Sums of Dirac masses and conditioned ubiquity

Sommes de masses de Dirac et ubiquité conditionnelle

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Abstract

Multifractal formalisms hold for certain classes of atomless measures μ obtained as limits of multiplicative processes. This naturally leads to ask whether non trivial discontinuous measures obey such formalisms. This is the case for a new kind of measures, whose construction combines additive and multiplicative chaos. This class is defined by $\nu_{\gamma,\sigma} = \sum_{j\geq 1} \frac{b^{-j\gamma}}{j^2} \sum_{k=0}^{b^j-1} \mu([kb^{-j},(k+1)b^{-j}))^{\sigma}\delta_{kb^{-j}}$ ($\mathrm{supp}(\mu)=[0,1]$, b integer ≥ 2, $\gamma\geq 0$, $\sigma\geq 1$). Under suitable assumptions on the initial measure μ, $\nu_{\gamma,\sigma}$ obeys some multifractal formalisms. Its Hausdorff multifractal spectrum $h\mapsto d_{\nu_{\gamma,\sigma}}(h)$ is composed of a linear part for h smaller than a critical value $h_{\gamma,\sigma}$, and then of a concave part when $h\geq h_{\gamma,\sigma}$. The same properties hold for the Hausdorff spectrum of some function series $f_{\gamma,\sigma}$ constructed according to a scheme similar to the one of $\nu_{\gamma,\sigma}$. These phenomena are the consequences of new results relating ubiquitous systems to the distribution of the mass of μ.

Résumé

Les formalismes multifractals sont vérifiés par certaines classes de mesures diffuses μ limites de processus multiplicatifs. Cela pose naturellement la question de savoir s'ils le sont encore pour des mesures non diffuses non triviales. C'est effectivement le cas pour des mesures d'un type nouveau, qui mêlent chaos additifs et multiplicatifs. Cette classe de mesures est définie par $\nu_{\gamma,\sigma} = \sum_{j\geq 1} \frac{b^{-j\gamma}}{j^2} \sum_{k=0}^{b^j-1} \mu([kb^{-j},(k+1)b^{-j}))^{\sigma}\delta_{kb^{-j}}$ ($\mathrm{supp}(\mu)=[0,1]$, b entier ≥ 2, $\gamma\geq 0$, $\sigma\geq 1$). Sous certaines hypothèses sur μ, plusieurs formalismes multifractals sont en effet satisfaits par $\nu_{\gamma,\sigma}$. De plus, son spectre multifractal de Hausdorff $h\mapsto d_{\nu_{\gamma,\sigma}}(h)$ se compose alors d'une partie linéaire pour h plus petit qu'une valeur critique $h_{\gamma,\sigma}$, puis d'une partie concave pour $h\geq h_{\gamma,\sigma}$. Cette propriété est partagée par le spectre de Hausdorff de séries de fonctions $f_{\gamma,\sigma}$ construites de façon analogue à $\nu_{\gamma,\sigma}$. L'analyse des singularités de ces objets fait appel à de nouveaux résultats combinant la notion d'ubiquité avec les propriétés d'auto-similarité de la mesure μ.

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Soit μ une mesure borélienne positive sur $[0,1]$. Soit b un entier ≥ 2, $\gamma \geq 0$ et $\sigma \geq 1$. Considérons la mesure constituée d'atomes $\nu_{\gamma,\sigma} = \sum_{j\geq 1} j^{-2} \sum_{k=0}^{b^j-1} b^{-j\gamma} \mu([kb^{-j},(k+1)b^{-j}))^\sigma \delta_{kb^{-j}}$. Nous nous intéressons au spectre de Hausdorff de $\nu_{\gamma,\sigma}$, c'est-à-dire à la fonction $d_{\nu_{\gamma,\sigma}}$ qui à un exposant $h \geq 0$ associe la dimension de Hausdorff $\dim E_h^{\nu_{\gamma,\sigma}}$ de l'ensemble $E_h^{\nu_{\gamma,\sigma}} = \{x : \liminf_{r\to 0^+} \frac{\log \nu_{\gamma,\sigma}(B(x,r))}{\log r} = h\}$.

À une mesure de Borel positive ν sur $[0,1]$, nous associons sa fonction d'échelle $\tau_\nu(q)$ définie relativement à la grille b-adique comme dans [6], et la transformée de Legendre $\tau_\nu^*(h) = \inf_{q\in\mathbb{R}}(qh - \tau_\nu(q))$. On dit que le formalisme multifractal est valide pour ν en h si $\dim E_h^\nu = \tau_\nu^*(h)$.

