Wavelet series built using multifractal measures

Séries d'ondelettes issues de mesures multifractales

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Abstract

Let μ be a positive locally finite Borel measure on $\mathbb{R}$. A natural way to construct multifractal wavelet series $F_\mu(x) = \sum_{j\geq 0, k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(x)$ is to set $|d_{j,k}| = 2^{-j(s_0-1/p_0)}\mu([k2^{-j},(k+1)2^{-j}))^{1/p_0}$, where $s_0, p_0 \geq 0$, $s_0 - 1/p_0 > 0$. Under suitable conditions, the function F_μ inherits the multifractal properties of μ. The transposition of multifractal properties works with most classes of statistically self-similar multifractal measures.

Several perturbations of the wavelet coefficients and their impact on the multifractal nature of F_μ are studied. As an application, the multifractal spectrum of the celebrated $\mathcal{W}$-cascades introduced by Arnéodo *et al* is obtained.

Résumé

Étant donnée une mesure borélienne positive μ définie sur $\mathbb{R}$, il est naturel de lui associer une série d'ondelettes $F_\mu(x) = \sum_{j\geq 0, k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(x)$ en prescrivant ses coefficients d'ondelettes de la façon suivante : On pose $|d_{j,k}| = 2^{-j(s_0-1/p_0)}\mu([k2^{-j},(k+1)2^{-j}))^{1/p_0}$, où $s_0, p_0 \geq 0$, $s_0 - 1/p_0 > 0$. Nous montrons comment les éventuelles propriétés multifractales de la mesure μ peuvent se transmettre à la série d'ondelettes F_μ.

Nous étudions la stabilité de cette construction après perturbation des coefficients d'ondelettes. Ce travail permet de calculer le spectre multifractal des cascades aléatoires d'ondelettes d'*Arnéodo et al.*

1. Introduction

In this note and in [2], we propose a natural construction of functions F_μ based on a measure μ and on a wavelet basis $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$. We focus for the exposition on the one-dimensional case, extensions to higher dimensions are immediate. Let ψ be a wavelet in the Schwartz class, as constructed for instance in [10]. The set of functions $\{\psi_{j,k} = \psi(2^j\cdot -k)\}$, where $(j,k)\in\mathbb{Z}^2$, forms an orthogonal basis of $L^2(\mathbb{R})$. Thus, any function $f\in L^2(\mathbb{R})$ can be written (note that we choose an L^∞ normalization for the wavelet basis and the wavelet coefficients)

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Preprint submitted to Elsevier Science *7th June 2005*

$$f=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}, \quad \text{where } d_{j,k} \text{ is the wavelet coefficient of } f\text{: } d_{j,k}:=2^j\int_{\mathbb{R}} f(t)\psi_{j,k}(t)dt. \tag{1}$$

Given a positive Borel measure μ on $\mathbb{R}$, $s_0, p_0 \geq 0$, $s_0 - 1/p_0 > 0$, the wavelet series F_μ is defined as

$$F_\mu(x)=\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} \pm 2^{-j(s_0-1/p_0)}\mu\big([k2^{-j},(k+1)2^{-j})\big)^{1/p_0}\ \psi_{j,k}(x), \tag{2}$$

For our purpose, we assume without loss of generality that the support of μ is included in $[0,1]$.

The notions of multifractal spectra and multifractal formalisms used in the following statement are defined in Section 2. The multifractal spectrum yields, thanks to their Hausdorff dimension, a geometrical information on the singularity sets of a function or a measure. We establish that the control of the Hausdorff multifractal spectrum d_μ of μ yields a control on the Hausdorff multifractal spectrum d_{F_μ} of F_μ:

Theorem 1.1 *Let μ be a positive Borel measure whose support is included in $[0,1]$. Let $s_0, p_0 \geq 0$, $s_0 - 1/p_0 > 0$, and consider the wavelet series F_μ (2) associated with μ.*

If μ obeys the multifractal formalism for measures at singularity $\alpha \geq 0$, then F_μ obeys the multifractal formalism for functions at $h = s_0 - 1/p_0 + \alpha/p_0$, and one has $d_{F_\mu}(h) = d_\mu(\alpha)$.

