The singularity spectrum of Lévy processes in multifractal time

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1 Introduction

The interest for multifractal stochastic processes is mainly motivated by the need for accurate models in the study of the variability of wild signals. These locally irregular signals come from physical phenomena such as fully developed turbulence, TCP Internet traffic, variations of financial prices, or heart beats.

Fractional Brownian Motions (FBM), Lévy processes and multiplicative cascades are frequently used when modeling these phenomena. However, these processes are partly satisfactory for different reasons. FBM are monofractal, and thus have the same Hölder exponents at every point. The two other models are multifractal, i.e. the pointwise Hölder exponents take several values, and the level sets of their Hölder exponents are dense fractal sets. Nevertheless the singularity spectra of the Lévy processes have a very specific linear increasing shape and, finally, the multifractal multiplicative cascades only generate non-decreasing processes.

Other kinds of multifractal models were thus studied to go beyond these limitations. For instance, Gaussian processes with non-constant prescribed Hölder exponents are introduced in [2]. Another approach consists in generating multifractal random wavelet series [23,7].

A third point of view consists in performing a (possibly multifractal) change of time in a given stochastic process $(X_t)_{t\geq 0}$. More precisely, given an atomless positive Borel measure μ on $\mathbb{R}_+$ supported by an interval of the form $[0,T]$ ($T \in (0,\infty)$), then the process $X \circ \mu([0,t])$ is considered. This process shall be viewed as the process X in (again, possibly multifractal) time μ.

The simplest situation lies in taking X equal to a monofractal process, like the FBM (see [32,3,14] and Section 6). In this case, due to the monofractality property, the multifractal nature of $X \circ \mu$ follows almost straightforward from the one of μ (see Section 6). In the situation when it is assumed that X also has multifractal sample paths, the multifractal time change creates more interesting structures, both from the modeling and mathematical viewpoints (see for instance [37] for preliminary results on this topic, especially concerning large deviation spectra). The fine local study of the sample paths multifractal properties is far more delicate than in the monofractal case. To our knowledge it has never been achieved in a non-trivial case.

This paper deals with the case when X is a Lévy process. We provide conditions on the measure μ under which the multifractal nature of the sample paths of the process $(Z_t = X \circ \mu([0,t]))_{t\geq 0}$ can be described. Before going further, we detail the reason which led us to consider this problem.

Let b be an integer ≥ 2 and $W = (W_0, \dots, W_{b-1})$ a positive random vector. Then consider in the space of Laplace transforms of probability distributions ϕ on $\mathbb{R}_+$ the equation

$$\phi(u) = \mathbb{E}\left(\prod_{i=0}^{b-1} \phi(uW_i)\right), \quad \forall\, u \geq 0. \tag{1}$$

This equation, referred to as the smoothing transformation, is solved in [15,18]. It comes from the modeling of fully developed turbulence [31,30] and of interacting particles systems. Subsequently, the problem is then to find all the non-trivial solutions (i.e. $\not\equiv 1$) of (1). The mapping

$$\varphi_W : q \in \mathbb{R} \mapsto -\log_b \mathbb{E}\Big(\sum_{i=0}^{b-1} W_i^q\Big) \in \mathbb{R} \cup \{-\infty\}. \tag{2}$$

naturally arises in the problem's solution. Indeed, under the assumption that $\varphi_W(p) > -\infty$ for some $p > 1$, it is proved by Durrett and Liggett in [15] that (1) has non-trivial solutions if and only if there exists $\beta \in (0,1]$ such that $\varphi_W(\beta) = 0$ and $\varphi_W'(\beta) \geq 0$. As a consequence of the concavity of the mapping φ_W, such a β is unique and

$$\beta = \inf\{\beta' \in [0,1] : \varphi_W(\beta') = 0\}.$$

It is worth noting that the existence of non-trivial solutions in the general

framework is almost entirely based on the existence of a non-trivial solution in the case $\beta = 1$ with $\varphi'_W(1) > 0$. Moreover, in this case, a fundamental non-trivial solution is given by the Laplace transform of the probability distribution of $\|\mu_W\|$, where μ_W is an independent multiplicative cascade on $[0,1]$ generated by the random vector $W = (W_0, \dots, W_{b-1})$ used in (1), see [31,26] and Section 7 for the construction of μ_W. This type of multiplicative cascade measures has been extensively studied in [25,19,16,34,1,4]. Their well-known multifractal properties are closely related to φ_W and (1).
Therefore, as soon as $\varphi_W(1) = 0$ and $\varphi'_W(1) > 0$, it is possible to naturally associate the non-trivial stochastic process $(Z_{W,t})_{t\in[0,1]} = \big(\mu_W([0,t])\big)_{t\in[0,1]}$ with (1) such that the Laplace transform of $Z_{W,1}$ resolves (1). Moreover, this process $Z_{W,\cdot}$ is completely characterized by a statistical self-similarity property (see (40) in Section 7).

This raises the problem of finding a natural process satisfying the same properties in the general case $\beta \in (0,1]$. In the case $\beta \in (0,1)$, $\varphi_W(\beta) = 0$ and $\varphi'_W(\beta) > 0$, we recall how the solution ϕ of (1) is deduced in [15,18] from the construction of $\|\mu_W\|$. First, the random vector $W_\beta = (W_0^\beta, \dots, W_{b-1}^\beta)$ is considered. By construction we get that $\varphi_{W_\beta}(1) = 0$ and $\varphi'_{W_\beta}(1) > 0$, and the situation is reduced to the one described above.
Let ϕ_β be the Laplace transform of $\|\mu_{W_\beta}\|$. A non-trivial solution of (1) is then given by the mapping $\phi : u \mapsto \phi_\beta(u^\beta)$. Let X_β be a β-stable Lévy subordinator independent of μ_{W_β}. Remark that the function ϕ is also the Laplace transform of the random variable $Z = X_\beta\big(\|\mu_{W_\beta}\|\big)$ ([15]). Hence, a method to construct a stochastic process $(Z_{W,t})_{t\in[0,1]}$ associated with ϕ and fulfilling the statistical self-similarity property (40) is then the following: Consider the stochastic process

$$Z_{W,t} = X_\beta\Big(\mu_{W_\beta}([0,t])\Big) = X_\beta(Z_{W_\beta,t}) \quad (t \in [0,1]). \tag{3}$$

This process has the form of a Lévy process in multifractal time, and it possesses the required properties. Indeed, the Laplace transform of $Z_{W,1}$ resolves (1), and in addition, since X_β has by construction independent increments and is independent of μ_{W_β}, the increments of $Z_{W,t}$ also satisfy the statistical self-similarity property (40). Surprizingly enough, stable Lévy subordinators and Mandelbrot multiplicative cascades thus appear as special elements of the same class of processes (obtained by subordinating the integral of a Mandelbrot cascade μ_W to an independent Lévy subordinator X_β) obeying a certain statistical self-similarity property.

Equation (1) can also be considered in the space of characteristic functions of probability distributions on $\mathbb{R}$. It is shown in [29] that if there exists $\beta \in (1,2]$ such that $\varphi_W(\beta) = 0$ and $\varphi'_W(\beta) \geq 0$, then (1) possesses a non-trivial non-positive solution. If $\varphi'_W(\beta) > 0$, we associate naturally with that solution the stochastic process $(Z_{W,t})_{t\geq 0}$ formally defined as in (3), but with a symmetric

β-stable Lévy process X_β (a Brownian motion without drift if $\beta = 2$). Again, the multifractal nature of $(Z_{W,t})_{t\geq 0}$ appears to be related to φ_W.

We now resume the problem we address (i.e. to perform the multifractal analysis of a Lévy process in multifractal time) and our results.

First, the local regularity of a function f is measured in this paper as follows. Let $d \geq 1$, I a non-trivial subinterval of $\mathbb{R}_+$, and $f : I \to \mathbb{R}^d$. If $x \in I$, the *pointwise Hölder exponent* $h_f(x)$ of f at x is defined [1] by

$$h_f(x) = \liminf_{\substack{y \to x \\ y \neq x}} \frac{\log |f(y) - f(x)|}{\log |y - x|}, \tag{4}$$

where $|\cdot|$ stands for the Euclidean norm, with the convention $|\log(0)| = \infty$.

Then the *multifractal nature* of f is expressed in terms of the size of the levels sets E_h^f of the function $h_f(\cdot)$ defined by $E_h^f = \{x \in I : h_f(x) = h\}$ ($h \geq 0$). This size is measured by the Hausdorff dimension (denoted dim, see Definition 3). Thus we focus on the estimation of the mapping

$$d_f : h \geq 0 \mapsto \dim E_h^f,$$

which is called *singularity spectrum* or *Hausdorff multifractal spectrum* of f. A function (*resp.* a process) is said to be multifractal when its singularity spectrum (*resp.* the singularity spectrum of its sample paths) is not reduced to a single point (*resp.* with probability 1).

The singularity spectrum of Lévy processes $(X_t)_{t\geq 0}$ – which corresponds in our context to the case where the measure μ equals the Lebesgue measure – is performed in [22] (see Theorem 1 below). There is no time change in this case: Lévy processes without Brownian part have with probability 1 a non-trivial linear multifractal spectrum. This typical shape is explained by the fact that the jump points of Lévy processes satisfy a *ubiquity* property with respect to the Lebesgue measure (the notion of ubiquity is detailed in Section 3.4).

In our context, when the measure μ is not monofractal, that is when the Hölder

[1] This exponent does not coincide with the usual pointwise exponent, that we denote $H_f(x)$, which involves a polynomial ([20]). If $h_f(x) \in \mathbb{R}^+ \setminus \mathbb{N}^*$, then $h_f(x) = H_f(x)$ but the two notions may differ if $h_f(x) \in \mathbb{N}^*$. Nevertheless $h_f(x)$ is the natural notion to be used here. Indeed, the study of (Z_t) requires information on the local behavior of $t \mapsto \mu([0,\cdot])$, i.e. on the Hölder exponents of the measure μ. These exponents are in general more tractable by using a definition similar to (4) than with the definition of [20].

exponent function of the measure μ

$$h_\mu : t \mapsto \liminf_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)} \tag{5}$$

possesses several non-trivial level sets, the situation becomes subtler. We prove that the local behavior of the process $(Z_t = X \circ \mu([0,t]))_{t\geq 0}$ is closely related to some *conditioned ubiquity* properties (see Section 3.4), which combine conditions on the jump points of (Z_t) with conditions on the local behavior of μ. Understanding these properties enables us to compute the singularity spectrum d_Z, under suitable assumptions. These technical assumptions are fulfilled by several classes of statistically self-similar measures μ with a construction based on multiplicative cascade schemes, for instance some $\mathbb{R}_+$-martingales (like μ_W above) in the sense of [24,6] or random Gibbs measures (see [9,10]).

Before summarizing our results, we start by recalling precisely the theorem obtained in [22]. Let $X = (X_t)_{t\geq 0}$ be a $\mathbb{R}^d$-valued Lévy process. Recall that X has stationary independent increments and that its characteristic function takes the form $\mathbb{E}\Big(e^{i\langle\lambda|X_t\rangle}\Big) = e^{-t\psi(\lambda)}$, where

$$\psi(\lambda) = i\langle a|\lambda\rangle + Q(\lambda)/2 + \int_{\mathbb{R}^d} \Big(1 - e^{i\langle\lambda|x\rangle} + i\langle\lambda|x\rangle \mathbf{1}_{|x|\leq 1}\Big)\, \pi(dx)$$

and where $a \in \mathbb{R}^d$, Q is a quadratic form, and π is a Radon measure on $\mathbb{R}^d \setminus \{0\}$, called the Lévy measure of X, satisfying

$$\int (1 \wedge |x|^2)\pi(dx) < \infty. \tag{6}$$

Define the Blumenthal-Getoor exponent of X as

$$\beta = \inf\Big\{\gamma \geq 0 : \int_{|x|\leq 1} |x|^\gamma \pi(dx) < \infty\Big\}.$$

We always have $\beta \in [0,2]$. Remark that

$$\beta = \sup\Big(0, \limsup_{j\to+\infty} j^{-1}\log_2 C_j\Big), \text{ where } C_j = \int_{2^{-j-1}\leq|x|\leq 2^{-j}} \pi(dx)\ (j \geq 1). \tag{7}$$

We focus on the pointwise Hölder exponents of sample paths of X, thus without loss of generality we omit the jump points generated by the compound process with intensity $\mathbf{1}_{\{|x|>1\}}\, \pi(dx)$. When $\int (1 \wedge |x|)\, \pi(dx) < \infty$, there are also several ways to write X as the sum of a Brownian motion B with drift $a' \in \mathbb{R}^d$ and covariance matrix Q and of a Lévy process $\widetilde{X}$ of Lévy measure $\mathbf{1}_{\{|x|\leq 1\}}\, \pi(dx)$, even when requiring that B and $\widetilde{X}$ are independent.

For $j \geq 0$, let $\pi_j(dx) = \mathbf{1}_{\{2^{-j-1}<|x|\leq 2^{-j}\}}\pi(dx)$. Then let $(Y_j)_{j\geq 0}$ be a sequence of independent compound Poisson processes such that the Lévy measure of Y_j

is π_j. We then choose $\widetilde{X}_.$ as follows:

$$\widetilde{X}_t = \sum_{j\geq 0} X_j(t) \text{ where } X_j(t) = \begin{cases} Y_j(t) & \text{if } \beta < 1, \\ Y_j(t) - \int x\pi_j(dx) & \text{if } \beta \geq 1. \end{cases} \tag{8}$$

Then a general Lévy process (with jumps of norm ≤ 1) has the form

$$X = \widetilde{X} + B(a', Q), \tag{9}$$

where $B(a', Q)$ is a Brownian motion with drift $a' \in \mathbb{R}^d$ and covariance matrix Q, independent of $\widetilde{X}$ (of course if $Q = 0$ then B is degenerate).

We now state the theorem of [22] using the pointwise Hölder exponent introduced above in (4) instead of the classical one.
By convention, $\dim E = -\infty$ means that the set E is empty.

Theorem 1 *Let X be a Lévy process decomposed in the form $\widetilde{X} + B(a', Q)$ as in (8) and (9), and consider the associated process $\widetilde{X}$. Suppose that $\beta \in (0, 2]$ and $\sum_{j\geq 1} 2^{-j}\sqrt{C_j \log(1 + C_j)} < +\infty$ (this holds as soon as $\beta < 2$).*
With probability 1, $d_{\widetilde{X}}(h) = \beta h$ if $h \in [0, 1/\beta]$ and $-\infty$ otherwise.

The influence of $B(a', Q)$ is also studied in [22], and the corresponding result is recalled in Theorem 3.

We now consider a positive Borel measure μ with a support equal to $[0, 1]$ and its integral F, i.e. F is the mapping $u \in [0, 1] \mapsto \mu([0, u])$. Let $(Z_u)_{u\in[0,1]}$ be the Lévy process in time F (or μ) given by $(Z_u = X_{F(u)})_{u\in[0,1]}$.

If μ is a multifractal measure, then F is a multifractal non-decreasing function. We are going to assume that μ is atomless, hence F is also continuous on $[0, 1]$. We use the pointwise exponent of μ defined in (5). If $h \geq 0$, the level sets E_h^μ of the measure μ are defined as $E_h^\mu = \{u : \ h_\mu(u) = h\}$. Finally, the singularity spectrum (or Hausdorff multifractal spectrum) of μ is the mapping $d_\mu : h \mapsto \dim E_h^\mu$.

