mu(q)+\gamma\eta_n(\tau'_\mu(q)-\gamma\varepsilon_n)-\tau_\mu(q+\gamma\eta_n)\big)}=2b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))},$$

where $O(\eta_n^2)$ is uniform over $q\in K$ if $\|\widetilde\eta\|_\infty$ is small enough.

Let g_k and d_k respectively stand for the word consisting of k consecutive zeros and the word consisting of k consecutive $b-1$. The estimation of $T_{n,2}(q)$ is achieved by using the following identity:

$$\bigcup_{v,w\in\mathcal{A}^n,\,\delta(v,w)=1}(v,w)=\bigcup_{m=0}^{n-1}\ \bigcup_{u\in\mathcal{A}^m}\ \bigcup_{r\in\{0,\dots,b-2\}}\big(u.r.d_{n-1-m},u.(r+1).g_{n-1-m}\big). \tag{4.4}$$

We have $T_{n,2}(q) = \mathcal{T}_n(q,-1) + \mathcal{T}_n(q,1)$ where for $\gamma \in \{-1,1\}$

$$\begin{aligned}
\mathcal{T}_n(q,\gamma) &= b_n(q) \sum_{\substack{v,w\in\mathcal{A}^n\\ \delta(v,w)=1}} \prod_{k=0}^{n-1} \mathbb{E}\big(W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big)\\
&= b_n(q) \sum_{m=0}^{n-1} \sum_{u\in\mathcal{A}^m} \sum_{r=0}^{b-2} \Theta_{n-1-m}(r) \prod_{k=0}^{m-1} \mathbb{E}\big(W_{u_{k+1}}(u|k)^{q+\gamma\eta_n}\big)\\
&= b_n(q) \sum_{m=0}^{n-1} b^{-m\tau_\mu(q+\gamma\eta_n)} \sum_{r=0}^{b-2} \Theta_{n-1-m}(r),
\end{aligned}$$

where $\Theta_m(r)$ is defined by

$$\begin{aligned}
\Theta_m(r) &= \mathbb{E}\big(W_r^q W_{r+1}^{\gamma\eta_n}\big)\big(\mathbb{E}(W_{b-1}^q)\big)^m \big(\mathbb{E}(W_0^{\gamma\eta_n})\big)^m\\
&\qquad + \mathbb{E}\big(W_r^{\gamma\eta_n} W_{r+1}^q\big)\big(\mathbb{E}(W_0^q)\big)^m \big(\mathbb{E}(W_{b-1}^{\gamma\eta_n})\big)^m.
\end{aligned}$$

All the components of W are positive almost surely. Thus, by definition (3.1) of $\widetilde{\tau}_\mu(q) = \tau_\mu(q)$, there is a constant $c_K \in (0,1)$ such that for all $q \in K$, $\max\big(\mathbb{E}(W_0^q), \mathbb{E}(W_{b-1}^q)\big) \le c_K b^{-\tau_\mu(q)}$. Moreover, if $\|\widetilde{\eta}\|_\infty$ is small enough, $\max\big(\mathbb{E}(W_0^{\gamma\eta_n}), \mathbb{E}(W_{b-1}^{\gamma\eta_n})\big) \le (c_K^{-1}+1)/2$ (this maximum goes to 1 when $\|\eta\|_\infty \to 0$). This yields (since $c_K(c_K^{-1}+1) = c_K+1$)

$$\begin{aligned}
\Theta_m(r) &\le \big(\mathbb{E}\big(W_r^q W_{r+1}^{\gamma\eta_n}\big) + (\mathbb{E}\big(W_r^{\gamma\eta_n} W_{r+1}^q\big)\big)\big((c_K+1)/2\big)^m b^{-m\tau_\mu(q)}\\
&\le C_K \big((c_K+1)/2\big)^m b^{-m\tau_\mu(q)}.
\end{aligned}$$

Consequently we get

$$\begin{aligned}
\mathcal{T}_n(q,\gamma) &\le C_K b^{n(\tau_\mu(q)+\gamma\eta_n(\tau_\mu'(q)-\gamma\varepsilon_n))} b^{-(n-1)\tau_\mu(q+\gamma\eta_n)}\\
&\qquad \times \sum_{m=0}^{n-1} \big((c_K+1)/2\big)^m b^{m(\tau_\mu(q+\gamma\eta_n)-\tau_\mu(q))}\\
&\le C_K b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))} \sum_{m=0}^{n-1} \big((c_K+1)/2\big)^m b^{m(\tau_\mu(q+\gamma\eta_n)-\tau_\mu(q))}.
\end{aligned}$$

The function τ_μ is continuously differentiable. Hence the sum $\sum_{m=0}^{n-1}((c_K+1)/2)^m$ $b^{m(\tau_\mu(q+\gamma\eta_n)-\tau_\mu(q))}$ is uniformly bounded over $n \ge 0$ and $q \in K$ if $\|\widetilde{\eta}\|_\infty$ is small enough. Finally, if $\|\widetilde{\eta}\|_\infty$ is small enough we also have $\mathcal{T}_n(q,\gamma) \le C_K b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))}$. Going back to $T_n(q)$, we get $T_n(q) \le C_K n b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))}$, $\forall q \in K$. This shows (4.2).

- **The case** $H = G$**:** The proof follows similar lines as for $F_n(q)$. The only additional required property in the computations is the boundedness of $\sup_{q\in K} \mathbb{E}(Y_q^{-h})$

for some $h > 0$. In fact, we shall need the following stronger property in the proof of Theorem 3.2.

Lemma 2. *1. If $K \subset \mathcal{O}$ is a compact interval, $\exists h > 0$ such that $\mathbb{E}\big(\sup_{q\in K} Y_q^{-h}\big) < \infty$.*
2. Assume that there exists $A > 1$ such that with probability one, $A^{-1} \le W_i$ (resp. $W_i \le A$) for all $0 \le i \le b-1$. Then, for any compact interval $K \subset \mathcal{O} \cap \mathbb{R}_+$ (resp. $\mathcal{O} \cap \mathbb{R}_-$), there are two constants $C_K > 0$ and $\gamma_K \in (0,1)$ depending on K such that

$$\text{for all } x > 0 \text{ small enough, } \mathbb{P}(\inf_{q\in K} Y_q \le x) \le \exp\big(-C_K x^{-\gamma_K/(1-\gamma_K)}\big).$$

Proof. 1. Fix K a compact subinterval of $\mathcal{O} \cap \mathbb{R}_+$ (resp. $\mathcal{O} \cap \mathbb{R}_-$). For $w \in \mathcal{A}^*$, we define $Z_K(w) = \inf_{q\in K} Y_q(w)$ ($Z_K(\emptyset)$ is denoted Z_K). We learn from [15, 5] that this infimum is positive since $q \mapsto Y_q(w)$ is almost surely positive and continuous.

For the random vector $W_q(w) = (W_{q,0}(w), W_{q,1}(w), ..., W_{q,b-1}(w))$, define $W_K(w) = \inf_{q\in K, 0\le i\le b-1} W_{q,i}(w)$ ($W_K(\emptyset)$ is denoted W_K). Since we assumed that $\mathcal{O}$ contains a neighborhood of 0, there exists $h > 0$ such that the moment of negative order $-h$ of this random variable $W_K(w)$ is finite.