On a alors le théorème suivant, qui dépend de deux conditions sur μ : **C1** demande que le support de μ soit $[0,1]$ et que $\sup_{x\in[0,1],r\in(0,1)} \frac{\log \mu(B(x,r))}{\log r} < \infty$. Etant donné un exposant $h \geq 0$, **C2**(h) est vérifiée lorsque la famille $\{(kb^{-j}, 2b^{-j})\}_{j\geq 1,k=0,..,b^j}$ forme un système hétérogène d'ubiquité par rapport à $\big(\mu, h, \tau_\mu^*(h)\big)$ au sens de [4].

Théorème 0.1 *Supposons que* **C1** *soit vérifiée par* μ. *Soit* $q_{\gamma,\sigma} = \inf\{q \in \mathbb{R} : \tau_\mu(\sigma q) + \gamma q = 0\}$ *et* $h_{\gamma,\sigma} = \sigma\tau'_\mu(\sigma q^-_{\gamma,\sigma}) + \gamma$. *On a* $q_{\gamma,\sigma} \in (0,1]$ *et* $0 \leq h_{\gamma,\sigma} \leq q^{-1}_{\gamma,\sigma}$.

(i) Si $h_{\gamma,\sigma} > 0$, *alors pour tout* $h \in [0, h_{\gamma,\sigma}]$, $d_{\nu_{\gamma,\sigma}}(h) \leq q_{\gamma,\sigma} h$. *De plus, si* **C2**$\big(\frac{h_{\gamma,\sigma}-\gamma}{\sigma}\big)$ *est vérifiée, alors pour tout* $h \in [0, h_{\gamma,\sigma}]$, $d_{\nu_{\gamma,\sigma}}(h) = q_{\gamma,\sigma} h$, *et le formalisme multifractal est valide en* h.

(ii) Si $h \geq h_{\gamma,\sigma}$, *alors* $d_{\nu_{\gamma,\sigma}}(h) \leq \tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)$. *De plus, si* **C2**$\big(\frac{h_{\gamma,\sigma}-\gamma}{\sigma}\big)$ *est vérifiée,* $d_{\nu_{\gamma,\sigma}}(h) = \tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)$, *et le formalisme multifractal est valide en* h.

Lorsqu'elle est vérifiée, **C2**(h) permet de calculer la dimension de Hausdorff d'ensembles limsup $S_\mu(\alpha,\delta,\widetilde{\varepsilon})$ étroitement liés à la fois à la distribution de la masse de μ et à la famille des points b-adiques $\{(kb^{-j}, 2b^{-j})\}_{j\geq 1,k=0,..,b^j}$. En effet, on a :

Théorème 0.2 *Soit* $h > 0$. *Étant donnés* $\delta \geq 1$ *et une suite positive* $\widetilde{\varepsilon} = \{\varepsilon_j\}_j$, *on définit l'ensemble*

$$S_\mu(h,\delta,\widetilde{\varepsilon}) = \bigcap_{N\geq 1} \quad \bigcup_{n\geq N: b^{-j(h+\varepsilon_j)} \leq \mu\left([kb^{-j},(k+1)b^{-j})\right) \leq b^{-j(h-\varepsilon_j)}} \quad [kb^{-j} - b^{-j\delta}, kb^{-j} + b^{-j\delta}].$$

Supposons que **C2**(h) *soit vérifiée par* μ. *Il existe une suite positive* $\widetilde{\varepsilon}$ *convergeant vers 0 telle que pour chaque* $\delta \geq 1$, *il existe une mesure de Borel positive* m_δ *sur* $[0,1]$ *telle que* $m_\delta(S_\mu(h,\delta,\widetilde{\varepsilon})) > 0$, *et pour tout borélien* $E \subset [0,1]$ *tel que* $\dim E < \tau_\mu^*(h)/\delta$, $m_\delta(E) = 0$. *En particulier, on a* $\dim S_\mu(h,\delta,\widetilde{\varepsilon}) \geq \tau_\mu^*(h)/\delta$.

La connaissance de la dimension de Hausdorff de ces ensembles $S_\mu(h,\delta,\widetilde{\varepsilon})$ permet le calcul de $d_{\nu_{\gamma,\sigma}}$.