The so-called scaling function τ_μ or L^q-spectrum associated with the measure μ is involved in our result. It is classically defined for positive Borel measures μ on $[0, 1]$ as

$$\tau_\mu : q \mapsto \liminf_{j\to+\infty} -j^{-1} \log_2 \sum_{0\leq k\leq 2^j-1} \mu\Big([k2^{-j}, (k + 1)2^{-j})\Big)^q. \tag{10}$$

The dyadic basis chosen in the definition (10) is not a restriction. Indeed, since $\text{supp}(\mu) = [0, 1]$, a different integer basis $b \geq 2$ would give the same value for τ_μ.

The Legendre transform f^* of a function $f:\mathbb{R}_+\to\mathbb{R}\cup\{-\infty\}$ is defined as $f^*: h\mapsto \inf_{q\in\mathbb{R}} hq - f(q)$.

Roughly speaking, our result yields the singularity spectrum d_Z of Z when the measure μ obeys the *multifractal formalism* in the sense that $d_\mu(h)=\tau_\mu^*(h)$ for all h (for detailed studies of multifractal formalisms for measures, the reader is referred to [13,35]). This property holds for many classes of statistically self-similar measures μ. These measures also satisfy three technical conditions **C1-3** invoked in our statement. For sake of shortness in this introduction, these conditions are specified later in Section 3.4. Among our assumptions, we shall keep this property in mind:

$$\tau_\mu'(1) \text{ exists and is strictly positive.} \tag{11}$$

This implies that the lower and upper Hausdorff dimensions of μ coincide with $\tau_\mu'(1)$ (see [33] for the corresponding definitions).

We shall prove the following result, which includes Theorem 1 as the special case where μ is the Lebesgue measure.

Theorem 2 *Let X be a Lévy process decomposed in the form $\widetilde{X}+B(a',Q)$ as in (8) and (9). Suppose that $\beta\in(0,2]$, and $\sum_{j\geq 1}2^{-j}\sqrt{C_j\log(1+C_j)}<+\infty$. Let μ be an atomless positive Borel measure whose support is $[0,1]$, such that (11) and* **C1** *hold true.*
We introduce the exponents $h_{\mu,\beta}=\tau_\mu'(1)/\beta$ and $\alpha_{\max}=\sup\{\alpha:\tau_\mu^*(\alpha)\geq 0\}$.

Let $(\widetilde{Z}_u)_{u\in[0,1]}$ be the stochastic process defined by $\widetilde{Z}(u)=\widetilde{X}_{\mu([0,u])}$ (i.e. the influence of $B(a',Q)$ in the decomposition (9) is not taken into account).

With probability 1:

(1) For every $h\in[0,h_{\mu,\beta})$, $d_{\widetilde{Z}}(h)\leq\beta h$.
Moreover, if **C2(**$h_{\mu,\beta}$**)** *holds, then for every $h\in[0,h_{\mu,\beta})$, $d_{\widetilde{Z}}(h)=\beta h$.*

(2) If $h\in[h_{\mu,\beta},\alpha_{\max}/\beta]$, $d_{\widetilde{Z}}(h)\leq\tau_\mu^*(\beta h)$.
Moreover, if **C3(**βh**)** *holds, then $d_{\widetilde{Z}}(h)=\tau_\mu^*(\beta h)$.*

(3) If $h>\alpha_{\max}/\beta$ then $E_{\widetilde{Z}}(h)=\emptyset$.

The singularity spectrum of $\widetilde{Z}$ is thus composed of two parts (see Figure 1): First a linear part of slope β, then a concave part which is a dilated and translated version of (a part of) the singularity spectrum of the initial measure μ. This shape reflects the combination of an additive structure (the Lévy process) with a multiplicative structure (the multifractal measure μ). Such a behavior is observed for the heterogeneous sums of Dirac masses studied in [8]. For the sequel, we note $D_{\mu,\beta}(h)$ the singularity spectrum obtained in Theorem

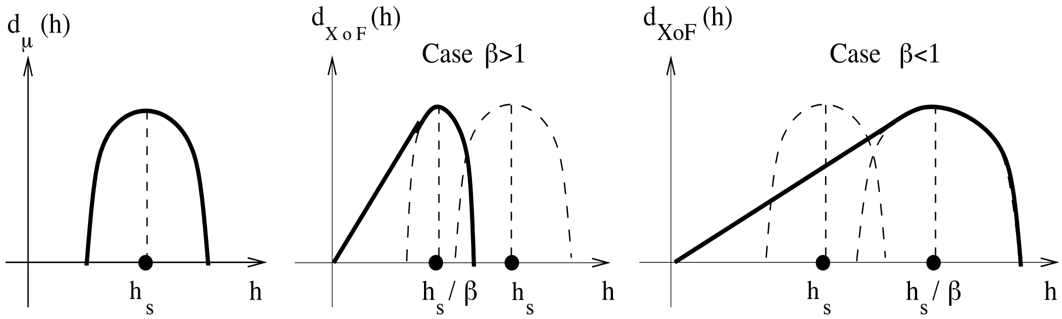

Fig. 1. Typical multifractal spectra of **Left:** a statistically self-similar measure μ, **Middle:** a Lévy process in multifractal time $\widetilde{X} \circ F$ when $\beta > 1$, and **Right:** when $\beta \leq 1$. Here h_s is the Lebesgue-almost sure exponent, i.e. $h_s = \tau'_\mu(0^+)$.

2, i.e. it is the mapping

$$D_{\mu,\beta}(h) = \begin{cases} \beta h & \text{if } h \in [0, h_{\mu,\beta}) \\ \tau_\mu^*(\beta h) & \text{if } h \in [h_{\mu,\beta}, \alpha_{\max}/\beta] \\ -\infty & \text{otherwise} \end{cases} \tag{12}$$

Remark that the singularity spectrum of $\widetilde{Z}$ is obtained as the Legendre transform of the function

$$\tau_{\mu,\beta}(q) = \begin{cases} \tau_\mu(q/\beta) & \text{if } q \leq \beta, \\ 0 & \text{otherwise} \end{cases}$$

as soon as $\mathbf{C2}(h_{\mu,\beta})$ and $\mathbf{C3}(h)$ hold true for all $h \in [\tau'_\mu(1), \alpha_{\max})$.

As said above, examples of measures illustrating our result are Gibbs measures and their random counterparts studied in [17,27,9], and of course the independent random cascades μ_W mentioned above in the study of the fixed points of the smoothing transformation (1). Other examples are the compound Poisson cascades and other $\mathbb{R}_+$-martingales studied in [5,3,6].

We now treat the general case, i.e. the influence of the drift and of the Brownian component.

Theorem 3 *Under the assumptions of Theorem 2, introduce the exponents* $\widetilde{h}_{\mu,\beta} = \inf\{h \geq 0 : \beta h < \tau_\mu^*(h)\}$ *if* $\beta < 1$ *and* $\overline{h}_{\mu,\beta} = \inf\{h \geq 0 : \beta h < \tau_\mu^*(2h)\}$. *We always have* $\widetilde{h}_{\mu,\beta} < h_{\mu,\beta}$ *and* $\overline{h}_{\mu,\beta} \leq \tau'_\mu(1)/2 \leq h_{\mu,\beta}$.

Consider the two mappings ($\widetilde{D}_{\mu,\beta}$ *is defined if* $\beta < 1$*)*

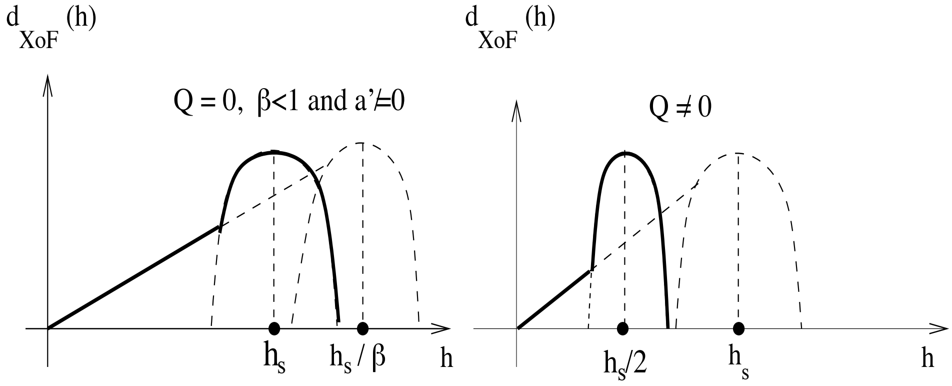

Fig. 2. Typical multifractal spectra of **Left:** a Lévy process in multifractal time $\widetilde{X} \circ F$ when $\beta < 1$, and **Right:** when $Q \neq 0$, here with $\beta = 1$.

$$\widetilde{D}_{\mu,\beta}(h) = \begin{cases} \beta h & \text{if } h \in [0, \widetilde{h}_{\mu,\beta}) \\ \tau_\mu^*(h) & \text{if } h \in [\widetilde{h}_{\mu,\beta}, \alpha_{\max}] \\ -\infty & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{D}_{\mu,\beta}(h) = \begin{cases} \beta h & \text{if } h \in [0, \overline{h}_{\mu,\beta}) \\ \tau_\mu^*(2h) & \text{if } h \in [\overline{h}_{\mu,\beta}, \frac{\alpha_{\max}}{2}] \\ -\infty & \text{otherwise.} \end{cases}$$

Let $(Z_u)_{u\in[0,1]}$ be the stochastic process defined by $Z(u) = X_{\mu([0,u])}$.

(1) Suppose that $Q = 0$ and ($a' = 0$ if $\beta < 1$). With probability 1, the same conclusions as for Theorem 2 occur here.

(2) Suppose that $Q = 0$, $\beta < 1$ and $a' \neq 0$. With probability 1,

(a) $d_Z \leq \widetilde{D}_{\mu,\beta}$.

*(b) If **C2**$(h_{\mu,\beta})$ holds, then for every $h \in [0, \widetilde{h}_{\mu,\beta})$, $d_Z(h) = \widetilde{D}_{\mu,\beta}(h)$.*

*(c) If ($\tau_\mu^*(\widetilde{h}_{\mu,\beta}) = \beta\widetilde{h}_{\mu,\beta}$ and **C2**$(h_{\mu,\beta})$ holds), or if ($\tau_\mu^*(\widetilde{h}_{\mu,\beta}) > \beta\widetilde{h}_{\mu,\beta}$ and **C3**$(\widetilde{h}_{\mu,\beta})$ holds), then $d_Z(\widetilde{h}_{\mu,\beta}) = \widetilde{D}_{\mu,\beta}(\widetilde{h}_{\mu,\beta})$.*

*(d) If $h \in (\widetilde{h}_{\mu,\beta}, \alpha_{\max}]$ and **C3**(h) holds, then $d_Z(h) = \widetilde{D}_{\mu,\beta}(h)$.*

(e) If $h > \alpha_{\max}$ then $E_Z(h) = \emptyset$.

(3) Suppose that $Q \neq 0$. With probability 1,

(a) $d_Z \leq \overline{D}_{\mu,\beta}$.

*(b) If **C2**$(h_{\mu,\beta})$ holds, then for every $h \in [0, \overline{h}_{\mu,\beta})$, $d_Z(h) = \overline{D}_{\mu,\beta}(h)$.*

*(c) If ($\tau_\mu^*(2\overline{h}_{\mu,\beta}) = \beta\overline{h}_{\mu,\beta}$ and **C2**$(h_{\mu,\beta})$ holds), or if ($\tau_\mu^*(2\overline{h}_{\mu,\beta}) > \beta\overline{h}_{\mu,\beta}$ and **C3**$(2\overline{h}_{\mu,\beta})$ holds), then we have $d_Z(\overline{h}_{\mu,\beta}) = \overline{D}_{\mu,\beta}(\overline{h}_{\mu,\beta})$.*

*(d) If $h \in (\overline{h}_{\mu,\beta}, \alpha_{\max}/2]$ and **C3**$(2h)$ holds, then $d_Z(h) = \overline{D}_{\mu,\beta}(h)$.*

(e) If $h > \alpha_{\max}/2$ then $E_Z(h) = \emptyset$.

The conclusions of items *(2)* and *(3)* are simple consequences of the fact that respectively a linear drift and a Brownian component are added to the "pure" Lévy process $\widetilde{X}$. The corresponding spectra are simply obtained as supremum of two spectra. This explains their non-concave shapes (see Figure 2).

The paper is organized as follows.
Section 2 recalls some useful properties of measures.
Section 3 introduces the main tools used in the proof of Theorem 2. Properties of Poisson point processes are discussed, and estimates for the increments of $\widetilde{X}$ obtained in [22] are recalled. Then, results on heterogeneous ubiquitous systems (introduced in [11]) are stated, and conditions **C1-3** are defined.
Section 4 is devoted to the proof of Theorem 2 when $B(a', Q) \equiv 0$. Sections 5 and 6 complete the proof to yield the general case $B(a', Q) \not\equiv 0$.
Section 7 deals with the validity of condition $\mathbf{C2}(h_{\mu,\beta})$ for independent multiplicative cascades, which play a central role in the fundamental example (3).

2 Local regularity of measures

For every $j \geq 1$ and $k \in [0, \dots, 2^j - 1]$, $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$. $I^+_{j,k}$ and $I^-_{j,k}$ denote the intervals $I_{j,k} + 2^{-j}$ and $I_{j,k} - 2^{-j}$.

If $u \in (0,1)$, $\forall j \geq 1$, $I_j(u)$ denotes the unique dyadic interval of length 2^{-j}, semi-open to the right, containing u. Then define $I^+_j(u) = I_j(u) + 2^{-j}$ and $I^-_j(u) = I_j(u) - 2^{-j}$.

The diameter of a set B is denoted by $|B|$. For the rest of the paper, the convention $\log(0) = -\infty$ is adopted.

Definition 1 *Let μ be a positive Borel measure on $[0,1]$. For $u_0 \in (0,1)$, the lower and upper Hölder exponents of μ at u_0 are respectively defined by*

$$\underline{\alpha}_\mu(u_0) = \liminf_{j \to +\infty} \frac{\log \mu(I_j(u_0))}{\log |I_j(u_0)|} \quad \text{and} \quad \overline{\alpha}_\mu(u_0) = \limsup_{j \to +\infty} \frac{\log \mu(I_j(u_0))}{\log |I_j(u_0)|}$$

When $\underline{\alpha}_\mu(u_0) = \overline{\alpha}_\mu(u_0)$, their common value is denoted $\alpha_\mu(u_0)$ and called the Hölder exponent of μ at u_0.

