CONTINUITY OF THE MULTIFRACTAL SPECTRUM OF A RANDOM STATISTICALLY SELF-SIMILAR MEASURE

JULIEN BARRAL

Université Paris-Sud

June 1999 (Revised version)

ABSTRACT. Until now ([19],[16],[14],[29],[28],[1],[3]), one determines the multifractal spectrum of a statistically self-similar positive measure of the type introduced, in particular by Mandelbrot in [26], [27], only in the following way: let μ be such a measure, for example on the boundary of a c-ary tree equipped with the standard ultrametric distance; for $\alpha \geq 0$, denote by E_α the set of the points where μ possesses a local Hölder exponent equal to α, and dim E_α the Hausdorff dimension of E_α; then, there exists a deterministic open interval $I \subset \mathbb{R}_+^*$ and a function $f : I \to \mathbb{R}_+^*$ such that for all α in I, with probability one, dim $E_\alpha = f(\alpha)$. This statement is not completely satisfactory. Indeed, the main result in this paper is: with probability one, for all $\alpha \in I$, dim $E_\alpha = f(\alpha)$. This holds also for a new type of statistically self-similar measures deduced from a result recently obtained by Liu [23]. We also study another problem left open in the previous works on the subject: if $\alpha = \inf(I)$ or $\alpha = \sup(I)$, one does not know whether E_α is empty or not. Under suitable assumptions, we show that $E_\alpha \neq \emptyset$ and calculate dim E_α.

1. INTRODUCTION.

The random multiplicative and statistically self-similar non negative measures introduced by Mandelbrot in [26] (1974) and their variants are limits of martingales obtained by multiplying in cascades i.i.d. non-negative random variables. These measures are constructed on the boundary of a c-ary tree equipped with a statistically self-similar distance ([18], [19], [16], [14], [29], [1], [3]) or on a geometrical projection in $\mathbb{R}^n$ of such a tree, like c-adic sub-cubes of $[0,1[^n$ ([26], [27], [20], [10], [28]), random intervals ([32], [4], [3]), or random self-similar Cantor sets ([14], [29], [1]). A fundamental point is that given μ such a measure, its properties depend essentially on the following equation (E) satisfied by the Laplace transform Φ_Y of the probability distribution of $Y = \|\mu\|$: for all $t \geq 0$, $\Phi_Y(t) = \mathbb{E}(\prod_{j=0}^{c-1} \Phi_Y(tW_j))$ (E) where c is an integer ≥ 2 and $(W_0, \dots, W_{c-1}) \in \mathbb{R}_+^c$ a random vector satisfying $E(\sum_{j=0}^{c-1} W_j) = 1$ and $P(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j>0\}} > 1) > 0$. The first important result

1991 *Mathematics Subject Classification.* 60 G 57, 28 A 75, 60 F 10, 60 D 05, 60 J 99, 60 G 17, 76 F 99.

Key words and phrases. Multifractal analysis, statistically self-similar measures, Mandelbrot's martingales, multiplicative cascades.

about μ is the following: $P(\mu \neq 0) > 0$, *i.e.* Φ_Y is non trivial if and only if $\mathbb{E}(\sum_{j=0}^{c-1} W_j \log W_j) < 0$ (Kahane [20], Durrett and Liggett [11]).

In the 80's, Durrett and Liggett [11] and Guivarc'h [15] studied the existence of non trivial solutions for (E) in the space of Laplace transforms of probability distributions. When $\mathbb{E}(\sum_{j=0}^{c-1} W_j \log W_j) = 0$, the critical situation where the Mandelbrot process degenerates, Durrett and Liggett proved (when the $W_j's$ are not all equal to 0 or 1) the existence of a non trivial but non explicit solution $\tilde{\Phi}$ of (E). Recently in [23], Liu constructed under some additional conditions an explicit modification of the Mandelbrot martingale, which converges to a non negative random variable $\tilde{Y}$ such that $\Phi_{\tilde{Y}} = \tilde{\Phi}$. It is remarkable that this construction provides a new type of random multiplicative and statistically self-similar non negative measures $\tilde{\mu}$. We extend to $\tilde{\mu}$ the second important result about μ due to Peyrière [31], [20]: there exists $D > 0$, (resp. $D = 0$), such that with probability one, μ (resp. $\tilde{\mu}$) is carried by the set E_D of the points of its support where the local Hölder exponent equals D; moreover the Hausdorff dimension of E_D is D.

