A class of multifractal semi-stable processes including Lévy subordinators and Mandelbrot multiplicative cascades

Une classe de processus multifractals semi-stables contenant subordinateurs de Lévy et cascades multiplicatives de Mandelbrot

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Abstract

We exhibit a class of statistically self-similar processes naturally associated with the so-called fixed points of the smoothing transformation [13,7,8]. This class includes stable subordinators and Mandelbrot multiplicative cascades. Both these processes are special exemples of Lévy processes in multifractal time, which are studied in [5]. We describe their multifractal nature.

Résumé

Nous présentons une classe de processus auto-similaires en loi naturellement associés aux généralisations des lois semi-stables considérées dans [13,7,8]. Cette classe contient en particulier les subordinateurs stables de Lévy ainsi que les cascades multiplicatives de Mandelbrot ; ses éléments sont des cas particuliers des processus de Lévy en temps multifractal étudiés dans [5]. Nous étudions leur nature multifractale.

1. Introduction

The best known fractal or multifractal stochastic processes are certainly Fractional Brownian Motions, Lévy processes, and Mandelbrot multiplicative cascades. It is natural to perform a multifractal change of time in such a stochastic process $(X_t)_{t\geq 0}$. More precisely, given an atomless multifractal positive Radon measure μ on $\mathbb{R}_+$ supported by an interval of the form $[0,T]$ ($T \in (0,\infty)$), then the process $X \circ \mu([0,t])$ is considered. The simplest situation lies in taking X equal to a monofractal process, like a FBM (see [14] for instance). In this case, the multifractal nature of $X \circ \mu$ follows almost straightforward from the one of μ. In the situation where X also has multifractal sample paths, the multifractal time change creates

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more interesting structures, both from the modeling and mathematical viewpoints (see for instance [16] for preliminary results on this topic, especially concerning large deviation spectra). To our knowledge, the study of the sample paths multifractal properties has not been achieved in a non-trivial case yet.

In this note, we focus on the case where X is a Lévy process and μ is a Mandelbrot measure on $[0,1]$. This choice illustrates the more general result obtained in [5]. Furthermore, it yields a link between the statistical self-similarity properties of stable Lévy processes and Mandelbrot measures.

2. Processes associated with generalized semi-stable laws

Let b be an integer ≥ 2 and $W=(W_0,\dots,W_{b-1})$ a positive random vector. Then consider in the space of Laplace transforms of probability distributions ϕ on $\mathbb{R}_+$ the equation

$$\phi(u)=\mathbb{E}\,\Big(\prod_{i=0}^{b-1}\phi(uW_i)\Big),\quad \forall\ u\geq 0. \tag{1}$$

This equation, solved in [7,8], comes from the modeling of fully developed turbulence [13] and of interacting particles systems. With (1) is naturally associated the structure function

$$\varphi_W: q\in\mathbb{R}\mapsto -\log_b \mathbb{E}\Big(\sum_{i=0}^{b-1}W_i^q\Big)\in\mathbb{R}\cup\{-\infty\}. \tag{2}$$

Under the assumption that $\varphi_W(p)>-\infty$ for some $p>1$, it is proved in [7] that (1) has non-trivial solutions if and only if there exists $\beta\in(0,1]$ such that $\varphi_W(\beta)=0$ and $\varphi_W'(\beta)\geq 0$. As a consequence of the concavity of the mapping φ_W, such a β is unique and $\beta=\inf\{\beta'\in[0,1]:\varphi_W(\beta')=0\}$.

Two special solutions of Equation (1) are:
- when $\beta=1$ and $\varphi_W(1^-)>0$, the probability distribution of $\|\mu_W\|$, where μ_W is an independent multiplicative cascade on $[0,1]$ generated by W (see [13,10]),
- when $\beta\in(0,1)$, the stable laws with Laplace exponent $-t^\beta$, and in this case W_i is constant and equal to $b^{-1/\beta}$ (see [11]).

When $\beta\in(0,1)$, $\varphi_W(\beta)=0$ and $\varphi_W(\beta^-)>0$, a non-trivial solution of (1) is $Z_\beta\|\mu_{W_\beta}\|^{1/\beta}$ [8], where Z_β is a positive stable law of index β and μ_{W_β} a Mandelbrot measure associated with $W_\beta=(W_0^\beta,\dots,W_{b-1}^\beta)$ and independent of X_β. Equivalently, if $(Z_t^{(\beta)})_{t\geq 0}$ is a stable Lévy subordinator of index β, which is independent of μ_{W_β}, then the law of $Z^{(\beta)}_{\|\mu_{W_\beta}\|}$ solves (1) (see [7]).