1. Introduction

Among the measures whose multifractal analysis can be performed, two families can be distinguished by the shape of their Hausdorff spectrum. On one side, measures built on an additive scheme, such as the Lebesgue-Stieljes measure associated with Lévy subordinators [10] for instance, classically exhibit linear increasing spectrum. On the other side, atomless measures with a construction involving a multiplicative scheme usually have a strictly concave spectrum, including a decreasing part when h is larger than a typical exponent h_t. Multinomial measures, Mandelbrot multiplicative cascades and their extensions, are classical examples of such measures, with the well-known $\bigcap$-shaped spectrum.

We consider a distinct construction scheme which mixes both additive and multiplicative structures.

Definition 1.1 *If* μ *is a positive Borel measure on* $[0,1]$ *and* b *is an integer greater than* 2, *let* $\nu_{\gamma,\sigma}$ *be the measure defined with the help of two parameters* $\gamma \geq 0$ *and* $\sigma \geq 1$ *by*

$$\nu_{\gamma,\sigma} = \sum_{j\geq 1} j^{-2} \sum_{k=0,\ldots,b^j-1} b^{-j\gamma}\mu([kb^{-j},(k+1)b^{-j}))^\sigma \delta_{kb^{-j}}. \tag{1}$$

The multifractal behavior of these measures combines the two typical multifractal behaviors described above. A part of their multifractal spectrum is closely related to the non-emptiness of a new kind of limsup-sets $S_\mu(\alpha,\delta,M)$ defined by (4). The computation of the Hausdorff dimension of these sets requires new ubiquity results (see Section 4 and [4]) which extend to the multifractal frame the classical ubiquity results only valid for a monofractal measure μ [7,11].

2. Multifractal spectrum of the new class of measures $\nu_{\gamma,\sigma}$

The local regularity of a measure μ at a point x is hereafter described by the Hölder exponent $h_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}$. The multifractal analysis of μ consists in computing the Hausdorff dimensions of the level sets of this Hölder exponent $E_h^\mu = \{x : h_\mu(x) = h\}$, $h \geq 0$. Then one tries to find the multifractal spectrum of μ, $h \mapsto d_\mu(h) = \dim(E_h^\mu)$, where $\dim(E)$ stands for the Hausdorff dimension of the set E.

In order to fully state our result, the notion of multifractal formalism for measures is required. A multifractal formalism is a formula which relates, via a Legendre transform, the multifractal spectrum d_μ to a scaling function associated with μ [6,15]. Here, we adopt the following definition for the scaling function [6]: $\forall\, q \in \mathbb{R}$, $\tau_\mu(q) = \liminf_{j\to+\infty} -\frac{1}{j}\log_b \sum_{0\leq k\leq b^j} \mu\big([kb^{-j},(k+1)b^{-j})\big)^q$, with the convention $0^q = 0$, $\forall\, q$. The multifractal formalism is said to hold for μ at exponent h when the multifractal spectrum of μ at h is equal to the Legendre transform τ_μ^* of τ_μ at h, i.e. when $d_\mu(h) = \tau_\mu^*(h) = \inf_{q\in\mathbb{R}}(qh - \tau_\mu(q))$.

Our result invokes two technical conditions. **C1** simply requires that the support of the measure μ is $[0,1]$ and that one has the control $\sup_{x\in[0,1], r\in(0,1)} \frac{\log \mu(B(x,r))}{\log r} < \infty$. $\mathbf{C2}(h)$ is said to hold for the measure μ and the exponent h when the family of points $\{(kb^{-j}, 2b^{-j})\}_{j\geq 1, k=0,..,b^j-1}$ forms an heterogeneous ubiquitous system with respect to $\big(\mu, h, \tau_\mu^*(h)\big)$ (see Section 4).

