DYNAMICS OF MANDELBROT CASCADES

JULIEN BARRAL*, JACQUES PEYRIÈRE†‡, AND ZHI-YING WEN†&

ABSTRACT. Mandelbrot multiplicative cascades provide a construction of a dynamical system on a set of probability measures defined by inequalities on moments. To be more specific, beyond the first iteration, the trajectories take values in the set of fixed points of smoothing transformations (i.e., some generalized stable laws).

Studying this system leads to a central limit theorem and to its functional version. The limit Gaussian process can also be obtained as limit of an 'additive cascade' of independent normal variables.

1. A DYNAMICAL SYSTEM

Consider the set $\mathscr{A} = \{0, \dots, b-1\}$, where $b \geq 2$. Set $\mathscr{A}^* = \bigcup_{n \geq 0} \mathscr{A}^n$, where, by convention, $\mathscr{A}^0$ is the singleton $\{\epsilon\}$ whose the only element is the empty word ϵ. If $w \in \mathscr{A}^*$, we denote by $|w|$ the integer such that $w \in \mathscr{A}^{|w|}$. If $n \geq 1$ and $w = w_1 \cdots w_n \in \mathscr{A}^n$ then for $1 \leq k \leq n$ the word $w_1 \cdots w_k$ is denoted by $w|_k$. By convention, $w|_0 = \epsilon$.

Given v and w in $\mathscr{A}^n$, $v \wedge w$ is defined to be the longest prefix common to both v and w, i.e., $v|_{n_0}$, where $n_0 = \sup\{0 \leq k \leq n \ : \ v|_k = w|_k\}$.

Let $\mathscr{A}^\omega$ stand for the set of infinite sequences $w = w_1 w_2 \cdots$ of elements of $\mathscr{A}$. Also, for $x \in \mathscr{A}^\omega$ and $n \geq 0$, let $x|_n$ stand for the projection of x on $\mathscr{A}^n$.

If $w \in \mathscr{A}^*$, we consider the cylinder $[w]$ consisting of infinite words in $\mathscr{A}^\omega$ whose w is a prefix.

We index the closed b-adic subintervals of $[0,1]$ by $\mathscr{A}^*$: for $w \in \mathscr{A}^*$, we set

$$I_w = \left[\sum_{1 \leq k \leq |w|} w_k b^{-k}, \sum_{1 \leq k \leq |w|} w_k b^{-k} + b^{-|w|} \right].$$

If $f \ : \ [0,1] \mapsto \mathbb{R}$ is bounded, for every sub-interval $I = [\alpha, \beta]$ of $[0,1]$, we denote by $\Delta(f, I)$ the increment $f(\beta) - f(\alpha)$ of f over the interval I.

Let $\mathcal{P}$ the set of Borel probability measures on $\mathbb{R}_+$. If $\mu \in \mathcal{P}$ and $p > 0$, we denote by $\mathbf{m}_p(\mu)$ the moment of order p of μ, i.e.,

$$\mathbf{m}_p(\mu) = \int_{\mathbb{R}_+} x^p \, \mu(\mathrm{d}x).$$

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* INRIA Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France. `Julien.Barral@inria.fr`.

† Dept. Math., Tsinghua University, Beijing, 100084, P. R. China. `wenzy@mail.tsinghua.edu.cn`.

‡ Université Paris-Sud, Mathématique bât. 425, CNRS UMR 8628, 91405 Orsay Cedex, France. `Jacques.Peyriere@math.u-psud.fr`.

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Then let $\mathcal{P}_1$ be the set of elements of $\mathcal{P}$ whose first moment equals 1:

$$\mathcal{P}_1 = \{\mu \in \mathcal{P} \ : \ \mathbf{m}_1(\mu) = 1\}.$$

The smoothing transformation S_μ associated with $\mu \in \mathcal{P}$ is the mapping from $\mathcal{P}$ to itself so defined: If $\nu \in \mathcal{P}$, one considers $2b$ independent random variables, $Y(0)$, $Y(1), \ldots, Y(b-1)$, whose common probability distribution is ν, and $W(0)$, $W(1), \ldots,$ $W(b-1)$ whose common probability distribution is μ; then $\mathsf{S}_\mu\nu$ is the probability distribution of $b^{-1} \sum_{0\le j<b} W(j)\,Y(j)$.

This transformation and its fixed points have been considered in several contexts, in particular by B. Mandelbrot who introduced it to construct a model for turbulence and intermittence (see [9, 10, 6, 11, 7, 4, 5]).

In this latter case, the measure μ is in $\mathcal{P}_1$ so that S_μ maps $\mathcal{P}_1$ into itself. It is known that the condition $\int x\log(x)\,\mu(\mathrm{d}x) < \log b$ is then necessary and sufficient for the weak convergence of the sequence $\mathsf{S}_\mu^n\delta_1$ (where δ_1 stands for the Dirac mass at point 1) towards a probability measure ν, which therefore is a fixed point of S_μ (see [9, 10, 6, 7, 4]). In other words, if $\int x\log(x)\,\mu(\mathrm{d}x) < \log b$ and if $\big(W(w)\big)_{w\in\mathscr{A}^*}$ is a family of independent random variables whose probability distribution is μ, then the non-negative martingale

$$Y_n = b^{-n} \sum_{w\in\mathscr{A}^n} W(w|_1)W(w|_2)\cdots W(w|_n) \tag{1}$$

is uniformly integrable and converges to a random variable Y whose probability distribution ν belongs to $\mathcal{P}_1$ and satisfies $\mathsf{S}_\mu\nu = \nu$. This means that there exists b copies $W(0), \ldots, W(b-1)$ of W and b copies $Y(0), \ldots, Y(b-1)$ of Y such that these $2b$ random variables are independent and

$$Y = b^{-1}\sum_{k=0}^{b-1} W(k)Y(k). \tag{2}$$

In this case, we denote the measure ν by $\mathsf{T}\mu$. It is natural to try and iterate T. But, in general this is not possible because $\nu = \mathsf{T}\mu$ may not inherit the property $\int x\log(x)\,\nu(\mathrm{d}x) < \log b$. So, we have to find a domain stable under the action of T. This will be done by imposing conditions on moments.