Theorem 1.1 is a satisfactory bridge between multifractal analysis of measures and multifractal analysis of functions. The wavelet series model F_μ possesses the remarkable property that its multifractal nature is still controlled after some natural multiplicative perturbations of its wavelet coefficients (see Section 3). This makes it possible to solve the problem of computing the Hausdorff multifractal spectra of the random cascades in wavelet dyadic trees (see [1] and Section 3). Indeed, these cascades, often used as models for instance in fluids mechanics and in traffic analysis, can be considered as perturbated versions of F_μ when μ is a canonical cascade measure ([9]), and their spectrum becomes accessible via this approach.

2. Definitions. Proof of the transposition of the multifractal properties from μ toward F_μ

2.1. *A multifractal formalism for functions*

Let $I \subset \mathbb{R}$ be a non-trivial open interval, a function $f \in L^\infty_{loc}(I)$, and $x_0 \in I$. The function f belongs to $C^h_{x_0}$ if there exists a polynomial P of degree smaller than $[h]$ such that there exists $C > 0$ such that $|f(x) - P(x-x_0)| \leq C|x-x_0|^h$ for all $x \in \mathbb{R}$ close enough to x_0. The *pointwise Hölder exponent* of f at x_0 is then $h_f(x_0) = \sup\{h : f \in C^h_{x_0}\}$. The level sets of the function h_f are denoted $E^f_h = \{x \in I : h_f(x) = h\}$, $h \geq 0$. Then the Hausdorff multifractal spectrum of f is defined as the mapping $d_f : h \mapsto \dim E^f_h$, where $\dim E$ stands for the Hausdorff dimension of a set E.

For any couple $(j,k) \in \mathbb{N}^* \times \mathbb{Z}$, set $I_{j,k} = [k2^{-j},(k+1)2^{-j})$. Then, if $x \in \mathbb{R}$, $\forall j \geq 1$, there exists a unique integer $k_{j,x}$ such that $x \in I_{j,k_{j,x}}$. Let us consider (as [7] does) for every $j \geq 0$ $k \in \mathbb{Z}$ and $x_0 \in \mathbb{R}$ the wavelet leaders of f defined by $L_{j,k} = \sup_{j'\geq j,\, k'2^{-j'}\in I_{j,k}} |d_{j',k'}|$, as well as $L_j(x_0) = \sup_{|k-k_{j,x_0}|\leq 1} L_{j,k}$. The wavelet leaders decay rate provides a pointwise Hölder exponent characterization.

Proposition 2.1 *[7] Let f be a function belonging to $C^\varepsilon(\mathbb{R})$, for some $\varepsilon > 0$, decomposed into (1). Then,* $\forall\ x_0 \in \mathbb{R}, \quad h_f(x_0) = \liminf_{j\to+\infty} \dfrac{\log L_j(x_0)}{\log 2^{-j}}$.

Recall that the Legendre transform of a concave function φ defined on an open interval $I \subset \mathbb{R}$ is the mapping $\varphi^* : h \in \mathbb{R} \mapsto \varphi^*(h) = \inf_{q\in I}(qh - \varphi(q)) \in \mathbb{R}\cup\{-\infty\}$.

The scaling function ξ_f associated with f is defined in [7] by the formula (with the convention $0^p = 0$ $\forall\ p \in \mathbb{R}$) $\xi_f : p \in \mathbb{R} \mapsto \xi_f(p) = \liminf_{j\to+\infty} -j^{-1}\log_2 \sum_{k\in\mathbb{Z}} |L_{j,k}|^p \ \in \mathbb{R}\cup\{-\infty,+\infty\}$. The following result yields an upper bound of d_f in terms of ξ_f^*.