From the begining of the 90's, many authors (Holley and Waymire [16], Kahane [19], Falconer [14], Olsen [29], Arbeiter and Patzshke [1], Molchan [28], Barral [3]), have taken an interest in the multifractal analysis of μ: given $\alpha \geq 0$, what is the Hausdorff dimension of the set E_α of the points of the support of μ where μ possesses a local Hölder exponent equal to α ? The answer is frequently formulated as follows: there exists I, a deterministic open subinterval of $\mathbb{R}_+$, and τ a convex function on $\mathbb{R}$, such that:
i) for every $\alpha \in I$, with probability 1, $\dim E_\alpha = \inf_{q\in\mathbb{R}} \alpha q + \tau(q) = \tau^*(\alpha) > 0$;
ii) with probability 1, if $\alpha \notin [\inf(I), \sup(I)]$ then $E_\alpha = \emptyset$.
The various works leading to this result differ of course by the choice of the space on which the measure is constructed, and by the variations in the construction ([4], [29]). But the most important difference is in the hypotheses on the W_j's. The weakest hypothesis are those of [19] (where Kahane determines only the lower bound of $\dim E_\alpha$), [28] and then [3], three independent papers.

However, if the statement *i)* is precise for a given $\alpha \in I$ with probability one, it is not satisfying because it does not give with probability 1 the Hausdorff dimension of E_α for all $\alpha \in I$. One reason for that is the following: given $\alpha \in I$, to compute $\dim E_\alpha$ it is crutial to construct an auxiliary measure μ_α with probability 1; but this construction does not insure the existence of the measures μ_α with probability 1 for all $\alpha \in I$. In [3], we proposed a solution to get round this difficulty by constructing a continuous modification of the process $\alpha \mapsto \|\mu_\alpha\|$, but we did not conclude.

In this paper, by showing that with probability one this modification is analytic, we obtain the stronger result: with probability 1, $\dim E_\alpha = \tau^*(\alpha)$ for all $\alpha \in I$. Our result on multifractal analysis also holds for the measure $\tilde{\mu}$. Moreover, under suitable conditions, we succeed to show that if $\alpha_0 \in \{\inf(I), \sup(I)\}$ then, with probability 1, $E_{\alpha_0} \neq \emptyset$, and we obtain $\dim E_{\alpha_0} = \tau^*(\alpha_0)$.

The paper is organized as follows. We end this part by proving our main result in a simple case. The rest of the paper is devoted to the general case, and the second result annouced in the abstract. In the second part, we give the construction of

Mandelbrot's and Liu's martingales; these martingales converge with probability one to non negative random variables. Theorem 2.2. recalls results on moments of positive orders of these variables; Theorem 2.3. completes the already known results on moments of negative orders and gives a bound for such finite moments. In the third part, we define on the boundary of a c-ary tree ∂T the two types of statististically self-similar measures deduced from the Mandelbrot and Liu constructions; we compare the supports of two such measures, and we notice that when one of these measures is continuous, it is possible to use it to define a statististically self-similar ultrametric distance on its support. In the fourth part, we are given three of these measures constructed simultaneously, μ_1, μ_2 and μ_3, such that their supports ∂T_i $(i = 1, 2, 3)$ satisfy $\partial T_1 \subset \partial T_2 \subset \partial T_3$, and μ_3 is continuous. Then ∂T_3 is equipped with the ultrametric distance d_3 defined with μ_3 in part 3., and we show at μ_1-almost every point of a subset of positive μ_1-measure of ∂T_3, the existence of a local Hölder exponent for μ_2. In the fifth part, we are given μ_2 and μ_3, $(\partial T_3, d_3)$, and we establish the results annonced in the previous paragraph for μ_2 on $(\partial T_3, d_3)$. In the sixth part, we extend our results to geometrical realizations of the measures on ∂T, by projecting them on statististically self-similar subsets of $\mathbb{R}^n$.

1.1. Definitions and notations.

Let c be an integer ≥ 2 and $\mathcal{C}$ be the set $\{0, \ldots, c-1\}$.

For all $n \in \mathbb{N}$, let T_n stand for the set $\mathcal{C}^n$ of words of length n on the alphabet $\mathcal{C}$. For $n = 0$, T_0 consists only in the empty word denoted by ϵ. Let $T = \bigcup_{n\in\mathbb{N}} T_n$. Let $a \in T$. The length of a is denoted by $|a|$. Moreover, if $a \neq \epsilon$ then it is spelt $a_1 \ldots a_{|a|}$ and for $1 \leq k \leq |a|$ one denotes by $a_{|k}$ the word $a_1 \ldots\ a_k$; $a_{|0} = \epsilon$. T is equipped with its natural structure of semi-group: if a and b are in T, the word ab is equal to a (resp. b) if b (resp. a) is equal to ϵ, and elsewhere it is equal to $a_1 \ldots a_{|a|} b_1 \ldots b_{|b|}$.

