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Threshold and Hausdorff spectrum of discontinuous measures

Abstract

Let χ be a finite Borel measure on $[0,1]^d$. Consider the L^q-spectrum of χ: $\tau_\chi(q) = \liminf_{n\to\infty} -n^{-1}\log_b \sum_{Q\in\mathcal{G}_n,\ \chi(Q)\neq 0} \chi(Q)^q$, where $\mathcal{G}_n$ is the set of b-adic cubes of generation n. Let $q_\tau = \inf\{q : \ \tau_\chi(q) = 0\}$ and $H_\tau = \tau'_\chi(q_\tau^-)$. When χ is a mono-dimensional continuous measure of information dimension D, $(q_\tau, H_\tau) = (1, D)$. When χ is purely discontinuous, its information dimension is $D = 0$, but the non-trivial pair (q_τ, H_τ) may contain relevant information on the distribution of χ. The connection between (q_τ, H_τ) and the large deviations spectrum of χ is studied in a companion paper. This paper shows that when a discontinuous measure χ possesses self-similarity properties, the pair (q_τ, H_τ) may store the main multifractal properties of χ, in particular the Hausdorff spectrum. This is observed thanks to a threshold performed on χ.

1 Introduction and statements of results

In a companion paper [5], we introduced new information parameters associated with any positive Borel measure χ on $[0,1]^d$. Let us recall their definitions. Let $b \geq 2$ be an integer and let $\mathcal{G}_n$ be the partition of $[0,1]^d$ into b-adic boxes $\prod_{i=1}^d [b^{-n}k_i, b^{-n}(k_i+1))$ with $(k_1, ..., k_d) \in \{0, 1, ..., b^n-1\}^d$. The L^q-spectrum of χ is the mapping defined for any $q \in \mathbb{R}$ by

$$\tau_\chi(q) = \liminf_{n\to\infty} -\frac{1}{n}\log_b s_n(q) \quad \text{where} \quad s_n(q) = \sum_{Q\in\mathcal{G}_n,\ \chi(Q)\neq 0} \chi(Q)^q. \qquad (1)$$

Key Words: Large deviations, Fractals

Mathematical Reviews subject classification: 60F10, 28A80

It is easy to see that the restriction to $\mathbb{R}_+$ of τ_χ does not depend on b. Two parameters are naturally associated with the measure χ:

$$q_\tau(\chi) = \inf\{q \in \mathbb{R} : \tau_\chi(q) = 0\} \quad \text{and} \quad H_\tau(\chi) = \tau'_\chi(q_\tau(\chi)^-). \tag{2}$$

The motivation of the introduction of these parameters was the following: For purely discontinuous measures, the classical measure dimensions vanish [25, 11, 16, 18, 7]. Nevertheless, these measures may have very interesting multifractal spectra [15, 1, 9, 14, 6, 24, 2, 4], and there is a need for other relevant parameters. The study of the pair $(q_\tau(\chi), H_\tau(\chi))$ and their relationships with the so-called large deviations spectrum are achieved in [5] and recalled below in Section 2. As we wished, these parameters are very pertinent for purely discontinuous measures χ, i.e. measures constituted only by positive Dirac masses of the form

$$\chi = \sum_{k\geq 1} m_k\, \delta_{x_k}, \tag{3}$$

with $\widetilde{m} = (m_k)_{k\geq 1} \in (\mathbb{R}^+)^{\mathbb{N}^*}$, $\sum_k m_k < \infty$ and $\widetilde{x} = (x_k)_{k\geq 1} \in ([0,1]^d)^{\mathbb{N}^*}$ such that the x_k's are pairwise distinct.

This paper aims at showing that for certain classes of purely discontinuous measures denoted ν in the following, these parameters not only store information about the large deviations spectrum of ν, but also they store essential information about the multifractal Hausdorff spectrum of ν. To achieve this, we apply a threshold procedure to such measures ν by keeping only the Dirac masses naturally associated with the information parameters introduced in [5]. We prove that the obtained measure, denoted $\nu^{\widetilde{\varepsilon}}$, has the same multifractal behavior as ν itself. Since the threshold procedure puts to zero the largest part of the Dirac masses of ν, it is thus very interesting to understand why the multifractal properties of ν are essentially the same as those of $\nu^{\widetilde{\varepsilon}}$.

From now on we shall work in the one-dimensional context. Extensions to higher dimensions are immediate, though more technical. Let us recall the definition of the Hausdorff spectrum of any measure χ. First, for $x \in \mathrm{Supp}(\chi)$ (the support of χ), the pointwise Hölder exponent of χ at x is defined by

$$h_\chi(x) = \liminf_{r\to 0^+} \frac{\log \chi(B(x,r))}{\log r}, \tag{4}$$

Then, for every $h \geq 0$ one defines the level sets of the pointwise Hölder exponent of χ and the multifractal Hausdorff spectrum of χ as

$$E_h^\chi = \{x \in \mathrm{Supp}(\chi) : h_\chi(x) = h\} \quad \text{and} \quad d_\chi : h \geq 0 \mapsto \dim E_h^\chi \tag{5}$$

where dim stands for the Hausdorff dimension. This spectrum is used to describe the geometric properties of measures at small scales. Recall that if

g is a function from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$, its Legendre transform is the mapping $g^* : h \mapsto \inf_{q \in \mathbb{R}}(hq - g(q)) \in \mathbb{R} \cup \{-\infty\}$. For every $h \geq 0$, one always has $d_\chi(h) \leq \tau_\chi^*(h)$ [8], and the multifractal formalism holds at h if $d_\chi(h) = \tau_\chi^*(h)$.

The measures ν we consider are introduced in [2]. Their construction's scheme is as follows: Let μ be a Borel probability measure on $[0,1]$ and let

$$\nu = \sum_{j \geq 1} \sum_{0 \leq k \leq b^j - 1:\, k \not\equiv 0 \mod b} \nu_{j,k} \delta_{kb^{-j}} \text{ , with } \nu_{j,k} = \frac{1}{j^2} \mu([kb^{-j}, (k+1)b^{-j})). \quad (6)$$

The jump points are located at the b-adic points, and an heterogeneity in the Dirac masses distribution is created by the measure μ. It turns out that this class of measures has a fruitful structure [4, 2].