Moreover, with probability one, $\forall\, q \in \mathcal{O}$ we have $Y_q = \sum_{i=0}^{b-1} W_{q,i}(\emptyset) Y_q(i)$. Hence

$$Z_K \ge W_K \sum_{i=0}^{b-1} Z_K(i). \tag{4.5}$$

By construction, the random variables $Z_K(i)$, $0 \le i \le b-1$, are i.i.d. with Z_K, and they are independent of the positive random variable W_K. Consequently, the Laplace transform of Z_K, denoted $L : t\ \ge 0 \mapsto \mathbb{E}(\exp(-tZ_K))$, satisfies the inequality

$$L(t) \le \mathbb{E}\Big(\prod_{i=0}^{b-1} L(W_K t)\Big) \quad (t \ge 0). \tag{4.6}$$

Since $\mathbb{E}(W_K^{-h}) < \infty$, using the approach of [43] to study the behavior at ∞ of Laplace transforms satisfying an inequality like (4.6) (see also [4, 38]), we get $\mathbb{E}(Z_K^{-h}) < \infty$.
2. It is a simple consequence of Theorem 2.5 in [38] (or Corollary 2.5 of [27]) and of the fact that the random variable W_K in (4.6) is lower bounded by a positive constant.

4.3. Proof of Theorem 3.2

Fix K a compact subinterval of $\mathcal{O}$. The computations performed to prove Theorem 3.1 yield (4.3). Thus there are two constants $C > 0$ and $\beta > 0$ as well as a sequence

$\widetilde{\eta} = (\eta_n)_{n\geq 1} \in \mathbb{R}_+^{\mathbb{N}^*}$ such that for every $j, n \geq 1$, $q \in K$ and $w \in \mathcal{A}^j$,

$$\mathbb{E}\Big(\sup_{q\in K} S^{N,\varepsilon_n,\eta_n}\big(\mu_q^w, \mu^w, \tau_\mu'(q)\big)\Big) \leq Cnb^{-\beta n\varepsilon_n^2}. \tag{4.7}$$

In order to apply Proposition 2.2, define

- $\Lambda = K$, $\{(m_\lambda^w, \mu_\lambda^w)\}_{w\in\mathcal{A}^*,\lambda\in\Lambda} = \{(\mu_q^w, \mu^w)\}_{w\in\mathcal{A}^*,q\in K}$ and $\{\alpha_\lambda\}_{\lambda\in\Lambda} = \{\tau_\mu'(q)\}_{q\in K}$,
- For $w \in \mathcal{A}^*$ and $n \geq 1$,

$$U^w = \inf_{q\in K} \|\mu_q^w\| \quad \text{and} \quad V_n^w = \sup_{q\in K} S^{N,\varepsilon_n,\eta_n}\big(\mu_q^w, \mu^w, \tau_\mu'(q)\big), \tag{4.8}$$

- For every $j \geq 1$, $\psi_j(\widetilde{\eta}) = 1$ and $\rho_j = \log(j)^{1+\eta}$.
- Fix $\eta > 0$ and $\eta' > 2\eta$. For every $j \geq 1$, we set $\mathcal{S}_j = \big[\exp\big(\big(j\log(j)^\eta\big)^{\frac{1}{1+2\eta}}\big)\big]$ if $\log(j)^{-\eta} \geq \varepsilon_j \geq j^{-1/2}\log(j)^{1/2+\eta}$ and $\mathcal{S}_j = \big[j\log(j)^{\eta'}\big]$ if $\varepsilon_j \geq \log(j)^{-\eta}$.

Now, on the one hand, Lemma 2.2 implies that

$$u_j := b^j\mathbb{P}(U^w \leq b^{-\rho_j}) \leq b^j \exp\big(-C_K b^{\frac{\gamma_K}{1-\gamma_K}(\log j)^{(1+\eta)}}\big). \tag{4.9}$$

Moreover, $\sum_{j\geq 1} u_j < \infty$. On the other hand, for some $\chi > 0$, for any $w \in \mathcal{A}^*$, we have

$$v_j := \sum_{n\geq\mathcal{S}_j} \mathbb{E}(V_n^w) \leq \sum_{n\geq\mathcal{S}_j} Cnb^{-\beta n\varepsilon_n^2} = O\big(b^{-j\log(j)^\chi}\big).$$

The sequence ρ_j has been chosen so that $\sum_{j\geq 1} b^j b^{\rho_j} v_j < \infty$. Consequently, Proposition 2.2 yields the desired upper bound for the growth speed $GS\big(\mu_q^w, \mu^w, \tau_\mu'(q), N, \widetilde{\varepsilon}\big)$.

Changing the measures $\{(m_q^w, \mu_q^w)\}_{w\in\mathcal{A}^*,q\in K}$ into $\{(\mu_q^w, \mu_q^w)\}_{w\in\mathcal{A}^*,q\in K}$ and the exponents $\{\tau_\mu'(q)\}_{q\in K}$ into $\{\tau_\mu'(q)q - \tau_\mu(q)\}_{q\in K}$, the same arguments yield the conclusion for $GS\big(\mu_q^w, \mu_q^w, \tau_\mu'(q)q - \tau_\mu(q)), N, \widetilde{\varepsilon}\big)$.

4.4. Proof of Theorem 3.3

We only prove the results for the control of $GS\big(\mu_q^w, \mu^w, \tau_\mu'(q), N, \widetilde{\varepsilon}\big)$ by $\mathcal{S}_j$, since $GS\big(\mu_q^w, \mu_q^w, \tau_\mu'(q)q - \tau_\mu(q)), N, \widetilde{\varepsilon}\big)$ is controlled by using the same approach.

1. Recall that $(\Omega, \mathcal{B}, \mathbb{P})$ denotes the probability space on which the random variables are defined. We consider on $\mathcal{B} \otimes \mathcal{B}([0,1])$ the so-called Peyrière probability $\mathcal{Q}_q$ [31]

$$\mathcal{Q}_q(A) = \mathbb{E}\Big(\int_{[0,1]} \mathbf{1}_A(\omega, t)\, \mu_q(dt)\Big) \quad (A \in \mathcal{B}\otimes\mathcal{B}([0,1])).$$

By construction $\mathcal{Q}_q$-almost surely means $\mathbb{P}$-almost surely, $\mu_q(\omega)$-almost everywhere.