The statistical self-similarity property expected to be satisfied by a process naturally associated with (1) will appear after the recall of the construction of μ_W. Let $\mathcal{A}$ be the alphabet $\{0,\dots,b-1\}$ and $\mathcal{A}^*=\bigcup_{n\geq 0}\mathcal{A}^n$ ($\mathcal{A}^0$ contains the empty word $\emptyset$). Consider a sequence $\big((W_0(w),\dots,W_{b-1}(w))\big)_{w\in\mathcal{A}^*}$ of independent copies of W. For $n\geq 1$, let $\mu_{W,n}$ be the measure defined on $[0,1]$ by uniformly distributing on every interval of the form $\big[\sum_{k=1}^n w_k b^{-k}, b^{-n}+\sum_{k=1}^n w_k b^{-k}\big]$ (where $w_1w_2\cdots w_n\in\mathcal{A}^n$) the mass $W_{w_1}(\emptyset)\cdot W_{w_2}(w_1)\cdots W_{w_n}(w_1w_2\cdots w_{n-1})$. Then, with probability one, $(\mu_{W,n})_{n\geq 1}$ converges weakly on $[0,1]$, as $n\to\infty$, to a measure μ_W called the independent multiplicative cascade measure associated with W. The self-similarity property of the process $Z_{W,t}=\mu([0,t])$ is then:

$$\forall\ n\geq 1,\quad \Big(Z_{W,(k+1)b^{-n}}-Z_{W,kb^{-n}}\Big)_{0\leq k<b^{-n}}\overset{d}{\equiv}\Big(Z_1(w)\prod_{k=1}^n W_{w_k}(w_1\cdots w_{k-1})\Big)_{w\in\mathcal{A}^n}, \tag{3}$$

where, on the right hand side, the set $\mathcal{A}^n$ is described in lexicographical order, the random vectors $(W_0(w),\dots,W_{b-1}(w))$'s are i.i.d. with W, and the random values $Z_1(w)$'s are i.i.d. with $Z_{W,1}$ and are independent of the $(W_0(w),\dots,W_{b-1}(w))$'s. Property (3) expresses the n^{th} iteration of (1).

Another fundamental process obeying (3) is the restriction to $[0,1]$ of any stable Lévy subordinator $Z^{(\beta)}$ of index $\beta \in (0,1]$ (by convention $Z_t^{(1)} = t$). In this case, the components of W satisfy $W_i \equiv b^{-1/\beta}$.

Finally, if there exists $\beta \in (0,1]$ such that $\varphi_W(\beta) = 0$ and $\varphi_W(\beta^-) > 0$, the general form of a statistically self-similar process in the sense of (3) is naturally obtained by considering the process

$$Z_t = Z^{(\beta)}_{\mu_{W_\beta}([0,t])} \quad (t \in [0,1]), \tag{4}$$

where μ_{W_β} is an independent multiplicative cascade measure associated with W_β, independently of $Z^{(\beta)}$.

3. Multifractal analysis of the stable Lévy subordinator in multifractal time

If $Z : [0,1] \mapsto \mathbb{R}_+$ is a non-decreasing function, we define its pointwise Hölder exponent at point t as the quantity $h_Z(t) = \liminf_{r\to 0, r\neq 0} \dfrac{\log|Z(t+r) - Z(t)|}{\log(r)}$. Then, the level sets of $h_Z(\cdot)$ are denoted $E_Z(h)$, $h \geq 0$. If $\varphi : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, its Legendre transform is $\varphi^* : h \mapsto \inf_{q\in\mathbb{R}} hq - \varphi(q)$. The Hausdorff dimension of a set E is denoted $\dim E$.

The general result obtained in [5] (for a general Lévy process in multifractal time) yields the following result unifying those obtained in [9] and [1] respectively for the multifractal natures of stable subordinators and Mandelbrot cascades. In order to avoid technicalities, let us assume that $\varphi_W > -\infty$ on $\mathbb{R}$.

Theorem 3.1 *Suppose that there exists $\beta \in (0,1]$ such that $\varphi_W(\beta) = 0$ and $\varphi'_W(\beta) > 0$. Let $(Z_t)_{t\in[0,1]}$ be the proces defined in (4). Let $\tau = \mathbf{1}_{\{(-\infty,\beta]\}}\varphi_W$ if $\beta < 1$ and $\tau = \varphi_W$ if $\beta = 1$. With probability one, $\dim E_Z(h) = \tau^*(h)$ for all h such that $\tau^*(h) \geq 0$, and $E_Z(h) = \emptyset$ for all h such that $\tau^*(h) < 0$.*