Theorem 1.1 *Let μ be a Gibbs measure as defined in Section 3.2. The measure ν (6) obeys the multifractal formalism at every $h > 0$ such that $\tau_\nu^*(h) > 0$, as well as at 0. More precisely, one has $H_\tau(\nu) = H_\tau(\mu)$ and*

$$\tau_\nu(q) = \begin{cases} \tau_\mu(q) & \textit{if } \tau_\nu(q) < 0, \\ 0 & \textit{otherwise,} \end{cases} \quad \textit{and} \quad d_\nu(h) = \begin{cases} h & \textit{if } 0 \leq h \leq H_\tau(\nu), \\ d_\mu(h) & \textit{otherwise.} \end{cases}$$

Let us explicit the thresholding procedure applied on ν. Let $\widetilde{\varepsilon} = (\varepsilon_j)_{j \geq 0}$ be a non-increasing positive sequence converging to 0. Consider ν (6) and let

$$\nu^{\widetilde{\varepsilon}} = \sum_{j \geq 1} \sum_{0 \leq k \leq b^j - 1:\, k \not\equiv 0 \mod b} t_{j,k}\, \nu_{j,k}\, \delta_{kb^{-j}} \quad (7)$$

$$\text{with} \qquad \forall\, j \geq 1, \ \ \forall\, k, \ \ t_{j,k} = \mathbf{1}_{[H_\tau(\nu) - \varepsilon_j, H_\tau(\nu) + \varepsilon_j]}\Big(\frac{\log \nu_{j,k}}{\log b^{-j}} \Big). \quad (8)$$

Heuristically, the measure $\nu^{\widetilde{\varepsilon}}$ contains only the Dirac masses $\nu_{j,k}\delta_{kb^{-j}}$ such that $\nu_{j,k} \sim b^{-jH_\tau(\nu)}$. A more complete explanation of such a formula comes from the companion paper [5], and is detailed in Section 2.

We obtain the following remarkable result which illustrates the amount of information potentially stored in the pair $(q_\tau(\nu), H_\tau(\nu))$.

Theorem 1.2 *Let μ be a Gibbs measure as in Section 3.2. Consider $\nu^{\widetilde{\varepsilon}}$ (7).*

There exists a non-increasing positive sequence $\widetilde{\varepsilon}$ converging to 0 such that $d_{\nu^{\widetilde{\varepsilon}}}(h) = d_\nu(h)$ for every $h > 0$ such that $\tau_\nu^*(h) > 0$. Moreover $\nu^{\widetilde{\varepsilon}}$ obeys the multifractal formalism at every $h > 0$ such that $\tau_\nu^*(h) > 0$, and at 0, and the L^q-spectra of ν and $\nu^{\widetilde{\varepsilon}}$ coincide ($\tau_\nu = \tau_{\nu^{\widetilde{\varepsilon}}}$).

Actually, a slightly more general result will be proved (Theorem 2.2).

Theorem 1.2 shows the role played by the information parameters $q_\tau(\nu)$ and $H_\tau(\nu)$ for discontinuous measures having a nice structure close to statistical self-similarity. There is no doubt on the fact that Theorem 1.2 can be extended to other nice families of measures, such as the inverse of Gibbs measures on cookie-cutters [21] and the self-similar sums of Dirac masses introduced in [24]. These measures will be studied in a forthcoming paper. However it seems difficult to get similar results for measures without any structure.

It will be justified in next section that at each scale j, approximately $b^{jH_\tau(\nu)}$ Dirac masses among b^j are kept after threshold. Since one generally has $H_\tau(\nu) < 1$ if μ is non trivial, the threshold we realize is very severe. The situation $H_\tau(\nu) = 1$ corresponds for instance to the choice $\mu = \ell$ (the Lebesgue measure). It is the typical example of an homogeneous sum of Dirac masses ν_ℓ, for which there exists a positive sequence $\widetilde{\varepsilon}$ going to 0 at ∞ such that $\nu_\ell^{\widetilde{\varepsilon}} = \nu_\ell$.

2 Detailed exposition of the result

2.1 More on the information parameters

The connection between $(q_\tau(\chi), H_\tau(\chi))$ and the more usual Hausdorff, packing or entropy dimensions of χ is the following: When $q_\tau(\chi) = 1$ and $H_\tau(\chi) = \tau'_\chi(1)$ exists, $H_\tau(\chi)$ defines without ambiguity the dimension of the measure χ [25, 16, 18, 11, 7].

The pair $(q_\tau(\chi), H_\tau(\chi))$ is also connected to the *large deviations spectrum* f_χ of χ. This spectrum describes the statistical distribution of χ at small scales in the following sense. This spectrum f_χ of χ is defined as

$$h \geq 0 \mapsto f_\chi(h) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log_b \# \mathcal{S}_n^\chi(h, \varepsilon),$$

where for $\varepsilon > 0$, $h \geq 0$ and $n \in \mathbb{N}$,

$$\mathcal{S}_n^\chi(h, \varepsilon) = \left\{ Q \in \mathcal{G}_n : b^{-n(h+\varepsilon)} \leq \chi(Q) \leq b^{-n(h-\varepsilon)} \right\} \tag{9}$$

Very classical considerations [12, 8, 22, 17, 5] show that $\forall\, h \geq 0$, one always has $d_\chi(h) \leq f_\chi(h) \leq \tau_\chi^*(h)$. Hence when the multifractal formalism holds at h, one also has $d_\chi(h) = f_\chi(h)$.

As a consequence of the fact that $f_\chi(\alpha) = \tau_\chi^*(\alpha)$ for all α of the form $\tau'(q^-)$ (see [23]), one always has $H_\tau(\chi) = \max\{h \geq 0 : f_\chi(h) = q_\tau(\chi) h\}$ if $q_\tau(\chi) > 0$. For a discontinuous measure $\chi = \sum_{k \geq 1} m_k\, \delta_{x_k}$ on $[0,1]^d$, the relationships between the large deviations spectrum f_χ restricted to $[0, H_\tau(\chi)]$ and the pair $(q_\tau(\chi), H_\tau(\chi))$ were investigated in [5]. Under a weak assumption on the distribution of the masses, it is shown that there exists a real number $H_g(\chi) \in (0, H_\tau(\chi)]$ depending on $(\widetilde{m}, \widetilde{x})$ such that $f_\chi(h) = q_\tau(\chi) h$ over $[0, H_g(\chi)]$. In addition, $H_g(\chi)$ is equal to $H_\tau(\chi)$ if $q_\tau(\chi) \in (0,1)$, but it may differ from $H_\tau(\chi)$ if $q_\tau(\chi) = 1$. We do not go into much details on $H_g(\chi)$ (this was the purpose of [5]). This linear increasing part in the large deviations spectrum is conform to the observations made on special classes of homogeneous and heterogeneous sums of Dirac masses studied in the last fifteen years [1, 15, 9, 23, 2, 4]. Moreover, the elements of these classes of measures, to which belong the measures (6) and (11), verify that $H_g(\chi) = H_\tau(\chi)$ even when $q_\tau(\chi) = 1$. This is always assumed hereafter.

The starting point of the threshold operation performed in this article is provided by two important remarks made in [5] (Proposition 3.3, [5]):

• For every $n \geq 1$, most of the cubes in $\mathcal{S}_n^\chi(H_\tau(\chi), \varepsilon)$ contain a point x_k such that $b^{-n(H_\tau(\chi)+\varepsilon)} \leq m_k \leq b^{-n(H_\tau(\chi)-\varepsilon)}$ (recall that $\mathcal{S}_n^\chi(H_\tau(\chi), \varepsilon)$ is the set of b-adic cubes Q of generation n such that $b^{-n(H_\tau(\chi)+\varepsilon)} \leq \chi(Q) \leq b^{-n(H_\tau(\chi)-\varepsilon)}$). Hence, the χ-mass of these cubes is approximately due to the presence of a single Dirac mass.