The left and right lower and upper Hölder exponents of μ at u_0 are defined by

$$\underline{\alpha}^-_\mu(u_0) = \liminf_{j \to +\infty} \frac{\log \mu(I^-_j(u_0))}{\log |I^-_j(u_0)|} \quad \text{and} \quad \underline{\alpha}^+_\mu(u_0) = \liminf_{j \to +\infty} \frac{\log \mu(I^+_j(u_0))}{\log |I^+_j(u_0)|}$$
$$\text{and} \quad \overline{\alpha}^-_\mu(u_0) = \limsup_{j \to +\infty} \frac{\log \mu(I^-_j(u_0))}{\log |I^-_j(u_0)|} \quad \text{and} \quad \overline{\alpha}^+_\mu(u_0) = \limsup_{j \to +\infty} \frac{\log \mu(I^+_j(u_0))}{\log |I^+_j(u_0)|}.$$

Similarly, when they coincide, $\alpha^-_\mu(u_0)$ and $\alpha^+_\mu(u_0)$ denote their common value. Finally, we define

$$\overline{h}_\mu(u_0) = \max(\overline{\alpha}^-_\mu(u_0), \overline{\alpha}_\mu(u_0), \overline{\alpha}^+_\mu(u_0))$$

and for $h \geq 0$

$$\overline{E}^\mu_h = \{u \in [0,1] : \overline{h}_\mu(u) = h\}.$$

We see that (the exponent $h_\mu(\cdot)$ and its level sets E^μ_h are defined in (5))

$$h_\mu(u_0) = \min(\underline{\alpha}^-_\mu(u_0), \underline{\alpha}_\mu(u_0), \underline{\alpha}^+_\mu(u_0)) = \liminf_{r\to 0^+} \frac{\log \mu(B(u_0,r))}{\log |B(u_0,r)|}.$$

Definition 2 *If μ is a positive Borel measure on $[0,1]$, and $\alpha \geq 0$, denote by $\widetilde{E}^\mu_\alpha$ the set $\{x : \alpha^-_\mu(x) = \alpha_\mu(x) = \alpha^+_\mu(x) = \alpha\}$.*

The following proposition puts together classical results derived from the multifractal formalism for measures (see [13,35]). It provides upper bounds for the Hausdorff dimension of union of level sets E^μ_α, $\widetilde{E}^\mu_\alpha$ and $\overline{E}^\mu_\alpha$. The singularity spectrum d_μ and the scaling function τ_μ were introduced in Section 1. For the reader's convenience we recall the definition of the Hausdorff dimension.

Definition 3 *Let $s \geq 0$. The s-dimensional Hausdorff measure of a set E, $\mathcal{H}^s(E)$, is defined as*

$$\mathcal{H}^s(E) = \lim_{r\searrow 0} \mathcal{H}^s_r(E), \quad \textit{with } \mathcal{H}^s_r(E) = \inf\Big\{\sum_i |E_i|^s\Big\},$$

the infimum being taken over all the countable families of sets E_i such that $|E_i| \leq r$ and $E \subset \bigcup_i E_i$. Then, the Hausdorff dimension of E, $\dim E$, is defined as $\dim E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = +\infty\}$.

Proposition 1 *Let μ be a positive Borel measure on $[0,1]$ and let $\alpha \geq 0$.*

(1) $\dim \widetilde{E}^\mu_\alpha \leq d_\mu(\alpha) \leq \tau^*_\mu(\alpha)$.
(2) If $\alpha \in [0, \tau'_\mu(0^+)]$, then $\dim \bigcup_{\alpha'\leq\alpha} E^\mu_{\alpha'} \leq \tau^*_\mu(\alpha)$.
(3) If $\alpha \geq \tau'_\mu(0^+)$, then $\dim \bigcup_{\alpha'\geq\alpha} \left(E^\mu_{\alpha'} \cup \overline{E}^\mu_{\alpha'}\right) \leq \tau^*_\mu(\alpha)$.
(4) If $\tau^*_\mu(\alpha) < 0$, then $E^\mu_\alpha = \emptyset$.

Next proposition follows from the definition of τ_μ and Tchernov inequalities.

Proposition 2 *Let μ be a positive Borel measure on $[0,1]$. For every $\alpha \geq 0$, $C > 0$ and $\varepsilon > 0$, there exists a scale J such that $j \geq J$ implies*

$$\frac{\log\Big(\#\Big\{k \in \{0,..,2^j-1\} : \mu(I_{j,k}) \geq C2^{-j(\alpha+\varepsilon)}\Big\}\Big)}{\log 2^j} \leq \sup_{\alpha'\leq\alpha+\varepsilon} \tau^*_\mu(\alpha') + \varepsilon.$$

3 Tools

In this section, we are given the Lévy process X, decomposed into the sum $X = \widetilde{X} + B(a', Q)$ described in (9).

3.1 Some notations

We denote by S the Poisson point process with intensity $\ell \otimes \pi$ associated with the Lévy process $X(t)$, where ℓ stands for the Lebesgue measure on $\mathbb{R}_+$ and π is the Lévy measure.

For every $j \geq 1$, let

$$G_j = \{t : (t, \lambda) \in S \text{ for some } \lambda \text{ such that } |\lambda| \in (2^{-j-1}, 2^{-j}]\}.$$

For $t \in G_j$, λ_t is the unique element $\lambda \in \mathbb{R}^d$ such that $(t, \lambda) \in S$. The jumps of the process $X_j(t)$ are thus exactly located at the points of G_j, and the value of the jump of X_j at $t \in G_j$ is λ_t.

For every $j \geq 1$ and for every $\delta > 0$, A^j_δ is the union of intervals

$$A^j_\delta = \bigcup_{t \in G_j} B(t, 2^{-(j+1)\delta}).$$

We clearly have $\bigcup_{t \in G_j} B(t, |\lambda_t|^\delta) \supset A^j_\delta$. Eventually, for every sequence $\widetilde{\delta} = \{\delta_j\}_j$ of non-negative numbers, we denote

$$A_{\widetilde{\delta}} = \limsup_{j \to +\infty} A^j_{\delta_j} = \bigcap_{J \geq 1} \bigcup_{j \geq J} A^j_{\delta_j}. \tag{13}$$

3.2 Coverings and weak redundancy properties associated with Poisson point processes

It is known [38,22] that with probability 1, for every $\delta < \beta$, if the sequence $\widetilde{\delta}$ is constantly equal to δ, then $A_{\widetilde{\delta}} = \mathbb{R}^+$ (recall (13)). An easy adaptation of the proof of Lemma 3 in [22] yields the following slightly stronger result.

Lemma 1 *With probability 1, there exists a non-decreasing non-negative sequence $\widetilde{\beta} = (\beta_j)_{j \geq 1}$ converging to β such that $A_{\widetilde{\beta}} = \mathbb{R}^+$.*

Notice that if the Lévy process is stable and if we can write in polar coordinates $\pi(dr, d\theta) = \alpha r^{-(1+\beta)}\, dr \nu(d\theta)$ with $\alpha \geq 1/2$ and ν a probability measure on the unit sphere, then the constant sequence $(\beta_j = \beta)_j$ can be chosen in the previous statement.

The problem of covering by Poisson intervals is connected with the problem of counting the number of points of S whose projection on $\mathbb{R}_+$ falls in a given dyadic interval $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$. Next Lemmas 2 and 4 are devoted to this question.

Lemma 2 *For $\delta > \beta$ and $\widetilde{\varepsilon} = \{\varepsilon_j\}_{j\geq 1}$ a sequence of positive numbers, and for every integers j and k, let*

$$K_{j,k}^{\delta,\widetilde{\varepsilon}} = \#\Big\{t \in I_{j,k} : t \in G_{j'} \text{ for some } j' \in [j/\delta, j/(\beta+\varepsilon_j)]\Big\}. \qquad (14)$$

There exist two sequences $\{\varepsilon_j\}_{j\geq 1}$ and $\{\eta_j\}_{j\geq 1}$ of positive real numbers converging to 0 such that for every integer $T > 0$, with probability 1: For every $\delta > \beta$, for every $j \geq 1$ large enough (depending on δ), for every $k \in \{0, \dots, 2^jT-1\}$, we have $K_{j,k}^{\delta,\widetilde{\varepsilon}} \leq 2^{j\eta_j}$.

PROOF. By definition of β, there exists a positive non-increasing sequence $\widetilde{\varepsilon}^{(1)} = \{\varepsilon_j^{(1)}\}_j$ converging to zero such that $C_j \leq 2^{j(\beta+\varepsilon_j^{(1)})}$.

Let T be a positive integer. Let $\delta > \beta$. For every $j \geq 1$ and $k \in \{0, \dots, 2^jT - 1\}$, the random variable $K_{j,k}^{\delta,\widetilde{\varepsilon}^{(1)}}$ is a Poisson variable with intensity $C_j^{\delta,\widetilde{\varepsilon}^{(1)}} = 2^{-j} \sum_{j/\delta \leq j' \leq j/(\beta+\varepsilon_j^{(1)})} C_{j'} \leq 2^{-j} \sum_{j/\delta \leq j' \leq j/(\beta+\varepsilon_j^{(1)})} 2^{j'(\beta+\varepsilon_{j'}^{(1)})} \leq M_{j,\delta} 2^{j((\beta+\varepsilon_{[j/\delta]}^{(1)})/(\beta+\varepsilon_j^{(1)})-1)}$, where $M_{j,\delta}$ is equal to $2\frac{2^{(\beta+\varepsilon_{[j/\delta]}^{(1)})/(\beta+\varepsilon_j^{(1)})}}{2^{\beta+\varepsilon_{[j/\delta]}^{(1)}}-1}$. In fact it is easily checked that the sequence $M_{j,\delta}$ can be bounded by a constant M_β independent of δ and j since the sequence $\{\varepsilon_j^{(1)}\}$ is bounded. Thus $C_j^{\delta,\widetilde{\varepsilon}^{(1)}} \leq M_\beta 2^{j((\beta+\varepsilon_{[j/\delta]}^{(1)})/(\beta+\varepsilon_j^{(1)})-1)}$, for every j and every δ.

Moreover, since the sequence $\{\varepsilon_j^{(1)}\}$ is non-increasing and converges to zero as $j \to +\infty$, $(\beta + \varepsilon_{[j/\delta]}^{(1)})/(\beta + \varepsilon_j^{(1)}) - 1$ is bounded by $\{\varepsilon_j^{(\delta)}\} = \{2\varepsilon_{[j/\delta]}^{(1)}/\beta\}$, which is a non-increasing sequence depending on δ. Thus $C_j^{\delta,\widetilde{\varepsilon}^{(1)}}$ is bounded by $M_\beta 2^{j\varepsilon_j^{(\delta)}}$.

We consider $\varepsilon_j = \varepsilon_{[j/\log(j+1)]}^{(1)}$ for all $j \geq 1$. For every $\delta > \beta$, for j large enough, we have $\varepsilon_j \geq \varepsilon_j^{(\delta)}$, and thus $C_j^{\delta,\widetilde{\varepsilon}} \leq C_j^{\delta,\widetilde{\varepsilon}^{(1)}} \leq 2^{j\varepsilon_j}$ (actually, without loss of generality, we can change a little bit the sequence $\{\varepsilon_j^{(\delta)}\}$ so that it takes into account the constant M_β).

We now use the following lemma which is a simple consequence of the Stirling formula

Lemma 3 *There exists an integer $r > 0$ such that for j large enough, for every Poisson random variable N of parameter $C > 0$, $\mathbb{P}(N > r(j + C)) \le 2^{-2j}$.*

Let $P_j^\delta = \mathbb{P}\Big(\exists k \in \{0, \dots, 2^jT - 1\} : K_{j,k}^{\delta,\widetilde{\varepsilon}} \ge r(j + C_j^{\delta,\widetilde{\varepsilon}})\Big)$. By Lemma 3 for j large enough we have $P_j^\delta \le 2^{-2j}2^jT$ so $\sum_{j\ge1} P_j^\delta < +\infty$. The Borel-Cantelli lemma implies that for every j large enough, for every $k \in \{0, \dots, 2^jT - 1\}$, $K_{j,k}^{\delta,\widetilde{\varepsilon}} \le 2^{j\eta_j}$, where η_j is the positive sequence converging to 0 at infinity defined by $2^{j\eta_j} = r(j + C_j^{\delta,\widetilde{\varepsilon}})$. This yields the uniform control over $k \in \{0, \dots, 2^jT - 1\}$ of $K_{j,k}^{\delta,\widetilde{\varepsilon}}$ for every $\delta > \beta$ with probability 1, and finally with probability 1 for all $\delta > \beta$ since the random functions $\delta \mapsto K_{j,k}^{\delta,\widetilde{\varepsilon}}$ are non-decreasing.

We need to introduce the notion of *weakly redundant system* in $\mathbb{R}_+$. This notion is later determinant to get upper bounds for the level sets of Hölder exponents.

Definition 4 *Let $(x_n)_{n\ge0} \in \mathbb{R}_+^{\mathbb{N}}$ and $(\lambda_n)_{n\ge0}$ a positive sequence converging to 0. For every $T > 0$ and $j \ge 0$, we introduce the sets of indices*

$$\mathcal{T}_j = \Big\{n : x_n \in [0, T],\ 2^{-(j+1)} < \lambda_n \le 2^{-j}\Big\}. \tag{15}$$

The family $\{(x_n, \lambda_n)\}_{n\in\mathbb{N}}$ is said to form a weakly redundant system if for every $T > 0$ there exists a sequence of integers $(N_{T,j})_{j\ge0}$ such that

(i) $\lim_{j\to\infty}(\log_2 N_{T,j})/j = 0$.

(ii) for every $j \ge 1$, $\mathcal{T}_j$ can be decomposed into $N_{T,j}$ pairwise disjoint subsets (denoted $\mathcal{T}_{j,1}, \dots, \mathcal{T}_{j,N_{T,j}}$) such that for each $1 \le i \le N_{T,j}$, the family $\Big\{B(x_n, \lambda_n) :\ n \in \mathcal{T}_{j,i}\Big\}$ is composed of disjoint balls.

Lemma 4 *Consider the Poisson point process $S = \bigcup_{j\ge0} G_j$. Let $(\beta_j)_{j\ge0}$ be a non-decreasing sequence converging to β.*

With probability 1, the family $\bigcup_{j\ge0}\{(t, |\lambda_t|^{\beta_j}) : t \in G_j\}$ forms a weakly redundant system.

PROOF. This is a direct consequence of the estimates obtained in the proofs of Lemmas 5 and 8 of [22] for the numbers $N_{j,k} = \#\{t \in G_j :\ t \in [k2^{-j}, (k+1)]2^{-j}\}$ when $\beta = 1$.

3.3 Local regularity of the Lévy process $\widetilde{X}$

As a consequence of the work achieved by Jaffard in [22], the increments of $\widetilde{X}$ satisfy the following almost-sure properties.

Proposition 3 *Let $\varepsilon > 0$. With probability 1:*

Let $t_0 \geq 0$ be not a jump point of $\widetilde{X}(t)$, and write $h_X(t_0) = 1/\delta$ for some $\delta \geq \beta$. For η small enough, there exists $\varepsilon' > 0$ such that for all $t \geq 0$,

$$\text{if } \ |t - t_0| \leq \eta, \ \text{ then } \sum_{j \geq \frac{\log_2 |t-t_0|^{-1}}{\beta+\varepsilon'}} |X_j(t) - X_j(t_0)| \leq |t - t_0|^{1/(\beta+\varepsilon)} \tag{16}$$

$$\text{and } \ |X(t) - X(t_0)| \leq |t - t_0|^{1/(\delta+\varepsilon)}. \tag{17}$$

Moreover, still for $|t-t_0| \leq \eta$, if $\sum_{j<\frac{\log_2 |t-t_0|^{-1}}{\beta+\varepsilon'}} X_j(\cdot)$ has no jump point between t and t_0, we get

$$\sum_{j<\frac{\log_2 |t-t_0|^{-1}}{\beta+\varepsilon'}} |X_j(t) - X_j(t_0)| \leq |t - t_0|^{1/(\beta+\varepsilon)}. \tag{18}$$

Equation (18) implies that when $\beta \geq 1$ the contribution of the sum of all the drifts associated with the processes $X_j(t)$, $j < \frac{\log_2 |t-t_0|^{-1}}{\beta+\varepsilon'}$, on a given interval $[t_0, t]$, is always less than $|t - t_0|^{1/(\beta+\varepsilon)}$.