Fix $\widetilde{\eta}$ as in the proof of Theorem 3.2. Also, for $j\geq 1$ let $\rho_j=\log(j)^{1+\eta}$, and let $\mathcal{S}_j=\left[j\log(j)^{-\kappa}\right]$. Now, for $j\geq 0$ and $n\geq 1$, define on $\Omega\times[0,1)$ the random variables

$$\begin{array}{rcl}\mathbf{U}^{(j)}(\omega,t) & = & \|\mu_q^{w^{(j)}(t)}(\omega)\|.\\ \text{and } \mathbf{V}_n^{(j)}(\omega,t) & = & S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q^{w^{(j)}(t)}(\omega),\mu^{w^{(j)}(t)}(\omega),\tau_\mu'(q)\big).\end{array}$$

To get Theorem 3.3, by Proposition 2.2, we claim that it is enough to prove that

$$\text{for some } h\in(0,1],\ \sum_{j\geq 0}\mathcal{Q}_q\big(\mathbf{U}^{(j)}\leq b^{-\rho_j}\big)<\infty \ \text{ and } \sum_{j\geq 0}b^{\rho_j h}\mathbb{E}_{\mathcal{Q}_q}\Big(\big(\sum_{n\geq\mathcal{S}_j}\mathbf{V}_n^{(j)}\big)^h\Big)<\infty,$$

where $\mathbb{E}_{\mathcal{Q}_q}$ means expectation with respect to Q_q.The main difference with the proofs of Proposition 2.2 and Theorem 3.2 is that here we do not seek for a result valid uniformly over the w of the same generation j, but only for a result valid for $w^{(j)}(t)$, for μ_q-almost every t. As a consequence we must control only one pair of random variables $(\mathbf{U}^{(j)},\mathbf{V}^{(j)})$ on each generation instead of b^j. This allows to slow the rate of increase of $\mathcal{S}_j$.

Fix $h\in(0,1)$. Since $x\mapsto x^h$ is sub-additive on $\mathbb{R}_+$, we have

$$\mathbb{E}_{\mathcal{Q}_q}\Big(\Big(\sum_{n\geq\mathcal{S}_j}\mathbf{V}_n^{(j)}\Big)^h\Big)\leq\sum_{n\geq\mathcal{S}_j}\mathbb{E}_{\mathcal{Q}_q}\Big(\big(\mathbf{V}_n^{(j)}\big)^h\Big).$$

For $\omega\in\Omega^*$, $j\geq 1$ and $n\geq 1$, by definition of the measures μ_q and μ_q^w, and since $\big(\mu_q^{w^{(j)}(t)}(\omega),\mu^{w^{(j)}(t)}(\omega)\big)$ does not depend on $t\in I_w$, we have

$$\begin{aligned}\int_{[0,1]}\big(\mathbf{V}_n^{(j)}(\omega,t)\big)^h\mu_q(\omega)(dt) &= \sum_{w\in\mathcal{A}^j}\int_{I_w}\big(\mathbf{V}_n^{(j)}(\omega,t)\big)^h\mu_q(\omega)(dt)\\ =\ & \sum_{w\in\mathcal{A}^j}\prod_{k=0}^{j-1}W_{q,w_{k+1}}(w|k)\int_{I_w}\big(\mathbf{V}_n^{(j)}(\omega,t)\big)^h\mu_q^w(\omega)\circ f_{I_w}^{-1}(dt)\\ =\ & \sum_{w\in\mathcal{A}^j}\Big(\prod_{k=0}^{j-1}W_{q,w_{k+1}}(w|k)\Big)\big(V_n^w\big)^h\|\mu_q^w\|,\end{aligned}$$

where $V_n^w=S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q^w(\omega),\mu^w(\omega),\tau_\mu'(q)\big)$ is defined as in the proof of Theorem 3.2. The above sum is a random variable on $(\Omega,\mathcal{B},\mathbb{P})$. In addition, in each of its terms, the product is independent of $\big(V_n^w\big)^h\|\mu_q^w\|$. Moreover, the probability distribution of $\big(V_n^w\big)^h\|\mu_q^w\|$ does not depend on w. Consequently, using the martingale property of the sequence $(\|\mu_{q,j}\|)_{j\geq 0}$, we get

$$\mathbb{E}_{\mathcal{Q}_q}\left(\big(\mathbf{V}_n^{(j)}\big)^h\right)=\mathbb{E}\left(\big(V_n^w\big)^h\|\mu_q^w\|\right),$$

where $w \in \mathcal{A}^j$. Let $p = 1/(1-h)$. The Hölder inequality yields

$$\mathbb{E}\Big(\big(V_n^w\big)^h \, \|\mu_q^w\|\Big) \le \big(\mathbb{E}\big(V_n^w\big)\big)^h \, \mathbb{E}\big(\|\mu_q\|^p\big)^{1/p}.$$

Finally, p is fixed close enough to 1 so that $\mathbb{E}\big(\|\mu_q\|^p\big) < \infty$ (see the proof of Lemma 3 for the existence of such a p). Then (4.7) implies that $\sum_{j\ge 1} b^{\rho_j h} \sum_{n\ge \mathcal{S}_j} \big(\mathbb{E}\big(V_n^w\big)\big)^h < \infty$, hence the conclusion.

Similar computations as above show that for every $j \ge 1$,

$$\mathcal{Q}_q\big(\mathbf{U}^{(j)} \le b^{-\rho_j}\big) = \mathbb{E}\big(\mathbf{1}_{\{Y_q \le b^{-\rho_j}\}} Y_q\big) \le b^{-\rho_j}\mathbb{P}(Y_q \le b^{-\rho_j}).$$

It follows from item 1. of Lemma 2 that for some $h > 0$, we have $\mathbb{P}(Y_q \le x) = O(x^h)$ as $x \to 0$. This implies $\sum_{j\ge 1} \mathcal{Q}_q\big(\mathbf{U}^{(j)} \le b^{-\rho_j}\big) < \infty$.

2. The proof is similar to the one of item *1.*. It is enough to prove the result for a compact subinterval K of $\mathcal{O}$ instead of $\mathcal{O}$. Fix such an interval K. The idea is now to consider on $\big(K \times \Omega \times [0,1], \mathcal{B}(K) \otimes \mathcal{B} \otimes \mathcal{B}([0,1])\big)$ the probability distribution $\mathcal{Q}_K$

$$\mathcal{Q}_K(A) = \int_K \big(\mathbb{E}_{\mathcal{Q}_q} \mathbf{1}_A(q,\omega,t)\big) \,\frac{dq}{|K|} \quad \big(A \in \mathcal{B}(K) \otimes \mathcal{B} \otimes \mathcal{B}([0,1])\big).$$

Then $\mathbf{U}^{(j)}(q,\omega,t)$ and $\mathbf{V}_n^{(j)}(q,\omega,t)$ are redefined as

$$\mathbf{U}^{(j)}(q,\omega,t) = \|\mu_q^{w^{(j)}(t)}(\omega)\| \text{ and } \mathbf{V}_n^{(j)}(q,\omega,t) = S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q^{w^{(j)}(t)}(\omega), \mu^{w^{(j)}(t)}(\omega), \tau_\mu'(q)\big).$$

Since there exists $p > 1$ such that $M = \sup_{q\in K} \mathbb{E}\big(\|\mu_q\|^p\big)^{1/p} < \infty$ (again, see the proof of Lemma 3), the computations performed above yield

$$\sum_{j\ge 0} b^{\rho_j h} \sum_{n \ge \mathcal{S}_j} \mathbb{E}_{\mathcal{Q}_K}\Big(\big(\mathbf{V}_n^{(j)}\big)^h\Big) \le |K|\, M \sum_{j\ge 1} b^{\rho_j h} \sum_{n\ge \mathcal{S}_j} \Big(\mathbb{E}\big(V_n^w\big)\Big)^h < \infty.$$

Finally, $\sum_{j\ge 0} \mathcal{Q}_K\big(\mathbf{U}^{(j)} \le b^{-\rho_j}\big) \le |K| \sum_{j\ge 1} b^{-\rho_j}\mathbb{P}\big(\inf_{q\in K} Y_q \le b^{-\rho_j}\big)$, which is finite by item 1. of Lemma 2.