• The b-adic cubes which contain such a point x_k are responsible for the linear shape of f_χ on $[0, H_\tau(\chi)]$.

Consequently, a certain amount of information is contained in the set of pairs (x_k, m_k) defined for any $\varepsilon > 0$ by

$$\mathcal{P}(H_\tau(\chi), \varepsilon) = \left\{ (m_k, x_k) : \begin{cases} \exists\, n \geq 1,\ \exists\, Q \in \mathcal{G}_n,\ Q \in \mathcal{S}_n^\chi(H_\tau(\chi), \varepsilon),\ x_k \in Q \\ \text{and } \ b^{-n(H_\tau(\chi)+\varepsilon)} \leq m_k \leq b^{-n(H_\tau(\chi)-\varepsilon)} \end{cases} \right\}.$$

A natural way to study this set of pairs (x_k, m_k) is to consider the measure

$$\chi^\varepsilon = \sum_{k\geq 1} \mathbf{1}_{\mathcal{P}(H_\tau(\chi),\varepsilon)}((m_k, x_k))\, m_k\, \delta_{x_k}. \tag{10}$$

This measure shall be viewed as a thresholded version of the initial measure χ (3). It can be deduced from [5] that the measure has the same large deviations spectrum as χ over $[0, H_\tau(\chi)]$.

This raises the following question: Do the measures χ^ε still contain enough Dirac masses to have the same Hausdorff spectrum as χ? This is the question investigated hereafter.

2.2 The measures $\nu_{\gamma,\sigma}$ and a more general result

Let μ be a Borel probability measure on $[0,1]$, $\gamma \geq 0$ and $\sigma \geq 1$, and

$$\nu_{\gamma,\sigma} = \sum_{j\geq 1} \sum_{\substack{0 \leq k \leq b^j - 1 \\ k \not\equiv 0 \mod b}} \nu_{j,k}\ \delta_{kb^{-j}}, \text{ with } \nu_{j,k} = \frac{b^{-j\gamma}}{j^2} \mu([kb^{-j}, (k+1)b^{-j}))^\sigma. \tag{11}$$

The condition $k \not\equiv 0 \mod b$ in the definition of $\nu_{\gamma,\sigma}$ (6) is not required in [2]. This is unessential, since the two measures (with or without the condition) are equivalent, and thus have the same multifractal nature.

Theorem 2.1 *[2] Let μ be a Gibbs measure as in Section 3.2, $\gamma \geq 0$ and $\sigma \geq 1$. The measure $\nu_{\gamma,\sigma}$ given by formula (11) obeys the multifractal formalism at every $h > 0$ such that $\tau^*_{\nu_{\gamma,\sigma}}(h) > 0$, as well as at 0. Moreover, one has*

$$\tau_{\nu_{\gamma,\sigma}}(q) = \begin{cases} \gamma q + \tau_\mu(\sigma q) & \text{if } \tau_{\nu_{\gamma,\sigma}}(q) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

These measures $\nu_{\gamma,\sigma}$ are generalized versions of the measures ν considered by Theorem 1.2 (indeed, $\nu_{0,1}$ is the measure ν of the introduction). Main Theorem 2.2 deals with $\nu_{\gamma,\sigma}$, and is thus more general and implies Theorem 1.2.

For $j \geq 1$ and $k \in [0, \ldots, b^j - 1]$, one sets $I_{j,k} = [kb^{-j}, (k+1)b^{-j})$. The measure $\nu_{\gamma,\sigma}$ is of the form (3) if one takes for the points x_k the b-adic numbers lb^{-j} with $l \not\equiv 0 \mod b$ and for the corresponding m_k the mass $m_{j,l} = \nu_{j,l}$. It is then easily seen that there exists a universal constant K such that $m_{j,l} \leq \nu_{\gamma,\sigma}(I_{j,l}) \leq K m_{j,l}$. Consequently, in this case, requiring that $I_{j,l} \in \mathcal{S}_j^{\nu_{\gamma,\sigma}}(H_\tau, \varepsilon)$ is equivalent to requiring that $b^{-j(H_\tau+\varepsilon)} \leq m_{j,l} \leq b^{-j(H_\tau-\varepsilon)}$.

We apply the threshold procedure (7-8) to the class of measures $\nu_{\gamma,\sigma}$ defined by (11). This procedure is finer than (10). Recall that, if $\widetilde{\varepsilon} = (\varepsilon_j)_{j\geq 0}$ be a positive sequence converging to 0, then we set

$$\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma} = \sum_{j\geq 1} \sum_{0\leq k\leq b^j-1:\, k\not\equiv 0 \mod b} t_{j,k}\, \nu_{j,k}\, \delta_{kb^{-j}} \tag{12}$$

with $t_{j,k} = \mathbf{1}_{[H_\tau(\nu_{\gamma,\sigma})-\varepsilon_j, H_\tau(\nu_{\gamma,\sigma})+\varepsilon_j]}\left(\frac{\log \nu_{j,k}}{\log b^{-j}}\right)$ defined as in (8) ($\nu_{\gamma,\sigma}$ is used instead of simply ν).

Theorem 2.2 *Let μ be a Gibbs measure as in Section 3.2, $\gamma \geq 0$ and $\sigma \geq 1$. Consider $\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}$ defined by (12).*

There exists a non-increasing positive sequence $\widetilde{\varepsilon}$ converging to 0 such that

1. *$(q_\tau(\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}), H_\tau(\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma})) = (q_\tau(\nu_{\gamma,\sigma}), H_\tau(\nu_{\gamma,\sigma}))$.*

2. *For every $0 \leq h \leq H_\tau(\nu_{\gamma,\sigma})$, $d_{\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}}(h) = d_{\nu_{\gamma,\sigma}}(h)$ and $\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}$ obeys the multifractal formalism at h.*

3. *If $\gamma = 0$ and $\sigma = 1$, then Theorem 1.2 applies.*

When $q_\tau(\nu) < 1$, the Hausdorff spectrum of $\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}$ may differ from $d_{\nu_{\gamma,\sigma}}$ on $(H_\tau(\nu_{\gamma,\sigma}), \infty)$. To see this heuristically, notice that the total mass conserved at each scale in $\nu^{\widetilde{\varepsilon}}_{\gamma,\sigma}$ is negligible with respect to the total mass of $\nu_{\gamma,\sigma}$, since approximatively there are at most $2^{jq_\tau H_\tau(\nu)}$ terms weighted by $2^{-jH_\tau(\nu)}$. Hence the amount of lost "information" is large. Nevertheless, it is remarkable that the Dirac masses we keep are enough to recover the spectrum on $[0, H_\tau(\nu_{\gamma,\sigma})]$.