3.4 Heterogeneous ubiquity and Hausdorff dimensions of limsup sets

General results of what we call "heterogeneous ubiquity" are obtained in [11] (see also [12]). Here, a simpler version adapted to our context is stated. It plays a similar role as the geometric Theorem 2 used in [22], but makes it possible to work out problems raised here by considering a multifractal time change. Some additional notations have to be introduced.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of points in $[0, 1]$ and $\{l_n\}_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to zero. Let $\delta > 1$. For every $n \in \mathbb{N}$ we set

$$I_n = [u_n - l_n, u_n + l_n], \ \widetilde{I}_n^+ = [u_n + l_n/4], \ \widetilde{I}_n^- = [u_n - l_n/4, u_n], \ I_n^\delta = [u_n - l_n^\delta, u_n + l_n^\delta].$$

In addition, given an integer $b \geq 2$, for $u \in [0, 1]$, we set

$$\mathcal{B}_j(u) = \Big\{ I_n : u \in I_n,\ l_n \in (b^{-(j+1)}, b^{-j}] \Big\}, \tag{19}$$

$$\mathcal{B}_j^\delta(u) = \Big\{ I_{j',k'} : \exists I_n \in \mathcal{B}_j(u) \text{ such that } I_{j',k'} \subset I_n^\delta \Big\}. \tag{20}$$

Definition 5 *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of points in $[0,1]$, and let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero.*

Let μ be a positive Borel measure such that supp$(\mu) = [0,1]$ and (11) holds.

The system $\{(u_n, l_n)\}_n$ is said to form an heterogeneous ubiquitous system with respect to $(\mu, \tau_\mu'(1))$ if the following holds true.

(1) There exists a non-increasing sequence $(\varphi_j)_{j\geq 0}$ with the properties:
   - *(a) $\lim_{j\to\infty} \varphi_j = 0$, $(j\varphi_j)_{j\geq 0}$ is non-decreasing at $+\infty$ and $\lim_{j\to\infty} j\varphi_j = +\infty$.*
   - *(b) $\forall\, \varepsilon > 0$, $(j(\varepsilon - \varphi_j))_{j\geq 0}$ is non-decreasing at $+\infty$,*
   - *(c) Properties (2), (3) and (4) below hold.*

(2) There exist an integer $b \geq 2$ such that
   - *(a) μ-almost every $t \in [0,1]$ belongs to $\bigcap_{N\geq 0} \bigcup_{n\geq N} [u_n - l_n/2, u_n + l_n/2]$.*
   - *(b) For μ-almost every $t \in [0,1]$, there exists an integer $j(t)$ such that $\forall j \geq j(t)$, $\forall\ k$ such that $|k - k_{j,t}^b| \leq 1$,*

$$b^{-j(\tau_\mu'(1)+\varphi_j)} \leq \mu([kb^{-j}, (k+1)b^{-j}]) \leq b^{-j(\tau_\mu'(1)-\varphi_j)},$$

   where $k_{j,t}^b$ is the unique integer k such that $t \in [kb^{-j}, (k+1)b^{-j})$. Thus (2)(b) implies for μ-a.e. $t \in [0,1]$ a precise control of the μ-mass of the three b-adic intervals around t.

(3) (Self-similarity of μ) For every b-adic subinterval L of $[0,1]$, let f_L denote the canonical affine mapping from L onto $[0,1]$. There exists a measure μ^L on L, equivalent to the restriction of μ to L, such that property (2)(b) holds for the measure $\mu^L \circ f_L^{-1}$ instead of the measure μ.

   Let $j_L = \log_b \left(|L|^{-1}\right)$ and for every $n \geq 1$, let

$$U_n^L = \left\{ t \in L : \left\{ \begin{array}{c} \forall\ j \geq n + j_L,\ \forall\ k,\ |k - k_{j,t}^b| \leq 1, \\ \mu^L\Big([kb^{-j}, (k+1)b^{-j}]\Big) \leq \left(\frac{b^{-j}}{|L|}\right)^{\tau_\mu'(1) - \varphi_{j-j_L}} \end{array} \right. \right\}.$$

   The sets U_n^L clearly form a non-decreasing sequence in $[0,1]$, and by (2)(b) and property (3), $\bigcup_{n\geq 1} U_n^L$ is of full μ^L-measure. Then define

$$n_L = \inf \Big\{ n \geq 1 :\ \mu^L(U_n^L) \geq \|\mu^L\|/2 \Big\}.$$

(4) (Control of the growth speed n_L and of the mass $\|\mu^L\|$)

There is a dense subset $\mathcal{D}$ of $(1,\infty)$ such that for every $\delta \in \mathcal{D}$, for μ-almost every $u \in [0,1]$, one can find an increasing sequence of integers $(j_k(u))_{k\geq 1}$ such that for every $k \geq 1$, there exists $L_k \in B^{\delta}_{j_k(u)}(u)$ satisfying $\lim_{k\to\infty} \frac{j_{L_k}}{j_k(u)} = \delta$ *and*

$$n_{L_k} \leq j_{L_k} \,.\, \varphi_{j_{L_k}} \quad \text{and} \quad |L_k|^{\varphi_{j_{L_k}}} \leq \|\mu^{L_k}\|. \tag{21}$$

The next result is established in [11].

Theorem 4 *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of points in $[0,1]$, let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero. Let μ be a positive Borel measure such that supp(μ) $= [0,1]$ and (11) holds.*

For every positive sequences $\widetilde{\varepsilon} = (\varepsilon_n)_{n\in\mathbb{N}}$ and $\widetilde{\delta} = (\delta_n)_{n\in\mathbb{N}}$, define the limsup set

$$S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) = \bigcap_{N\geq 0} \bigcup_{n\geq N:\, |l_n|^{\tau'_\mu(1)+\varepsilon_n} \leq \mu(\widetilde{I}_n^+), \mu(\widetilde{I}_n^-) \leq \mu(I_n) \leq |l_n|^{\tau'_\mu(1)-\varepsilon_n}} I_n^{\delta_n}.$$

Suppose that $\{(u_n, l_n)\}_n$ forms an heterogeneous ubiquitous system with respect to $(\mu, \tau'_\mu(1))$.

There exists a positive sequence $\widetilde{\varepsilon}$ converging to 0 such that for every $\delta \geq 1$, there exists a non-decreasing sequence $\widetilde{\delta}$ converging to δ as well as a positive Borel measure m_δ such that:

- $m_\delta(E) = 0$ *for every Borel set E such that* $\dim E < \tau'_\mu(1)/\delta$,
- $m_\delta\Big(S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})\Big) > 0.$

In particular, $\dim S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \geq \tau'_\mu(1)/\delta$.

Moreover, if the system $\{(u_n, l_n)\}_{n\in\mathbb{N}}$ is weakly redundant (see Definition 4), we precisely have $\dim S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) = \tau'_\mu(1)/\delta$.

The set $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})$ is constituted by points which are well approximated at rate $\delta > 1$ by some points u_n, these points being selected according to the behavior of μ around u_n. Thus Theorem 4 emphasizes a ubiquity property *conditioned* by a measure μ, and shows the existence of exceptional points related simultaneously to the local behavior of the measure μ and to the approximation rate by the system $\{(u_n, l_n)\}_n$. The condition $|l_n|^{\tau'_\mu(1)+\varepsilon_n} \leq \mu(\widetilde{I}_n^+), \mu(\widetilde{I}_n^-) \leq \mu(I_n) \leq |l_n|^{\tau'_\mu(1)-\varepsilon_n}$ involved in the definition of the set $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})$ appears in the weaker form $|l_n|^{\tau'_\mu(1)+\varepsilon_n} \leq \mu(I_n) \leq |l_n|^{\tau'_\mu(1)-\varepsilon_n}$ in [11], but due to property *(2)(b)* the work achieved in [11] makes it possible to add automatically the condition on $\mu(\widetilde{I}_n^+)$ and $\mu(\widetilde{I}_n^-)$ and it yields Theorem 4.

Remark 1 *For some classes of measures μ, it turns out that property* (4) *can be simplified in the stronger one: There exists $j_0 \geq 0$ such that (21) holds for all b-adic interval L of generation larger than j_0. This is the case for instance for the class of random Gibbs measures described in [9]. Unfortunately independent random cascades do not satisfy this uniform property, and their study required working with the weaker condition* (4) *(see Sections 1 and 7 as well as [10]).*

3.5 Conditions **C1-3**

Let μ be an atomless positive Borel measure with a support equal to $[0, 1]$.

Condition **C1**

There exist two positive constants γ_1 and γ_2 such that for every small enough sub-interval I of $[0, 1]$, $|I|^{\gamma_1} \leq \mu(I) \leq |I|^{\gamma_2}$.

Condition **C2**$(h_{\mu,\beta})$

Recall that $h_{\mu,\beta} = \tau_\mu'(1)/\beta$. By assumption the function $F : t \in [0, 1] \mapsto \mu([0, t])$ is increasing and continuous on $[0, 1]$.

The Poisson point process S can be written $S = \{(t_n, \lambda_n)\}_{n\geq 1}$, with $|\lambda_n| \searrow 0$. Let $\{\beta_j\}_{j\geq 1}$ be a sequence as found in Lemma 1.

For every $(t_n, \lambda_n) \in S$ such that $t_n \in G_j$, we set $u_n = F^{-1}(t_n)$, and we define the sequence l_n as $2\Big|F^{-1}\Big(B\Big(t_n, |\lambda_n|^{\beta_j}\Big)\Big)\Big|$. This ensures that $(0, 1) \subset \limsup_{n\to\infty} B(u_n, l_n/2)$.

Condition **C2**$(h_{\mu,\beta})$ is said to hold when (11) holds and when $\{(u_n, l_n)\}_{n\geq 1}$ forms an heterogeneous ubiquitous system with respect to $(\mu, \tau_\mu'(1))$.

We shall see in Section 7 that this holds under suitable assumptions when μ is an independent multiplicative cascade. Consequently, the assertions of Theorem 2 concerning the linear parts of the spectra apply to the process Z_W defined in (3).

Condition **C3**(h)

There exists a positive Borel measure m_h on $[0,1]$ such that $m_h(\widetilde{E}^\mu_h) > 0$ and for every Borel set $E \subset [0,1]$ such that $\dim E < \tau^*_\mu(h)$, $m_h(E) = 0$.

Suppose that μ is a independent multiplicative cascade. It is shown in [10] that if the function φ_W is everywhere finite, then with probability 1, condition **C3**(h) holds for all h such that $\tau^*_\mu(h) > 0$. Consequently, the assertions of Theorem 2 concerning the strictly concave parts of the spectra apply to the process Z_W defined in (3).

4 Computation of the Hausdorff spectrum of $\widetilde{X} \circ F$: Theorem 2

In this section, in order to simplify the notations, we assume that $X = \widetilde{X}$, i.e. $B(a', Q) = 0$ in (9), so that $\widetilde{X}$ and $\widetilde{Z}$ in Theorem 2 are simply denoted X and Z.

By Lemma 1, there exists a non-decreasing sequence of positive real numbers $\widetilde{\beta} = \{\beta_j\}_{j\geq 1}$ converging to β such that, with probability 1, the set $A_{\widetilde{\beta}}$ (defined in (13)) equals $\mathbb{R}_+$. Such a sequence is fixed.

4.1 Characterization of the Hölder exponents of $Z = X \circ F$

For every $j \geq 1$, for every $t \in G_j$, let $l_t = 2|F^{-1}([t - |\lambda_t|^{\beta_j}, t + |\lambda_t|^{\beta_j}])|$ and $I_t = [F^{-1}(t) - l_t, F^{-1}(t) + l_t]$. These intervals were considered in condition **C2**$(h_{\mu,\beta})$ in Section 3.5. By construction of the $\{\beta_j\}_j$, we have

$$[0,1] \subset \bigcap_{J\geq 1} \bigcup_{j\geq J} \bigcup_{t\in G_j} [F^{-1}(t) - l_t/2, F^{-1}(t) + l_t/2].$$

Definition 6 *Let $\alpha \geq 0$, $\delta \geq 1$ and $\varepsilon > 0$.*

A real number u_0 is said to satisfy the property $\mathcal{P}(\alpha, \delta, \varepsilon)$ if there exist an infinite number of jump points u of Z satisfying

$$|u - u_0| \leq l_{F(u)}^{\delta-\varepsilon} \text{ and } l_{F(u)}^{\alpha+\varepsilon} \leq \mu\Big(I_{F(u)}\Big) \leq l_{F(u)}^{\alpha-\varepsilon}. \tag{22}$$

Remark that, by construction, if $t = F(u)$ and $t \in G_j$ for some integer $j \geq 1$, then under (22) we also have $2^{-j} \leq l_{F(u)}^{\frac{\alpha-\varepsilon}{\beta+\varepsilon}}$ if j is large enough.

A real number u_0 is said to satisfy the property $\widetilde{\mathcal{P}}(\alpha, \delta, \varepsilon)$ if there exist an *infinite number of jump points u of Z which satisfy (22) together with $l_{F(u)}^{\frac{\alpha+\varepsilon}{\beta-\varepsilon}} \leq 2^{-j}$ if $F(u) \in G_j$ (notice that here 2^{-j} is approximately equal to the size of the jump of Z at u).*

We then set for $h > 0$

$$\mathcal{T}_{\beta,h} = \left\{ u \in [0,1] : \begin{cases} \forall \varepsilon > 0, \ \exists\, \alpha \geq 0, \ \exists\, \delta \geq 1 \text{ such that} \\ \frac{\alpha}{\beta\delta} \leq h + \varepsilon \text{ and } u \text{ satisfies } \mathcal{P}(\alpha,\delta,\varepsilon) \end{cases} \right\}, \tag{23}$$

$$\widetilde{\mathcal{T}}_{\beta,h} = \left\{ u \in [0,1] : \begin{cases} \forall \varepsilon > 0, \ \exists\, \alpha \geq 0, \ \exists\, \delta \geq 1 \text{ such that} \\ \frac{\alpha}{\beta\delta} \leq h + \varepsilon \text{ and } u \text{ satisfies } \widetilde{\mathcal{P}}(\alpha,\delta,\varepsilon) \end{cases} \right\}. \tag{24}$$

Heuristically, the point u_0 satisfies $\mathcal{P}(\alpha,\delta,\varepsilon)$ or $\widetilde{\mathcal{P}}(\alpha,\delta,\varepsilon)$ when it is well-approximated by jump points u of Z, at rate δ relatively to $I_F(u)$, these points being selected so that they satisfy $\mu(I_F(u))) \sim l^{\alpha}_{F(u)}$.

Remark that if $0 < h' \leq h$, then we clearly obtain $\widetilde{\mathcal{T}}_{\beta,h'} \subset \mathcal{T}_{\beta,h'} \subset \mathcal{T}_{\beta,h}$.

We denote $\overline{S} = \left\{ t \in \mathbb{R}_+ : \exists\, \lambda \in \mathbb{R}^d, \ (t,\lambda) \in S \right\}$, i.e. $\overline{S}$ is the projection on $\mathbb{R}_+$ of the Poisson point process S associated with $X(t)$, as well as the set of jump points of X.

This section is devoted to the proof of the following result, which is a simple consequence of next Propositions 4, 5 and 6.