Indeed, it is easily seen that the sequence $(Y_n)_{n\ge1}$ defined by (1) remains bounded in L^2 norm if and only if $\mathbb{E}(W^2) = \mathbf{m}_2(\mu) < b$, and that in this case Formula (2) yields

$$\mathbb{E}\,Y^2 = \frac{b-1}{b-\mathbb{E}\,W^2} \tag{3}$$

(since the random variables $W(j)$ and $Y(j)$ are independent and of expectation 1, squaring both sides of Formula (2) yields $b^2\,\mathbb{E}\,Y^2 = b\,\mathbb{E}\,W^2\,\mathbb{E}\,Y^2 + b(b-1)$). It follows that if $b \ge 3$ and $1 \le \mathbb{E}\,W^2 < b-1$, we have $\mathbb{E}\,Y^2 \le \mathbb{E}\,W^2$ (the equality holding only if $W = 1$). Therefore, since the condition $\mathbb{E}\,W^2 < b$ is stronger than $\mathbb{E}(W\log W) < \log b$ when $\mathbb{E}\,W = 1$ (since the function $t \mapsto \log\mathbb{E}\,W^t$ is convex), T is a transformation on the subset of $\mathcal{P}_1$ defined by

$$\mathscr{P}_b = \big\{\mu \in \mathcal{P}_1 \ : \ 1 < \mathbf{m}_2(\mu) < b-1\big\}.$$

If $\mu \in \mathscr{P}_b$, due to (2), we can associate with each $n \ge 0$ a random variable W_{n+1} as well as $2b$ independent random variables $W_n(0), \ldots, W_n(b-1)$ and $W_{n+1}(0), \ldots, W_{n+1}(b-1)$ such that

$$W_{n+1} = \frac{1}{b}\sum_{k=0}^{b-1} W_n(k)W_{n+1}(k), \tag{4}$$

$\mathsf{T}^n\mu$ is the probability distribution of $W_n(k)$ for every k such that $0 \le k \le b-1$, and $\mathsf{T}^{n+1}\mu$ is the probability distribution of W_{n+1} and $W_{n+1}(k)$ for every $0 \le k \le b-1$. We advise the reader that if one writes Formula (4) with $n-1$ instead of n, the variables $W_n(k)$ which then appear are different from the previous ones.

In Mandelbrot [9, 10], the random variable Y represents the increment between 0 and 1 of the non-decreasing continuous function h on $[0,1]$ obtained as the almost sure uniform limit of the sequence of non-decreasing continuous functions ϕ_n defined by

$$\phi_n(u) = \int_0^u \prod_{k=1}^n W(\tilde{t}|_k)\, dt, \tag{5}$$

where $\tilde{t}$ stands for the sequence of digits in the base b expansion of t (of course the ambiguity for countably many t's is harmless). In other words, for $w \in \mathscr{A}^*$, we have

$$\Delta(\phi, I_w) = b^{-|w|} Y(w) \prod_{1\le j\le |w|} W(w|_j), \tag{6}$$

where $Y(w)$ has the same distribution as Y and is independent of the variables $W(w|_j)$.

Let us denote by $F(\mu)$ the probability distribution of the limit ϕ, considered as a random continuous function.

We are going to study the dynamical system $(\mathscr{P}_b, \mathsf{T})$. This will lead to a description of the asymptotic behavior of $\big(\mathsf{T}^n\mu, F(\mathsf{T}^{n-1}\mu)\big)_{n\ge 1}$ as n goes to ∞.

We need some more definitions. For $b \ge 3$, set

$$w_2(b) = \min\left(b-1,\, b\frac{b^4-4b^2+12b-8}{b^4+8b^2-12b+4}\right)$$

and, for t such that $1 < t < w_2(b)$,

$$w_3(b,t) = \frac{b^2}{2} + \frac{1}{2}\sqrt{\frac{b(b^4-4b^2+12b-8) - t\big(b^4+8b^2-12b+4\big)}{b-t}}.$$

One always has $w_3(b,t) < b^2-1$.

Also set

$$\mathscr{D}_b = \Big\{\mu \in \mathcal{P} \;:\; \mathbf{m}_1(\mu) = 1,\, 1 < \mathbf{m}_2(\mu) < w_2(b),\ \text{and}\ \mathbf{m}_3(\mu) < w_3\big(b, \mathbf{m}_2(\mu)\big)\Big\}.$$

Theorem 1 (Central limit theorem)**.** *Suppose $b \ge 3$. Let $\mu \in \mathscr{P}_b$, and, for $n \ge 0$, define* $\sigma_n = \left(\int (x-1)^2\, \mathsf{T}^n\mu(\mathrm{d}x)\right)^{1/2}$. *Then*

(1) *The limit of $(b-1)^{n/2}\sigma_n$ exists and is positive; so $\lim_{n\to\infty} \mathsf{T}^n\mu = \delta_1$.*
(2) *If $\mu \in \mathscr{D}_b$, then,* $\sup_{n\ge 0} \int \left(\frac{|x-1|}{\sigma_n}\right)^3 \mathsf{T}^n\mu(\mathrm{d}x) < \infty$.
(3) *Suppose that there exists $p > 2$ such that* $\sup_{n\ge 0} \int \left(\frac{|x-1|}{\sigma_n}\right)^p \mathsf{T}^n\mu(\mathrm{d}x) < \infty$.
Then, if W_n is a variable whose distribution is $\mathsf{T}^n\mu$, $\frac{W_n - 1}{\sigma_n}$ *converges in distribution towards $\mathcal{N}(0,1)$.*

Theorem 2 (Functional CLT)**.** *Suppose $b \ge 3$. Let $\mu \in \mathscr{P}_b$. Then*

(1) *The probability distributions $F(\mathsf{T}^n\mu)$ weakly converges towards δ_{Id}.*
(2) *Suppose that there exists $p > 2$ such that*

$$\sup_{n\ge 1} \int \left(\frac{|x-1|}{\sigma_n}\right)^p \mathsf{T}^n\mu(\mathrm{d}x) < \infty.$$

(In particular, this holds if μ lies in the domain of attraction $\mathscr{D}_b$.) Then, if h_n is a random function distributed according to $F(\mathsf{T}^{n-1}\mu)$, the distribution of $\dfrac{h_n - \mathrm{Id}}{\sigma_n}$ weakly converges towards the distribution of the unique continuous Gaussian process $(X_t)_{t\in[0,1]}$, such that $X(0) = 0$ and, for all $j \geq 1$, the covariance matrix M_j of the vector $\big(\Delta(X, I_w)\big)_{w\in\mathscr{A}^j}$ is given by

$$\mathsf{M}_j(w, w') = \begin{cases} b^{-2j}\big(1 + (b-1)|w|\big) & \text{if } w = w', \\ b^{-2j}(b-1)|w \wedge w'| & \text{otherwise.} \end{cases}$$

In Section 4, we will give an alternate construction of this Gaussian process X: It will be obtained as the almost sure limit of an additive cascade of normal variables.

Remark 1. It would be interesting to know whether the new limit process and central limit theorems provided in this paper could be useful for modeling in any area.

2. Proof of Theorem 1

Throughout this section and the next one, we assume $b \geq 3$.