4.5. Proof of Theorem 3.4

We assume without loss of generality that K contains the point 1. Define $q_K = \max\{|q| : q \in K\}$. Recall that for $j \ge 0$ and $n \ge 1$, if $(w,v) \in \mathcal{A}^j \times \mathcal{A}^n$ and $q \in K$ then

$$\mu^w(I_v)^q = \mu_q^w(I_v) b^{-n\widetilde{\tau}_\mu(q)} \frac{Y(wv)^q}{Y_q(wv)}.$$

Then summing over $v \in \mathcal{A}^n$ yields

$$Y_q(w) b^{-n\widetilde{\tau}_\mu(q)} \inf_{q\in K, v\in\mathcal{A}^n} \frac{Y(wv)^q}{Y_q(wv)} \le b^{-n\tau_n^w(q)} \tag{4.10}$$

$$\text{and} \quad b^{-n\tau_n^w(q)} \le Y_q(w) b^{-n\widetilde{\tau}_\mu(q)} \sup_{q\in K, v\in\mathcal{A}^n} \frac{Y(wv)^q}{Y_q(wv)}. \tag{4.11}$$

Fix $\delta \in (0,1)$ and $\theta > 0$ such that the conclusions of Propositions 4.1 and 4.2 below hold. Then, with probability one, for j large enough, for every $w \in \mathcal{A}^j$, $q \in K$ and $n \ge j^\delta$, we have $b^{-n\tau_n^w(q)} \le Y_q(w) b^{-n\widetilde{\tau}_\mu(q)} n^{(q_K+1)\theta}$. Now, remarking that $\mathcal{N}_n(\mu^w, \widetilde{\tau}_\mu'(q), \varepsilon_n) \min(b^{-nq(\widetilde{\tau}_\mu'(q)+\varepsilon_n)}, b^{-nq(\widetilde{\tau}_\mu'(q)-\varepsilon_n)}) \le b^{-n\tau_n^w(q)}$, we obtain

$$\begin{aligned} \mathcal{N}_n(\mu^w, \widetilde{\tau}_\mu'(q), \varepsilon_n) &\le b^{n\left(\widetilde{\tau}_\mu'(q)q - \tau_n^w(q) + sgn(q)q\varepsilon_n\right)} \\ &\le Y_q(w) b^{n\left(\widetilde{\tau}_\mu'(q)q - \widetilde{\tau}_\mu(q) + sgn(q)q\varepsilon_n\right)} n^{(q_K+1)\theta}. \end{aligned}$$

On the other hand, due to Theorem 3.2 and Proposition 4.1, there exists $\theta > 0$ such that, with probability one, for j large enough, for all $w \in \mathcal{A}^*$ and $q \in K$

$$\mu_q^w \left(E^{\mu^w}_{\widetilde{\tau}_\mu'(q), \mathcal{S}_j}(0, \widetilde{\varepsilon}) \bigcap E^{\mu_q^w}_{\widetilde{\tau}_\mu'(q)q - \widetilde{\tau}_\mu(q), \mathcal{S}_j}(0, \widetilde{\varepsilon}) \right) \ge \|\mu_q^w\|/2 = Y_q(w)/2.$$

This implies that $b^{n(\widetilde{\tau}_\mu'(q)q - \widetilde{\tau}_\mu(q) - \varepsilon_n)} Y_q(w)/2 \le \mathcal{N}_n(\mu^w, \widetilde{\tau}_\mu'(q), \varepsilon_n)$ for every $n \ge \mathcal{S}_j$. Moreover, for j large enough, $j^\delta \le \mathcal{S}_j$. Then for n large enough, $\sup_{q\in K} sgn(q)q\varepsilon_n + (q_K+1)\theta \log_b(n)/n$ is controlled by $(1+q_K)\varepsilon_n$. The conclusion follows.

4.6. Proof of Theorem 3.5

We begin with three technical lemmas.

Lemma 3. *Assume that $\mathcal{O} = \mathbb{R}$. For every compact subinterval K of $\mathbb{R}$, there exist $C_K, c_K > 0$ such that*

$$\text{for every } x \ge 1, \ \sup_{q\in K} \mathbb{P}(Y_q \ge x) \le C_K \exp\big(-c_K x\big).$$

Proof. We recall the following properties involved in the proofs of several statements of Section 3: it is known (see [31, 20]) that if $h > 1$ then $\mathbb{E}(Y_1^h) < \infty$ if and only if $\mathbb{E}\Big(\sum_{k=0}^{b-1} W_k^h\Big) < 1$. Consequently, if $q \in \mathcal{O} = \{q \in \mathbb{R} : \widetilde{\tau}_\mu'(q)q - \widetilde{\tau}_\mu(q) > 0\}$ and $h > 1$, then $\mathbb{E}(Y_q^h) < \infty$ if and only if $\mathbb{E}\Big(\sum_{k=0}^{b-1} W_{q,k}^h\Big) < 1$, that is $\widetilde{\tau}_\mu(qh) - h\widetilde{\tau}_\mu(q) > 0$. Moreover, it follows from Theorem VI.A.b.i) of [4] that for every compact subinterval K of $\mathcal{O}$, there exists $h > 1$ such that $\sup_{q\in K} \mathbb{E}(Y_q^h) < \infty$.

The fact that the mapping $q \mapsto \widetilde{\tau}_\mu(q)/q$ is increasing on $\mathbb{R}_-^*$ and $\mathbb{R}_+^*$ is equivalent to $\mathcal{O} = \mathbb{R}$. As a consequence, we obtain $\widetilde{\tau}_\mu(qh) - h\widetilde{\tau}_\mu(q) > 0$ for all $q \in \mathbb{R}$ and $h > 1$, that is $\mathbb{E}(Y_q^h) < \infty$. We also have $\|W_{q,k}\|_\infty \leq 1$ for all $q \in \mathbb{R}$ and $0 \leq k \leq b-1$.