Theorem 2.1 *Let μ be a positive Borel measure supported by $[0,1]$, and assume that* **C1** *holds for μ. Let $\gamma \geq 0$ and $\sigma \geq 1$, and consider the measure $\nu_{\gamma,\sigma}$ defined in (1). Let $q_{\gamma,\sigma} = \inf\{q \in \mathbb{R} : \tau_\mu(\sigma q) + \gamma q = 0\}$, and $h_{\gamma,\sigma} = \sigma\tau'_\mu(\sigma q_{\gamma,\sigma}^-) + \gamma$. One has $q_{\gamma,\sigma} \in (0,1]$ and $0 \leq h_{\gamma,\sigma} \leq q_{\gamma,\sigma}^{-1}$.*

(i) If $h_{\gamma,\sigma} > 0$, for every $h \in [0, h_{\gamma,\sigma}]$, $d_{\nu_{\gamma,\sigma}}(h) \leq q_{\gamma,\sigma} h$. If moreover $\mathbf{C2}\big(\frac{h_{\gamma,\sigma}-\gamma}{\sigma}\big)$ holds, then for every $h \in [0, h_{\gamma,\sigma}]$ $d_{\nu_{\gamma,\sigma}}(h) = q_{\gamma,\sigma} h$, and the multifractal formalism holds at h.

(ii) If $h \geq h_{\gamma,\sigma}$, then $d_{\nu_{\gamma,\sigma}}(h) \leq \tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)$. If moreover $\mathbf{C2}\big(\frac{h_{\gamma,\sigma}-\gamma}{\sigma}\big)$ holds, $d_{\nu_{\gamma,\sigma}}(h) = \tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)$, and the multifractal formalism holds at h.

Theorem 2.1 applies to several classes of atomless statistically self-similar measures μ (see [2,5]), as well as to the measure $\nu_{0,1}$ itself: In this case the process can be iterated, the spectrum being unchanged.

It is shown in [2] that multifractal formalisms that focus on level sets such as $\{x : \lim_{r\to 0} \frac{\log \nu_{\gamma,\sigma}(B(x,r))}{\log r} = h\}$ (defined using a limit instead of a lim inf) do not hold for these measures $\nu_{\gamma,\sigma}$ at h when $0 < h < h_{\gamma,\sigma}$. Theorem 2.1 thus pleads for the choice of the sets $E_h^{\nu_{\gamma,\sigma}}$ defined using the Hölder exponent $h_{\nu_{\gamma,\sigma}}(x)$.

Also, the measures $\nu_{\gamma,\sigma}$ provide new examples of measures that may have a scaling function $\tau_{\nu_{\gamma,\sigma}}$ whose derivative does not exist at some point (here $q_{\gamma,\sigma}$), while satisfying a multifractal formalism. Such a non differentiability is called a phase transition in the thermodynamical terminology.

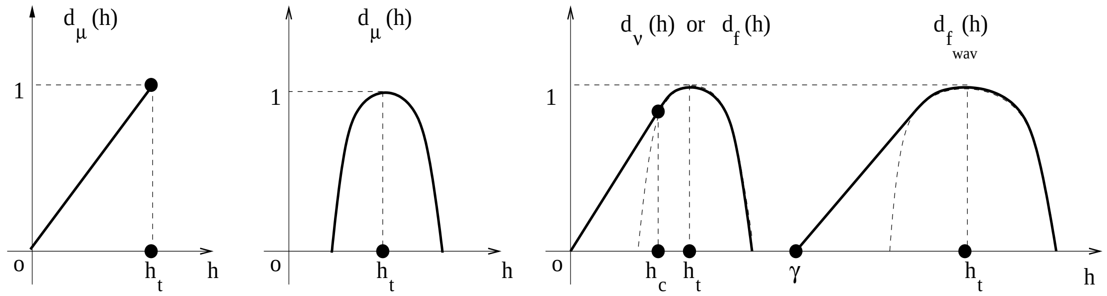

Fig.1: Typical multifractal spectrum of a measure μ built on an (**Left**) additive or (**Middle**) multiplicative scheme, and (**Right**) of $\nu_{\gamma,\sigma}$ (Theorem 2.1) and $f_{\gamma,\sigma}$ (Theorem 3.1) with $\gamma = 0$ and $\sigma = 1$; of f_{wav} (Theorem 3.2) with arbitrary γ, σ.

Fig.1: Spectre multifractal typique d'une mesure μ construite sur un schéma (**gauche**) additif ou (**centre**) multiplicatif, et (**droite**) de $\nu_{\gamma,\sigma}$ (Théorème 2.1) ou $f_{\gamma,\sigma}$ (Théorème 3.1) avec $\gamma = 0$, $\sigma = 1$; de f_{wav} (Théorème 3.2) avec γ, σ quelconques.