Theorem 5 *Assume that* **C1** *holds. With probability 1, for every $h > 0$, we have $\mathcal{A}_h \subset E^Z_h \subset \mathcal{B}_h$, where*

$$\mathcal{A}_h = \begin{cases} \widetilde{\mathcal{T}}_{\beta,h} \setminus \left[\left(\bigcup_{h'<h} E^{\mu}_{\beta h'} \right) \cup \left(\bigcup_{h'<h} \mathcal{T}_{\beta,h'} \right) \cup \overline{S} \right] & \text{if } 0 \leq h < h_{\mu,\beta} \\ \widetilde{E}^{\mu}_{\beta h} \setminus \left(F^{-1}(\overline{S}) \cup \bigcup_{\delta > \beta} F^{-1}(A_\delta) \right) & \text{if } h \geq h_{\mu,\beta}. \end{cases} \tag{25}$$

$$\mathcal{B}_h = \begin{cases} \left(\mathcal{T}_{\beta,h} \setminus \bigcup_{h'<h} \widetilde{\mathcal{T}}_{\beta,h'} \right) \cup \bigcup_{h' \leq h} E^{\mu}_{\beta h'} & \text{if } 0 \leq h \leq \tau'_\mu(0^+)/\beta, \\ \bigcup_{h' \geq h} \overline{E}^{\mu}_{\beta h'} & \text{if } h \geq \tau'_\mu(0^+)/\beta. \end{cases} \tag{26}$$

Consequently, in order to compute the singularity spectrum of Z, it remains for us to find an upper bound for $\dim \mathcal{B}_h$ and a lower bound for $\dim \mathcal{A}_h$. This is achieved in the next sections.

Proposition 4 *Assume that* **C1** *holds. With probability 1:*

For every $u_0 \in [0,1]$ not a jump point of Z, let $h_\mu(u_0) = \alpha \geq 0$ and $\overline{h}_\mu(u_0) = \overline{\alpha}$, *and write* $t_0 = F(u_0) \in [0, \|\mu\|]$ *and* $h_X(t_0) = 1/\delta_{u_0}$ *where* $\delta_{u_0} \geq \beta$. *Then*

$$\alpha/\delta_{u_0} \leq h_Z(u_0) \leq \overline{\alpha}/\delta_{u_0}. \tag{27}$$

PROOF. Let $\varepsilon > 0$. By the definition of $h_\mu(u_0)$, there exists $\eta_1 > 0$ such that

$$\text{for every } 0 < r \leq \eta_1,\ \mu(B(u_0, r)) \leq r^{\alpha-\varepsilon}. \tag{28}$$

Let j_r be the unique integer such that $2^{-j_r} \leq r \leq 2^{-j_r+1}$.

By definition of $\overline{\alpha}$, we can also choose η_1 small enough so that

$$\text{for every } 0 < r \leq \eta_1, \text{ if } I \in \{I^-_{j_r+2}(u_0), I_{j_r+2}(u_0), I^+_{j_r+2}(u_0)\},\ |\mu(I)| \geq r^{\overline{\alpha}+\varepsilon}. \tag{29}$$

Remark that $I^-_{j_r+2}(u_0) \cup I_{j_r+2}(u_0) \cup I^+_{j_r+2}(u_0) \subset B(u_0, r)$. Similarly, using the definition of $h_X(t_0) = 1/\delta_{u_0}$ and Proposition 3, there exists η_2 such that

$$\text{for every number } s \text{ such that } |s| \leq \eta_2, |X(t_0 + s) - X(t_0)| \leq s^{1/\delta_{u_0}-\varepsilon}, \tag{30}$$

and for some sequence $(h_j)_{j\geq 1}$ such that $|h_j| \searrow 0$,

$$|X(t_0 + h_j) - X(t_0)| \geq |h_j|^{1/(\delta_{u_0}+\varepsilon)}. \tag{31}$$

Since the function F is continuous on $[0, 1]$, we can thus choose η_1 small enough so that $F(B(u_0, \eta_1)) \subset B(t_0, \eta_2)$.

• Let $-\eta_1 \leq r \leq \eta_1$. By (30) and then (28), we have

$$\begin{aligned} |Z(u_0 + r) - Z(u_0)| &= |X \circ F(u_0 + r) - X \circ F(u_0)| \\ &\leq |F(u_0 + r) - F(u_0)|^{1/\delta_{u_0}-\varepsilon} \leq |r|^{(\alpha+\varepsilon)/\delta_{u_0}-(\alpha+\varepsilon)\varepsilon} \end{aligned}$$

since $|F(u_0 + r) - F(u_0)| \leq \mu(B(u_0, |r|))$. This holds for every $\varepsilon > 0$, hence the lower bound of (27).

• Let j be such that (31) holds, and let r_j be the unique real number such that $F(u_0 + r_j) = t_0 + h_j$. We get

$$|Z(u_0 + r_j) - Z(u_0)| = |X(t_0 + h_j) - X(t_0)| \geq |h_j|^{1/(\delta_{u_0}+\varepsilon)}.$$

By (29), $\mu([u_0, u_0 + r_j]) \geq \mu(I^+_{j_{r_j}+2}(u_0)) \geq |r_j|^{\overline{\alpha}+\varepsilon}$. Since $F(u_0 + r_j) - F(u_0) = h_j$, we otain $|h_j| \geq |r_j|^{\overline{\alpha}+\varepsilon}$, and thus

$$|Z(u_0 + r_j) - Z(u_0)| \geq |r_j|^{(\overline{\alpha}+\varepsilon)/\delta_{u_0}+(\overline{\alpha}+\varepsilon)\varepsilon}.$$

Since this holds for an infinite number of r_j converging to zero and then for every $\varepsilon > 0$, the conclusion follows.

Proposition 5 *Assume that* **C1** *holds and $u_0 \in \widetilde{\mathcal{T}}_{\beta,h}$ for some $h \geq 0$. Then $h_Z(u_0) \leq h$.*

PROOF. Let $\varepsilon \in (0, \beta)$. The proof uses the following Lemma of [21].

Lemma 5 *Assume that a function f is discontinuous on a dense set of $\mathbb{R}$. For a fixed $x \in \mathbb{R}$, assume also that there exists a sequence $\{r_n\}_n$ converging to x such that for every n, f has right and left limits $f(r_n^+)$ and $f(r_n^-)$ at r_n, and $|f(r_n^+) - f(r_n^-)| = s_n > 0$. Then*

$$h_f(x) \leq \liminf_{n \to +\infty} \frac{|\log s_n|}{|\log |r_n - x_0||}.$$

Let $(u_n)_{n \geq 1}$ be an infinite sequence of jump points of Z that verifies (22) for u_0 as well as the fact that the size of the jump of Z at u_n is greater than $l_{F(u_n)}^{\frac{\alpha+\varepsilon}{\beta-\varepsilon}}$. Lemma 5 yields then

$$h_Z(u_0) \leq \liminf_{n \to +\infty} \frac{|\log l_{F(u_n)}^{\frac{\alpha+\varepsilon}{\beta-\varepsilon}}|}{|\log |l_{F(u_n)}|^{\delta-\varepsilon}|} \leq \frac{\alpha+\varepsilon}{(\delta-\varepsilon)(\beta-\varepsilon)} \leq (h+\varepsilon)\frac{\alpha+\varepsilon}{\alpha}\frac{\delta}{\delta-\varepsilon}\frac{\beta}{\beta-\varepsilon}.$$

Let γ_2 be as in **C1**. Since $\lim_{\varepsilon \to 0^+} \sup_{\delta > 1, \alpha \geq \gamma_2/2} \frac{\alpha+\varepsilon}{\alpha}\frac{\delta}{\delta-\varepsilon}\frac{\beta}{\beta-\varepsilon} = 1$, the conclusion follows.

Proposition 6 *Assume that* **C1** *holds. With probability 1, we have the following property: For every $u_0 \in [0, 1]$ not a jump point of Z, if $h_Z(u_0) < h_\mu(u_0)/\beta$, then $u_0 \in \mathcal{T}_{\beta, h_Z(u_0)}$.*

PROOF. Set $h = h_Z(u_0)$, $\alpha = h_\mu(u_0)$, $t_0 = F(u_0)$ and $h_X(t_0) = 1/\delta_{u_0}$ for some $\delta_{u_0} \geq \beta$. Necessarily, $\delta_{u_0} > \beta$ otherwise, if $\delta_{u_0} = \beta$, then by Proposition 4 we would have $h \geq \alpha/\beta$.

Let $\varepsilon > 0$. By definition of h, there exists a sequence $(r_n)_{n \geq 1}$ such that $|r_n| \searrow 0$ and $|Z(u_0 + r_n) - Z(u_0)| \geq |r_n|^{h+\varepsilon}$. We set $u_n = u_0 + r_n$, and $t_n = F(u_n)$. We have $|X(t_n) - X(t_0)| \geq |r_n|^{h+\varepsilon}$, and $|t_n - t_0| = \mu([u_0, u_n]) \leq \mu(B(u_0, |r_n|)) \leq |r_n|^{\alpha-\varepsilon}$ by (28).

We denote by j_n the unique integer such that $2^{-j_n} \leq |t_n - t_0| < 2^{-j_n+1}$. For every $\varepsilon' > 0$ we can write

$$X(t_n) - X(t_0) = \sum_{j<[j_n/(\beta+\varepsilon')]} X_j(t_n) - X_j(t_0) + \sum_{j\geq[j_n/(\beta+\varepsilon')]} X_j(t_n) - X_j(t_0).$$

By Proposition 3, there exists $\varepsilon' > 0$ such that (16) and (18) hold. We thus have

$$\begin{aligned}\sum_{j<[j_n/(\beta+\varepsilon')]} |X_j(t_n) - X_j(t_0)| &\geq \Big| \sum_{j<[j_n/(\beta+\varepsilon')]} X_j(t_n) - X_j(t_0) \Big| \\ &\geq |X(t_n) - X(t_0)| - |t_n - t_0|^{1/(\beta+\varepsilon)} \\ &\geq |r_n|^{h+\varepsilon} - |r_n|^{\frac{\alpha-\varepsilon}{\beta+\varepsilon}}.\end{aligned}$$

The parameter ε can be chosen small enough so that $(h+\varepsilon)(\beta+\varepsilon) < \alpha - \varepsilon$. Then there exists $C > 0$ such that for n large enough

$$\sum_{j<[j_n/(\beta+\varepsilon')]} |X_j(t_n) - X_j(t_0)| \geq C|r_n|^{h+\varepsilon}. \qquad (32)$$

Remembering (18) and using again that $|t_n - t_0|^{1/(\beta+\varepsilon)} \leq |r_n|^{\frac{\alpha-\varepsilon}{\beta+\varepsilon}}$, we conclude that $\sum_{j<[j_n/(\beta+\varepsilon')]} X_j(\cdot)$ has a jump point between t_n and t_0 (since the contribution of the drift is not large enough to explain (32)).

Consider one among the jump points with tallest size, i.e. a real number T_n in $[t_0, t_n]$ such that T_n is a jump point for X_{J_n} for some $J_n < [j_n/(\beta+\varepsilon')]$ and there is no jump point of $X(t)$ in $[t_0, t_n]$ belonging to some $G_{j'}$, $j' < J_n$. Remark that since $h_X(t_0) = 1/\delta_{u_0}$, for n large enough $j_n/(\delta_{u_0}+\varepsilon) \leq J_n \leq j_n/(\beta+\varepsilon')$.

We now apply Lemma 2 with $T = [\mu([0,1]) + 1]$ and $\delta = \delta_{u_0}$. We choose j_n large enough so that ε_{j_n} and η_{j_n} are less than $\varepsilon/2$. Let k be the unique integer such that $t_0 \in [k2^{-j_n}, (k+1)2^{-j_n})$. We get $[t_0, t_n] \subset I = \bigcup_{l=k-2,\dots,k+2} I_{j_n,l}$. By Lemma 2 applied to the five intervals contained in I, the number of jumps in the interval $[t_0, t_n]$ of all the X_j's, $j < \left[\frac{j_n}{\beta+\varepsilon'}\right]$, is less than $5 \cdot 2^{j_n \eta_{j_n}}$.

Using (32) and the existence of T_n, we obtain

$$|D| + 5 \cdot 2^{j_n\eta_{j_n}} 2^{-J_n} \geq \sum_{j<[j_n/\beta+\varepsilon']} |X_j(t_n) - X_j(t_0)| \geq C|r_n|^{h+\varepsilon},$$

where D stands for the contribution of the drift of all the X_j's, $j < \left[\frac{j_n}{\beta+\varepsilon'}\right]$, on the interval $[t_0, t_n]$. But, again by (18), $|D| \leq |t_n - t_0|^{1/(\beta+\varepsilon)} \leq |r_n|^{\frac{\alpha-\varepsilon}{\beta+\varepsilon}}$. As above, since $\frac{\alpha-\varepsilon}{\beta+\varepsilon} > h$, for n large enough $5 \cdot 2^{j_n\eta_{j_n}} 2^{-J_n} \geq C|r_n|^{h+\varepsilon}$, for another constant C. This enables to compare 2^{-J_n} with $|r_n|$. Indeed, since **C1** yields $j_n = O\Big(|\log(|r_n|)|\Big)$ and η_{j_n} goes to 0 when $n \to +\infty$, we obtain

$$2^{-J_n} \geq C|r_n|^{h+2\varepsilon} \geq |r_n|^{h+3\varepsilon}. \qquad (33)$$

Denote by U_n the real number $F^{-1}(T_n)$, and consider $I_{T_n} = I_{F(U_n)}$ (the intervals I_t for $t \in G_j$ were defined at the beginning of Section 4.1). By construction this interval satisfies $\mu(I_{T_n}) \geq 2 \cdot 2^{-J_n \beta_{J_n}}$. Thus $u_0 \in I_{T_n}$ for n large enough because $\beta_{J_n} J_n \leq \frac{\beta_{J_n}}{\beta+\varepsilon'} j_n < j_n$. Thus by (28) $2 \cdot 2^{-J_n \beta_{J_n}} \leq \mu(I_{T_n}) \leq l_{T_n}^{\alpha-\varepsilon}$ for n large enough. We write $\mu(I_{T_n}) = l_{T_n}^{\alpha_n}$ for some $\alpha_n \geq \alpha - 2\varepsilon$.

Now, we know that $|u_0 - U_n| \leq |r_n|$. But $|r_n| \leq 2^{-J_n \frac{1}{h+3\varepsilon}} \leq C l_{T_n}^{\frac{\alpha_n}{\beta_{J_n}(h+3\varepsilon)}}$ by (33). Define $\delta_n = \frac{\alpha_n}{\beta_{J_n}(h+3\varepsilon)}$. For ε small enough and n large enough, we see that $\delta_n \geq 1$ (since $h < \alpha/\beta$).

If γ_1 is the constant of condition **C1**, for every n large enough, the couple (α_n, δ_n) belongs to the square $[0, \gamma_1] \times [1, \delta_{u_0} + \varepsilon]$. Without loss of generality by extracting a subsequence, we can assume that (α_n, δ_n) converges to (α_0, δ_0). By construction $\frac{\alpha_0}{\beta\delta_0} \leq h + 4\varepsilon$. Hence $\mathcal{P}(\alpha_0, \delta_0, 4\varepsilon)$ holds.

PROOF OF THEOREM 5. Let $h \geq 0$ and $u_0 \in E_h^Z$. By Propositions 5 and 6, $u_0 \in \bigcup_{h' \leq h} E^\mu_{\beta h'} \cup \mathcal{T}_{\beta,h} \setminus \bigcup_{h'<h} \widetilde{\mathcal{T}}_{\beta,h'}$. Also, by Proposition 4 $u_0 \in \bigcup_{h' \geq h} \overline{E}^\mu_{\beta h'}$. Consequently $E_h^Z \subset \mathcal{B}_h$.

Propositions 5 and 6 clearly imply that $\widetilde{\mathcal{T}}_{\beta,h} \setminus \left[\left(\bigcup_{h'<h} E^\mu_{\beta h'} \right) \cup \left(\bigcup_{h'<h} \mathcal{T}_{\beta,h'} \right) \cup \overline{S} \right] \subset E_h^Z$. Thus $\mathcal{A}_h \subset E_h^Z$ when $h < h_{\mu,\beta}$.