Proposition 3. *If $\mu \in \mathscr{P}_b$ and $\sigma_n^2 = \displaystyle\int (x-1)^2 \mathsf{T}^n\mu(\mathrm{d}x)$, the sequence $(b-1)^{n/2}\sigma_n$ converges to $\sigma_0\sqrt{\dfrac{b-2}{b-2-\sigma_0^2}}$.*

Proof. Equations (4) and (3) yield $\mathbb{E}\,W_{n+1}^2 = \dfrac{b-1}{b - \mathbb{E}\,W_n^2}$, from which we get the formula

$$\sigma_{n+1}^2 = \mathbb{E}\,W_{n+1}^2 - 1 = \frac{\sigma_n^2}{b-1-\sigma_n^2}, \tag{7}$$

which can be written as

$$\frac{\sigma_{n+1}^2}{b-2-\sigma_{n+1}^2} = \frac{(b-1)^{-1}\sigma_n^2}{b-2-\sigma_n^2}.$$

This yields

$$\frac{\sigma_n^2}{b-2-\sigma_n^2} = \frac{\sigma_0^2}{b-2-\sigma_0^2}(b-1)^{-n}. \tag{8}$$

□

Proposition 4. *If $\mu \in \mathscr{D}_b$, then both sequences $\big(\mathbf{m}_2(\mathsf{T}^n\mu)\big)_{n\geq 1}$ and $\big(\mathbf{m}_3(\mathsf{T}^n\mu)\big)_{n\geq 1}$ are non-increasing and converge to 1 as n goes to ∞.*

Proof. For $n \geq 0$, we set $u_n = \mathbf{m}_2(\mathsf{T}^n\mu)$ and $v_n = \mathbf{m}_3(\mathsf{T}^n\mu)$ and deduce from (4) that, for all $n \geq 0$, we have

$$u_{n+1} = \frac{b-1}{b-u_n} \tag{9}$$

$$v_{n+1} = \frac{(b-1)\big(3u_n u_{n+1} + b - 2\big)}{b^2 - v_n} \tag{10}$$

if $u_n < b$ and $v_n < b^2$

(Formula (9) is a restatement of (3), and taking cubes in Formula (4) yields $b^3\,\mathbb{E}\,W_{n+1}^3 = b\,\mathbb{E}\,W_n^3\,\mathbb{E}\,W_{n+1}^3 + 3b(b-1)\,\mathbb{E}\,W_n^2\,\mathbb{E}\,W_{n+1}^2 + b(b-1)(b-2)$ hence Formula (10)). Since $1 \leq u_0 < b-1$, as we already saw it, Equation (9) implies that u_n decreases, except in the trivial case $\mu = \delta_1$. Moreover u_n converges towards 1, the stable fixed point of $t \mapsto (b-1)/(b-t)$.

The conditions $\mathbf{m}_2(\mu) \le w_2(b)$ and $\mathbf{m}_3(\mu) < w_3\big(b, \mathbf{m_2}(\mu)\big)$ are optimal to ensure that $v_1 \le v_0$, and also they impose $v_0 < b^2 - 1$. We conclude by recursion: if $v_{n+1} \le v_n < b^2$, then we have

$$\begin{aligned} v_{n+2} &\le \frac{(b-1)\big(3u_{n+1}u_{n+2} + b - 2\big)}{b^2 - v_n} \\ &\le \frac{(b-1)\big(3u_n u_{n+1} + b - 2\big)}{b^2 - v_n} = v_{n+1}. \end{aligned}$$

Thus $(v_n)_{n\ge 0}$ is non-increasing and $1 \le v_n < b^2 - 1$, so we deduce from (10) and the fact that u_n converges to 1 that v_n converges to the smallest fixed point of the mapping $x \mapsto (b^2-1)/(b^2-x)$, namely 1. □

Proposition 5. *There exists $C > 0$ such that, for $\mu \in \mathscr{D}_b$ and $n \ge 1$, we have*

$$\big(b^2 - \mathbb{E}\, W_n^3\big)\, \mathbb{E}\, |Z_{n+1}^3| \le (b-1)^{3/2}\, \mathbb{E}\, |Z_n^3| + C\big((\mathbb{E}\, |Z_n^3|)^{2/3} + (\mathbb{E}\, |Z_n^3|)^{1/3} + 1\big),$$

where $Z_n = \dfrac{W_n - 1}{\sigma_n}$.

Proof. We use the following simplified notations: $W = W_n$, $Y = W_{n+1}$, σ_Y and σ_W stand for the standard deviations of Y and W, $Z_Y = \sigma_Y^{-1}\,|Y-1|$, $Z_W = \sigma_W^{-1}\,|W-1|$, and $r = \sigma_W/\sigma_Y$.

Then Equation (2) becomes $\quad b\,|Y-1| \le \sum_{i=0}^{b-1} W(i)\,|Y(i)-1| + \sum_{i=0}^{b-1} |W(i)-1|$, i.e.,

$$b\, Z_Y \le \sum_{i=0}^{b-1} W(i)\, Z_{Y(i)} + r \sum_{i=0}^{b-1} Z_{W(i)}, \tag{11}$$

which yields

$$\big(b^2 - \mathbb{E}(W^3)\big)\mathbb{E}(Z_Y^3) \le r^3 \mathbb{E}(Z_W^3) + \sum_{i=0}^{3} \binom{3}{i} r^i\, T_i,$$

where

$$\begin{aligned} T_0 &= 3(b-1)\,\mathbb{E}\, W^2\, \mathbb{E}\, Z_Y^2\, \mathbb{E}\, Z_Y + (b-1)(b-2)(\mathbb{E}\, Z_Y)^3, \\ T_1 &= \mathbb{E}\,(W^2 Z_W)\, \mathbb{E}\, Z_Y^2 + 2(b-1)\, \mathbb{E}(W Z_W)(\mathbb{E}\, Z_Y)^2 \\ &\quad + (b-1)\,\mathbb{E}\, Z_W\, \mathbb{E}\, W^2\, \mathbb{E}\, Z_Y^2 + (b-1)(b-2)\,\mathbb{E}\, Z_W (\mathbb{E}\, Z_Y)^2, \\ T_2 &= \mathbb{E}\, Z_Y\, \mathbb{E}\,(W Z_W^2) + 2(b-1)\,\mathbb{E}(W Z_W)\, \mathbb{E}\, Z_Y\, \mathbb{E}\, Z_W \\ &\quad + (b-1)\,\mathbb{E}\, Z_Y\, \mathbb{E}\, Z_W^2 + (b-1)(b-2)\,\mathbb{E}\, Z_Y (\mathbb{E}\, Z_W)^2, \\ T_3 &= 3(b-1)\,\mathbb{E}\, Z_W\, \mathbb{E}\, Z_W^2 + (b-1)(b-2)(\mathbb{E}\, Z_W)^3. \end{aligned}$$