If $(V(a))_{a\in T}$ is a sequence of elements of $\mathbb{C}^c$, one defines for a in T

$$v(a) = \prod_{0\leq k<|a|} V_{a_{k+1}}(a_{|k}).$$

Let ∂T be the set of infinite words written with $\mathcal{C}$. An element x of ∂T is spelt $x_1 x_2 \ldots$ and for all $n \in \mathbb{N}$, $x_{|n}$ denotes $x_1 \ldots x_n$ if $n \geq 1$; $x_{|0} = \epsilon$.
If $(x, y) \in \partial T^2$ and $x \neq y$, $x \wedge y$ denotes $x_{|n}$ where $n = \max\{p \geq 0;\ x_{|p} = y_{|p}\}$; $x \wedge x = x$.

For $n \in \mathbb{N}$ and $a \in T_n$, $I_a = \{x \in \partial T;\ x_{|n} = a\}$ is the cylinder generated by a, and for $x \in \partial T$, $I_n(x)$ and $I_{x\wedge x}$ denote respectively $I_{x_{|n}}$ and $\{x\}$.
We denote by $\mathcal{T}$ the σ-algebra generated in ∂T by the I_a's, $a \in T$.

If μ is a positive measure on ∂T, one calls *support* of μ and denote by supp(μ) the set of the points of ∂T, all neighbourhoods of which carry a piece of μ.

If J is a finite set, we denote by $\#J$ the number of its elements.

If $(p_0, \ldots, p_{c-1})$ is in $\mathbb{R}_+^c$ and satisfy $\sum_{j=0}^{c-1} p_j = 1$ then $\partial T(p_0, \ldots, p_{c-1})$ denotes the set of the elements x of ∂T such that for all $j \in \mathcal{C}$, $\lim_{n\to\infty} \frac{\#\{1\leq k\leq n;\ x_k = j\}}{n} = p_j$.

If I is a subset of $\mathbb{R}^n$, we denote by $\mathrm{Int}(I)$ its interior, $\mathrm{Cl}(I)$ its closure, and if $n = 1$, $\mathrm{Cl}^+(I)$ the set $\mathrm{Cl}(I) \setminus \{\inf(I)\}$.

Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a function. One calls Legendre transform of f and denotes by f^* the concave function: $\alpha \in \mathbb{R} \mapsto f^*(\alpha) = \inf_{q \in \mathbb{R}} \alpha\, q + f(q)$.
The following geometrical remarks will be usefull in 5.:
1) For α and q_0 in $\mathbb{R}$, $\alpha\, q + f(q)$ is the ordinate of the intersection of the line $q \mapsto -\alpha(q - q_0) + f(q)$ and the line $\{q = 0\}$.

2) If f is convex on $\mathbb{R}_+$, non increasing on $\mathbb{R}_-$, and $f(0) < \infty$ then: if $0 \leq \alpha \leq -f'(0^+)$ then $f^*(\alpha) = \inf_{q \in \mathbb{R}_+} \alpha\, q + f(q)$; if $\alpha \in [-f'(0^+), -f'(0^-)]$ then $f^*(\alpha) = f(0)$; if $\alpha \geq -f'(0^-)$ then $f^*(\alpha) = \inf_{q \in \mathbb{R}_-} \alpha\, q + f(q)$.

3) Assume that f is C^1 on $\mathbb{R}$, strictly convex, $f(0) > 0$, $f(1) = 0$ and define $\alpha_{\inf} = \lim_{q \mapsto \infty} -f'(q)$ (resp. $\alpha_{\sup} = \lim_{q \mapsto -\infty} -f'(q)$) and $J = \{q \in \mathbb{R};\ f^*(-f'(q)) > 0\}$.

If $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) $\subset J$ then $f^*(\alpha_{\inf}) = \lim_{q \mapsto \infty} f^*(-f'(q))$ (resp. if $\alpha_{\sup} < \infty$ then $f^*(\alpha_{\sup}) = \lim_{q \mapsto -\infty} f^*(-f'(q))$). Moreover, the graph of f and the line $y = \alpha_{\inf} q + f^*(\alpha_{\inf})$ (resp. $y = \alpha_{\sup} q + f^*(\alpha_{\sup})$) are asymptotics at ∞ (resp. $-\infty$).
If $\mathbb{R}_+ \not\subset J$ (resp. $\mathbb{R}_- \not\subset J$) then $f^*(-f'(\sup(J))) = 0$ (resp. $f^*(-f'(\inf(J))) = 0$).