3. Multifractal spectrum of function series $f_{\gamma,\sigma}$ naturally associated with $\nu_{\gamma,\sigma}$

The multifractal analysis of functions constitutes a companion domain of the multifractal analysis of measures. Given a bounded function f on $[0,1]$, $x_0 \in (0,1)$ and $h \geq 0$, $f \in C^h(x_0)$ if there exist a constant C and a polynomial P of degree $\leq h$ such that $\big|f(x) - P(x-x_0)\big| \leq C|x-x_0|^h$ for x close enough to x_0. The pointwise Hölder exponent of f at x_0 [9] is then defined as $h_f(x_0) = \sup\{h : f \in C^h(x_0)$. The multifractal analysis of f consists in computing its Hausdorff spectrum $d_f : h \mapsto \dim\{x : h_f(x) = h\}$.

Inspired by the construction of the measures $\nu_{\gamma,\sigma}$, it is natural to build functions with a structure comparable with the one of $\nu_{\gamma,\sigma}$. Let π be a positive Borel measure supported by $[0,1]$, ϕ a bounded function on $\mathbb{R}$, $\gamma \geq 0$ and $\sigma \geq 1$. In two situations, we study the function series defined by

$$f(x) = \sum_{j=1}^{+\infty} f_j(x) \text{ where } f_j(x) = \frac{b^{-j\gamma}}{j^2} \sum_{k=0,\dots,b^j-1} \pi([kb^{-j},(k+1)b^{-j}))^{\sigma} \phi(b^j x - k). \tag{2}$$

The first choice of ϕ creates a discontinuity at each b-adic number, the second one uses a wavelet.

Theorem 3.1 *Under the assumptions of Theorem 2.1, and if $\phi(x) = (x-1)1\!\!1_{(x-1)\in[0,1)}$ and $\pi = \mu$, the corresponding function series (2), here denoted by $f_{\gamma,\sigma}$, satisfy:*

(i) If $h_{\gamma,\sigma} > 0$ and $\mathbf{C2}\big(\frac{h_{\gamma,\sigma}-\gamma}{\sigma}\big)$ holds, for any $h \in \big[0, \frac{h_{\gamma,\sigma}-\gamma}{\sigma}\big]$, $d_{f_{\gamma,\sigma}}(h) = q_{\gamma,\sigma} h$.

(ii) If $h \geq h_{\gamma,\sigma}$ and $\mathbf{C2}\big(\frac{h-\gamma}{\sigma}\big)$ holds, then $d_{f_{\gamma,\sigma}}(h) = d_\mu\big(\frac{h-\gamma}{\sigma}\big) = \tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)$.

Theorem 3.2 $b=2$. *Under the assumptions of Theorem 2.1, if ϕ is a C^∞ wavelet as constructed for instance in [14] and if $\pi = \nu_{0,1}$, the corresponding function series (2), denoted here f_{wav}, satisfy:*

(i) If $h_{0,1} > 0$ and $\mathbf{C2(}h_{0,1}\mathbf{)}$ holds, for any $h \in [\gamma, \gamma + \sigma h_{0,1}]$, one has $d_{f_{wav}}(h) = q_{0,1}\frac{h-\gamma}{\sigma}$.

(ii) If $h \geq \gamma + \sigma h_{0,1}$ and $\mathbf{C2}\big(\frac{h-\gamma}{\sigma}\big)$ holds, then $d_{f_{wav}}(h) = d_\mu\big(\frac{h-\gamma}{\sigma}\big) = \tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)$.

Theorem 3.1 is established in [3], and Theorem 3.2 is a consequence of a work achieved in [1].

The interest in function series with bounded variations and having a dense set of discontinuities goes back to the note of C. Jordan [12], where the space BV was introduced.

4. Ubiquity revisited via multifractality

Along the proof of Theorem 2.1, the non-emptiness of limsup-sets defined as $S_\mu(\alpha,\delta,M)$ in (4) is crucial. The study of this kind of sets is comparable with classical problems of ubiquity [7], but here the sets

$S_\mu(\alpha,\delta,M)$ are conditioned by a property $\mathcal{P}_M$ which depends on a (statistically self-similar) measure μ. We thus need a refinement of the ubiquity results in order to be able to combine the approximation rate by some family of points with the multifractal properties of a measure μ. This is achieved by Theorem 4.2.