Theorem 2.1 *[7] Let f and ψ as above. The scaling function ξ_f does not depend on ψ, and for any $h \geq 0$, $d_f(h) \leq (\xi_f)^*(h)$.*

Definition 2.2 *The function f is said to obey the multifractal formalism at $h \geq 0$ if $d_f(h) = \xi_f^*(h)$.*

2.2. *A slight modification of the box multifractal formalism for measures*

Definition 2.3 *Let μ be a positive Borel measure on $[0,1]$, and $x_0 \in (0,1)$.*

- The lower and upper Hölder exponent of μ at x_0 are $\underline{\alpha}_\mu(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x_0}})}{\log 2^{-j}}$ and $\overline{\alpha}_\mu(x_0) = \limsup_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x_0}})}{\log 2^{-j}}$. When $\underline{\alpha}_\mu(x_0) = \overline{\alpha}_\mu(x_0)$, their common value is denoted $\alpha_\mu(x_0)$. Then, the left and right lower Hölder exponents of μ at x_0 are defined by $\underline{\alpha}_\mu^-(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x_0}-1})}{\log 2^{-j}}$ and $\underline{\alpha}_\mu^+(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x_0}+1})}{\log 2^{-j}}$.

- For every $\alpha \geq 0$, let us introduce $E_\alpha^\mu = \left\{x \in (0,1) \cap supp(\mu) : \alpha_\mu(x) = \alpha,\ \underline{\alpha}_\mu^-(x) \geq \alpha,\ \underline{\alpha}_\mu^+(x) \geq \alpha\right\}$.

- The mapping $d_\mu :\ \alpha \geq 0 \mapsto \dim(E_\alpha^\mu)$ is called the multifractal spectrum of μ.

Let μ be a positive Borel measure on $[0,1]$. As for functions, a scaling function τ_μ can be associated with μ as the mapping $\tau_\mu : q \in \mathbb{R} \mapsto \liminf_{j\to+\infty} -j^{-1}\log_2 \sum_{0\leq k\leq 2^j} \mu(I_{j,k})^q$. It follows from [4] that $\dim(E_\alpha^\mu) \leq \tau_\mu^*(\alpha)$.

Definition 2.4 *The measure μ is said to obey the multifractal formalism at $\alpha \geq 0$ if $\dim(E_\alpha^\mu) = \tau_\mu^*(\alpha)$.*

The difference between this multifractal formalism and the one of [4] is located in the restrictive definition of the level sets E_α^μ. We choose this definition for E_α^μ to ensure some stability properties after perturbations of wavelet coefficients (see Section 3). Large classes of statistically self-similar measures fulfill this formalism (see [3]), and thus Theorem 1.1 applies to the corresponding wavelet series F_μ.

2.3. *Sketch of the proof of Theorem 1.1*

The proof we propose is based on Proposition 2.1 and Theorem 2.1. An alternative proof can be found in [11]. Let $\alpha \geq 0$ and $h = s_0 - 1/p_0 + \alpha/p_0$. For the wavelet series F_μ, one remarks that for every couple (j,k), $L_{j,k} = d_{j,k}$. Hence, as a consequence of Proposition 2.1, one sees that $E_\alpha^\mu \subset E_h^{F_\mu}$. Since μ is supposed to verify the multifractal formalism at α, one gets that $\tau_\mu^*(\alpha) = d_\mu(\alpha) \leq d_{F_\mu}(h)$. For the upper bound, notice that for the wavelet series F_μ, one has $\xi_{F_\mu}(p) = s_0 - 1/p_0 + \tau_\mu(p/p_0)$. Thus, Theorem 2.1 implies that $d_{F_\mu}(h) \leq \xi_{F_\mu}^*(h) = (s_0 - 1/p_0 + \tau_\mu(p/p_0))^*(h) = d_\mu(\alpha)$.