Sections 3 gives some background necessary to establish Theorem 2.2, while Sections 4 is devoted to the proof of Theorem 2.2.

3 Scaling properties of Gibbs measures

If $x \in (0,1)$, $I_j(x)$ is the unique b-adic interval of scale $j \geq 1$, semi-open to the right, containing x, and for every $\epsilon \in \{-1,0,1\}$, $I_j^{(\epsilon)}(x) = I_j(x) + \epsilon b^{-j}$. In the following, $|B|$ always denotes the diameter of the set B. Eventually, for the rest of the paper, the convention $\log(0) = -\infty$ is adopted.

3.1 Some dimension and large deviations bounds

Definition 3.1 *Let μ be a positive Borel measure on $[0,1]$. For $x \in (0,1)$, recall the definition (4) of the Hölder exponent of μ at x and of the level sets E_α^μ defined for every $\alpha \geq 0$ by $E_\alpha^\mu = \{x : h_\mu(x) = \alpha\}$.*
For $\tilde{\xi} = (\xi_j)_{j\geq 1}$ a positive non-increasing sequence converging to zero, one sets

$$\widetilde{E}^\mu_{\alpha,\tilde{\xi}} = \left\{ x : \begin{cases} \text{there is a scale } J_x \text{ such that for every } j \geq J_x, \\ \forall \epsilon \in \{-1,0,1\},\ b^{-j(\alpha+\xi_j)} \leq \mu(I_j^{(\epsilon)}(x)) \leq b^{-j(\alpha-\xi_j)} \end{cases} \right\}. \tag{13}$$

For any $\tilde{\xi}$, It is obvious that $\widetilde{E}^\mu_{\alpha,\tilde{\xi}} \subset E^\mu_\alpha$. The level sets $\widetilde{E}^\mu_{\alpha,\tilde{\xi}}$ contain points around which the local μ-behavior can be very precisely controlled.

As a simple consequence of [8, 17], one gets

Proposition 3.2 *Let μ be a positive Borel measure on $[0,1]$, and let $(h_{\min}, h_{\max})$ be the maximal open interval on which $\tau^*_\mu > 0$.*

1. *For every $\alpha \geq 0$ such that $\tau^*_\mu(\alpha) \geq 0$ and for any non-increasing sequence $\tilde{\xi}$ converging to zero, $\dim \widetilde{E}^\mu_{\alpha,\tilde{\xi}} \leq d_\mu(\alpha) \leq f_\mu(\alpha) \leq \tau^*_\mu(\alpha)$.*

2. *If μ obeys the multifractal formalism at every $\alpha \in (h_{\min}, \tau'_\mu(0^+)]$, then for every $\alpha \leq (h_{\min}, \tau'_\mu(0^+)]$, $\dim \left(\bigcup_{\alpha' \leq \alpha} E^\mu_{\alpha'} \right) = \dim E^\mu_\alpha$.*

3. *If μ obeys the multifractal formalism at every $\alpha \in [\tau'_\mu(0^+), h_{\max})$, then for every $\alpha \in [\tau'_\mu(0^+), h_{\max})$, $\dim \left(\bigcup_{\alpha' \geq \alpha} E^\mu_{\alpha'} \right) = \dim E^\mu_\alpha$.*

A last quantity will be needed.

Definition 3.3 *Let λ be a positive Borel measure on $\mathbb{R}$. Let us define, $\forall\alpha \geq 0$, $J \geq 0$ and $K \in \{0, ..., b^J - 1\}$, $\eta > 0$, $j \geq J+1$,*

$$N_{J,K}(\lambda, j, \eta, \alpha) = \# \left\{ k \not\equiv 0 \bmod b : \begin{cases} I_{j,k} \subset I_{J,K}, \\ b^{-(j-J)(\alpha+\eta)} \leq \frac{\lambda(I_{j,k})}{\lambda(I_{J,K})} \leq b^{-(j-J)(\alpha-\eta)} \end{cases} \right\}.$$

Heuristically, $N_{J,K}(\lambda, j, \eta, \alpha)$ is the number of intervals $I_{j,k} \subset I_{J,K}$ such that, when forgetting what happens before j, the rescaled μ-measure of $I_{j,k}$, $\frac{\lambda(I_{j,k})}{\lambda(I_{J,K})}$, is approximately equal to $b^{-(j-J)\alpha} = \left(\frac{|I_{j,k}|}{|I_{J,K}|}\right)^{\alpha}$.

3.2 Gibbs measures and their multifractal properties

Here are defined the Gibbs measures used in Theorems 2.1 and 2.2. We summarize some of their scaling and multifractal properties. We consider deterministic Gibbs measures, the same properties hold for random Gibbs measures.

3.3 Definition

Let c an integer ≥ 2 and let ℓ stand for the Lebesgue measure on $[0,1]$.

Let ϕ be a 1-periodic Hölder continuous function on $\mathbb{R}$ and $\omega = (\omega_n)_{n\geq 0}$ be a sequence of independent random phases uniformly distributed in $[0,1]$.

Let T be the shift transformation on $[0,1)$: $T(t) = ct \mod 1$.

For $n \geq 1$ and $t \in [0,1)$ let us consider the Birkhoff sums

$$S_n(\phi)(t) = \sum_{k=0}^{n-1} \phi(T^k t) \quad \text{and} \quad S_n(\phi,\omega)(t) = \sum_{k=0}^{n-1} \phi(T^k t + \omega_k).$$

Let also

$$Q_n(t) = \frac{\exp\big(S_j(\phi)(t)\big)}{\int_{[0,1]} \exp\big(S_j(\phi)(u)\big)\,du} \quad \text{and} \quad Q_n(t,\omega) = \frac{\exp\big(S_j(\phi,\omega)(t)\big)}{\int_{[0,1]} \exp\big(S_j(\phi,\omega)(u)\big)\,du}.$$

It follows from the thermodynamic formalism [19, 13] that $\mu_n = Q_n(\cdot)\cdot\ell$ (resp. $\mu_n^\omega = Q_n(\cdot,\omega)\cdot\ell$) converges (resp. almost surely), as $n \to \infty$, to a deterministic Gibbs (resp. random Gibbs) measure denoted μ (resp. μ^ω).

The multifractal analysis of μ and μ^ω is performed for instance in [8, 20, 10, 13]. With ϕ and ω are associated the analytic functions

$$P: \qquad q \mapsto \log(c) + \lim_{n\to\infty} j^{-1} \log \int_{[0,1)} \exp(qS_n(\phi(t))\,du$$

$$\text{and} \quad \widetilde{P}: \qquad q \mapsto \log(c) + \lim_{n\to\infty} j^{-1} \mathbb{E} \log \int_{[0,1)} \exp(qS_n(\phi(t,\omega))\,du,$$

which respectively are the topological pressures of ϕ relative to T and $\widetilde{T}$: $(t,\omega) \mapsto \big(T(t), \theta(\omega)$, where $\theta(\omega) = (\omega_{n+1})_{n\geq 0}\big)$. One has $\tau_\mu(q) = \frac{qP(1)-P(q)}{\log(c)}$, and a.s. $\tau_{\mu^\omega}(q) = \frac{q\widetilde{P}(1)-\widetilde{P}(q)}{\log(c)}$.