We fix K a compact subset of $\mathbb{R}$. Then consider the quantity $t_k(q) = \mathbb{E}(Y_q^k)/k!$ for $q \in K$ and $k \geq 1$, and $t_0(q) = 1$. Using Equation (4.6) in [36] (our random variable Y_q is denoted there W), we get that for every $k \geq 2$, for every $q \in K$,

$$t_k(q) \leq \ c_K \sum_{(k_0,\dots,k_{b-1}):\, 0\leq k_i \leq k-1 \text{ and } k_0+\dots+k_{b-1}=k} \ \prod_{i=0}^{b-1} t_{k_i}(q),$$

where $c_K = \sup_{q\in K}\sup_{k\geq 2}\big(1 - b^{-\widetilde{\tau}_\mu(kq)+k\widetilde{\tau}_\mu(q)}\big)^{-1}$. We see that $c_K = \sup_{q\in K}\big(1 - b^{-\widetilde{\tau}_\mu(2q)+2\widetilde{\tau}_\mu(q)}\big)^{-1} < \infty$. Hence, if $\widetilde{t}_k = \sup_{q\in K} t_k(q)$, we have

$$\forall\, k \geq 2, \ \ \widetilde{t}_k \leq \ c_K \sum_{(k_0,\dots,k_{b-1}):\, 0\leq k_i \leq k-1 \text{ and } k_0+\dots+k_{b-1}=k} \ \prod_{i=0}^{b-1} \widetilde{t}_{k_i}.$$

Since $\widetilde{t}_0 = \widetilde{t}_1 = 1$, Lemma 2.6 of [25] yields $\limsup_{k\to\infty} \widetilde{t}_k^{\frac{1}{k}} < \infty$. This implies the existence of a constant $C > 0$ such that

$$\forall\, k \geq 1, \quad \sup_{q\in K} \mathbb{E}(Y_q^k) \leq C^k k!.$$

Now, fix $c_K \in (0, C^{-1})$. For $x > 0$, we have

$$\begin{aligned} \sup_{q\in K} \mathbb{P}(Y_q \geq x) \leq e^{-c_K x} \sup_{q\in K} \mathbb{E}\big(e^{c_K Y_q}\big) \quad &\leq \quad e^{-c_K x} \sum_{k=0}^{\infty} c_K^k \sup_{q\in K} \mathbb{E}(Y_q^k)/k! \\ &\leq \quad (1 - c_K C)^{-1} e^{-c_K x}. \end{aligned}$$

Remark 4.2. We are not able to control $\mathbb{P}(\sup_{q\in K} Y_q \geq x)$ at ∞. This is the reason why the next Lemmas 4 and 5 are needed to obtain Proposition 4.2.

For $n \geq 1$, let Q_n be the set of dyadic numbers of generation n.

Lemma 4. *Let K be a compact subinterval of $\mathcal{O}$. Let $\eta > 0$. There exists $\chi \in (0,1)$ and $\delta \in (0,1)$ such that, with probability 1,*

1. *for j large enough, $\forall\, w \in \mathcal{A}^j$, $\forall\, n \geq [j^{1+\eta}]$, $\forall\, q, q' \in Q_n$ such that $|q - q'| = 2^{-n}$, we have $\big|Y_q^w - Y_{q'}^w\big| \leq |q' - q|^\chi$.*

2. *for j large enough, $\forall n \ge j^\delta$, $\forall\, w \in \mathcal{A}^j$, $\forall v \in \mathcal{A}^n$, $\forall\, m \ge [n^{1+\eta}]$, for all $q, q' \in Q_m$ such that $|q'-q| = 2^{-m}$, we get $\left|Y_q^{wv} - Y_{q'}^{wv}\right| \le |q'-q|^\chi$.*

Proof. By Theorem VI.A.b. *i*) of [4], $\exists\ h > 1,\ C_K > 0$ such that

$$\text{for all } (q,q') \in K^2,\ \mathbb{E}\left(|Y_q - Y_{q'}|^h\right) \le C_K |q-q'|^h. \tag{4.12}$$

For $n \ge 1$, let $\widetilde{Q}_n$ be the set of pairs $(q,q') \in Q_n$ such that $|q-q'| = 2^{-n}$, and let $\chi \in (0,(h-1)/h)$. Using (4.12) and the Markov inequality, we obtain

$$\begin{aligned} p_n &:= \mathbb{P}\left(\exists\ (q,q') \in \widetilde{Q}_n,\ |Y_q - Y_{q'}| \ge |q-q'|^\chi\right) \\ &\le \sum_{(q,q')\in\widetilde{Q}_n} \mathbb{P}\Big(|Y_q - Y_{q'}| \ge |q-q'|^\chi\Big) \le 2|K| 2^n C_K 2^{n\chi h} 2^{-nh}. \end{aligned}$$

Fix $\eta > 0$ and $\delta \in ((1+\eta)^{-1}, 1)$. $\sum_{j\ge1} b^j \sum_{n \ge [j^{1+\eta}]} p_n < \infty$ implies item **1.** of Lemma 4 by the Borel-Cantelli lemma. Also, item **2.** follows from the fact that $\sum_{j\ge1} b^j \sum_{n\ge j^\delta} b^n \sum_{m\ge[n^{1+\eta}]} p_m < \infty$.

Lemma 5. *Under the assumptions of Theorem 3.4, let $K \subset \mathbb{R}$ be a compact interval. Let $\eta > 0$. There exist $\delta \in (0,1)$, $\theta > 1$ such that, with probability 1, for j large enough,*

1. *$\forall\, w \in \mathcal{A}^j$, $\sup_{q \in Q_{[j^{1+\eta}]}\cap K} Y_q(w) \le j^\theta$.*

2. *$\forall\, n \ge j^\delta$, $\forall\,(v,w) \in \mathcal{A}^n \times \mathcal{A}^j$, $\sup_{q\in Q_{[n^{1+\eta}]}\cap K} Y_q(vw) \le\ n^\theta$.*

Proof. Fix $\theta > 1+\eta$. For $q \in K$ and $j \ge 1$ define $p_j(q) = \mathbb{P}(Y_q \ge j^\theta)$. By Lemma 3,

$$\begin{aligned} \forall\, j \ge 1,\ \ \mathbb{P}\Big(\sup_{q\in Q_{[j^{1+\eta}]}\cap K} Y_q \ge j^\theta\Big) &\le \sum_{q\in Q_{[j^{1+\eta}]}\cap K} p_j(q) \\ &\le p_j := 2C_K|K| 2^{j^{1+\eta}} \exp\big(-c_K j^{\theta_K}\big). \end{aligned}$$

We let the reader verify that $\sum_{j\ge1} b^j p_j < \infty$ and $\sum_{j\ge1} b^j \sum_{n\ge j^\delta} b^n p_n < \infty$ if $\delta \in (\theta_K^{-1}, 1)$. This yields items 1. and 2. of Lemma 5.

The two following Propositions are needed to control the inequality (4.10).

Proposition 4.1. *Under the assumptions of Theorem 3.2, let K be a compact subinterval of $\mathcal{O} \cap \mathbb{R}_+$ (resp. $\mathcal{O} \cap \mathbb{R}_-$). There exist $\theta > 0$ and $\delta \in (0,1)$ such that, with probability 1, for j large enough*

1. *$\forall\ w \in \mathcal{A}^j$, $\inf_{q\in K} Y_q(w) \ge j^{-\theta}$.*

2. $\forall n \geq j^\delta$, $\forall\ w \in \mathcal{A}^j$, $\forall v \in \mathcal{A}^n$, $\inf_{q\in K} Y_q(wv) \geq\ n^{-\theta}$.