Let $(\Omega, \mathcal{A}, P)$ stand for the probability space on which the random variables in this paper are defined.

We adopt the following conventions: $\log 0 = -\infty$ and $0 \times \infty = 0$. If A and B are two random variables (r. v.), we write $A \sim B$ to mean that they are identically distributed.

1.2 The main result in the simplest case.

We give the proof of the main result announced in the introduction, in a case which covers the one studied in [16] and a part of those studied in [14] and [1].

∂T is equipped with the standard ultrametric distance d given by $d(x, y) = c^{-|x \wedge y|}$. Let $W = (W_0, \dots, W_{c-1}) \in {\mathbb{R}_+^*}^c$ be a random vector such that for some positive real numbers a and b, with probability one, for all $j \in \mathcal{C}$, $a \leq W_j \leq b$. Define $\tilde{\tau}(q) = \log_c \mathbb{E}(\sum_{j=0}^{c-1} W_j^q)$ for $q \in \mathbb{R}$ and $\langle W^z \rangle = \mathbb{E}(\sum_{j=0}^{c-1} W_j^z)$ for $z \in \mathbb{C}$.

Suppose that $\tilde{\tau}(1) = 0$ and $\tilde{\tau}'(1) < 0$. Let $(W(a))_{a \in T}$ be independent copies of W, and assume $W(\epsilon) = W$. Then, for all $a \in T$, the sequence

$$Y_n(a) = \sum_{b \in T_n} W_{b_1}(a)\, W_{b_2}(ab_1)\, \dots\, W_{b_n}(ab_1 \dots b_{n-1})$$

is a positive martingale which (by [20] or [11]), converges with probability one to a positive r. v. $Y(a)$ of expectation 1; the convergence also holds in L^h norm for all $h > 1$ such that $\mathbb{E}(\sum_{j=0}^{c-1} W_j^h) < 1$. Moreover, for all $a \in T$, one has with probability one

$$(E_a) \qquad Y(a) = \textstyle\sum_{j=0}^{c-1} W_j(a)\, Y(aj),$$

$Y(a) \sim Y(\epsilon)$, and for all $n \geq 1$ the $Y(a)$'s, $a \in T_n$, are mutually independent and they are independent of the $W(b)$'s, $|b| \leq n - 1$.

By using equations (E_a) one defines with probability one on $(\partial T, \mathcal{T})$ an unique positive measure μ by the relations:

$$\text{for all } a \in T,\ \ \mu(I_a) = \Big(\prod_{0 \le k < |a|} W_{a_{k+1}}(a_{|k}) \Big)\, Y(a);$$

then define for all $q \in \mathbb{R}$,

$$\tau(q) = \sup\{t \in \mathbb{R};\ \limsup_{n\to\infty} c^{-nt} \sum_{a \in T_n} \mu^q(I_a) = \infty\}$$

and define $J = \{q \in \mathbb{R};\ \tilde\tau^*(-\tilde\tau'(q)) = -\tilde\tau'(q)q + \tilde\tau(q) > 0\}$ and $I = -\tilde\tau'(J)$. I and J are intervals, $I = \{\alpha > 0;\ \tilde\tau^*(\alpha) > 0\}$. Our result is

Theorem 1.1. *For $\alpha \in \mathbb{R}_+$, define $E_\alpha = \{x \in \partial T;\ \lim_{n\to\infty} \dfrac{\log\,\mu(I_n(x))}{\log\,|I_n(x)|} = \alpha\}$, where $|\ |$ denotes the diameter with respect to d.*
With probability one:
i) for all $q \in J$, $\tau(q) = \tilde\tau(q)$.
ii) For all $\alpha \in I$, $\dim\, E_\alpha = \tau^*(\alpha) = \tilde\tau^*(\alpha)$.
iii) If $0 \le \alpha < \inf(I)$ or $\alpha > \sup(I)$, then $E_\alpha = \emptyset$.