Finally, when $h \geq h_{\mu,\beta}$, if $u_0 \in \mathcal{A}_h$, by Proposition 4 $h_Z(u_0) = h_\mu(u_0)/\beta$ (since $h_\mu(u_0) = \overline{h}_\mu(u_0)$). Hence $\mathcal{A}_h \subset E_h^Z$.

4.2 Upper bound for the singularity spectrum of Z

Let us start by the decreasing part of the spectrum.

Proposition 7 *With probability 1, for every $h \geq \tau_\mu'(0^+)/\beta$, $\dim E_h^Z \leq \tau_\mu^*(\beta h)$ and $E_h^Z = \emptyset$ if $h > \alpha_{\max}/\beta$.*

PROOF. This Proposition 7 directly follows from Theorem 5 used when $h \geq \tau_\mu'(0^+)/\beta$ (which yields $E_h^Z \subset \mathcal{B}_h$), and then from item *(3)* of Proposition 1 to find an upper bound for $\dim \mathcal{B}_h$.

In order to get an upper bound for the increasing part of the multifractal spectrum of Z, some notations and new sets are needed.

For every $j \geq 1$, $t \in G_j$ and $\delta \geq 1$, let

$$I_t^{(\delta)} = B(F^{-1}(t), l_t^\delta). \tag{34}$$

We consider, for $\alpha \geq 0$, $\varepsilon > 0$ and $\delta \geq 1$, the sets

$$T_{\alpha,\delta,\varepsilon} = \bigcap_{J\geq 1} \bigcup_{j\geq J} \bigcup_{t\in G_j:\, l_t^{\alpha+\varepsilon}\leq \mu(I_t)\leq l_t^{\alpha-\varepsilon}} I_t^{(\delta)}. \tag{35}$$

The Hausdorff dimension of the sets $T_{\alpha,\delta,\varepsilon}$ is easily tractable (as shown by the following proposition). Moreover, these sets are closely related with the sets $\mathcal{T}_{\beta,h}$.

Lemma 6 *Assume that* **C1** *holds for* μ. *For every* $\alpha > 0$ *such that* $\tau_\mu^*(\alpha) \geq 0$, $\delta \geq 1$ *and* $\varepsilon > 0$

$$\dim T_{\alpha,\delta,\varepsilon} \leq \frac{\sup_{\alpha'\leq\alpha+\varepsilon} \tau_\mu^*(\alpha) + \varepsilon}{\delta}. \tag{36}$$

PROOF. We first use Lemma 4. Due to the definition of I_t, the weak redundancy property of $S = \bigcup_{j\geq 0}\{(t, |\lambda_t|^{\beta_j}) : t \in G_j\}$ implies the existence of a non-negative sequence $(\xi_j)_{j\geq 0}$ converging to 0 such that as soon as $G_j \neq \emptyset$, the set $\{I_t : t \in G_j\}$ can be written as a union of $2^{j\xi_j}$ families $\mathcal{G}_{j,i}$ of pairwise disjoint intervals.

We have $T_{\alpha,\delta,\varepsilon} = \bigcap_{J\geq 1} \bigcup_{j\geq J} S_j$, where

$$S_j = \bigcup_{t\in G_j:\, l_t^{\alpha+\varepsilon}\leq \mu(I_t)\leq l_t^{\alpha-\varepsilon}} I_t^{(\delta)}. \tag{37}$$

Fix $\alpha_0 \in (0, \tau_\mu'(0^+))$. Let $\alpha \in [\alpha_0, \tau_\mu'(0^+))$ and $\varepsilon \in (0, \alpha_0/2)$. Let $J \geq 1$ and $j \geq J$. Let $t \in G_j$ and let J_t denotes the unique integer such that $2^{-J_t} < |I_t| \leq 2^{-J_t+1}$. If $l_t^{\alpha+\varepsilon} \leq \mu(I_t) \leq l_t^{\alpha-\varepsilon}$, then at least one of the intervals $I_{J_t+2,k}$ such that $I_{J_t+2,k} \cap I_t \neq \emptyset$ must satisfy $\mu(I_{J_t+2,k}) \geq \frac{1}{16} l_t^{\alpha+\varepsilon} \geq C2^{-(J_t+2)(\alpha+\varepsilon)}$, where C is a constant depending only on α. Moreover, due to **C1** and the definition of the interval I_t, there exists two positive constants γ and γ' independent of t such that for j large enough, $\gamma j \leq J_t + 2 \leq \gamma' j$.

For every integer $m \geq 1$, let $F_m = \{I_{m,k} : \mu(I_{m,k}) \geq C2^{-m(\alpha+\varepsilon)}\}$ for every i. We deduce from the last considerations that every I_t belonging to some $\mathcal{G}_{j,i}$ and satisfying $\mu(I_t) \geq l_t^{\alpha+\varepsilon}$ must intersect an element I of $\bigcup_{\gamma j\leq m\leq\gamma' j} F_m$. In this case, $|I|^\delta \leq |I_t^{(\delta)}| \leq C|I|^\delta$ for some constant C depending only on δ.
Moreover, since the elements of $\mathcal{G}_{j,i}$ are pairwise disjoint, the intervals I of $\bigcup_{\gamma j\leq m\leq\gamma' j} F_m$ previously selected intersect at most two elements of $\mathcal{G}_{j,i}$. Also, we learn from Proposition 2 that for m large enough, the cardinality of F_m is less than or equal to $2^{m(\sup_{\alpha'\leq\alpha+\varepsilon} \tau_\mu^*(\alpha')+\varepsilon)}$.

Now let $s > \Big(\sup_{\alpha'\leq\alpha+\varepsilon}\tau^*(\alpha)+\varepsilon\Big)/\delta$. Recall Definition 3. It follows from the previous remarks that for some constant $C'>0$,

$$\begin{aligned}
\mathcal{H}^s_{C'2^{-\gamma J}}(T_{\alpha,\delta,\varepsilon}) &\leq \sum_{j\geq J}\ \sum_{t\in G_j:\, l_t^{\alpha+\varepsilon}\leq\mu(I_t)\leq l_t^{\alpha-\varepsilon}} |I_t^{(\delta)}|^s \\
&\leq \sum_{j\geq J}\sum_i \sum_{I_t\in\mathcal{G}_{j,i}: l_t^{\alpha+\varepsilon}\leq\mu(I_t)} |I_t^{(\delta)}|^s \leq \sum_{j\geq J}\sum_i \sum_{\gamma j\leq m\leq\gamma' j} 2\sum_{I\in F_m} C|I|^{s\delta} \\
&\leq 2C\sum_{j\geq J} 2^{j\xi_j}\sum_{\gamma j\leq m\leq \gamma' j} 2^{-s\delta m}2^{m(\sup_{\alpha'\leq\alpha+\varepsilon}\tau^*_\mu(\alpha')+\varepsilon)},
\end{aligned}$$

Since $\xi_j\to 0$ when $j\to+\infty$, $\lim_{J\to\infty}\mathcal{H}^s_{C'2^{-\gamma J}}(T_{\alpha,\delta,\varepsilon})=0$, thus $\dim T_{\alpha,\delta,\varepsilon}\leq s$.

Proposition 8 *Assume that* **C1** *holds. With probability 1, for every exponent $h\in[0,\tau'_\mu(0^+)/\beta)$, $\dim E^Z_h\leq D_{\mu,\beta}(h)$ (recall that $D_{\mu,\beta}$ is defined in (12)).*

PROOF. If $h=0$, then it follows from Proposition 4 that E^Z_h is contained in the set $F^{-1}(\overline{S})\cup E^\mu_0\cup(\cap_{\delta>1}A_\delta)$. Thus $\dim E^Z_h=0$.

Fix now $h\in(0,\frac{\tau'_\mu(0^+)}{\beta})$. Item *(2)* of Theorem 5 implies that $\dim E^Z_h$ is bounded by $\dim E^Z_h\leq\max\Big(\dim\mathcal{T}_{\beta,h}\setminus\bigcup_{h'<h}\widetilde{\mathcal{T}}_{\beta,h'},\dim\bigcup_{\alpha\leq\beta h}E^\mu_\alpha\Big)$. Item *(2)* of Proposition 1 yields $\dim\bigcup_{\alpha\leq\beta h}E^\mu_\alpha\leq\tau^*_\mu(\beta h)$. It remains to find an upper bound for $\dim\mathcal{T}_{\beta,h}$.

For every $\varepsilon>0$, $\mathcal{T}_{\beta,h}\subset\bigcup_{\substack{(\alpha,\delta)\in\mathbb{Q}\times\mathbb{Q}\\ \alpha>0,\,\tau^*_\mu(\alpha)\geq 0,\,\delta\geq 1,\,\alpha/\beta\delta\leq h+\varepsilon}} T_{\alpha,\delta,\varepsilon}$. Lemma 6 yields

$$\begin{aligned}
\dim\mathcal{T}_{\beta,h} &\leq \sup_{\substack{(\alpha,\delta)\in\mathbb{Q}\times\mathbb{Q}\\ \alpha>0,\,\tau^*_\mu(\alpha)\geq0,\,\delta\geq1,\,\alpha/\beta\delta\leq h+\varepsilon}} \dim T_{\alpha,\delta,\varepsilon} \\
&\leq \sup_{\substack{(\alpha,\delta)\in\mathbb{Q}\times\mathbb{Q}\\ \alpha\geq0,\,\tau^*_\mu(\alpha)\geq0,\,\delta\geq1,\,\alpha/\beta\delta\leq h+\varepsilon}} \frac{\sup_{\alpha'\leq\alpha+\varepsilon}\tau^*_\mu(\alpha')+\varepsilon}{\delta} \\
&\leq \max(\beta(h+\varepsilon)d_1(h,\varepsilon),d_2(h,\varepsilon)),
\end{aligned}$$

where $\begin{cases} d_1(h,\varepsilon)=\sup_{\alpha\geq\beta h}\frac{\sup_{\alpha'\leq\alpha+\varepsilon}\tau^*_\mu(\alpha')+\varepsilon}{\alpha}, \\ d_2(h,\varepsilon)=\sup_{0\leq\alpha<\beta h,\,\tau^*_\mu(\alpha)\geq0,\,\delta\geq1,\,\alpha/\beta\delta\leq h+\varepsilon}\frac{\sup_{\alpha'\leq\alpha+\varepsilon}\tau^*_\mu(\alpha')+\varepsilon}{\delta}. \end{cases}$

Since $\beta h\leq\tau'_\mu(0^+)$, $\lim_{\varepsilon\to0}d_2(h,\varepsilon)=\tau^*_\mu(\beta h)$.
The next observations are already done in [8] (they are easy to check using the continuity of τ^*_μ on its support and the fact that $\sup_{\alpha\geq0:\,\tau^*_\mu(\alpha)\geq0}\tau^*_\mu(\alpha)/\alpha$ is reached for $\alpha=\tau'_\mu(1^-)$):

- If $h \leq \tau_\mu'(1)/\beta$, then $\lim_{\varepsilon\to 0} d_1(h,\varepsilon) = 1$.
- If $h \geq \tau_\mu'(1)/\beta$, then $\lim_{\varepsilon\to 0} d_1(h,\varepsilon) = \tau_\mu^*(\beta h)/\beta h$.

We finally get the desired upper bound for $\dim \mathcal{T}_{\beta,h}$ and thus also for $\dim E_h^Z$.

4.3 Lower bound for the singularity spectrum of Z

Proposition 9 *Suppose that* **C1** *holds. With probability 1, for every $h \geq h_{\mu,\beta}$ such that* **C3(**βh**)** *holds,* $\dim E_h^Z \geq \tau_\mu^*(\beta h)$.

PROOF. Fix a realization of Z and $h \geq h_{\mu,\beta}$ such that **C3(**βh**)** holds.

Let $m_{\beta h}$ be the measure given by **C3(**βh**)**. Combining **C3(**βh**)** and item *(1)* of Theorem 5, it is enough to prove that $m_{\beta h}\Big(\bigcup_{\delta>\beta} E_\delta\Big) = 0$ and $m_{\beta h}\Big(\widetilde{E}_{\beta h}^\mu \cap F^{-1}(\overline{S})\Big) = 0$, where $E_\delta = \widetilde{E}_{\beta h}^\mu \cap (F^{-1}(A_\delta) \setminus F^{-1}(\overline{S}))$.

Since S is countable and the family of sets A_δ is monotonic, it remains to show that $\dim E_\delta < \tau^*(\beta h)$ for every $\delta > \beta$. Fix such a δ and let $u \in E_\delta$.

Let $\delta_{F(u)} = \limsup_{j\to\infty} \sup_{t\in G_j} \frac{\log|t-F(u)|}{\log|\lambda_t|}$. Since $F(u) \in A_\delta$, $\delta_{F(u)} \geq \delta$. Let $(t_n)_{n\geq 1}$ be a sequence of points of S verifying $\lim_{n\to\infty} \frac{\log|t_n - F(u)|}{|\lambda_{t_n}|} = \delta_{F(u)}$.

Denote $u_n = F^{-1}(t_n)$. Since $u \in \widetilde{E}_{\beta h}$, we get

$$\limsup_{n\to\infty} \frac{\log|u-u_n|}{\log l_{t_n}} = \frac{1}{\beta h} \limsup_{n\to\infty} \frac{\log|F(u)-F(u_n)|}{\log l_{t_n}}.$$

Moreover, since $u \in I_{t_n} \cap \widetilde{E}_{\beta h}$, we also have $\lim_{n\to\infty} \frac{\log l_{t_n}}{\log|F(I_{t_n})|} = \frac{1}{\beta h}$. But by construction of the I_{t_n}'s we know that $\lim_{n\to\infty} \frac{\log|F(I_{t_n})|}{\log|\lambda_{t_n}|} = \beta$. Consequently,

$$\limsup_{n\to\infty} \frac{\log|u-u_n|}{\log l_{t_n}} = \frac{\delta_{F(u)}}{\beta} \geq \frac{\delta}{\beta} > 1.$$

It follows from these remarks that $E_\delta \subset T_{\beta h,\delta/\beta,\varepsilon}$ for all $\varepsilon > 0$. Lemma 6 yields that $\dim E_\delta \leq \beta\tau^*(\beta h)/\delta < \tau^*(\beta h)$.

Proposition 10 *Suppose that* **C1** *and* **C2(**$h_{\mu,\beta}$**)** *hold. Then, with probability 1, for every $\delta > 1$, $\dim E^Z_{\tau_\mu'(1)/(\beta\delta)} \geq \tau'(1)/\delta$; equivalently, for every $0 < h < h_\beta$, $\dim E_h^Z = d_Z(h) \geq \beta h$.*

PROOF. Let $\delta > 1$, $h = h_{\mu,\beta}/\delta$ and $d = \tau_\mu'(1)/\delta$.

Fix a realization of Z and S such that the properties involved in condition $\mathbf{C2}(h_{\mu,\beta})$ are satisfied. Theorem 4 provides us with the non-decreasing sequence $\widetilde{\delta}$ converging to δ, the positive sequence $\widetilde{\varepsilon}$ converging to 0, the set $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})$, and the measure m_δ.

By construction, all the points of $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})$ satisfy $\widetilde{\mathcal{P}}(\tau'_\mu(1), \delta, \varepsilon)$ for all $\varepsilon > 0$. So $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \subset \widetilde{\mathcal{T}}_{\beta,h}$. Moreover, $m_\delta(S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})) > 0$, which, by Theorem 4, implies that $\dim S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \geq \tau'(1)/\delta = \beta h$.