As, for $X \in \{W, Y\}$ we have $\mathbb{E}\, Z_X \le (\mathbb{E}\, Z_X^2)^{1/2} = 1$, and $\mathbb{E}\, X^2 < b - 1$, we get the simpler bound

$$\big(b^2 - \mathbb{E}\, W^3\big)\, \mathbb{E}\, Z_Y^3 \le r^3\, \mathbb{E}\, Z_W^3 + \sum_{i=0}^{3} \binom{3}{i} r^i\, T_i',$$

where

$$\begin{aligned} T_0' &= (b-1)(4b-5), \\ T_1' &= \mathbb{E}\,(W^2 Z_W) + 2(b-1)\,\mathbb{E}\,(W Z_W) + (b-1)(2b-3), \\ T_2' &= \mathbb{E}\,(W Z_W^2) + 2(b-1)\,\mathbb{E}\,(W Z_W) + (b-1)^2, \\ T_3' &= b^2 - 1. \end{aligned}$$

Since $\mathbb{E}\, W^3 < b^2 - 1$, the Hölder inequality yields

$$\mathbb{E}\,(W^2 Z_W) \le (\mathbb{E}\, W^3)^{2/3} (\mathbb{E}\, Z_W^3)^{1/3} \le (b^2-1)^{2/3} (\mathbb{E}\, Z_W^3)^{1/3}$$

and

$$\mathbb{E}\,(W Z_W^2) \le (\mathbb{E}\,W^3)^{1/3}(\mathbb{E}\,Z_W^3)^{2/3} \le (b^2-1)^{1/3}(\mathbb{E}\,Z_W^3)^{2/3}.$$

Furthermore, $\mathbb{E}\,(W Z_W) \le \left(\mathbb{E}\,W^2\,\mathbb{E}\,Z_W^2\right)^{1/2} \le \sqrt{b-1}$.

We know from (7) that $r < \sqrt{b-1}$. Therefore, there exists a constant $C > 0$ independent of μ such that

$$\left(b^2 - \mathbb{E}\,W^3\right)\mathbb{E}\,Z_Y^3 \le (b-1)^{3/2}\,\mathbb{E}\,Z_W^3 + C\left((\mathbb{E}\,Z_W^3)^{2/3} + (\mathbb{E}\,Z_W^3)^{1/3} + 1\right).$$

□

Corollary 6. *If $\mu \in \mathscr{D}_b$ then* $\sup_{n\ge 1} \int \sigma_n^{-3}\,|x-1|^3\,\mathsf{T}^n\mu(\mathrm{d}x) < \infty.$

Proof. Since $b^2 - \mathbb{E}\,W_n^3$ converges towards b^2-1, and $b^2-1 > (b-1)^{3/2}$, the bound in the last proposition yields that Z_n is bounded in L^3. □

This accounts for the first two assertions of Theorem 1. Proving its last assertion requires a careful iteration of Formula (4). Recall that we set $Z_n = \dfrac{W_n - 1}{\sigma_n}$. Equation (4) yields

$$Z_{n+1} = \frac{1}{b}\sum_{k=0}^{b-1}\left[\sigma_n\,Z_n(k)\,Z_{n+1}(k) + \frac{\sigma_n}{\sigma_{n+1}}\,Z_n(k) + Z_{n+1}(k)\right]. \tag{12}$$

If we set

$$R_n = \frac{1}{b}\sum_{j=0}^{b-1} Z_n(j)Z_{n-1}(j)\sigma_{n-1} + \frac{1}{b}\left(\frac{\sigma_{n-1}}{\sigma_n} - \sqrt{b-1}\right)\sum_{j=0}^{b-1} Z_{n-1}(j), \tag{13}$$

then Equation (12) rewrites as

$$Z_{n+1} = R_{n+1} + \frac{\sqrt{b-1}}{b}\sum_{k=0}^{b-1} Z_n(k) + \frac{1}{b}\sum_{k=0}^{b-1} Z_{n+1}(k). \tag{14}$$

We are going to use repeatedly Formula (14). Let ϵ stand for empty word on any alphabet. Fix $n > 1$, define $R_n(\epsilon,\epsilon) = R_n$ as well as $Z_n(\epsilon,\epsilon) = Z_n$, and write using (14)

$$Z_n = Z_n(\epsilon,\epsilon) = R_n(\epsilon,\epsilon) + \frac{\sqrt{b-1}}{b}\sum_{j\in\mathscr{A}} Z_{n-1}(j,0) + \frac{1}{b}\sum_{j\in\mathscr{A}} Z_n(j,1). \tag{15}$$

Since we are interested in distributions only, we can take copies of these variables so that we can write

$$\begin{aligned}
Z_n(j,1) &= R_n(j,1) + \frac{\sqrt{b-1}}{b}\sum_{k\in\mathscr{A}} Z_{n-1}(jk,10) + \frac{1}{b}\sum_{k\in\mathscr{A}} Z_n(jk,11)\\
Z_{n-1}(j,0) &= R_{n-1}(j,0) + \frac{\sqrt{b-1}}{b}\sum_{k\in\mathscr{A}} Z_{n-2}(jk,00) + \frac{1}{b}\sum_{k\in\mathscr{A}} Z_{n-1}(jk,01).
\end{aligned}$$

Notice that since by definition in Formula (15) the random variables of the form $Z_{n-1}(j,w)$ and $Z_n(j,w)$, $(j,w) \in \mathscr{A}\times\{0,1\}$, are mutually independent, and the same holds for the random variables $R_n(j,w)$ and $R_{n-1}(j,w)$, $(j,w) \in \mathscr{A}\times\{0,1\}$, as well as for the random variables $Z_{n-2}(jk,w)$, $Z_{n-1}(jk,w)$ and $Z_n(jk,w)$, $(jk,w) \in \mathscr{A}^2\times\{0,1\}^2$.

Then Formula (15) rewrites as

$$Z_n(\epsilon,\epsilon) = R_n(\epsilon,\epsilon) + b^{-1}\sum_{j\in\mathscr{A}}\Big(\sqrt{b-1}\,R_{n-1}(j,0) + R_n(j,1)\Big) +$$

$$b^{-2}\sum_{w\in\mathscr{A}^2}\Big((b-1)\,Z_{n-2}(w,00) + \sqrt{b-1}\,\big(Z_{n-1}(w,01) + Z_{n-1}(w,10)\big) + Z_n(w,11)\Big)$$

and so on. At last we get $Z_n = T_{1,n} + T_{2,n}$, with

$$T_{1,n} = \sum_{k=0}^{n-1} b^{-k} \sum_{\substack{m\in\{0,1\}^k\\ w\in\mathscr{A}^k}} (b-1)^{(k-\varsigma(m))/2} R_{n-k+\varsigma(m)}(w,m) \tag{16}$$

$$T_{2,n} = b^{-n} \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{A}^n}} (b-1)^{(n-\varsigma(m))/2} Z_{\varsigma(m)}(w,m), \tag{17}$$

where $\varsigma(m)$ stands for the sum of the components of m. Moreover, all variables in Equation (17) are independent, and in Equation (16), the variables corresponding to the same k are independent.