Gibbs measures considered here obey the multifractal formalism. In particular, for every $h \geq 0$, $d_\mu(h) = \tau_\mu^*(h)$ as soon as $\tau_\mu^*(h) > 0$.

Actually, Theorems 1.1, 1.2, 2.1 and 2.2 hold for the elements of a larger class of measures described in [3], which also contains the multinomial measures their random counterpart.

3.4 Properties of Gibbs measures

In this section, we fix a Gibbs measure μ as defined above. In the random case, μ is a realization of μ^ω, that is to say the following results hold almost surely. We fix another integer $b \geq 2$ in order to consider the b-adic grid defined in Section 1.

Fine properties on the measure μ are required to prove Theorem 2.2. Let $(h_{\min}, h_{\max})$ as in Proposition 3.2.

• **Property P1 (lower and upper bound for the scaling properties):** One has $h_{\min} > 0$ and $h_{\max} < +\infty$. The measure μ obeys the multifractal formalism at any $h \in (h_{\min}, h_{\max})$. For j large enough, for every $0 \leq k \leq b^j - 1$, $b^{-2h_{\max}j} \leq \mu(I_{j,k}) \leq b^{-h_{\min}j/2}$.

• **Property P2 (Gibbs states as analyzing measures):** Let $\mathcal{L}$ be a compact subset of $(h_{\min}, h_{\max})$. There is a sequence $\widetilde{\xi} = (\xi_j)_j$ such that for every $\alpha \in \mathcal{L}$, one can find a Borel measure m_α on $[0,1]$ such that $m_\alpha(\widetilde{E}^\mu_{\alpha,\widetilde{\xi}}) > 0$ and $m_\alpha(E) = 0$ for every Borel set $E \subset [0,1]$ such that $\dim E < \tau_\mu^*(\alpha)$ (this yields $\dim \widetilde{E}_{\alpha,\widetilde{\xi}} = \dim E^\mu_\alpha = \tau_\mu^*(\alpha)$). Let q_α be the unique $q \in \mathbb{R}$ such that $\alpha = \tau_\mu'(q)$. A possible choice for m_α is the Gibbs measure μ_{q_α} constructed as μ with the potential $q_\alpha\phi$. One has $\tau_\mu^*(\alpha) = \tau'_{\mu_{q_\alpha}}(1)$.

• **Property P3 (Heterogeneous ubiquity):** It follows from [2]. For $\rho \geq 1$, $\alpha > 0$ and for a positive sequence $\widetilde{\xi} = (\widetilde{\xi}_j)_{j\geq 1}$ define the limsup set

$$S_\mu(\rho,\alpha,\widetilde{\xi}) = \bigcap_{J\geq 0} \bigcup_{j\geq J} \bigcup_{\substack{k\in\{0,\ldots,b^j-1\}:\, k\not\equiv 0 \mod b \\ b^{-j(\alpha+\xi_j)} \leq \mu(I_{j,k}) \leq b^{-j(\alpha-\xi_j)}}} [kb^{-j}, kb^{-j} + b^{-j\rho}]. \tag{14}$$

Let $\mathcal{L}$ be a compact subset of $(h_{\min}, h_{\max})$. There exists a positive sequence $\widetilde{\xi}$ converging to 0 such that for every $\rho \geq 1$ and $\alpha \in \mathcal{L}$ one can find a positive Borel measure $m_{\alpha,\rho}$ such that:

- $m_{\alpha,\rho}(E) = 0$ for every Borel set E such that $\dim E < \tau_\mu^*(\alpha)/\rho$,
- $m_{\alpha,\rho}\big(S_\mu(\rho,\alpha,\widetilde{\xi})\big) > 0$.

In particular, dim $S_\mu(\rho,\alpha,\widetilde{\xi}) \geq \tau_\mu^*(\alpha)/\rho$.

• **Property P4 (Uniform renewal speed of large deviations spectrum):** This property is proved in [3].

Let $\mathcal{L}$ be a compact subinterval of $(h_{\min}, h_{\max})$. Let $\eta > 0$, and let us consider the sequence $\gamma_j := \sqrt{\dfrac{\log(j)^{1+\eta}}{j^{1/4}}}$, for $j \geq 1$. There exists a constant $M > 0$ and a scale $J_0 \geq 1$ such that for every $J \geq J_0$ and $K \in \{0, ..., b^J - 1\}$, for every $j \geq J + \exp(\sqrt{(1+\eta)\log(J)})$ and $\alpha \in \mathcal{L}$

$$b^{(j-J)(\tau_\mu^*(\alpha) - M\gamma_{j-J})} \leq N_{J,K}(\mu, j, \gamma_{j-J}, \alpha) \leq b^{(j-J)(\tau_\mu^*(\alpha) + M\gamma_{j-J})}. \tag{15}$$

Remark 3.4 *Properties* **P1** *and* **P2** *are well known for Gibbs measures associated with a smooth enough potential [8, 10, 20]. Properties* **P3** *and* **P4** *rely on finer properties without the restriction $k \not\equiv 0 \mod b$, but simple verifications show that the results also hold with this restriction.*

It is important for the sequel to precise that in Property **P2** *and* **P3***, one can take $\widetilde{\xi}$ equal to the sequence $(\gamma_j)_{j\geq 1}$ of* **P4***.*

4 Proof of Theorem 2.2

4.1 Proof of Theorem 2.2(3)

We begin by the last assertion. In this section, $\gamma = 0$ and $\sigma = 1$, thus $\nu_{0,1}$ is simply denoed ν. A b-adic number kb^{-j} is said to be *irreducible* if the fraction k/b^j is irreducible.

Let $\widetilde{\gamma} = (\gamma_j)_{j\geq 1}$ be the sequence defined in Section 3.4. For $j \geq 1$ define

$$\varepsilon_j = 2\gamma_{\frac{j}{\log j}} + 6\frac{h_{\max}}{\log j}. \tag{16}$$

Due to the last remark of Section 3.4, Property **P2** and **P3** hold with $\widetilde{\xi} := \widetilde{\varepsilon}/2$.

For simplicity of notations, we consider the measure $\nu^t := \nu^{\widetilde{\varepsilon}} = \nu_{0,1}^{\widetilde{\varepsilon}}$ (7) associated with this sequence $\widetilde{\varepsilon} = (\varepsilon_j)_j$. We also denote $t_{j,k}\nu_{j,k}$ by $\nu_{j,k}^t$.