3. Wavelet coefficients perturbation and application to the $\mathcal{W}$-cascades of Arnéodo *et al*

The perturbation we consider consists in multiplying the wavelet coefficients by the terms of a real sequence $(\pi(j,k))_{j\geq 0, 0\leq k<2^j}$. Consider the wavelet series F_μ (2) and define, whenever it exists,

$$F_\mu^{pert}(x) = \sum_{j\geq 0,\, 0\leq k<2^j} d_{j,k}(F_\mu^{pert})\psi_{j,k}(x) \ \text{ with } \ d_{j,k}(F_\mu^{pert}) = d_{j,k}\cdot\pi(j,k) = \pm 2^{-j(s_0-1/p_0)}\mu(I_{j,k})^{1/p_0}\pi(j,k).$$

3.1. *Principles of the multiplicative perturbations*

Let us consider the following properties for $(\pi(j,k))_{j,k}$:

$$(\mathcal{P}_1) : \limsup_{j\to\infty} j^{-1} \max_{0\leq k\leq 2^j-1} \log |\pi(j,k)| \leq 0. \qquad (\mathcal{P}_2) : \liminf_{j\to\infty} j^{-1} \min_{0\leq k\leq 2^j-1} \log |\pi(j,k)| \geq 0.$$

$$(\mathcal{P}_3) : T = \big\{x : \limsup_{j\to+\infty} j^{-1}\log|\pi(j,k_{j,x})| < 0\big\} = \emptyset. \qquad (\mathcal{P}_4(d)) : 0 \leq d < 1 \text{ and } \dim T \leq d.$$

Proposition 3.1 *[2] Let μ be a positive Borel measure on $[0,1]$.*

If $(\pi(j,k))_{j,k}$ satisfies $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, then the two wavelet series F_μ and F_μ^{pert} have the same exponents at every point x_0. Moreover $\xi_{F_\mu^{pert}} \equiv \xi_{F_\mu}$.

If $(\pi(j,k))_{j,k}$ satisfies $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$, then $\forall\, \alpha \geq 0$, $d_\mu(\alpha) \leq d_{F_\mu^{pert}}(s_0 - 1/p_0 + \alpha/p_0)$ with equality if $\alpha \leq \tau_\mu'(0^+)$ and μ obeys the multifractal formalism at α.

If $(\pi(j,k))_{j,k}$ satisfies $(\mathcal{P}_1)$ and $(\mathcal{P}_4(d))$ for some $d \in [0,1)$, then $\forall \alpha \geq 0$ such that $d_\mu(\alpha) > d$, $d_\mu(\alpha) \leq d_{F_\mu^{pert}}(s_0 - 1/p_0 + \alpha/p_0)$, with equality if $\alpha \leq \tau_\mu'(0^+)$ and μ obeys the multifractal formalism at α.

3.2. *Examples of perturbation of wavelet series*

- Uniform control on $\pi(j,k)$: $(\mathcal{P}_1)$ (resp. $(\mathcal{P}_2)$) holds almost surely if the $\pi(j,k)$ are identically distributed with a random variable with finite moments of every positive (resp. negative) order.

- Gaussian $\pi(j,k)$: Both $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$ hold almost surely if the $\pi(j,k)$ are independent centered Gaussian random variables with variance $\sigma(j,k)$ such that $\lim_{j\to\infty} j^{-1}\max_{0\leq k\leq 2^j-1} |\log \sigma(j,k)| = 0$. Then F_μ^{pert} yields a Gaussian process with controlled Hausdorff multifractal spectrum (thanks to d_μ) in its increasing part. If, moreover, $\pi(j,k) \sim \mathcal{N}(0,1)$ and μ is quasi-Bernoulli ([4]) relatively to the basis 2, then the conclusions of the first assertion of Proposition 3.1 hold.