By [3]VI., it is enough to prove that with probability one, for all $\alpha \in I$, $\dim\, E_\alpha \ge \tilde\tau^*(\alpha)$. We need

Theorem 1.2. *In a complex neighbourhood of J, with probability one, the sequence of analytic functions $\big(Y_{z,n}\ :\ z \mapsto \langle W^z\rangle^{-n} \sum_{a \in T_n} (W_{a_1}(\epsilon))^z (W_{a_2}(a_1))^z \ \dots\ (W_{a_n}(a_1 \dots a_{n-1}))^z\big)_{n\ge 1}$ converges, uniformly on the compact sets, towards an analytic function $z \mapsto Y_z$. Moreover, with probability one, $q \mapsto Y_q$ has no zero in J.*

Corollary 1.1. *With probability one, for all $a \in T$, the sequence $(Y_{q,n}(a)\ :\ q \mapsto \langle W^q\rangle^{-n} \sum_{b \in T_n} (W_{b_1}(a))^q (W_{b_2}(ab_1))^q \ \dots\ (W_{b_n}(ab_1 \dots b_{n-1}))^q)_{n\ge 1}$ converges on J towards a positive analytic function $q \mapsto Y_q(a)$.*
So, with probability one, one defines for all $q \in J$ an unique measure μ_q on ∂T by the relations: for all $a \in T$, $\mu_q(I_a) = \langle W^q\rangle^{-n}\big(\prod_{0\le k<|a|} (W_{a_{k+1}}(a_{|k}))^q\big)\, Y_q(a)$.
Then, with probability one, for all $q \in J$, for μ_q-almost every x,

$$\lim_{n\to\infty} \frac{\log\,\mu_q(I_n(x))}{\log\,|I_n(x)|} = \tilde\tau^*(\alpha_q) \ \text{ and } \lim_{n\to\infty} \frac{\log\,\mu(I_n(x))}{\log\,|I_n(x)|} = \alpha_q, \text{ where } \alpha_q = -\tilde\tau'(q).$$

Then, the proof of Theorem 1.1. follows immediately from Billingsley's Theorem ([6], p 136-145) and the fact that with probability one, for all $q \in J$, E_{α_q} carries the measure μ_q.

Proof of Theorem 1.2. *i*) It suffices to prove that for all $q \in J$, the result is true in a complex neighbourhood of q.
Fix $q \in J$. In a deterministic bounded neighbourhood V_q of q, $|\langle W^z\rangle| > 0$ and then the maps $z \mapsto Y_{z,n}$ are analytic. Moreover, by [3] Lemma VI.A., there exists $h \in]1, 2]$ such that $\langle W^q\rangle^{-h}\mathbb{E}(\sum_{j=0}^{c-1} W_j^{qh}) < 1$, so one can choose V_q such that $\sup_{z\in V_q} |\langle W^z\rangle|^{-h}\mathbb{E}(\sum_{j=0}^{c-1} |W_j^z|^h) < 1$.

Then, using the complex version of a key result of [34] (see [5] Lemma 1) yields the uniform and exponential convergence on V_q of $\|Y_{z,p}-Y_{z,n}\|_h$ to 0 as $n,p\to\infty$. So, if D is a closed disc contained in V_q, the previous uniform and exponential convergence together with the Cauchy integral formula applied, as in [5], to the boundary of a largest disc containing D imply that $(z\mapsto Y_{z,n})_{n\geq 1}$ converges uniformly on D (the same approach is used in [5] to study a more general family of analytic martingales related to some branching random walks; but the families of analytic martingales that we shall study in part 5. are not covered by [5], this is why we give some details. Notice that the result of [34] is also crutial in [3] to establish the continuity of the Mandelbrot process). By writing the Cauchy integral formula for $\frac{d}{dz}Y_z$ one obtains also $\sup_{z\in D}\mathbb{E}(|\frac{d}{dz}Y_z|^h)<\infty$.