Let $d\geq 1$ and μ a positive Borel measure on $[0,1]^d$. The analysis of the local structure of μ may be naturally done on a c-adic grid for some $c\geq 2$. This is the case for instance for multinomial measures or Mandelbrot cascades. We shall thus need the following notations. For every integer $j\geq 1$ and for every multi-integer $\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{N}^d$, let us denote by $I_{j,\mathbf{k}}$ the c-adic box $\Pi_{i=1}^d[k_ic^{-j},(k_i+1)c^{-j})$. For every point $x\in[0,1)^d$, $\forall j\geq 1$, we denote $\mathbf{k}_{j,x}$ the unique $\mathbf{k}\in\mathbb{N}^d$ such that $x\in I_{j,\mathbf{k}_{j,x}}$.

Definition 4.1 *Let μ be a positive Borel measure supported by $[0,1]^d$, and α, β two positive real numbers. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,1]^d$, and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a non-increasing sequence of positive real numbers converging to 0. The family $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ is said to form an heterogeneous ubiquitous system with respect to (μ,α,β) if the following conditions* **(1-4)** *are fulfilled ($|I|$ stands for the diameter of the set I).*

(1) *There exist ϕ and ψ, two non-decreasing continuous functions defined on $\mathbb{R}_+$ such that (i) $\varphi(0)=\psi(0)=0$, $r\mapsto r^{-\varphi(r)}$ and $r\mapsto r^{-\psi(r)}$ are non-increasing near 0^+, and $\lim_{r\to 0^+}r^{-\varphi(r)}=+\infty$; (ii) $\forall\varepsilon>0$, $r\mapsto r^{\varepsilon-\varphi(r)}$ is non-decreasing near 0; (iii)* **(2)**, **(3)** *and* **(4)** *hold.*

(2) *There exist a measure m with a support equal to $[0,1]^d$, which satisfies the following properties:*
(i) m-almost every $y\in[0,1]^d$ belongs to $\bigcap_{N\geq 1}\bigcup_{n\geq N}B(x_n,\frac{\lambda_n}{2})$.
(ii) For m-almost every $y\in[0,1]^d$, there exists an integer $j(y)$ such that

$$\forall j\geq j(y),\ \forall\ \mathbf{k}\text{ such that }\|\mathbf{k}-\mathbf{k}_{j,y}\|_\infty\leq 1,\ m\big(I_{j,\mathbf{k}}\big)\leq|I_{j,\mathbf{k}}|^{\beta-\varphi(|I_{j,\mathbf{k}}|)}\tag{3}$$

(iii) There exists a constant M (depending on c and μ) such that for m-almost every $y\in[0,1]^d$, one can find an integer $j(y)$ such that $\forall j\geq j(y)$, $\forall$ $\mathbf{k}$ such that $\|\mathbf{k}-\mathbf{k}_{j,y}\|_\infty\leq 1$, $\mathcal{P}_M(I_{j,\mathbf{k}})$ holds, where $\mathcal{P}_M(I)$ is said to hold for a set I and a constant M when $\frac{1}{M}|I|^{\alpha+\psi(|I|)}\leq\mu\big(I\big)\leq M|I|^{\alpha-\psi(|I|)}$.

(3) *(Self-similarity of m) For every c-adic box I of $[0,1]^d$, let f_I denote the canonical affine mapping from I onto $[0,1]^d$. There exists a measure m^I on I, equivalent to the restriction of m to I, such that the measure $m^I\circ f_I^{-1}$ satisfies (3), and with the same exponent β.*
Let us define the non-decreasing sequence $\{E_n^I\}_{n\geq 1}$ of sets of $[0,1]^d$

$$E_n^I=\left\{y\in I:\left\{\begin{array}{c}\forall\ j\geq n+\log_c\left(|I|^{-1}\right),\\ \forall\ \mathbf{k}\text{ such that }\|\mathbf{k}-\mathbf{k}_{j,y}\|_\infty\leq 1,\end{array}\right.\ m^I\left(I_{j,\mathbf{k}}\right)\leq\left(\frac{|I_{j,\mathbf{k}}|}{|I|}\right)^{\beta-\varphi\left(\frac{|I_{j,\mathbf{k}}|}{|I|}\right)}\right\}.$$

By (3), $\bigcup_{n\geq 1}E_n^I$ is of full m^I-measure. Define $n_I=\inf\left\{n\geq 1:\ m^I(E_n^I)\geq\frac{1}{2}\|m^I\|\right\}$.