When proving Proposition 8, we established that every set of the non-decreasing sequence $(\mathcal{T}_{\beta,h'})_{h'<h}$ is of Hausdorff dimension less than βh. Thus $m_\delta(\cup_{h'<h}\mathcal{T}_{\beta,\delta}) = 0$. Also $m_\delta(\cup_{h'<h}E^\mu_{\beta h'}) = 0$ by Proposition 1. Thus

$$m_\delta\left((S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \setminus \left[\Big(\bigcup_{h'<h} E^\mu_{\beta h'}\Big) \cup \Big(\bigcup_{h'<h} \mathcal{T}_{\beta,h'}\Big) \cup \overline{S}\right]\right) > 0.$$

Using Theorem 5*(1)* and the fact that $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \subset \widetilde{\mathcal{T}}_{\beta,h}$, we get that $m_\delta(E^Z_h) > 0$, hence the conclusion.

5 The case $a' \neq 0$ and $Q = 0$: Item *(1)* and *(2)* of Theorem 3

In this section, we use the decomposition (9) with $a' \neq 0$ and $Q = 0$ to write $Z(t) = \widetilde{X}(F(t)) + F(t)a$, with $a \in \mathbb{R}^d \setminus \{0\}$. We write $\widetilde{Z} = \widetilde{X} \circ F$.

We begin by relating the function h_Z with $h_{\widetilde{Z}}$ and h_F. We first notice that $h_F = h_\mu$. Then, equation (4) implies that $h_Z(u) \geq \min\Big(h_{\widetilde{Z}}(u), h_\mu(u)\Big)$ for every $u \in [0,1]$ with equality if $h_{\widetilde{Z}}(u) \neq h_\mu(u)$. Also, the study achieved in [22] yields $h_{\widetilde{X}+a'Id} = \min(h_{\widetilde{X}}, 1)$. This implies that:

- When $\beta \leq 1$, for every $u \in [0,1]$, $h_Z(u) \leq \overline{h}_\mu(u)$.
- When $\beta > 1$, for every $u \in [0,1]$, $h_Z(u) \leq h_{\widetilde{X}}(F(u)) \cdot \overline{h}_\mu(u) \leq \overline{h}_\mu(u)/\beta$.

From the previous discussion, we deduce that when $\beta \leq 1$

$$E^Z_h \subset \begin{cases} \bigcup_{h' \leq h} E^{\widetilde{Z}}_{h'} \cup E^\mu_{h'} & \text{if } h \leq \tau'_\mu(0^+), \\ \bigcup_{h' \geq h} \overline{E}^\mu_{h'} & \text{otherwise} \end{cases}$$

and when $\beta > 1$

$$E^Z_h \subset \begin{cases} \bigcup_{h' \leq h} E^{\widetilde{Z}}_{h'} \cup E^\mu_{h'} & \text{if } h \leq \tau'_\mu(0^+)/\beta, \\ \bigcup_{h' \geq \beta h} \overline{E}^\mu_{h'} & \text{otherwise} \end{cases}.$$

By using Theorem 5*(2)*, Proposition 1 and the estimates obtained in the proof of Proposition 8, we conclude that

$$\forall\ h \geq 0,\ d_Z(h) \leq \begin{cases} \widetilde{D}_{\mu,\beta}(h) & \text{if } \beta < 1, \\ D_{\mu,\beta}(h) & \text{otherwise.} \end{cases}$$

The following remarks yield the lower bound.

• Suppose that $\beta < 1$. Let $h \geq \widetilde{h}_{\mu,\beta}$. If **C3**$(h)$ holds, then it follows from the proof of Proposition 9 that for m_h-almost every $u \in [0,1]$, $h_{\widetilde{Z}}(u) = h_\mu(u)/\beta > h_\mu(u)$. Consequently, $h_Z(u) = h_\mu(u)$ m_h-almost everywhere. This yields $\dim E_h^Z \geq \tau_\mu^*(h)$.

Suppose now that **C2**$(h_{\mu,\beta})$ holds. If $0 < h \leq \widetilde{h}_{\mu,\beta}$, then let $\delta = h_{\mu,\beta}/h$. Lemma 5 combined with the continuity of F yield that the set $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \setminus \left[\left(\bigcup_{h'<h} E^\mu_{\beta h'}\right) \cup \left(\bigcup_{h'<h} \mathcal{T}_{\beta,h'}\right) \cup \overline{S}\right]$ is included in E_h^Z. We conclude that $\dim E_h^Z \geq \beta h$, as in the proof of Proposition 10.

• Suppose that $\beta \geq 1$. The case $h < h_{\mu,\beta}$ is treated as the case $h < \widetilde{h}_{\mu,\beta}$ when $\beta < 1$. If $h \geq h_{\mu,\beta}$, then Lemma 5 combined with the continuity of F yield $\widetilde{E}^\mu_{\beta h} \setminus \left(F^{-1}(\overline{S}) \cup \bigcup_{\delta > \beta} F^{-1}(A_\delta)\right) \subset E_h^Z$. We conclude as in the proof of Proposition 9.

6 The case $Q \neq 0$: Item *(3)* of Theorem 3

We begin with a proposition which takes care of the Brownian part $B \circ F$.

Proposition 11 *Let μ be a positive measure on $[0,1]$ and $B_{1/2}$ a Brownian motion. With probability 1, $\forall u_0 \in [0,1]$, $h_\mu(u_0)/2 \leq h_{B_{1/2}\circ F}(u_0) \leq \overline{h}_\mu(u_0)/2$.*

PROOF. Let $\varepsilon > 0$. For almost every sample path of $B_{1/2}$,

$$\forall t_0,\ \forall t \text{ close enough to } t_0,\ |B_{1/2}(t) - B_{1/2}(t_0)| \leq |t - t_0|^{1/2-\varepsilon}, \tag{38}$$

and there is an infinite number of t_n converging to t_0 such that

$$|B_{1/2}(t) - B_{1/2}(t_0)| \geq |t - t_0|^{1/2+\varepsilon}. \tag{39}$$

Let $u_0 \in [0,1]$. For u close enough to u_0, (38) implies that

$$|B_{1/2} \circ F(u) - B_{1/2} \circ F(u_0)| \le |F(u) - F(u_0)|^{1/2-\varepsilon} \le |u - u_0|^{(h_\mu(u_0)-\varepsilon)(1/2-\varepsilon)}.$$

for some constant C. Moreover, by (39) there is an infinite number of points $u_n = F^{-1}(t_n)$ such that

$$\begin{aligned} |B_{1/2} \circ F(u_n) - B_{1/2} \circ F(u_0)| &\ge |F(u_n) - F(u_0)|^{1/2+\varepsilon} \\ &\ge |u_n - u_0|^{(\overline{h}_\mu(u_0)+\varepsilon)(1/2+\varepsilon)}. \end{aligned}$$

The result follows.

As a consequence of Proposition 11, we obtain (see [37] and references therein for results of the same kind on $B \circ \mu$).

Proposition 12 *Let μ be a positive Borel measure on $[0,1]$, let $B_{1/2}$ be a Brownian motion. With probability 1, for every $h \ge 0$, $d_{B\circ F}(h) \le \tau_\mu^*(2h)$ and $E_h^{B\circ F} = \emptyset$ if $\tau_\mu^*(2h) > 0$. Moreover, if* **C3(**$2h$**)** *holds, $d_{B\circ F}(h) = \tau_\mu^*(2h)$.*

PROOF. Let $h \ge \tau_\mu^*(0^+)/2$. By Proposition 11, $E_h^{B\circ F} \subset \bigcup_{h' \ge 2h} \overline{E}_{h'}^\mu$, and by Proposition 1 $\dim \bigcup_{h' \ge 2h} \overline{E}_{h'}^\mu \le \tau_\mu^*(2h)$.

Let $h \le \tau_\mu^*(0^+)/2$. By Proposition 11, $E_h^{B\circ F} \subset \bigcup_{h' \le 2h} E_{h'}^\mu$, and by Proposition 1, we get $\dim \bigcup_{h' \le 2h} E_{h'}^\mu \le \tau_\mu^*(2h)$.

If **C3(**$2h$**)** holds, $\widetilde{E}_{2h}^\mu \subset E_h^{B\circ F}$ and $\dim \widetilde{E}_{2h}^\mu = \tau_\mu^*(2h)$.

Theorem 2., item *(3)* is obtained using the same arguments as in Section 5.

7 Back to the fixed points of the smoothing transformation (1)

7.1 Recalls on Mandelbrot multiplicative cascades μ_W, and some self-similarity properties of $X \circ \mu$

Recall how the measure μ_W on $[0,1]$ is obtained. Let $\mathcal{A}$ be the alphabet $\{0, \dots, b-1\}$ and $\mathcal{A}^* = \bigcup_{n \ge 0} \mathcal{A}^n$ ($\mathcal{A}^0$ contains the empty word $\emptyset$). Consider a sequence $\Big((W_0(w), \dots, W_{b-1}(w))\Big)_{w \in \mathcal{A}^*}$ of independent copies of W. For $n \ge 1$, let $\mu_{W,n}$ be the measure defined on $[0,1]$ by uniformly distributing on every b-adic interval of the form $\left[\sum_{k=1}^n w_k b^{-k}, b^{-n} + \sum_{k=1}^n w_k b^{-k}\right]$, $w_1 w_2 \cdots w_n \in \mathcal{A}^n$, the mass $W_{w_1}(\emptyset) \cdot W_{w_2}(w_1) \cdots W_{w_n}(w_1 w_2 \cdots w_{n-1})$. Then, with probability 1, the sequence of multiplicative cascades $(\mu_{W,n})_{n\geq 1}$ converges weakly on $[0,1]$, as $n\to\infty$, to a measure μ_W called the independent multiplicative cascade measure associated with W.

The real number $\varphi_W'(1)$ has a geometric interpretation: Both the lower and upper Hausdorff dimensions of μ_W equal $\varphi_W'(1)$ (for the definitions of these dimensions, see [36,26]).

Consider such a measure $\mu=\mu_W$, and assume that μ and the Lévy process X are independent. The probability space $(\Omega,\mathbb{P})$ can be written as a product $(\Omega_S\times\Omega_\mu,\mathbb{P}_S\otimes\mathbb{P}_\mu)$, where $(\Omega_S,\mathbb{P}_S)$ and $(\Omega_\mu,\mathbb{P}_\mu)$ are the probability spaces on which are respectively defined the Poisson point process S and the measure μ.

If, moreover, $X=X_\beta$ and $\mu=\mu_{W_\beta}$ as in Section 1, the reader can check that the following property holds: $\forall n\geq 1$

$$\Big(Z_{W,(k+1)b^{-n}}-Z_{W,kb^{-n}}\Big)_{0\leq k<b^{-n}} \overset{d}{\equiv} \Big(Z(w)\prod_{k=1}^{n}W_{w_k}(w_1\cdots w_{k-1})\Big)_{w\in\mathcal{A}^n}, \quad (40)$$

where, on the right hand side,

- the set $\mathcal{A}^n$ is described in lexicographical order,
- the random vectors $(W_0(w),\dots,W_{b-1}(w))$'s are i.i.d. with W,
- the random values $Z(w)$'s are i.i.d. with $Z_{W,1}$ and are independent of the $(W_0(w),\dots,W_{b-1}(w))$'s.

Also, if the function φ_W defined in (2) is not equal to $-\infty$ on a neighborhood of $(-\infty,2]$ and $\varphi_W'(\beta)>0$, then it follows from [34,1,4] that $\tau_\mu=\varphi_{W_\beta}$ on the interval $J=\{q\leq 1:\varphi^*_{W_\beta}(\varphi'_{W_\beta}(q))\geq 0\}$ almost surely. This yields $\tau_{\mu,\beta}\equiv\varphi_W$ on the interval $J_\beta=\beta\cdot J$.

7.2 The validity of $\mathbf{C2}(h_{\mu,\beta})$ *when* μ *is a Mandelbrot measure*

Let $\varphi_j=j^{-1/2}\log^2(j)$ for every $j\geq 1$ and let $(j_p)_{p\geq 1}$ be an increasing sequence such that $\lim_{p\to\infty} j_p^{-1}\log_2 C_{j_p}=\beta$ (recall (7)). Let $(n_p)_{p\geq 1}$ be the sequence of integers defined by $n_p=\inf\{k: b^{-k(\tau_\mu'(1)-\varphi_k)}C_{j_p}\leq 1\}$. We can choose the sequence $(\beta_j)_{j\geq 1}$ of Lemma 1 so that $2^{-(j_p+2)\beta_{j_p}}\geq b^{-n_p(\tau_\mu'(1)-\varphi_{n_p})}$. This last technical point is used at the end of the proof of Proposition 14.

It is shown in [10] that properties *(1)* and *(2)(b)* of Definition 5 are fulfilled $\mathbb{P}_\mu$-almost surely by μ with our choice of φ_j. Moreover, by our choice of $(\beta_j)_{j\geq 1}$ in Lemma 1 and $\{(u_n,l_n)\}$ in $\mathbf{C2}(h_{\mu,\beta})$, property *(2)(a)* of Definition 5 is automatically fulfilled. So it remains to show that properties *(3)* and *(4)* of Definition 5 are satisfied $\mathbb{P}_\mu$-almost surely and $\mathbb{P}_S \otimes \mathbb{P}_\mu$-almost surely respectively.

Property *(3)* comes from the statistical self-similarity of μ: For $v \in \mathcal{A}^*$, let μ^v be the measure constructed on $[0,1]$ in the same way as μ is, but with the family of random vectors $\Big((W_0^v(w),\dots,W_{b-1}^v(w))\Big)_{w\in\mathcal{A}^*} = \Big((W_0(v\cdot w),\dots,W_{b-1}(v\cdot w))\Big)_{w\in\mathcal{A}^*}$ instead of $\Big((W_0 w),\dots,W_{b-1}(w))\Big)_{w\in\mathcal{A}^*}$. Let $|v|$ stand for the length of the word v and define $L_v = \Big[\sum_{k=1}^{|v|} v_k b^{-k}, b^{-|v|} + \sum_{k=1}^{|v|} v_k b^{-k}\Big]$. By construction, $\mathbb{P}_\mu$-almost surely, the restriction of the measure μ to L_v is equal to $W_{v_1}(\emptyset)W_{v_2}(v_1)\cdots W_{v_{|v|}}(v_1\cdots v_{|v|-1})\cdot \mu^v \circ f_{L_v}$ (the invertible function f_{L_v} is defined in Definition 5 *(3)*). Consequently, property *(3)* holds $\mathbb{P}_\mu$-almost surely with the choice $\mu^{L_v} = \mu^v \circ f_{L_v}$.