Proof. Fix $\theta > 1$ such that $\theta_K = \theta \frac{\gamma_K}{1-\gamma_K} > 1$, where γ_K is as in Lemma 2. Also define $p_j = \mathbb{P}\left(\inf_{q\in K} Y_q < j^{-\theta}\right)$.

We let the reader verify, using Lemma 2, that $\sum_{j\geq 1} b^j p_j < \infty$ and if $\delta \in (\theta_K^{-1}, 1)$ then $\sum_{j\geq 1} b^j \sum_{n\geq j^\delta} b^n p_n < \infty$. This yields *1.* and *2.*.

Proposition 4.2. *Under the assumptions of Theorem 3.4, let $K \subset \mathcal{O}$ be a compact interval. There is $\theta > 0$, $\delta \in (0,1)$ such that, with probability 1, for j large enough*

1. $\forall\ w \in \mathcal{A}^j$, *we have* $\sup_{q\in K} Y_q(w) \leq j^\theta$.

2. $\forall n \geq j^\delta$, $\forall\ w \in \mathcal{A}^j$, $\forall v \in \mathcal{A}^n$, $\sup_{q\in K} Y_q(wv) \leq n^\theta$.

Proof. We assume without loss of generality that the end points of K are dyadic numbers. It is standard (see the proof of Kolmogorov theorem in [33]) that Lemma 4 implies that there is a constant $C_K > 0$ such that, with probability one

1. for j large enough, $\forall\ w \in \mathcal{A}^j$, $\forall\ q, q' \in K$ such that $|q - q'| \leq 2^{-[j^{1+\eta}]}$, we have $\left|Y_q(w) - Y_{q'}(w)\right| \leq C_K |q' - q|^\chi$.

2. for j large enough, $\forall n \geq j^\delta$, $\forall\ (v, w) \in \mathcal{A}^n \times \mathcal{A}^j$, for all $q, q' \in K$ such that $|q' - q| \leq 2^{-[n^{1+\eta}]}$, we have $\left|Y_q(wv) - Y_{q'}(wv)\right| \leq C_K |q' - q|^\chi$.

Then, Lemma 5 concludes the proof.

Finally, Theorem 3.5 is a consequence of (4.10) and of Propositions 4.1 and 4.2.

5. Growth speed as a tool for conditioned ubiquity results

Let $\{(x_n, \lambda_n)\}_{n\geq 1}$ be a sequence in $[0,1] \times (0,1]$ such that $\lim_{n\to\infty} \lambda_n = 0$.

For every $t \in (0,1)$, $k \geq 1$ and $r \in (0,1)$, we consider the set of balls

$$\mathcal{B}_{k,r}(t) = \left\{B(x_n, \lambda_n) :\ t \in B(x_n, r\lambda_n),\ \lambda_n \in (b^{-(k+1)}, b^{-k}]\right\}$$

($B(y, r')$ is the closed interval centered at y with radius r'). Notice that this set may be empty. Then, if $\xi > 1$ and $B(x_n, \lambda_n) \in \mathcal{B}_{k,1/2}(t)$, let $\mathcal{B}_k^\xi(t)$ be the set of b-adic intervals of maximal length included in $B(x_n, \lambda_n^\xi)$.

The next result is key to build a generalized Cantor set of Hausdorff dimension $\geq \tau_\mu^*(h)/\xi$ in the set $K(\mu, h, \xi, \widetilde{x}, \widetilde{\lambda}, \widetilde{\varepsilon})$ (1.6), when μ is a Mandelbrot measure. All along this Cantor set's construction, a property related to the control of the growth speed is used. This property is the following: If $q \in \mathcal{O}$ and $h = \tau_\mu'(q)$, then each measure μ_q^u is carried by the set $E^{\mu_q^u}_{\tau_\mu^*(h)}(N, \widetilde{\varepsilon})$ and, roughly speaking, if $GS\big(\mu_q^u, \mu_q^u, \tau_\mu^*(h), N, \widetilde{\varepsilon}\big)$ is not too large, the measure μ_q^u restricted to $E^{\mu_q^u}_{\tau_\mu^*(h)}(N, \widetilde{\varepsilon})$ can be viewed as being monofractal of exponent $\tau_\mu^*(h)$.

Theorem 5.1. *Suppose that* $\limsup_{n\to\infty} B(x_n, \lambda_n/4) = (0,1)$*. Let* μ *be an independent random cascade. Fix* $\kappa > 0$*. For* $j \geq 2$*, let* $\mathcal{S}_j = j \log(j)^{-\kappa}$ *and* $\rho_j = \log(j)^\chi$ *with* $\chi > 1$*. Assume that (3.4) holds.*

For every $q \in \mathcal{O}$ *and* $\xi > 1$*, with probability one, the property* $\mathcal{P}(\xi, q)$ *holds, where* $\mathcal{P}(\xi, q)$ *is: For* μ_q*-almost every* t*, there are infinitely many* $k \geq 1$ *such that* $\mathcal{B}_{k,1/2}(t) \neq \emptyset$ *and there exists* $u \in \{v \in \mathcal{A}^* : \exists\, I \in \mathcal{B}_k^\xi(t),\ I = I_v\}$ *such that*

$$GS\big(\mu_q^u, \mu_q^u, \tau_\mu'(q)q - \tau_\mu(q), N, \widetilde{\varepsilon}\big) \leq \mathcal{S}_{|u|}, \quad \textit{and} \quad \|\mu_q^u\| \geq b^{-\rho_{|u|}}. \tag{5.1}$$

Remark 5.1.

1. Under the assumptions of Theorem 5.1, Theorems 3.1 and 5.1 associated with the main result on heterogeneous ubiquity established in [10] imply that for every $q \in \mathcal{O}$ and $\xi > 1$, with probability one, dim $K(\mu, \tau_\mu'(q), \xi, \widetilde{x}, \widetilde{\lambda}, \widetilde{\varepsilon}) \geq \big(\tau_\mu'(q)q - \tau_\mu(q)\big)/\xi$.
2. If q is fixed in $\mathcal{O}$, the assumption $\limsup_{n\to\infty} B(x_n, \lambda_n/4) = (0,1)$ can be weakened by requiring only that $\limsup_{n\to\infty} B(x_n, \lambda_n/4)$ is of full μ_q-measure.
3. The result in [12] on ubiquity conditioned by Mandelbrot measures concerns the case where $\{(x_n, \lambda_n)\}_n = \{(kb^{-j}, b^{-j})\}_{j\geq 1, 0\leq k < b^{-j}}$. There, a slightly different version of $\mathcal{P}(\xi, q)$ is invoked, whose proof is easily deduced from the one of Theorem 5.1.

Proof. For $k \geq 1$ and $w \in \mathcal{A}^{k+3}$, notice that $\mathcal{B}_{k,1/4}(t) \subset \mathcal{B}_{k,1/2}(s)$ for all $t, s \in I_w$. Let $\mathcal{R}_w = \{n : \exists t \in I_w,\ B(x_n, \lambda_n) \in \mathcal{B}_{k,1/4}(t)\}$. Define $n(w) = \inf\big\{n : x_n = \min\{x_m : m \in \mathcal{R}_w\}\big\}$ if $\mathcal{R}_w \neq \emptyset$ and $n(w) = 0$ otherwise.