Now we prove that with probability one, the process $q\mapsto Y_q$ is positive. Let K be a compact subinterval of J. We claim that $M=\sup_{q\in K}\mathbb{E}(Y_q^{-\frac{1}{2}})<\infty$. Assume this result. If the event $A_K=\{\exists\, q\in K;\, Y_q=0\}$ is of positive probability, as a zero of $q\mapsto Y_q$, which is non negative and analytic, is of even positive order, $\mathbb{E}(\mathbb{1}_{A_K}\int_K Y_q^{-\frac{1}{2}}\,dq)=\infty$. On the other hand, $\mathbb{E}(\int_K Y_q^{-\frac{1}{2}}\,dq)\leq\int_K M dq<\infty$, a contradiction. So $P(A_K)=0$, which gives the conclusion.
To obtain $\sup_{q\in K}\mathbb{E}(Y_q^{-\frac{1}{2}})<\infty$, for $q\in J$, let ψ_q denote the function $t\geq 0\mapsto \mathbb{E}(e^{-tY_q})$. The corresponding version of (E_ϵ) for Y_q gives for all $t\geq 0$, $\psi_q(t)=\mathbb{E}\big(\prod_{j=0}^{c-1}\psi_q(t\langle W^q\rangle^{-1}(W_j)^q)\big)$. Let $\delta_q=\operatorname{ess\,inf}\min_{j\in\mathcal{C}}\langle W^q\rangle^{-1}W_j^q$; $\psi_q(t)\leq\psi_q^c(t\delta_q)$. Then, a sequence of iterations, the almost sure continuity of $q\mapsto Y_q$ and the one of $q\mapsto\delta_q$ yield β and $\gamma>0$ such that for all $t\geq 1$ and $q\in K$, $\psi_q(t)\leq e^{-\beta\, t^\gamma}$. The conclusion comes from the equality $\mathbb{E}(Y_q^{-\frac{1}{2}})=\frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty t^{-\frac{1}{2}}\psi_q(t)\,dt$ for all $q\in K$.

Proof of Corollary 1.1. The part concerning the processes and the construction of the measures is an immediate consequence of Theorem 1.2.
It is sufficient to establish the results on the limit for any compact subinterval of J, instead of J. Let K be a compact subinterval of J.

We prove that with probability 1, for all $q\in K$, for μ_q-almost every x, $\limsup_{n\to\infty}\dfrac{\log\,\mu_q(I_n(x))}{\log\,|I_n(x)|}\leq\tilde\tau^*(\alpha_q)$, which is the most delicate inequality to obtain.

For all $q\in K$, $\varepsilon>0$ and $n\geq 1$, define $E_{q,n,\varepsilon}=\{x\in\partial T;\ \frac{\log\,\mu_q(I_n(x))}{-n\log\,c}\geq\tilde\tau^*(\alpha_q)+\varepsilon\}$.
Fix $\varepsilon>0$ and $\eta\in]0,1[$; for $n\geq 1$, define on K the function
$f_n\,:\,q\mapsto c^{-n((1-\eta)\tilde\tau(q)+\eta(\tilde\tau^*(\alpha_q)+\varepsilon))}\sum_{a\in T_n}\Big(\prod_{0\leq k<|a|}(W_{a_{k+1}}(a_{|k}))^{(1-\eta)q}\Big)\,Y_q^{1-\eta}(a)$. Then
$\sum_{n\geq 1}\mu_q(E_{q,n,\varepsilon})\leq\sum_{n\geq 1}\sum_{a\in T_n,\,I_a\subset E_{q,n,\varepsilon}}\mu_q(I_a)$, so
$\sum_{n\geq 1}\mu_q(E_{q,n,\varepsilon})\leq\sum_{n\geq 1}\sum_{a\in T_n}\mu_q^{1-\eta}(I_a)\,c^{-n(\tau^*(\alpha_q)+\varepsilon)\eta}=\sum_{n\geq 1}f_n(q)$.

A simple calculus using the hypotheses on the $W(a)$'s gives
$\mathbb{E}(f_n(q))=\big(c^{\tilde\tau((1-\eta)q)-((1-\eta)\tilde\tau(q)+\eta(\tilde\tau^*(\alpha_q)+\varepsilon))}\big)^n\,\mathbb{E}(Y_q^{1-\eta})$. Moreover, for all $q\in K$, $\mathbb{E}(Y_q^{1-\eta})\leq\mathbb{E}(Y_q)=1$ (see [3]). Thus, as for η small enough,
$C_K=\sup_{q\in K}c^{\tilde\tau((1-\eta)q)-((1-\eta)\tilde\tau(q)+\eta(\tilde\tau^*(\alpha_q)+\varepsilon))}<1$, for such an η

$\sup_{q\in K}\mathbb{E}\sum_{n\geq 1}f_n(q)\leq\sum_{n\geq 1}C_K^n<\infty$, so for all $q\in K$, with probability one, $\sum_{n\geq 1}\mu_q(E_{q,n,\varepsilon})<\infty$. This is not sufficient to conclude.