- Lacunary $\pi(j,k)$: Fix $d \in [0,1)$. $(\mathcal{P}_1)$ and $(\mathcal{P}_4(d))$ hold almost surely if the $\pi(j,k)$ are i.i.d random variables equal to 0 with probability $p = 2^{d-1}$ and 1 with probability $1-p$ (if $p < 1/2$, then $T = [0,1)$, see [5]). These lacunary wavelet series and those studied in [6] are of very different nature.

3.3. *Applications to wavelet cascades on the dyadic tree of [1]*

Let $\mathcal{A} = \{0,1\}$. For every $w \in \mathcal{A}^* = \cup_{j\geq 0}\mathcal{A}^j$ ($\mathcal{A}^0 := \{\emptyset\}$), let I_w be the b-adic subinterval of $[0,1]$, semi-open to the right, naturally encoded by w.

On the one hand, in [1], a random variable $\mathcal{W}$ is chosen as follows: $\mathbb{P}(|\mathcal{W}| > 0) = 1$, $-\infty < \mathbb{E}(\log|\mathcal{W}|) < 0$, and there exists $\eta > 0$ such that for every $h \in [0,\eta]$, $f(h) = \inf_{q\in\mathbb{R}} \big(hq + 1 + \log_2 \mathbb{E}(|\mathcal{W}|^q)\big) < 0$. Then, a sequence $(\mathcal{W}_w)_{w\in\mathcal{A}^*}$ of independent copies of $\mathcal{W}$ is chosen, and a random wavelet series F is defined by its wavelet coefficients as follows: $d_{j,k}(F) = \mathcal{W}_{w_1}\mathcal{W}_{w_1w_2}\dots\mathcal{W}_{w_1w_2\dots w_j}$ if $j \geq 0$, $0 \leq k < 2^j$ and $I_w = I_{j,k}$.

On the other hand, for every $n \geq 1$, let us consider the sequence of weights $\{W_{w_1\dots w_j}\}_{w\in\mathcal{A}^*} = \big\{|\mathcal{W}_{w_1\dots w_j}|/2\mathbb{E}(|\mathcal{W}|)\big\}_{w\in\mathcal{A}^*}$, and the random measure μ_j on $\mathbb{R}$ with density with respect to the Lebesgue measure given on every interval I_w, $w = w_1w_2\dots w_j$, by $2^j\, W_{w_1}W_{w_1w_2}\dots W_{w_1w_2\dots w_j}$ and such that $\mu_j = 0$ outside $[0,1]$. With probability one, the sequence μ_j converges vaguely to a dyadic random multiplicative cascade measure μ when j goes to infinity. The assumptions on $\mathcal{W}$ imply $\mathbb{E}(W\log W) < 0$. Hence, with probability one [8], $\text{supp}(\mu) = [0,1]$. Let us then introduce the series F_μ with parameters $s_0 = 2$ and $p_0 = 1$ and its perturbation F_μ^{pert} associated with the sequence $\pi(j,k) = (\mu_j(I_{j,k})/\mu(I_{j,k}))^{1/p_0}$. One has

$$|d_{j,k}(F)| = 2^{(s_0-1/p_0)j}\big(2\mathbb{E}(|\mathcal{W}|)\big)^j 2^{-(s_0-1/p_0)j} W_{w_1}\dots W_{w_1\dots w_j} = 2^{(2+\log_2\mathbb{E}(|\mathcal{W}|))j}|d_{j,k}(F_\mu^{pert})|.$$

This enables to establish the following result as a consequence of the first assertion of Proposition 3.1.

Theorem 3.1 *[2] Suppose that $W \leq 1$, $\mathbb{P}(W = 1) < 1/2$ and all the moments of $\mathcal{W}$ are finite. Let $[h_{\min}, h_{\max}] = \{h : f(h) \geq 0\}$. With probability 1, one has $d_F(h) = f(h)$ for every $h \in (h_{\min}, h_{\max})$.*

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