If $\xi > 1$ and $n(w) > 0$, let $u(w)$ be the word encoding the b-adic interval of maximal length included in $B(x_n, \lambda_n^\xi)$ and whose left end point is minimal. If $\xi > 1$ and

$n(w)=0$, let $u(w)$ be the word of generation $[\xi|w|]$ with prefix w and its $[\xi|w|]-|w|$ last digits equal to 0.

Now, $w^{(j)}(t)$ being defined as in the statement of Theorem 3.3, we prove a slightly stronger result than Theorem 5.1: For every $q\in\mathcal{O}$ and $\xi>1$, with probability one, the property $\widetilde{\mathcal{P}}(\xi,q)$ holds, where $\widetilde{\mathcal{P}}(\xi,q)$ is: For μ_q-almost every t, if j is large enough, for all $k\ge j$ such that $n\big(w_{k+3}(t)\big)>0$, $u=u(w_{k+3}(t))$ satisfies (5.1).

In the sequel we denote $u(w_{k+3}(t))$ by $u_{k,\xi}(t)$.

Fx $\xi>1$ and $q\in K$. For $j\ge 0$ and $n\ge 1$ define on $\Omega\times[0,1)$ the random variables

$$\begin{aligned} \mathbf{U}^{(j)}(\omega,t) &= \|\mu_q^{u_{j,\xi}(t)}(\omega)\|. \\ \text{and}\quad \mathbf{V}_n^{(j)}(\omega,t) &= S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q^{u_{j,\xi}(t)}(\omega),\mu_q^{u_{j,\xi}(t)}(\omega),q\tau_\mu'(q)-\tau_\mu'(q)\big). \end{aligned}$$

We can use the proof of Proposition 2.2 to deduce that it is enough to prove

$$\sum_{j\ge1}\mathcal{Q}_q\Big(\Big\{\exists\ k\ge j,\ b^{\rho_{|u_{k,\xi}(t)|}}\sum_{n\ge\mathcal{S}_{|u_{k,\xi}(t)|}}\mathbf{V}_n^{(k)}(\omega,t)\ge 1/2\Big\}\Big)<\infty \tag{5.2}$$

$$\text{and}\quad \sum_{j\ge1}\mathcal{Q}_q\Big(\Big\{\exists\ k\ge j,\ \mathbf{U}^{(j)}(\omega,t)\le b^{-\rho_{|u_{k,\xi}(t)|}}\Big\}\Big)<\infty. \tag{5.3}$$

Since there exist $c>c'>0$ such that $c'\xi k\le|u_{k,\xi}(t)|\le c\xi k$ for all t, denoting $\bar{k}=[c\xi k]+1$ and $\widetilde{k}=[c'\xi k]$, in order to get (5.2) and (5.3), it is enough to show that

$$\begin{cases}\mathcal{T}=\sum_{j\ge1}\sum_{k\ge j}\mathcal{Q}_q\big(b^{\rho_{\bar{k}}}\sum_{n\ge\mathcal{S}_{\widetilde{k}}}\mathbf{V}_n^{(k)}(\omega,t)\ge 1/2\big)<\infty,\\ \mathcal{T}'=\sum_{j\ge1}\sum_{k\ge j}\mathcal{Q}_q(\mathbf{U}^{(k)}\le b^{-\rho_{\widetilde{k}}})<\infty.\end{cases}$$

Notice that $\mathcal{T}\le 2^h\displaystyle\sum_{j\ge1}\sum_{k\ge j}\sum_{n\ge\mathcal{S}_{\widetilde{k}}}b^{\rho_{\bar{k}}h}\mathbb{E}_{\mathcal{Q}_q}\Big(\big(\mathbf{V}_n^{(k)}\big)^h\Big)$ if $h\in(0,1)$.

Mimicking the computations performed in the proof of Theorem 3.3, we get

$$\int_{[0,1]}\big(V_n^{u_{k,\xi}(t)}(\omega)\big)^h\,\mu_q(\omega)(dt)=\sum_{w\in\mathcal{A}^{k+3}}\big(\prod_{k=0}^{k-1}W_{q,w_{k+1}}(w|k)\big)\big(V_n^{u(w)}\big)^h\,\|\mu_q^w\|.$$

Using the independences as well as p and h as in the proof of Theorem 3.3, we obtain

$$\mathbb{E}_{\mathcal{Q}_q}\big(\big(\mathbf{V}_n^{(k)}\big)^h\big)\le\big(\mathbb{E}\big(V_n^{u(w)}\big)\big)^h\,\mathbb{E}\big(\|\mu_q^w\|^p\big)^{1/p}$$

where w is any element of $\mathcal{A}^*$. Then our choice for ρ_j and $\mathcal{S}_j$ ensures that $\mathcal{T}$ is finite.

For any $h'>0$, $\mathcal{T}'\le\sum_{j\ge1}\sum_{k\ge j}b^{-\rho_{\widetilde{k}}h}\mathbb{E}_{\mathcal{Q}_q}\big(\big(\mathbf{U}^{(k)}\big)^{-h'}\big)$. The same computation as above yields that with the same h and p, $\mathbb{E}_{\mathcal{Q}_q}\big(\big(\mathbf{U}^{(k)}\big)^{-h'}\big)\le\big(\mathbb{E}\big(Y_q^{u(w)}\big)^{-h'/h}\big)^h\big(\mathbb{E}(\|\mu_q^w\|^p)\big)^{1/p}$

for any element w of $\mathcal{A}^*$. If h' is small enough, the right hand side is bounded by Lemma 2 independently of k and the conclusion follows from our choice for ρ_j.

Acknowledgements

The authors are grateful to the referee for valuable comments.

References

[1] Arbeiter, M., Patzschke, N. (1996). *Random self-similar multifractals*, Math. Nachr. **181**, 5–42.

[2] Arnol'd, V.I. (1963). *Small denominators and problems of stability of motion in classical and celestial mechanics*, Ups. Mat. Nauk. **18**, 91–192, Eng. transl. in Russ. Math. Surveys **18**, 85-191.

[3] Bacry, E., Muzy, J.-F. (2003). *Log-infinitely divisible multifractal processes*, Commum. Math. Phys. **236**, 449-475.

[4] Barral, J. (1999). *Moments, continuité et analyse multifractale des martingales de Mandelbrot*, Probab. Theory Relat. Fields, 113: 535-569.

[5] Barral, J. (2000). *Continuity of the multifractal spectrum of a random statistically self-similar measure*, J. Theoretic probab. **13** , 1027-1060.

[6] Barral, J., Ben Nasr, F., Peyrière, J. (2003) *Comparing multifractal formalisms: The neighboring boxes conditions*, Asian J. Math. **7**, 149-165.

[7] Barral, J., Coppens, M.O. and Mandelbrot, B.B. (2003). *Multiperiodic Multifractal Martingale Measures*, J. Math. Pures Appl. 82: 1555-1589.