For $n \geq 1$ let

$$U_n^v = \left\{ t\in[0,1] : \left\{ \begin{array}{c} \forall\ j\geq n,\ \forall\ k,\ |k-k_{j,t}^b|\leq 1, \\ \mu^v\Big([kb^{-j},(k+1)b^{-j}]\Big) \leq b^{-j(\tau_\mu'(1)-\varphi_j)} \end{array} \right\} \right\}.$$

Then let

$$n_v = \inf\Big\{n\geq 1:\ \mu^v(U_n^v) \geq \|\mu^v\|/2\Big\}.$$

It remains us to show that $\mathbb{P}_S \otimes \mathbb{P}_\mu$ almost surely, there exists a dense subset $\mathcal{D}$ of $(1,\infty)$ such that for every $\delta\in\mathcal{D}$, for μ-almost every $u\in[0,1]$, there exists an increasing sequence of integers $(j_k(u))_{k\geq 1}$ such that for every $k\geq 1$ there exists $L_{v_k} \in B^\delta_{j_k(u)}(u)$ satisfying $\lim_{k\to\infty} \frac{|v_k|}{j_k(u)} = \delta$ and

$$n_{v_k} \leq |v_k|\,.\,\varphi_{|v_k|} \ \text{ and }\ b^{-|v_k|\varphi_{|v_k|}} \leq \|\mu^{v_k}\|. \tag{41}$$

The function F is still defined by $F(t) = \mu([0,t])$. For every $w\in\mathcal{A}^{n_p}$, let $N_w(\omega_S,\omega_\mu)$ be the number of points of the Poisson point process S falling in $F(L_w)\times(2^{-(j_p+1)},2^{-j_p}]$. Conditionally on μ, the variable N_w is a Poisson variable with intensity $\mu(L_w)C_{j_p}$. Then, the orthogonal projection of $S\cap(F(L_w)\times(2^{-(j_p+1)},2^{-j_p}])$ onto $F(L_w)$ is equal to $\{\zeta_1,\dots,\zeta_{N_w}\}$, where $(\zeta_i)_{i\geq 1}$ is a sequence of independent random variables (under $\mathbb{P}_S$), uniformly distributed in $F(L_w)$.

We set $\zeta_w = \zeta_1$ and $\widetilde{\zeta}_w = F^{-1}(\zeta_w)$. If $\delta > 1$, $v(\delta,\widetilde{\zeta}_w)$ stands for the word of generation $[\delta|w|]+1$ such that $\widetilde{\zeta}_w \in L_{v(\delta,\widetilde{\zeta}_w)}$.

If $t\in[0,1)$ and $n\geq 1$, then we denote by $w_n(t)$ the element w of $\mathcal{A}^n$ such that $t\in L_w$.

The validity of *(4)* is a consequence of the following propositions.

Proposition 13 *Let $\delta > 1$. With $\mathbb{P}$-probability 1, for μ-almost every t, if p is large enough, then (41) holds with $v_k = v(\delta, \widetilde{\zeta}_{w_{n_p}(t)})$.*

Proposition 14 *With $\mathbb{P}$-probability 1, for μ-almost every t, there are infinitely many p's such that $N_{w_{n_p}(t)} \geq 1$, that is $\zeta_{w_{n_p}(t)}$ is a jump point of X.*

For $n \geq 1$ and $v \in \mathcal{A}^*$ let $R_n(v) = \mu^v\Big((U_n^v)^c\Big)$. The proof of Proposition 13 uses the following result which is a consequence of our choice for φ_j and Lemma 1 in [10].

Lemma 7 *For every $n \geq 1$, the random variables $R_n(v)$, $v \in \mathcal{A}^*$, are identically distributed. Denote $R_n(\emptyset) = R_n$. Then, for all $h \in (0,1)$, $\mathbb{E}\Big((R_n)^h\Big) = O(b^{-\log^2(n)})$.*

PROOF OF PROPOSITION 13. Let $\mathcal{Q}$ be the probability measure defined on $\mathcal{B}(\Omega_S) \otimes \mathcal{B}(\Omega_\mu) \otimes \mathcal{B}([0,1])$ by

$$\mathcal{Q}(A) = \mathbb{E}\Big(\int_{[0,1]} \mathbf{1}_A(\omega_S, \omega_\mu, t)\, \mu(dt)\Big).$$

Notice that $\mathcal{Q}$-almost surely means for $\mathbb{P}_S \otimes \mathbb{P}_\mu$-almost every (ω_S, ω_μ), for μ_{ω_μ}-almost every t. Let $\psi_j = j\varphi_j$, $r_p = [\delta n_p] + 1$ and $\rho_p = \log^{3/2}(n_p)$. By the Borel-Cantelli lemma, and since $\rho_p \leq \psi_{r_p}$ for p large enough, it is enough to prove that

$$\sum_{p\geq 1} \mathcal{Q}\left(b^{\rho_p} R_{\psi_{r_p}}\Big(v(\delta, \widetilde{\zeta}_{w_{n_p}(t)})\Big) \geq 1/2\right) < \infty \tag{42}$$

$$\sum_{p\geq 1} \mathcal{Q}\left(\|\mu^{v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})}\| \leq b^{-\rho_p}\right) < \infty. \tag{43}$$

We establish (42). For $p \geq 1$ and $h \in (0,1)$, we have

$$\mathcal{Q}\left(b^{\rho_p} R_{\psi_{r_p}}\Big(v(\delta, \widetilde{\zeta}_{w_{n_p}(t)})\Big) \geq 1/2\right) \leq 2^h b^{\rho_p h} \mathbb{E}_{\mathcal{Q}}\left(R_{\psi_{r_p}}\Big(v(\delta, \widetilde{\zeta}_{w_{n_p}(t)})\Big)^h\right). \tag{44}$$

In addition, $\mathbb{E}_{\mathcal{Q}}\left(R_{\psi_{r_p}}\Big(v(\delta, \widetilde{\zeta}_{w_{n_p}(t)})\Big)^h\right) = \mathbb{E}\Big(\sum_{w\in\mathcal{A}^{n_p}} R_{\psi_{r_p}}\Big(v(\delta, \widetilde{\zeta}_w)\Big)^h \mu(L_w)\Big).$

Given $u, w \in \mathcal{A}^*$, $w \preceq u$ means that $L_u \subset L_w$. We obtain

$$\mathbb{E}\left(R_{\psi_{r_p}}\left(v(\delta,\widetilde{\zeta}_w)\right)^h\mu(L_w)\right) = \sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u} \mathbb{E}\left(\mathbf{1}_{L_u}(\widetilde{\zeta}_w)R_{\psi_{r_p}}(u)\mu(L_w)\right)$$

$$= \sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u} \mathbb{E}\left(\mathbf{1}_{F(L_u)}(\zeta_w)R_{\psi_{r_p}}(u)^h\mu(L_w)\right)$$

$$= \sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u} \mathbb{E}_{\mathbb{P}_\mu}\left(\mathbb{P}_S\left(\zeta_w\in F(L_u)\right)R_{\psi_{r_p}}(u)^h\mu(L_w)\right)$$

$$= \sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u} \mathbb{E}\left(\frac{|F(L_u)|}{|F(L_w)|}R_{\psi_{r_p}}(u)^h\mu(L_w)\right) = \sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u} \mathbb{E}\left(|F(L_u)|R_{\psi_{r_p}}(u)^h\right).$$

It follows from the previous equality and the structure of μ that

$$\mathbb{E}_{\mathcal{Q}}\Big(R_{\psi_{r_p}}\left(v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})\right)^h\Big) = \mathbb{E}(R_{\psi_{r_p}}(u)^h\|\mu^u\|),$$

where u is any element of $\mathcal{A}^*$. Since it is assumed that μ is positive with probability 1 as well as $\mathbb{E}(\sum_{k=0}^{b-1}W_k^\alpha)<\infty$ for some $\alpha>1$, it follows from [15] that α can be chosen so that $\mathbb{E}(\|\mu\|^\alpha)<\infty$. Consequently, the Hölder inequality yields $\mathbb{E}(R_{\psi_{r_p}}(u)^h\|\mu^u\|)\le\mathbb{E}(\|\mu\|^\alpha)^{1/\alpha}\mathbb{E}(R_{\psi_{r_p}}^{h\alpha'})^{1/\alpha'}$, where $\alpha^{-1}+\alpha'^{-1}=1$. The conclusion follows by using (44) together with Lemma 7 applied with h small enough.

We move to (43). For $p\ge1$ and $h\in(0,1)$, we have

$$\mathcal{Q}\left(\|\mu^{v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})}\|\le b^{-\rho_p}\right)\le b^{-\rho_p h}\mathbb{E}_{\mathcal{Q}}\left(\|\mu^{v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})}\|^{-h}\right).$$

Computations comparable to those used in establishing (42) show that

$$\mathbb{E}_{\mathcal{Q}}\left(\|\mu^{v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})}\|^{-h}\right)=\mathbb{E}\left(\|\mu\|^{1-h}\right)<\infty.$$

The conclusion follows from our choice for ρ_p.

PROOF OF PROPOSITION 14. Let $\omega_\mu\in\Omega_\mu$ such that $\mu=\mu(\omega_\mu)$ is defined and positive, and let $t\in(0,1)$ in the set of full μ-measure described in property *(2)(b)* of Definition 5. The random variables $N_{w_{n_p}(t)}(\cdot,\omega_\mu)$, $p\ge1$, are $\mathbb{P}_S$ independent, and

$$\mathbb{P}_S\Big(N_{w_{n_p}(t)}(\cdot,\omega_\mu)\ge1\Big)=1-\exp\Big(-\mu(L_{w_{n_p}(t)})C_{j_p}\Big).$$

Due to the definition of n_p and property *(2)(b)*, for p large enough, we have $1-\exp\Big(-\mu(L_{w_{n_p}(t)})C_{j_p}\Big)\ge1-\exp(-1)$, so $\sum_{p\ge1}\mathbb{P}_S(N_{w_{n_p}(t)}(\cdot,\omega_\mu)\ge1)=\infty$. The Borel-Cantelli lemma allows to conclude that $\mathbb{P}_S$-almost surely $N_{w_{n_p}(t)}(\omega_S,\omega_\mu)\ge1$ for infinitely many p. Since this holds $\mathbb{P}_\mu$-almost surely, for μ-almost every t, we get the desired result by the Fubini theorem.

A final important remark is that the constraint $2^{-(j_p+1)\beta_{j_p}} \geq b^{-n_p(\tau_\mu'(1)-\varphi_{n_p})}$ imposed on β_{j_p} ensures that $t \in [u_n - l_n/2, u_n, +l_n/2]$ if u_n stands for $\widetilde{\zeta}_{w_{n_p}(t)}$.

References

[1] M. Arbeiter and N. Patzschke, Random self-similar multifractals, Math. Nachr. **181**, (1996) 5–42.

[2] A. Ayache, J. Lévy Véhel, Generalized Multifractional Brownian Motion: Definition and Preliminary Results, Fractals - Theory and Applications in Engineering (1999).

[3] E. Bacry and J.-F. Muzy, Log-infinitely divisible multifractal processes, Commun. Math. Phys. **236**, (2003) 449–475.

[4] J. Barral, Continuity of the multifractal spectrum of a statistically self-similar measure, J. Theoretic. Probab. **13**, (2000) 1027–1060.

[5] J. Barral and B. Mandelbrot, Multifractal products of cylindrical pulses, Probab. Theory Relat. Fields **124**, (2002) 409–430.

[6] J. Barral and B. Mandelbrot, Random multiplicative multifractal measures, Parts I,II,III Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot, Proc. Symp. Pure Math., **72, 2**, 3-90, AMS, Providence, RI (2004).

[7] J. Barral and S. Seuret, From multifractal measures to multifractal wavelet series, J. Fourier Anal. Appl. **11**, (2005) 589–614.

[8] J. Barral and S. Seuret, Combining multifractal additive and multiplicative chaos, Commun. Math. Phys. **257**, (2005) 473–497.

[9] J. Barral and S. Seuret, Inside singularity sets of random Gibbs measures, J. Stat. Phys. **120**, (2005) 1101–1124.

[10] J. Barral and S. Seuret, Renewal of singularity sets of random self-similar measures, Preprint (2004) http://fr.arxiv.org/abs/math.PR/0503421, accepted for publication in Adv. Appl. Probab.

[11] J. Barral and S. Seuret, Heterogeneous ubiquitous systems in $\mathbb{R}^d$ and Hausdorff dimensions, Preprint (2005), http://fr.arxiv.org/abs/math.GM/0503419.

[12] J. Barral and S. Seuret, Sums of Dirac masses and conditioned ubiquity, C. R. Acad. Sci. Paris, Sér. I **339**, (2004) 787–792.

[13] G. Brown, G. Michon, J. Peyrière, On the multifractal analysis of measures, J. Stat. Phys. **66(3-4)**, (1992) 775–790.

[14] P. Chainais, R. Riedi, P. Abry, On scale invariant infinitely divisible cascades, IEEE Trans. on Info. Theory, vol. 51, no. 3, March 2005.

[15] R. Durrett and T. Liggett, Fixed points of the smoothing transformation, Z. Wahrsch. verw. Gebiete **64**, (1983) 275–301.

[16] Falconer, K.J., The multifractal spectrum of statistically self-similar measures, J. Theor. Prob. **7**, (1994) 681–702 .

[17] A.H. Fan, Multifractal analysis of infinite products, J. Stat. Phys. **86(5/6)**, (1997) 1313–1336.

[18] Y. Guivarc'h, Sur une extension de la notion de loi semi-stable, Ann. Inst. H. Poincaré, Probab. et Statist. **26**, (1990) 261–285.

[19] R. Holley and E.C. Waymire, Multifractal dimensions and scaling exponents for strongly bounded random fractals, Ann. Appl. Probab. **2**, (1992), 819–845.

[20] S. Jaffard, Exposants de Hölder en des points donnés et coefficients d'ondelettes, C. R. Acad. Sci. Paris, Sér. I **308**, (1989), 79–81.

[21] S. Jaffard, Old friends revisited: The multifractal nature of some classical functions, J. Fourier Anal. Appl. **3(1)**, (1997) 1–22.

[22] S. Jaffard, The multifractal nature of Lévy processes, Probab. Theory Relat. Fields **114(2)**, (1999) 207–227.

[23] S. Jaffard, On lacunary wavelet series, Ann. Appl. Prob. **10(1)**, (2000) 313–329.

[24] J.-P. Kahane, Positive martingales and random measures, Chi. Ann. of Math **8B1**, (1987) 1–12.

[25] J.-P. Kahane, Produits de poids aléatoires et indépendants et applications. Fractal Geometry and Analysis, J. Bélair and S. Dubuc (eds.), (1991) 277–324.

[26] J.-P. Kahane and J. Peyrière, Sur certaines martingales de Benoît Mandelbrot, Adv. Math. **22**, (1976) 131–145.

[27] Y. Kifer, Fractals via random iterated function systems and random geometric constructions, Fractal geometry and stochastics (Finsterbergen, 1994) Progr. Probab. **37**, (1995) 145–164, Birkhäuser, Basel.

[28] Q. Liu, On generalized multiplicative cascades, Stoch. Proc. Appl. **86**, (2000) 263–286.

[29] Q. Liu, Asymptotic properties and absolute continuity of laws stable by random weighted mean, *Stoch. Proc. Appl.* **95**, (2001) 83–107.

[30] B.B. Mandelbrot, Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire, C. R. Acad. Sci. Paris **278**, (1974) 289–282 and 355–358.

[31] B.B. Mandelbrot, Intermittent turbulence in self-similar cascades: divergence of hight moments and dimension of the carrier, J. Fluid. Mech. **62**, (1974) 331–358.

[32] B. Mandelbrot, A. Fischer , L. Calvet, A multifractal model of asset returns, Cowles Foundation Discussion Paper **#1164** (1997).

[33] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Univ. Press (1995).

[34] G.M. Molchan, Scaling exponents and multifractal dimensions for independent random cascades, Commun. Math. Phys., **179**, (1996) 681–702.

[35] L. Olsen, A multifractal formalism, Adv. Math. **116**, (1995) 92–195.

[36] J. Peyrière, Turbulence et dimension de Hausdorff, C. R. Acad. Sci. Paris Sér. A **278**, (1974) 567–569.

[37] R. Riedi, Multifractal processes, *Long Range Dependence: Theory and Applications*, eds. Doukhan, Oppenheim, Taqqu, (Birkhäuser 2002), pp 625–715.

[38] L.A. Shepp, Covering the line with random intervals, Z. Wahrsch. Verw. Gebiete **23**, (1972) 163–170.