To improve this, the idea is to show that if η is small enough then

$\int_K \sum_{n\geq 1} |f_n'(q)|\,dq < \infty$. Then, one concludes as follows: with probability one, $\sum_{n\geq 1} f_n$ converges uniformly on K, so with probability one, for all $q \in K$, $\sum_{n\geq 1} \mu_q(E_{q,n,\varepsilon}) < \infty$. Fix D a dense countable subset of $]0,\infty[$; as D is countable, with probability one, for all $q \in K$, for all $\varepsilon \in D$, $\sum_{n\geq 1} \mu_q(E_{q,n,\varepsilon}) < \infty$, and by the Borel-Cantelli lemma, for μ_q-almost every x in ∂T, for all $\varepsilon \in D$, $\limsup_{n\to\infty} \frac{\log\,\mu_q(I_n(x))}{\log\,|I_n(x)|} \leq \tilde{\tau}^*(\alpha_q) + \varepsilon$.

With probability one, for all $n \geq 1$ and $q \in K$, $|f_n'(q)| \leq g_n(q) + h_n(q)$, with

$$g_n(q) = \sum_{a\in T_n} |\frac{d}{dq}[c^{n((1-\eta)\tilde{\tau}(q)+\eta(\tilde{\tau}^*(\alpha_q)+\varepsilon))} \prod_{0\leq k<|a|} (W_{a_{k+1}}(a_{|k}))^{(1-\eta)q}]|\, Y_q^{1-\eta}(a) \text{ and}$$

$$h_n(q) = \sum_{a\in T_n} c^{n((1-\eta)\tilde{\tau}(q)+\eta(\tilde{\tau}^*(\alpha_q)+\varepsilon))} \prod_{0\leq k<|a|} (W_{a_{k+1}}(a_{|k}))^{(1-\eta)q}\, |\frac{d}{dq} Y_q(a)|\, Y_q^{-\eta}(a).$$

As $\sup_{q\in K} |\frac{d}{dq}\tilde{\tau}^*(\alpha_q)|$ and $\sup_{q\in K} \mathbb{E}(\sum_{j=0}^{c-1} |\frac{d}{dq}[\langle W^q\rangle^{-1} W_j^q]| \langle W^q\rangle^{\eta} W_j^{-\eta q})$ are finite, new calculus show that there exists a constant $C > 0$ such that for all $n \geq 1$, $\sup_{q\in K} \mathbb{E}(g_n(q)) \leq C\,n\,C_K^n$ and $\sup_{q\in K} \mathbb{E}(h_n(q)) \leq C_K^n \sup_{q\in K} \mathbb{E}(|\frac{d}{dq} Y_q|\, Y_q^{-\eta})$.

We saw in the proof of Theorem 1.2. that $\sup_{q\in K} \mathbb{E}(Y_q^{-\frac{1}{2}}) < \infty$ and that there exists $h > 1$ such that $\sup_{q\in K} \mathbb{E}(|\frac{d}{dq} Y_q|^h) < \infty$. Thus, if η is small enough, an Hölder inequality yields $C' = \sup_{q\in K} \mathbb{E}(|\frac{d}{dq} Y_q|\, Y_q^{-\eta}) < \infty$.
So $\mathbb{E}(\int_K \sum_{n\geq 1} |f_n'(q)|\,dq) \leq \sum_{n\geq 1} C\,n\,C_K^n + C'C_K^n < \infty$.

$\liminf_{n\to\infty} \frac{\log\,\mu_q(I_n(x))}{\log\,|I_n(x)|}$, $\limsup_{n\to\infty} \frac{\log\,\mu(I_n(x))}{\log\,|I_n(x)|}$ and $\liminf_{n\to\infty} \frac{\log\,\mu(I_n(x))}{\log\,|I_n(x)|}$ can be studied similarly.

2. TWO MARTINGALES OBTAINED BY MULTIPLICATIVE CASCADES.

If $W = (W_0, \dots, W_{c-1}) \in \mathbb{R}_+^c$ is a random vector, then we say that W satisfy the hypothesis (H_0) if

$$(H_0)\quad \mathbb{E}(\sum_{j=0}^{c-1} W_j) = 1,\ P(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j>0\}} > 1) > 0 \text{ and } P(\sum_{j=0}^{c-1} W_j|1-W_j| = 0) < 1.$$

We denote by $\mathcal{E}$ the set of such W's, and define
$\mathcal{E}_1 = \{W \in \mathcal{E};\ \mathbb{E}(\sum_{j=0}^{c-1} W_j \log W_j) < 0\}$ and
$\mathcal{E}_2 = \{W \in \mathcal{E};\ \mathbb{E}(\sum_{j=0}^{c-1} W_j \log W_j) = 0,\ \exists\,\delta > 0,\ \mathbb{E}((\sum_{j=0}^{c-1} W_j)^{1+\delta}) < \infty\}$.