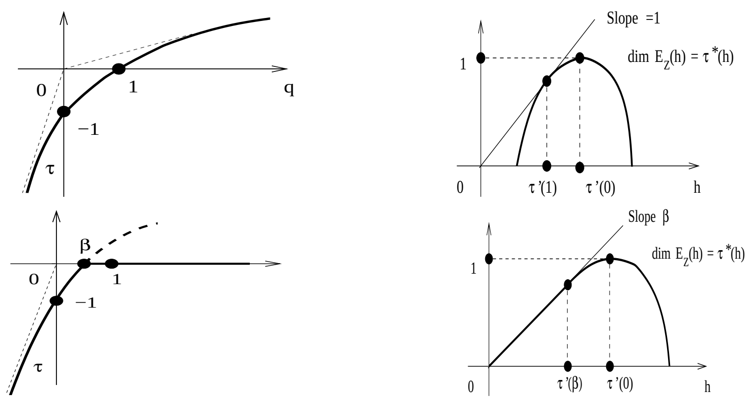

Fig. 1: Upper figures: Case $\beta = 1$: $Z_t = \mu_W([0,t])$, and $\tau(q) = \varphi_W(q)$ when $\varphi_W^*(\varphi'_W(q)) \geq 0$.
Lower figures: General case $\beta \in (0,1)$. $\tau(q) = \varphi(q)$ when $q \leq \beta$ and $\varphi_W^*(\varphi'_W(q)) \geq 0$, and otherwise $\tau(q) = 0$ on $[\beta, \infty)$.

Comments on Theorem 3.1

The proof when $\beta < 1$: (the case $\beta = 1$ follows from [1]) One uses tools from [9], [2], [3] and [4].

The characterization of the level sets of $h_Z(\cdot)$ uses results for the increments of $Z^{(\beta)}$ in [9] and adapts the approach used in [2].

Let S be the Poisson point process such that $Z^{(\beta)\prime} = \sum_{(s,\lambda)\in S} \lambda\, \delta_s$. Let $F_\beta : t \mapsto \mu_{W_\beta}([0,t])$. Also denote by $\{(x_n, \lambda_n)\}$ the family $\Big\{\Big(F_\beta^{-1}(s), 2.\big|F_\beta^{-1}\big([s-\lambda^\beta, s+\lambda^\beta]\big)\big|\Big)\Big\}_{(s,\lambda)\in S}$.

The linear part in the spectrum $d_Z : h \mapsto \dim E_Z(h)$ reflects conditioned ubiquity properties associated with the jump points x_n of Z, relatively to the family λ_n and the measure μ_{W_β} (see [3] for the notion of heterogeneous ubiquity). Roughly speaking, for every $h \in (0, \tau'(\beta))$, up to a "small" set, the set $E_Z(h)$ consists of those points t for which there exists an increasing sequence n_j such that $t \in [x_{n_j} - \lambda_{n_j}^{\tau'(\beta)/h}, x_{n_j} + \lambda_{n_j}^{\tau'(\beta)/h}]$ for all j and $\lim_{j\to\infty} \frac{\log \mu_{W_\beta}([x_{n_j}-\lambda_{n_j}, x_{n_j}+\lambda_{n_j}])}{\log \lambda_{n_j}} = \beta\tau'(\beta)$. The Hausdorff dimension of such sets is estimated thanks to the main result of [3].

The strictly concave part of d_Z reflects the multifractal structure of μ_{W_β}. Indeed, if $h \geq \tau'(\beta)$ one proves that $E_Z(h)$ is equal to $E_{F_\beta}(\beta h)$ again up to a "small" set.

The validity of the multifractal formalism: The derivative ν of Z obeys the standard multifractal formalisms for measures associated with the level sets $E_\nu(h) = \left\{t : \liminf_{r\to 0^+} \frac{\log \mu([t-r,t+r])}{\log(r)} = h\right\}$. In particular, the scaling functions associated with ν (see [6,15,2]) all equal τ on the interval where $\tau^*(\tau') > 0$.

Extension of Theorem 3.1: The multifractal analysis of a Lévy processes X with drift and Brownian component is performed in [9] under some minor restriction on the Lévy measure. Under the same assumptions as in [9], [5] provides general conditions on a positive continuous measure μ on $[0,1]$ under which the multifractal analysis of the process X in multifractal time μ can be performed. The result applies to large classes of statistically self-similar measures.

Equation (1) can also be expressed in terms of characteristic function instead of Laplace transform; it has been partially studied in [12]. If $\varphi_W(\beta) = 0$ and $\varphi'_W(\beta) > 0$ for some $\beta \in (1,2)$, then the stochastic process naturally associated with (1) is a symetric β-stable process in multifractal time μ_{W_β}.

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