For reader's convenience, we also provide a more constructive approach to obtain the previous decomposition of Z_n. At first, we notice that the meaning of Equation (14) is the following: given independent variables $Z_n(k)$ and $Z_{n+1}(k)$ (for $0 \le k < b$) equidistributed with Z_n and Z_{n+1}, if we define R_n by Equation (13), then the right hand side of Equation (14) is equidistributed with Z_{n+1}.

Let n be fixed larger than 2. One starts with a collection

$$\big\{Z_l(w,m)\big\}_{0\le l\le n,\, w\in\mathscr{A}^n,\, \{0,1\}^n}$$

of independent random variables such that the $Z_l(\cdot,\cdot)$ have the same distribution as Z_l.

One defines by recursion

$$R_l(w,m) = \frac{1}{b}\sum_{j=0}^{b-1} Z_{l-1}(wj,m0)Z_l(wj,m1)\sigma_{l-1} + \frac{1}{b}\left(\frac{\sigma_{l-1}}{\sigma_l} - \sqrt{b-1}\right)\sum_{j=0}^{b-1} Z_{l-1}(wj,m0)$$

and

$$Z_l(w,m) = R_l(w,m) + \frac{\sqrt{b-1}}{b}\sum_{j=0}^{b-1} Z_{l-1}(wj,m0) + \frac{1}{b}\sum_{j=0}^{b-1} Z_l(wj,m1),$$

for $0 \le l \le n$, $(w,m) \in \mathscr{A}^j \times \{0,1\}^j$ with $j \ge n-l$.

Due to (14), all these new variables $Z_l(\cdot.\cdot)$ are equidistributed with Z_l, and we get $Z_n(\epsilon,\epsilon) = T_{1,n} + T_{2,n}$.

Proposition 7. *We have* $\lim_{n\to\infty}\mathbb{E}\,T_{1,n}^2 = 0$*, so* $T_{1,n}$ *converges in distribution to 0.*

Proof. Set $r_n^2 = \mathbb{E}\,R_n^2$. We have

$$b\,r_n^2 = \sigma_{n-1}^2 + \left(\frac{\sigma_{n-1}}{\sigma_n} - \sqrt{b-1}\right)^2,$$

which together with Formulae (7) and (8) implies that there exists $C > 0$ such that $r_n^2 \le C(b-1)^{-n}$ for all $n \ge 1$. By using the independence properties of random variables in (16) as well as the triangle inequality, we obtain

$$\begin{aligned}
\left(\mathbb{E}\,T_{1,n}^2\right)^{1/2} &\le \sum_{0\le k<n} b^{-k}\left(\sum_{0\le j\le k}\binom{k}{j} b^k(b-1)^j\, r_{n-j}^2\right)^{1/2}\\
&\le C\sum_{0\le k<n} b^{-k}\left(\sum_{0\le j\le k}\binom{k}{j} b^k(b-1)^j\,(b-1)^{j-n}\right)^{1/2}\\
&\le C\sum_{0\le k<n} b^{-k/2}\big((b-1)^2+1\big)^{k/2}(b-1)^{-n/2}\\
&= C\,(b-1)^{-n/2}\sum_{0\le k<n}\left(\frac{(b-1)^2+1}{b}\right)^{k/2}\\
&= \mathrm{O}\left(\left(1-\frac{b-2}{b(b-1)}\right)^{n/2}\right).
\end{aligned}$$

□

Proposition 8. *If there exists $p>2$ such that*

$$\sup_{n\ge1}\int\left(\frac{|x-1|}{\sigma_n}\right)^p \mathsf{T}^n\mu(\mathrm{d}x)<\infty,$$

(i.e., $(|Z_n|)_{n\ge1}$ is bounded in L^p), then $T_{2,n}$ converges in distribution to $\mathcal{N}(0,1)$.

Proof. If Y is a positive random variable, a, p and ε are positive numbers with $p>2$, we have

$$\begin{aligned}
\mathbb{E}\left(a^2Y^2\,\mathbf{1}_{\{aY>\varepsilon\}}\right) &\le a^2\,(\mathbb{E}\,Y^p)^{2/p}\,\big(\mathbb{P}(aY>\varepsilon)\big)^{1-2/p}\\
&\le a^2\,(\mathbb{E}\,Y^p)^{2/p}\,\left(\varepsilon^{-p}a^p\,\mathbb{E}\,Y^p\right)^{1-2/p} = a^p\varepsilon^{2-p}\,\mathbb{E}\,Y^p.
\end{aligned}$$

So, we have

$$\sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{A}^n}} b^{-2n}(b-1)^{(n-\varsigma(m))}\,\mathbb{E}\left(Z_{\varsigma(m)}(w,m)^2\mathbf{1}_{\left\{b^{-n}(b-1)^{(n-\varsigma(m))/2}|Z_{\varsigma(m)}(w,m)|>\varepsilon\right\}}\right)$$

$$\le\sum_{k=0}^n\binom{n}{k} b^{n-np}(b-1)^{p(n-k)/2}\varepsilon^{2-p}\,\mathbb{E}\,|Z_k|^p\le\varepsilon^{2-p}\left(\frac{(b-1)^{p/2}+1}{b^{p-1}}\right)^n\sup_{k\ge0}\mathbb{E}\,|Z_k|^p,$$

and this last expression converges towards 0 as n goes to ∞. But, as we have

$$\mathbb{E}\,T_{2,n}^2=\sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{A}^n}} b^{-2n}(b-1)^{n-\varsigma(m)}=\sum_{k=0}^n\binom{n}{k}b^{-n}(b-1)^{n-k}=1,$$

the Lindeberg theorem yields the conclusion. □

3. Proof of Theorem 2

We begin by the following observation: for any real function f on $[0,1]$, one has

$$\omega(f,\delta)\le 2(b-1)\sum_{j\ge-\frac{\log\delta}{\log b}}\ \sup_{w\in\mathscr{A}^j}\Delta(f,I_w), \tag{18}$$

where, $\omega(f,\delta)$ stands for the modulus of continuity of a function f on $[0,1]$:

$$\omega(f,\delta)=\sup_{\substack{t,s\in[0,1]\\ |t-s|\le\delta}}|f(t)-f(s)|.$$

Proposition 9. *Suppose that $\mu \in \mathscr{P}_b$. If h_n is a random continuous function distributed according to $F(\mathsf{T}^{n-1}\mu)$, set $\mathcal{Z}_n = \dfrac{h_n - \mathrm{Id}}{\sigma_n}$. The probability distributions of the random continuous functions $\mathcal{Z}_n$, $n \geq 1$, form a tight sequence.*

Proof. By Theorem 7.3 of [3], since $(h_n - \mathrm{Id})(0) = 0$ almost surely for all $n \geq 1$, it is enough to show that for each positive ε

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\big(\omega(\mathcal{Z}_n, \delta) \geq 2(b-1)\,\varepsilon\big) = 0, \tag{19}$$

We first establish the following lemma.