(4) *(Speed of renewal of level sets and control of the mass $\|m^I\|$) There exists J_m such that for every $j\geq J_m$, for every c-adic box $I=I_{j,\mathbf{k}}$, $n_I\leq\log_c\left(|I|^{-1}\right)\varphi(|I|)$ and $|I|^{\varphi(|I|)}\leq\|m^I\|$.*

Theorem 4.2 *Let us assume that $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ forms an heterogeneous ubiquitous system with respect to (μ,α,β). For δ and M two real numbers greater than 1, let*

$$S_\mu(\alpha,\delta,M)=\bigcap_{N\geq 1}\ \bigcup_{n\geq N:\,\mathcal{P}_M(B(x_n,\lambda_n))\text{ holds}}B(x_n,\lambda_n^\delta).\tag{4}$$

There exists a constant M such that for every $\delta\geq 1$, one can find a positive measure m_δ such that $m_\delta(S_\mu(\alpha,\delta,M))>0$, and for every $x\in S_\mu(\alpha,\delta,M)$, $\limsup_{r\to 0^+}\frac{m_\delta\left(B(x,r)\right)}{r^{\beta/\delta-(4+d)\varphi(r)}}<\infty$. In particular, $\dim S_\mu(\alpha,\delta,M)\geq\beta/\delta$.

Theorem 4.2 is established in [4]. Heuristically, since $\mathcal{P}_M(I)$ ensures that $\mu(I)\sim|I|^\alpha$ up to a small correction, Theorem 4.2 allows the computation of the Hausdorff dimension of sets of points x that are infinitely often close at rate δ to points x_n (i.e. such that $\|x-x_n\|_\infty\leq\lambda_n^\delta$) that verify $\mu(B(x_n,\lambda_n))\sim\lambda_n^\alpha$.

Theorem 4.2 is referred to as "measure-conditioned ubiquity" because it involves an ubiquity property of a family of points that must satisfy some property. Here this property is related to the behavior of $\mu\big(B(x_n,\lambda_n)\big)$. The "usual" ubiquity theorems [7,11] must be seen as Theorem 4.2 in the particular case where the measure μ is the Lebesgue measure, that is the condition is empty since μ is strictly monofractal.

• **Examples of families** $\{(x_n,\lambda_n)\}$**:** - The b-adic family $\big\{(\mathbf{k}b^{-j},2b^{-j})\big\}_{\mathbf{k}\in\mathbb{N}^d,\, j\in\mathbb{N}^*}$ (where $b\in\mathbb{N}\backslash\{0,1\}$ can differ from c) and the family of rational numbers $\big\{(\mathbf{p}/q,2/q^{1+1/d})\big\}_{\mathbf{p}\in\mathbb{N}^d,\, q\in\mathbb{N}^*}$ satisfy (i) of assumption **(2)** for any measure m, because $\limsup_{n\to\infty}B(x_n,\lambda_n/2)=[0,1]^d$ in these cases. If m is atomless, $d=1$ and $\alpha\in\mathbb{R}\backslash\mathbb{Q}$, the family $\{(\{n\alpha\},2/n)\}_{n\in\mathbb{N}^*}$ satisfies (i) of **(2)** ($\{x\}$ is the fractional part of x).

- Item (i) of **(2)** can also be satisfied almost surely for some random families of points. Let $\{x_n\}$ be a sequence of points independently and uniformly distributed in $[0,1]^d$, and $\{\lambda_n\}_n$ be a non-increasing sequence of positive numbers. If $\limsup_{n\to+\infty}\big(\sum_{p=1}^n\lambda_p/2-d\log n\big)=+\infty$ then, with probability one, $\limsup_{n\to\infty}B(x_n,\lambda_n/2)=[0,1]^d$ (see [13]).

• **Examples of measures** μ**:** Suppose that $\limsup_{n\to\infty}B(x_n,\lambda_n/2)=[0,1]^d$. In [5], the properties required by Definition 4.1 are shown to hold for several classes of statistically self-similar measures (multinomial measures, Gibbs measures and their random counterparts) with $\alpha>0$ and $\beta=\tau_\mu^*(\alpha)$ as soon as $\tau_\mu^*(\alpha)>0$. We mention that, in order to treat the Mandelbrot multiplicative cascades, a slight stronger version of Theorem 4.2 is required, see [4,5].