[8] Barral, J., Mandelbrot, B.B. (2002). *Multifractal products of cylindrical pulses*, Probab. Theory. Rel. Fields, 124: 409-430.

[9] Barral, J., Mandelbrot, B.B. (2004). *Random multiplicative multifractal measures*, in Fractal Geometry and Applications: a Jubilee of Benoît Mandelbrot, M. L. Lapidus and M. van Frankenhuijsen eds., Proceedings of Symposia in Pure Mathematics, AMS, Providence.

[10] Barral, J., Seuret S. (2004). *Heterogeneous ubiquitous systems and Hausdorff dimension in* $\mathbb{R}^d$, ArXiv math.GM/0503419, submitted.

[11] Barral, J., Seuret S. (2004). *Sums of Dirac masses and conditioned ubiquity*, C. R. Acad. Sci. Paris, Ser. I **339**, 787–792.

[12] Barral, J., Seuret S. (2005) *Combining multifractal additive and multiplicative chaos*, Commun. Math. Phys. **257**, 473–497.

[13] Barral, J., Seuret S. (2005). *The singularity spectrum of Lévy processes in multifractal time*, accepted for publication in Advances in Mathematics.

[14] Barral, J., Seuret S. (2005). *Inside singularities sets of random Gibbs measures*, J. Stat. Phys. **120**, 1101–1124.

[15] Biggins, J.D. (1992). *Uniform convergence of martingales in the branching random walk*, Ann. Prob. **20**, 137-151.

[16] Billingsley, P. (1965). Ergodic Theory and Information, John Wiley, New York.

[17] Brown, G., Michon, G. and Peyrière, J. (1992). *On the multifractal analysis of measures*, J. Stat. Phys., **66**(3-4),775-790.

[18] Collet, P., Koukiou, F. (1992). *Large deviations for multiplicative chaos*, Commun. Math. Phys. **147**, 329–342.

[19] Dodson, M.M. (2002). *Exceptional sets in dynamical systems and Diophantine approximation*, Rigidity in Dynamics and Geometry (Cambridge, 2000), 77–98, Springer.

[20] Durrett, R., Liggett, T. (1983). *Fixed points of the smoothing transformation*, Z. Wahrsch. verw. Gebiete **64** , 275–301.

[21] Falconer, K.J. (1994). *The multifractal spectrum of statistically self-similar measures*, J. Theor. Prob. **7**:681–702.

[22] Falconer, K.J. (2000). *Representation of families of sets by measures, dimension spectra and Diophantine approximation*, Math. Proc. Camb. Phil. Soc. **128**, 111–121.

[23] Fan, A.H. (1997). *Multifractal analysis of infinite products*, J. Stat. Phys., 86(5/6):1313-1336.

[24] Frisch, U., Parisi, G. (1985). *Fully developped turbulence and intermittency*, Proc. International Summer school Phys., Enrico Fermi, 84-88, North Holland.

[25] Graf S., Mauldin, R.D., Williams, S.C. (1988) *The exact Hausdorff dimension in random recursive constructions*, Mem. amer. Math. Soc. **71**, 381.

[26] Halsey, T.C., Jensen, M.H., Kadanoff, L.P., Procaccia, I., Shraiman, B.I. (1986). *Fractal measures and their singularities: The characterization of strange sets*, Phys. Rev. A, **33**(2), 1141–115.

[27] Holley R., Waymire, E.C. (1992). *Multifractal dimensions and scaling exponents for strongly bounded random fractals*, Ann. Appl. Probab. **2**, 819–845.

[28] Jaffard, S. (1999.) *T*he multifractal nature of Lévy processes, Probab. Theory Relat. Fields **114(2)**, 207–227.

[29] Kahane, J.-P. (1985). *Sur le chaos multiplicatif* , Ann. Sci. Math. Québec, **9**, 105-150.

[30] Kahane, J.-P. (1987). *Positives martingales and random measures*, Chi. Ann. of Math, (1) **8B**, 1–12.

[31] Kahane, J.-P., Peyrière J. (1976). *Sur certaines martingales de Benoît Mandelbrot*, Adv. Math. **22**, 131-145.

[32] Khanin, K., Kifer, Y. (1996) *Thermodynamic formalism for random transformations and statistical mechanics*, in: Amer. Math. Soc. Transl. (2) **171**, 107–140

[33] Karatzas, I., Shreve, S.E. (1988) Brownian Motion and Stochastic Calculus, Springer-Verlag New-York.

[34] Kifer, Y. (1995). *Fractals via random iterated function systems and random geometric constructions*, Fractal geometry and stochastics (Finsterbergen, 1994) Progr. Probab. **37**, 145–164, Birkhäuser, Basel.

[35] Lévy Véhel, J., Riedi, R.H. (1997). *TCP traffic is multifractal: a numerical study*, INRIA research report, RR-3129.

[36] Liu, Q. (1996). *The growth of an entire characteristic function and the tail probabilities of the limit of a tree martingale*, in: Progress in Probability **40**, Birkhäuser-Verlag Basel.

[37] Liu, Q. (2000). *On generalized multiplicative cascades*, Stoch. Proc. Appl., **86**, 263–286.

[38] Liu, Q. (2001). *Asymptotic properties and absolute continuity of laws stable by random weighted mean*, Stoch. Proc. Appl. **95**, 83–107.

[39] Mandelbrot, B.B. (1974). *Intermittent turbulence in self-similar cascades: divergence of hight moments and dimension of the carrier*, J. Fluid. Mech. **62** 331–358.

[40] Mandelbrot, B.B. (1997). Fractals and Scaling in Finance: Discontinuity, Concentration, Risk, Springer.

[41] Mandelbrot, B.B. (2001). *Scaling in financial prices: III. Cartoon Brownian motions in multifractal time*, Quantitative Finance, **1**, 427–440.

[42] Mandelbrot, B.B., Fisher, L., Calvet, A. (1997). *A multifractal model for asset returns*, Cowles foundation discussion paper # 1164.

[43] Molchan, G.M. (1996). *Scaling exponents and multifractal dimensions for independent random cascades*, Commun. Math. Phys., **179**, 781–702.

[44] Muzy, J.-F., Bacry, E. (2002). *Multifractal stationary random measures and multifractal random walks with log-infinitely divisible scaling laws*, Phys. Rev. E **66**.

[45] Olsen, L. (1994). Random Geometrically Graph Directed Self-similar Multifractals, *Pitman Res. Notes Math. Ser.* **307**.

[46] Olsen, L. (1995). *A multifractal formalism*, Adv. Math. **116**, 92-195.

[47] Ossiander, M., Waymire, E.C. (2000) *Statistical estimation for multiplicative cascades*, Ann. Stat. **28**, 1533–1560.

[48] Parry W., Policott M. (1990) *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Société Mathématique de France, Astérisque, 187–188.

[49] Riedi, R. (1995). *An improved multifractal formalism and self-similar measures*, J. Math. Anal. Appl. **189**, 462–490.

[50] Riedi, R. (2003). *Multifractal processes*, in Theory and Applications of Long-Range Dependence, eds. Doukhan et al., Birkhäuser Boston, Boston, MA, 625–716.