Lemma 10. *Let γ and H be two positive numbers such that $2H + \gamma - 1 < 0$. Also let $n_0 \geq 1$ be such that $\sup_{n \geq n_0 - 1} \mathbb{E}\, W_n^2 \leq b^\gamma$. For $j \geq 1$, $n \geq n_0$ and $t > 0$ we have*

$$\mathbb{P}\Big(\sup_{w \in \mathscr{A}^j} \Delta\big(\mathcal{Z}_n, I_w\big) \geq t\, b^{-jH}\Big) \leq (b-1)\, t^{-2} (j+1)^3 b^{j(2H+\gamma-1)}.$$

Proof. Let $j \geq 1$, $w \in \mathscr{A}^j$ and $n \geq n_0$. Formula (6) shows that the increment $\Delta_n(w) = \Delta(\mathcal{Z}_n, I_w)$ takes the form

$$\begin{aligned} \Delta_n(w) &= b^{-j}\sigma_n^{-1}\left[W_n(w)\prod_{k=1}^{j} W_{n-1}(w|_k) - 1\right] \\ &= b^{-j} Z_n(w) \prod_{k=1}^{j} W_{n-1}(w|_k) \\ &\quad + b^{-j} \sum_{l=1}^{j} \frac{\sigma_{n-1}}{\sigma_n} Z_{n-1}(w|_l) \prod_{k=1}^{l-1} W_{n-1}(w|_k). \end{aligned} \tag{20}$$

Consequently,

$$\begin{aligned} \mathbb{P}\left(|\Delta_n(w)| \geq t\, b^{-jH}\right) &\leq \mathbb{P}\left(b^{-j} Z_n(w) \prod_{k=1}^{j} W_{n-1}(w|_k) \geq \frac{t\, b^{-jH}}{j+1}\right) \\ &\quad + \sum_{l=1}^{j} \mathbb{P}\left(b^{-j} \frac{\sigma_{n-1}}{\sigma_n} Z_{n-1}(w|_l) \prod_{k=1}^{l-1} W_{n-1}(w|_k) \geq \frac{t\, b^{-jH}}{j+1}\right). \end{aligned}$$

By using the Markov inequality, the equality $\mathbb{E}\, Z_k^2 = 1$, and the fact that $\mathbb{E}\, W_{n-1}^2 \geq 1$, we obtain that each probability in the previous sum is less than

$$(b-1)\, t^{-2} (j+1)^2 b^{-2(1-H)j} \left(\mathbb{E}\, W_{n-1}^2\right)^j,$$

so that the sum of these probabilities is bounded by $(b-1)\, t^{-2}(j+1)^3 b^{j(\gamma - 2(1-H))}$. Consequently,

$$\begin{aligned} \mathbb{P}\left(\exists\ w \in \mathscr{A}^j,\ |\Delta_n(w)| \geq t\, b^{-jH}\right) &\leq (b-1)\, t^{-2}(j+1)^3 b^{j(\gamma - 2(1-H)+1)} \\ &= (b-1)\, t^{-2}(j+1)^3 b^{j(2H+\gamma-1)}. \end{aligned}$$

□

Now, we can continue the proof of Proposition 9. Fix H, γ, and n_0 as in Lemma 10, set $j_\delta = -\log_b \delta$, and assume that $n \geq n_0$. Due to (18) and Lemma 10, we have

$$\begin{aligned} \big\{\omega(\mathcal{Z}_n, \delta) \geq 2(b-1)\,\varepsilon\big\} &\subset \left\{\sum_{j \geq j_\delta} \sup_{w \in \mathscr{A}^j} \Delta(\mathcal{Z}_n, I_w) > \varepsilon\right\} \\ &\subset \bigcup_{j \geq j_\delta} \left\{\sup_{w \in \mathscr{A}^j} \Delta(\mathcal{Z}, I_w) > (1 - b^{-H})\, b^{j_\delta H}\, \varepsilon\, b^{-jH}\right\}, \end{aligned}$$

so

$$\mathbb{P}\big(\omega(\mathscr{Z}_n,\delta)\geq 2(b-1)\,\varepsilon\big)\leq \frac{(b-1)\,b^{-2j_\delta H}}{(1-b^{-H})^2\varepsilon^2}\sum_{j\geq j_\delta}(j+1)^3 b^{(2H+\gamma-1)\,j}.$$

Consequently,

$$\lim_{\delta\to 0}\sup_{n\geq n_0}\mathcal{P}\big(\omega(\mathscr{Z}_n,\delta)>2(b-1)\,\varepsilon\big)=0.$$

□

Proposition 11. *Suppose that $\mu\in\mathscr{D}_b$. For every $n\geq 1$ let h_n be a random continuous function whose probability distribution is $F(\mathsf{T}^n\mu)$. Fix $j\geq 1$. The probability distribution of the vector $\Big(\Delta(\frac{h_n-\mathrm{Id}}{\sigma_n},I_w)\Big)_{w\in\mathscr{A}^j}$ converges, as n goes to ∞, to that of a Gaussian vector whose covariance matrix M_j is given by*

$$\mathsf{M}_j(w,w')=\begin{cases} b^{-2j}(1+(b-1)|w|) & \textit{if } w=w',\\ b^{-2j}(b-1)|w\wedge w'| & \textit{otherwise}\end{cases}.$$

Proof. We use the same notations as in the proof of Lemma 10. Let $j\geq 1$ and $w\in\mathscr{A}^j$. In the right hand side of (20), the random variables $Z_n(w)$ and $Z_{n-1}(w|_l)$, $1\leq l\leq j$, are independent and their probability distribution converge weakly to $\mathcal{N}(0,1)$, while the common probability distribution of the $W_{n-1}(w|_l)$, $1\leq l\leq j$, converges to δ_1, and $\frac{\sigma_{n-1}}{\sigma_n}$ converges to $\sqrt{b-1}$.

This implies that there exist $\big(\mathcal{N}(v)\big)_{v\in\bigcup_{k=1}^j\mathscr{A}^k}$ and $\big(\widetilde{\mathcal{N}}(w)\big)_{w\in\mathscr{A}^j}$ two families of $\mathcal{N}(0,1)$ random variables so that all the random variables involved in these families are independent, and

$$\lim_{n\to\infty}\big(\Delta_n(w)\big)_{w\in\mathscr{A}^j}\stackrel{\text{dist}}{=} b^{-j}\left(\widetilde{\mathcal{N}}(w)+\sqrt{b-1}\sum_{k=1}^{j}\mathcal{N}(w|_k)\right)_{w\in\mathscr{A}^j}. \tag{21}$$

The fact that the vector in the right hand side of (21) is Gaussian is an immediate consequence of the independence between the normal laws involved in its definition. The computation of the covariance matrix is left to the reader. □

4. The limit process as the limit of an additive cascade

Recall that, if $v\in\mathscr{A}^*$, $[v]$ stands for the cylinder in $\mathscr{A}^\omega$ consisting of sequences beginning by v. Let $\mathscr{A}^+$ stand for the set of non-empty words on the alphabet $\mathscr{A}$.