By Theorem 2.1, one has $H_\tau := H_\tau(\nu) = \tau_\mu'(1)$ and thus by construction $\tau_\mu^*(H_\tau) = H_\tau$. We are going to show that $d_{\nu^t}(h) = d_\nu(h) (= \tau_\nu^*(h))$ for all $h \in [0, h_{\max})$. Since $\tau_\nu \leq \tau_{\nu^t}$, this yields $\tau_\nu^* = \tau_{\nu^t}^*$ on $\mathbb{R}_+$ and thus $\tau_\nu = \tau_{\nu^t}$ (remember that τ_ν and τ_{ν^t} are non-decreasing).

4.1.1 First results on the local regularity of ν^t

It is easy to verify that for every $x \in [0,1]$,

$$h_\nu(x) \leq h_{\nu^t}(x) \quad \text{and} \quad h_\nu(x) \leq h_\mu(x). \tag{17}$$

The first inequality is due to the fact that by construction, for any Borel set $B \subset [0,1]$, $\nu^t(B) \le \nu(B)$. The second one follows from the fact that for any b-adic interval $I_{j,k}$, $\nu(I_{j,k}) \ge j^{-2}\mu(I_{j,k})$.

Proposition 4.1 *For every $\varepsilon > 0$, there is an integer J_ε such that for any $\beta \in [h_{\min}/2, 2h_{\max}]$, $\forall\, j \ge J_\varepsilon$, for every integer K such that Kb^{-J} is irreducible,*

$$\mu(I_{J,K}) = b^{-J\beta} \Rightarrow b^{-J(\beta+\varepsilon)} \le \nu^t(I_{J,K}) \le b^{-J(\beta-\varepsilon)}.$$

Proof: Let $\varepsilon > 0$. Let J_1 be large enough so that $j \ge J_1$ implies $0 < \max(\gamma_j, \varepsilon_j) \le \varepsilon/2$ and $b^{-2jh_{\max}} \le \mu(I_{j,k}) \le b^{-jh_{\min}/2}$ for all $0 \le k \le b^j - 1$. Let Kb^{-J} be an irreducible b-adic number such that $J \ge J_1$, and let β be defined by $\mu(I_{J,K}) = b^{-J\beta}$.

• Let us first notice that (recall the definition of the measure ν (6))

$$\begin{aligned}
\nu(I_{J,K}) &= \frac{1}{J^2}\mu(I_{J,K}) + \sum_{j \ge J+1} \frac{1}{j^2} \sum_{\substack{k=0,\ldots,b^j-1:\\ k \not\equiv 0 \mod b,\ kb^{-j} \in I_{J,K}}} \mu([kb^{-j}, (k+1)b^{-j})) \\
&\le \frac{1}{J^2}\mu(I_{J,K}) + \sum_{j \ge J+1} \frac{1}{j^2}\mu(I_{J,K}).
\end{aligned}$$

If J is greater than some fixed integer $J_2 \ge 1$ large enough, then $\nu(I_{J,K}) \le \mu(I_{J,K})b^{J\varepsilon/2} \le b^{-J(\beta-\varepsilon/2)}$. Now it is obvious that by construction, for any subset B of $[0,1]$, $\nu^t(B) \le \nu(B)$, hence we get the first inequality $\mu(I_{J,K}) = b^{-J\beta} \Rightarrow \nu^t(I_{J,K}) \le b^{-J(\beta-\varepsilon)}$ for any $J \ge \max(J_1, J_2)$.

• The converse inequality is more difficult to obtain. Let us show that $\nu^t(I_{J,K}) \ge b^{-J(\beta+\varepsilon)}$. One has by definition

$$\nu^t(I_{J,K}) = \nu^t_{J,K} + \sum_{j \ge J+1} \sum_{\substack{k=0,\ldots,b^j-1:\\ k \not\equiv 0 \mod b,\ kb^{-j} \in I_{J,K}}} \nu^t_{j,k}\delta_{kb^{-j}} \tag{18}$$

1. If $\beta = H_\tau$: By construction, for J large enough one has $\nu^t_{J,K} = J^{-2}\mu(I_{J,K})$, and $\nu^t(I_{J,K}) \ge \nu^t_{J,K}) \ge b^{-J(\beta+\varepsilon)}$.

2. If $\beta > H_\tau$: Let us recall (18). To find a lower bound for $\nu^t(I_{J,K})$, one has to look for non-zero Dirac masses (after threshold) in the sum (18).

Let us use Property **P4** applied with $\alpha = H_\tau$. Let $\eta > 0$. There exists a constant $M > 0$ and a scale $J_0 \ge 1$ such that with probability one, for every $J \ge J_0$ and $K \in \{0, ..., b^J - 1\}$, for every $j \ge J + \exp(\sqrt{(1+\eta)\log(J)})$, (15) holds with $\alpha = H_\tau$. In particular, one has for every $j \ge J + \exp(\sqrt{(1+\eta)\log(J)})$

$$N_{J,K}(\mu, j, \gamma_{j-J}, H_\tau) \ge b^{(j-J)(H_\tau - M\gamma_{j-J})}. \tag{19}$$

Let $I_{j,k}$ be any of the intervals such that $I_{j,k} \subset I_{J,K}$, $k \not\equiv 0 \mod b$, and

$$\mu(I_{J,K}) b^{-(j-J)(H_\tau+\gamma_{j-J})} \leq \mu(I_{j,k}) \leq \mu(I_{J,K}) b^{-(j-J)(H_\tau-\gamma_{j-J})}.$$

One has $b^{-j\alpha^1_{j,J}} \leq \mu(I_{j,k}) \leq b^{-j\alpha^2_{j,J}}$ with

$$\alpha^1_{j,J} = H_\tau + \gamma_{j-J} - \frac{J}{j}(H_\tau - \beta + \gamma_{j-J}), \quad \alpha^2_{j,J} = H_\tau - \gamma_{j-J} - \frac{J}{j}(H_\tau - \beta + \gamma_{j-J}).$$

In order to have $\nu^t_{j,k} \neq 0$, one must have $[\alpha^2_{j,J}, \alpha^1_{j,J}] \subset [H_\tau - \varepsilon_j + \log_b(j^{-2})/j, H_\tau + \varepsilon_j + \log_b(j^{-2})/j]$. This is achieved as follows.