5. Another application of Theorem 4.2: Conditioned Diophantine approximation

Consider for the family $\{(x_n,\lambda_n)\}_n$ the pairs $\{p/q,1/q^2\}_{p,q\in\mathbb{N}^2,\,p<q}$. For any $x\in[0,1]$, consider the b-adic expansion of $x=\sum_{m=1}^\infty x_m b^{-m}$, where $\forall m,\ x_m\in\{0,1,\dots,b-1\}$. Let $\phi_{i,n}(x)$ be the function $x\mapsto\phi_{i,n}(x)=\frac{\#\{m\le n: x_m=i\}}{n}$. Given $(\rho_0,\rho_1,\dots,\rho_{b-1})\in(0,1)^b$ such that $\sum_{i=0}^{b-1}\rho_i=1$, and $\delta>1$, let

$$E_\delta^{\rho_0,\rho_1,\dots,\rho_{b-1}}=\left\{x\in[0,1]:\begin{array}{c}\text{there is an infinite number of integers } p_n,\ q_n \text{ such that } |x-p_n/q_n|\le q_n^{-2\delta}\\ \text{and } \forall i\in\{0,1,\dots,b-1\},\ \lim_{n\to+\infty}\phi_{i,[\log_b q_n^2]}\big(p_n/q_n\big)=\rho_i\end{array}\right\}.$$

Besicovitch and later Eggleston [8] found $\dim E_1^{\rho_0,\rho_1,\dots,\rho_{b-1}}=\sum_{i=0}^{b-1}-\rho_i\log_b\rho_i$. We address the problem of the computation of the Hausdorff dimension of the sets $E_\delta^{\rho_0,\rho_1,\dots,\rho_{b-1}}$, which is the set of points which are well-approximated by rational numbers that have given frequencies of digits. This problem is not covered by the usual ubiquity results. Applying Theorem 4.2 with the multinomial measure μ associated with the weights $(\rho_0,\rho_1,\dots,\rho_{b-1})$ and with the family $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}=\{(p/q,1/q^2)\}_{p,q\in\mathbb{N}\times\mathbb{N}^*}$ yields $\dim E_\delta^{\rho_0,\rho_1,\dots,\rho_{b-1}}\ge\delta^{-1}\sum_{i=0}^{b-1}-\rho_i\log_b\rho_i$. The opposite inequality follows from standard arguments.

References

[1] J. Barral, S. Seuret, From multifractal measures to multifractal wavelet series, Preprint, 2002.

[2] J. Barral, S. Seuret, Combining multifractal additive and multiplicative chaos, Preprint, 2004.

[3] J. Barral, S. Seuret, Functions with multifractal variations, Math. Nachr. (2004) 274–275, 3–18.

[4] J. Barral, S. Seuret, Heterogeneous ubiquitous systems in $\mathbb{R}^d$ and Hausdorff dimension, Preprint, 2004.

[5] J. Barral, S. Seuret, Speed of renewal of level sets for statistically self-similar measures, Preprint, 2004.

[6] G. Brown, G. Michon, J. Peyrière, On the multifractal analysis of measures, J. Stat. Phys. (1992) 66(3-4), 775–790.

[7] M.M. Dodson, M.V. Melián, D. Pestane, S.L. Vélani, Patterson measure and Ubiquity, Ann. Acad. Sci. Fenn. Ser. A I Math. (1995) 20(1), 37–60.

[8] H. Eggleston, The fractional dimension of a set defined by decimal properties, Quart. J. Math. Oxford Ser. (1949) 20, 31–36.

[9] S. Jaffard, Exposants de Hölder en des points donnés et coefficients d'ondelettes, C. R. Acad. Sci. Paris, **308** Série I (1989), 79–81.

[10] S. Jaffard, The multifractal nature of Lévy processes, Prob. Th. and Rel. Fields (1999) 114(2), 207–227.

[11] S. Jaffard, On lacunary wavelet series, Ann. of Appl. Prob. (2000) 10(1), 313–329.

[12] C. Jordan, Sur la série de Fourier, C. R. Acad. Sci. Paris, 92 (1881), 228-230.

[13] J.P. Kahane, Some random series of functions, 2nd Ed. Cambridge Univ. Pres, 1985.

[14] Y. Meyer, Ondelettes et Opérateurs, Hermann, 1990.

[15] L. Olsen, A multifractal formalism, Adv. Math. (1995) 116, 92-195.