We are going to show that there exists a finitely additive random measure M on $\mathscr{A}^\omega$ satisfying almost surely for all $w\in\mathscr{A}^+$

$$M([w])=b^{-j}\left(\zeta(w)+\sqrt{b-1}\sum_{k=1}^{j}\xi(w|_k)\right),\text{where } j=|w| \tag{22}$$

instead of (21), where the variables $\big(\xi(w)\big)_{w\in\mathscr{A}^+}$ are independent with common distribution $\mathcal{N}(0,1)$ and the variable $\zeta(w)$ is $\mathcal{N}(0,1)$ and independent of $\big(\xi(w|_j)\big)_{1\leq j\leq |w|}$.

Indeed, if we set $S(w)=\sum_{k=1}^{j}\xi(w|_k)$, since $M([w])=\sum_{\ell\in\mathscr{A}}M([w\ell])$ we should have

$$\begin{aligned} b\left(\zeta(w)+\sqrt{b-1}\,S(w)\right) &= \sum_{\ell\in\mathscr{A}}\left(\zeta(w\ell)+\sqrt{b-1}\sum_{k=1}^{j+1}\xi\big((w\ell)|_k\big)\right)\\ &= \sum_{\ell\in\mathscr{A}}\Big(\zeta(w\ell)+\sqrt{b-1}\,\xi(w\ell)\Big)+b\sqrt{b-1}\,S(w).\end{aligned}$$

Iterating this last formula, gives

$$\zeta(w)=b^{-n}\sum_{v\in\mathscr{A}^n}\zeta(wv)+\sqrt{b-1}\sum_{j=1}^{n}b^{-j}\sum_{v\in\mathscr{A}^j}\xi(wv).$$

The first term of the right hand side converges to 0 with probability 1 since its L^2 norm is $b^{-n/2}$. The second term is a martingale bounded in L^2 norm. Therefore its limit, a $\mathcal{N}(0,1)$ variable, is a.s. equal to $\zeta(w)$.

Finally, we get a finitely additive Gaussian random measure defined on the cylinders of $\mathscr{A}^\omega$ by

$$M([w])=b^{-|w|}\sqrt{b-1}\left(\lim_{n\to\infty}\sum_{v\in\bigcup_{k=1}^n\mathscr{A}^k}b^{-|v|}\xi(wv)+\sum_{1\le k\le|w|}\xi(w|_k)\right). \tag{23}$$

Then, the limit process of the previous sections can be seen as the primitive of the projection of M on $[0,1]$.

Of course (23) makes sense even for $b=2$.

It is easy to compute covariances:

$$\mathbb{E}\big(M([v])M([w])\big)=\begin{cases}b^{-2|v|}(1+(b-1)|v|) & \text{if } w=v,\\ (b-1)\,b^{-(|v|+|w|)}|v\wedge w| & \text{otherwise.}\end{cases}$$

It is then straighforward to check that, with probability 1, for all $\varepsilon>0$ we have $\sup_{v\in\mathscr{A}^n}|M([v])|=\mathrm{o}(b^{-n(1-\varepsilon)})$. This can be refined, in particular thanks to the multifractal analysis of the branching random walk $S(w)=\sum_{1\le j\le|w|}\xi(w|_j)$. In term of the associated Gaussian process $(X_t)_{t\in[0,1]}$, it is natural to consider for all $\alpha\in\mathbb{R}$ the sets

$$\overline{E}_\alpha=\left\{t\in[0,1):\limsup_{n\to\infty}\frac{\Delta(X,I_n(t))}{nb^{-n}}=\alpha\sqrt{b-1}\right\},$$

$$\underline{E}_\alpha=\left\{t\in[0,1):\liminf_{n\to\infty}\frac{\Delta(X,I_n(t))}{nb^{-n}}=\alpha\sqrt{b-1}\right\},$$

and

$$E_\alpha=\underline{E}_\alpha\bigcap\overline{E}_\alpha,$$

where $I_n(t)$ stands for the semi-open to the right b-adic interval of generation n containing t.

In the next statement, dim E stands for the Hausdorff dimension of the set E.

Theorem 12. *With probability 1,*

1. *the modulus of continuity of X is a $\mathrm{O}\big(\delta\log(1/\delta)\big)$,*
2. *X does not belong to the Zygmund class,*
3. *the set E_0 contains a set of full Lebesgue measure at each point of which X is not differentiable,*
4. *$\dim E_\alpha=\dim\underline{E}_\alpha=\dim\overline{E}_\alpha=1-\dfrac{\alpha^2}{2\log b}$ if $|\alpha|\le\sqrt{2\log b}$, and $E_\alpha=\emptyset$ if $|\alpha|>\sqrt{2\log b}$. Furthermore, $E_{-\sqrt{2\log b}}$ and $E_{\sqrt{2\log b}}$ are nonempty.*

Remark 2. We do not know whether there are points in E_0 at which X is differentiable. We do not either if the pointwise regularity of X is 1 everywhere.

Proof. For $w\in\mathscr{A}^*$, as above we set $\zeta(w)=\sqrt{b-1}\lim_{n\to\infty}\sum_{v\in\bigcup_{k=1}^n\mathscr{A}^k}b^{-|v|}\xi(wv)$.

We have $\sum_{n\ge1}\mathbb{P}\big(\exists\ w\in\mathscr{A}^n,\ |\zeta(w)|>2\sqrt{2\log b}\sqrt{n}\big)<\infty$, hence, with probability 1, $\sup_{w\in\mathscr{A}^n}|\zeta(w)|=\mathrm{O}(\sqrt{n})$.

Also, $\sum_{n\geq 1}\mathbb{P}\big(\exists\ w\in\mathscr{A}^n,\ |S(w)|>2n\sqrt{2\log b}\big)<\infty$ and, with probability 1, $\sup_{w\in\mathscr{A}^n}|S(w)|=\mathrm{O}(n)$. This yields the property regarding the modulus of continuity thanks to (18). This proves the first assertion.