Let $\theta > 0$. There exists a scale J_3 such that for every $J \geq J_3$, for every $j \geq J + J^{1+\theta}$, one has

$$\frac{j}{\log j} \leq j - J \quad \text{and} \quad \frac{6h_{\max}}{\log j} \geq \frac{J}{j}(2h_{\max} + H_\tau + \gamma_{j-J}) + \frac{\log_b(j^2)}{j}$$

Let $J_4 = \max(J_1, J_2, J_3)$ (J_4 is independent of β). Then remembering (16), for every $J \geq J_4$, as soon as $j \geq J + J^{1+\theta}$ one has

$$H_\tau - \varepsilon_j - \log_b(j^2)/j \leq \alpha^2_{j,J} \leq \alpha^1_{j,J} \leq H_\tau + \varepsilon_j - \log_b(j^2)/j.$$

Hence those intervals $I_{j,k} \subset I_{J,K}$ (with $j \geq J + J^{1+\theta}$) such that $k \not\equiv 0 \mod b$ and $b^{-j\alpha^1_{j,J}} \leq \mu(I_{j,k}) \leq b^{-j\alpha^2_{j,J}}$ give rise to non-zero masses in the sum (18).

Using (18) and (19), one obtains that for every $J \geq J_4$, for every K such that Kb^{-J} is irreducible, for every $j_0 = J + J^{1+\theta}$,

$$\begin{aligned}
\nu^t(I_{J,K}) &\geq \sum_{\substack{k=0,\dots,b^{j_0}-1: \\ k\not\equiv 0 \mod b,\ kb^{-j_0} \in I_{J,K}}} \nu^t_{j_0,k} \geq \frac{1}{j_0^2} N_{J,K}(\mu, j_0, \gamma_{j_0-J}, H_\tau) b^{-j_0\alpha^1_{j_0,J}} \\
&\geq \frac{1}{j_0^2} b^{(j_0-J)(H_\tau - M\gamma_{j_0-J})} b^{-j_0(H_\tau + \gamma_{j_0-J} - \frac{J}{j_0}(H_\tau - \beta + \gamma_{j_0-J}))}
\end{aligned}$$

Hence $\nu^t(I_{J,K}) \geq b^{-J\beta} \frac{b^{(j_0-J)(M+1)\gamma_{j_0-J}}}{j_0^2}$. Since $\gamma_j = \sqrt{\frac{\log(j)^{1+\eta}}{j^{1/4}}}$, one gets

$$\begin{aligned}
(j_0 - J)(M+1)\gamma_{j_0-J} &= J^{1+\theta}(M+1)\gamma_{J^{1+\theta}} = (M+1)J^{1+\theta}\sqrt{\frac{\log(J^{1+\theta})^{1+\eta}}{J^{(1+\theta)/4}}} \\
&= (M+1)(1+\theta)^{(1+\eta)/2} J^{7(1+\theta)/8} \log(J)^{(1+\eta)/2}.
\end{aligned}$$

Choosing $\theta \in (0, 1/7)$, there is a scale J_5 (independent of β) such that for every $J \geq J_5$, one has $(j_0 - J)(M+1)\gamma_{j_0-J} \leq \varepsilon J$. It is also obvious that $\frac{1}{j_0^2} \geq b^{-\varepsilon J}$ for J large enough. Finally, for J large enough, one obtains

$$\nu^t(I_{J,K}) \geq b^{-J(\beta+2\varepsilon)}.$$

3. If $\beta < H_\tau$: The same arguments as above yield the same result. ∎

We emphasize that the order of magnitude of the sequence γ_j plays a crucial role in the previous computation.

Proposition 4.2 *For every $\varepsilon > 0$, there is an integer J_ε such that for any $\beta \in [h_{\min}/2, 2h_{\max}]$, for every $j \geq J_\varepsilon$, for every integer K such that Kb^{-J} is reducible,*

$$\mu(I_{J,K}) = b^{-J\beta} \Rightarrow b^{-J(\beta+\varepsilon)} \leq \nu^t(I_{J,K}) \leq \nu(I_{J,K}).$$

Proof: The right inequality is immediate, and the left one is a consequence of Proposition 4.1. ∎

Using Propositions 4.1 and 4.2, one can specify (17) and assert that,

$$\text{for every } x \in (0,1), \quad h_\nu(x) \leq h_{\nu^t}(x) \leq h_\mu(x). \tag{20}$$

Proposition 4.3 *Let $\rho \geq 1$. Consider the limsup set $S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2)$ defined in (14). For every $x \in S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2)$, $h_{\nu^t}(x) \leq H_\tau/\rho$.*

Proof: By definition of $S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2)$, for every $x \in S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2)$, there is an infinite number of scales j_n such that for some $k_n \in \{0, ..., b^{j_n} - 1\}$ with $k_n b^{-j_n}$ irreducible, $|x - k_n b^{-j_n}| \leq b^{-j_n \rho_{j_n}}$ and simultaneously $b^{-j_n(H_\tau + \varepsilon_{j_n}/2)} \leq \mu(I_{j_n,k_n})) \leq b^{-j_n(H_\tau - \varepsilon_{j_n/2})}$. This implies that

$$\nu^t(B(x, b^{-j_n\rho_{j_n}})) \geq \frac{1}{j_n^2}\mu(I_{j_n,k_n})) \geq b^{-j_n(H_\tau+\varepsilon_{j_n})} = b^{-j_n\rho_{j_n}(\frac{H_\tau+\varepsilon_{j_n}}{\rho_{j_n}})}.$$

Hence $\frac{\log \nu^t(B(x,b^{-j_n\rho_{j_n}}))}{\log b^{-j_n\rho_{j_n}}} \leq \frac{H_\tau+\varepsilon_{j_n}}{\rho_{j_n}}$. Since $\varepsilon_j \to 0$ and $\rho_j \to \rho$, by letting j_n go to $+\infty$, one obtains that $h_{\nu^t}(x) \leq H_\tau/\rho$. ∎

4.1.2 Upper bound for the multifractal spectrum of d_{ν^t}

Proposition 4.4 *For every $h \in [0, \tau_\mu'(0)]$, $d_{\nu^t}(h) \leq d_\nu(h) = \tau_\mu^*(h)$.*

Proof: Let us use (20): For every $x \in [0,1]$, $h_{\nu^t}(x) \geq h_\nu(x)$. This implies that for every $h \in [0, \tau_\mu'(0)]$, $E_h^{\nu^t} \subset \bigcup_{h' \leq h} E_{h'}^\nu$.

By Proposition 3.2 and Theorem 2.1, for every $h \in [0, \tau_\mu'(0)]$, one has $\dim \bigcup_{h' \leq h} E_{h'}^\nu \leq \dim E_h^\nu = d_\nu(h)$, hence the result. ∎

Proposition 4.5 *For every $h \in (\tau_\mu'(0), h_{\max}]$, $d_{\nu^t}(h) \leq d_\nu(h) = \tau_\mu^*(h)$.*

Proof: Let $h \in (\tau_\mu'(0), h_{\max}]$. By (20), $E_h^{\nu^t} \subset \bigcup_{h' \geq h} E_h^\mu$. By Proposition 3.2, $d_{\nu^t}(h) = \dim E_h^{\nu^t} \leq \dim \bigcup_{h' \geq h} E_h^\mu \leq \tau_\mu^*(h)$. ∎

4.1.3 Lower bound for the multifractal spectrum of d_{ν^t}

Proposition 4.6 *For every $h \in [0, H_\tau]$, $d_{\nu^t}(h) \geq d_\nu(h) = h$.*

Proof: We first apply property **P3**. Let $h \in (0, H_\tau]$, and let us consider $\rho = H_\tau/h$. Consider the measure m_ρ of **P3**. Let $S = S_\mu(\rho, H_\tau, \widetilde{\varepsilon}/2))$.

By Proposition 4.3, every $x \in S$ verifies $h_{\nu^t}(x) \leq H_\tau/\rho = h$. Hence $S \subset \bigcup_{h' \leq h} E_{h'}^{\nu^t}$. By Proposition 3.2, for all $i \geq 1$, $\dim \bigcup_{h' \leq h-1/i} E_{h'}^{\nu^t} \leq \tau_{\nu^t}^*(h-1/i)$. Moreover, $\tau_{\nu^t} \geq \tau_\nu$ so $\tau_{\nu^t}^*(h-1/i) \leq \tau_\nu^*(h-1/i) < \tau_\nu^*(h) = H_\tau/\rho$. Hence $m_\rho(\bigcup_{i \geq 1} \bigcup_{h' \leq h-1/i} E_{h'}^{\nu^t}) = 0$.

One has $S \setminus \bigcup_{i \geq 1} \bigcup_{h' \leq h-1/i} E_{h'}^{\nu^t} \subset E_h^{\nu^t}$. Since $\varepsilon_j \geq \xi_j$, by construction $m_\rho(S) > 0$, and $m_\rho(S \setminus \bigcup_{i \geq 1} \bigcup_{h' \leq h-1/i} E_{h'}^{\nu^t}) > 0$. Hence $\dim E_h^{\nu^t} \geq h$. ∎

Proposition 4.7 *For every $h \in (H_\tau, h_{\max})$, $d_{\nu^t}(h) \geq d_\nu(h)$.*

Proof: We need a lemma extracted from the proof of Proposition 8 in [2].

Lemma 4.8 *Let $h \in [H_\tau, h_{\max})$. Let m_h be a measure as in* **P2**. *Then there exists a subset S of $\widetilde{E}^\mu_{h,\widetilde{\xi}}$ such that $m_h(S) > 0$ and $S \subset E_h^\nu$.*

Let $h \in (H_\tau, h_{\max})$. Consider a set S and a measure m_h as in Lemma 4.8. Let $x \in S \subset \widetilde{E}^\mu_{h,\widetilde{\xi}} \cap E_h^\nu$. Remark that at every scale j, at least one of $I_j^{(-1)}(x)$, $I_j^0(x)$, $I_j^{(+1)}(x)$, is irreducible. Hence, for this irreducible b-adic interval I, by Proposition 4.1 one has $\nu^t(I) \geq \mu(I) b^{-j\varepsilon} \geq b^{-j(h+\xi_j+\varepsilon)}$. This holds for every j large enough, and then for every ε small enough. Hence $h_{\nu^t}(x) \leq h$.

But $h_{\nu^t}(x)$ is always larger than $h_\nu(x)$, which equals h since $S \subset E_h^\nu$. Hence $h_{\nu^t}(x) = h$, and $S \subset E_h^{\nu^t}$.

As a consequence, $m_h(E_h^{\nu^t}) \geq m_h(S) > 0$, and $\dim E_h^{\nu^t} \geq d_\nu(h)$. ∎

4.2 Proof of Theorem 2.2(2)

We come back to the general measures $\nu_{\gamma,\sigma}$ and to the general form of its thresholded version $\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}$. Let $\widetilde{\xi}$ be a sequence such that **P3** holds with $\alpha = \tau_\mu'(\sigma q_\tau(\nu))$, and let $\widetilde{\varepsilon} = (\varepsilon_j)_{j \geq 1}$ be defined by $\varepsilon_j = \sigma \widetilde{\xi}_j + 2\log_b(j)/j$.

Let $\mu_{\sigma q_\tau}$ be the Gibbs measure constructed as μ, but with the potential $\sigma q_\tau(\nu)\phi$. One deduces from Theorem 2.1 that $q_\tau(\nu_{\gamma,\sigma}) H_\tau(\nu_{\gamma,\sigma}) = \tau_\mu^*(\sigma q_\tau(\nu_{\gamma,\sigma}))$. Then, the same arguments as those used in the proof of Proposition 4.3 show that for $\rho \geq 1$, if $x \in S_\mu(\rho, \tau_\mu'(\sigma q_\tau(\nu_{\gamma,\sigma})), \widetilde{\xi})$, then $h_{\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}}(x) \leq H_\tau(\nu_{\gamma,\sigma})/\rho$.

Since by construction $\tau_{\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}} \geq \tau_{\nu_{\gamma,\sigma}}$, for every $h \in [0, H_\tau(\nu_{\gamma,\sigma})]$ one has $\dim \bigcup_{h' \leq h} E_{h'}^{\nu_{\gamma,\sigma}^{\widetilde{\varepsilon}}} \leq \tau_{\nu_{\gamma,\sigma}}^*(h) = q_\tau(\nu_{\gamma,\sigma}) h$. This is enough to conclude as in the proof of Proposition 4.6.

4.3 Proof of Theorem 2.2(1)

It follows from (2) that $d_{\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}}(H_\tau(\nu_{\gamma,\sigma})) = q_\tau(\nu_{\gamma,\sigma})H_\tau(\nu_{\gamma,\sigma})$. Moreover, due to the threshold operation ona has $\tau_{\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}} \geq \tau_\nu$ on $\mathbb{R}_+$, so that has $q_\tau(\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}) \leq q_\tau(\nu_{\gamma,\sigma})$. On the other hand, $d_{\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}}(H_\tau(\nu_{\gamma,\sigma})) \leq \tau^*_{\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}}(H_\tau(\nu_{\gamma,\sigma})) \leq q_\tau(\nu^{\tilde{\varepsilon}}_{\gamma,\sigma})H_\tau(\nu_{\gamma,\sigma})$. This yields $q_\tau(\nu^{H_\tau(\nu),\tilde{\varepsilon}}) = q_\tau(\nu)$ and then $H_\tau(\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}) \leq H_\tau(\nu_{\gamma,\sigma})$ again because $\tau_{\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}} \geq \tau_{\nu_{\gamma,\sigma}}$ and these functions are concave. Coming back to $d_{\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}}(H_\tau(\nu_{\gamma,\sigma})) = q_\tau(\nu_{\gamma,\sigma})H_\tau(\nu_{\gamma,\sigma})$ and the fact that for any positive Borel measure on $[0,1]$ one has $d_\chi(h) \leq \tau^*_\chi(h) < q_\tau(\chi)h$ if $h > H_\tau(\chi)$, one gets $H_\tau(\nu^{\tilde{\varepsilon}}_{\gamma,\sigma}) = H_\tau(\nu_{\gamma,\sigma})$.

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