To see that X is not in the Zygmund class, it is enough to find $t\in(0,1)$ such that $\limsup_{h\to 0,h\neq 0}\left|\dfrac{f(t+h)+f(t-h)-2f(h)}{h}\right|=\infty$. Take $t=b^{-1}$ and $h=b^{-n}$. Let $\underline{t}$ and $\overline{t}$ stand for the infinite words $0(b-1)(b-1)\dots(b-1)\cdots$ and $1(b-1)(b-1)\dots(b-1)\cdots$. We have

$$\left|\frac{f(t+h)+f(t-h)-2f(h)}{h\sqrt{b-1}}\right|=\left|\frac{\zeta(\overline{t}_{|n})-\zeta(\underline{t}_{|n})}{\sqrt{b-1}}+S(\overline{t}_{|n})-S(\underline{t}_{|n})\right|.$$

Since $|\zeta(\overline{t}_{|n})-\zeta(\underline{t}_{|n})|=\mathrm{O}(\sqrt{n})$ and since the random walks $S(\overline{t}_{|n})$ and $S(\underline{t}_{|n})$ are independent, the law of the iterated logarithm yields the desired behavior as $h=b^{-n}$ goes to 0.

The fact that E_0 contains a set of full Lebesgue measure on which X is nowhere differentiable is a consequence of the Fubini theorem combined with the property $|\zeta(\widetilde{t}_{|n})|=O(\sqrt{n})$ and the law of the iterated logarithm which almost surely holds for the random walk $(S(\widetilde{t}_{|n}))_{n\geq 1}$ for each $t\in[0,1]$.

Since, with probability 1, we have $\sup_{w\in\mathscr{A}^n}|\zeta(w)|=\mathrm{O}(\sqrt{n})$, we only have to take into account the term $S(w)$ in the asymptotic behavior of $\dfrac{\Delta(X,I_w)}{|w|b^{|w|}}$ as $|w|$ goes to ∞. Thus, in the definition of the sets $\underline{E}_\alpha$, $\overline{E}_\alpha$ and E_α, $\dfrac{\Delta(X,I_w)}{\sqrt{b-1}|w|b^{|w|}}$ can be replaced by $\dfrac{S(w)}{|w|}$. Then the result is mainly a consequence of the work [2] on the multifractal analysis of Mandelbrot measures.

To get an upper bound for the Hausdorff dimensions, we set

$$\beta(q)=\liminf_{n\to\infty}-\frac{1}{n}\log_b\sum_{w\in\mathscr{A}^n}\exp(qS(w))$$

for $q\in\mathbb{R}$. Standard large deviation estimates show that $\dim F_\alpha\leq\inf_{q\in\mathbb{R}}\dfrac{-\alpha q}{\log b}-\beta(q)$ for all $\alpha\in\mathbb{R}$ and $F\in\{\underline{E},\overline{E},E\}$ (the occurrence of a negative dimension meaning that the corresponding set is empty). Also, using the fact that $\beta(q)$ is the supremum of those numbers t such that $\limsup_{n\to\infty}b^{nt}\sum_{w\in\mathscr{A}^n}\exp(qS(w))<\infty$ yields

$$\beta(q)\geq\lim_{n\to\infty}-\frac{1}{n}\log_b\mathbb{E}\sum_{w\in\mathscr{A}^n}\exp(qS(w))=-1-\frac{q^2}{2\log b}.$$

Since both sides of this inequality are concave functions, we actually have, with probability 1, $\beta(q)\geq -1-q^2/2\log b$ for all $q\in\mathbb{R}$. Consequently, the upper bound for the dimension used with $\alpha=-\beta'(q)\log b=q$ yields, with probability 1, $\dim F_q\leq 1-q^2/2$ for all $q\in[-\sqrt{2\log b},\sqrt{2\log b}]$ and $F_q=\emptyset$ if $|q|>\sqrt{2\log b}$.

For the lower bounds, we only have to consider the sets E_α.

If $q\in[-\sqrt{2\log b},\sqrt{2\log b}]$, let ϕ_q be the non-decreasing continuous function associated with the family $\big(W_q(w)=\exp(qW(w)-q^2/2)\big)_{w\in\mathscr{A}^+}$ as ϕ was with $\big(W(w)\big)_{w\in\mathscr{A}^+}$ in Section 1. We learn from [2] that, with probability 1, all the functions ϕ_q, $q\in(-\sqrt{2\log b},\sqrt{2\log b})$ are simultaneously defined; their derivatives (in the sense of distributions) are positive measures denoted by μ_q. Then, computations very similar to those used to perform the multifractal analysis of μ_1 in [2] show that, with probability 1, for all $q\in(-\sqrt{2\log b},\sqrt{2\log b})$ the dimension of μ_q is $1-q^2/2\log b$ and $\mu_q(E_q)>0$.

For $q \in \{-\sqrt{2\log b}, \sqrt{2\log b}\}$ it turns out (see [8, 2]) that the formula

$$\mu_q(I_w) = \lim_{p\to\infty} -\sum_{v\in\mathscr{A}^p} \Pi(wv)\log\Pi(wv),$$

where

$$\Pi(wv) = b^{-|w|+p}\prod_{k=1}^{|w|} W_q(w_{|k})\prod_{j=1}^{p} W_q(wv_{|j}),$$

defines almost surely a positive measure carried by E_q. □

5. Other random Gaussian measures and processes

This time $b \geq 2$, $\big(\xi(w)\big)_{w\in\mathscr{A}^+}$ is a sequence of independent $\mathcal{N}(0,1)$ variables, and $\big(\alpha(w)\big)_{w\in\mathscr{A}^+}$ and $\big(\beta(w)\big)_{w\in\mathscr{A}^+}$ are sequences of numbers subject to the conditions

$$\alpha(w) = \sum_{\ell\in\mathscr{A}} \alpha(w\ell),$$

$$\sum_{v\in\mathscr{A}^+} |\alpha(wv)|^p|\beta(wv)|^p < \infty \quad \text{for some } p\in(1,2].$$

Then, for all $w \in \mathscr{A}^+$, the martingale $\sum_{v\in\mathscr{A}^n} \alpha(wv)\,\beta(wv)\,\xi(wv)$ is bounded L^p norm (if $p < 2$ this uses an inequality from [1]) and the formula

$$M([w]) = \lim_{n\to\infty} \sum_{v\in\bigcup_{k=1}^n \mathscr{A}^k} \alpha(wv)\,\beta(wv)\,\xi(wv) + \alpha(w)\sum_{1\leq j\leq|w|} \beta(w|_j)\,\xi(w|_j) \tag{24}$$

almost surely defines a random measure which generalizes the one considered in the previous sections. Here again, the primitive of the projection on $[0,1]$ of this measure defines a continuous process, which is Gaussian if $p = 2$.

Remark 3. The hypotheses under which this last construction can be performed can be relaxed: if the random variables $\xi(w)$, $w \in \mathscr{A}^+$, are independent, centered, and $\sum_{v\in\mathscr{A}^+} |\alpha(wv)|^p|\beta(wv)|^p\mathbb{E}(|\xi(wv)|^p) < \infty$, Formula (24) still yields a random measure.

The fine study of the associate process as well as some improvement of Theorem 12 will be achieved in a further work.

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