Let W be in $\mathcal{E}_1 \cup \mathcal{E}_2$ and $(W(a))_{a\in T}$ be independent copies of W. For $a \in T$ and $n \geq 1$, let

$$Y_n(a) = \sum_{b\in T_n} W_{b_1}(a)\, W_{b_2}(ab_1) \dots W_{b_n}(ab_1 \dots b_{n-1}),\ \text{if } W \in \mathcal{E}_1$$

and

$$Y_n(a) = -\sum_{b\in T_n} \prod_{0\leq k<|b|} (W_{b_{k+1}}(ab_{|k}))\ \log \prod_{0\leq k<|b|} (W_{b_{k+1}}(ab_{|k})),\ \text{if } W \in \mathcal{E}_2.$$

In the case $W \in \mathcal{E}_1$ (resp. $\mathcal{E}_2$), the sequence $(Y_n(a))_{n\geq 1}$ was introduced in [26], [27] (resp. [23]). From [26], [27], [20], [11], [21] and [23], we can state the following

Theorem 2.1. *For all $a \in T$, the sequence $(Y_n(a))_{n\geq 1}$ is a martingale which converges with probability one to a non negative r. v. $Y(a)$ of positive mean. For all*

$n \geq 1$, the families $(W(a))_{a\in\bigcup_{0\leq k\leq n-1} T_k}$ and $(Y(a))_{a\in T_n}$ are independent and the $Y(a)$'s, $a \in T_n$ are mutually independent and have the same probability distribution as $Y(\epsilon) = Y$.

With probability one, for all $a \in T$, $Y(a) = \sum_{j=0}^{c-1} W_j(a)\, Y(aj)$ (E_a). $P(Y = 0)$ is the unique fixed point in $[0, 1[$ of $f : x \mapsto \sum_{k=0}^{c} P\big(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j>0\}} = k\big)\, x^k$.

We shall need some results about the moments of $Y = Y(\epsilon)$. The moments of positive orders were studied in [20], [11] and [21] when $W \in \mathcal{E}_1$, and when $W \in \mathcal{E}_2$ their study is easy to deduce from results of [11] and [23].

Theorem 2.2. *i) If $W \in \mathcal{E}_1$ then $\mathbb{E}(Y) = 1$. Moreover, for all $h > 1$, one has $\mathbb{E}(Y^h) < \infty$ if and only if $\mathbb{E}(\sum_{j=0}^{c-1} W_j^h) < 1$.*

ii) If $W \in \mathcal{E}_2$ then $\lim_{t\to 0} \frac{1-\mathbb{E}(e^{-tY})}{t \log \frac{1}{t}} = 1$ ([11], [23]). So $\mathbb{E}(Y) = \infty$ and for all $h \in]0, 1[$, $\mathbb{E}(Y^h) < \infty$.

The moments of negative orders were studied successively in [19], [28], [3] and [24], the study being always based on some estimation of the Laplace transform of the probability distribution of Y. But in these works, there are no explicit estimations of these moments when they are finite, and we shall need to control them in 5.

Define $\tilde{c} = \sum_{k=0}^{c-1} \mathbb{1}_{\{W_j>0\}}$ and for $0 \leq k \leq c$, $p_k = P(\tilde{c} = k)$; $m = \inf\{2 \leq k \leq c;\, p_k > 0\}$; $p = P(Y = 0)$; $s = \sum_{k=0}^{c} k p^{k-1} p_k$; $C(p) = \max_{2\leq k\leq c} \sum_{j=2}^{k} \binom{k}{j} p^{k-j}$;

$A = \min_{j\in\mathcal{C},\, W_j>0} W_j$ if $\tilde{c} \geq 1$ and $A = 0$ if $\tilde{c} = 0$; $\delta = \text{ess inf } A_{|\{\tilde{c}\geq 1\}}$ ($\delta < 1$ by (H_0)); $\tilde{Y}$ a random variable independent of A, which satisfies $\tilde{Y} \sim Y$; for $t > 0$, $p(t) = P(0 < A\tilde{Y} < t^{-1/2}) + e^{-t^{1/2}}$; $\psi : \mathbb{R}_+ \to \mathbb{R}_+^*$, $t \mapsto \mathbb{E}(\mathbb{1}_{\{Y>0\}}\, e^{-tY})$; if $\alpha > 1$, α^* its conjugate: $\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$.

Theorem 2.3. *1) Suppose $\delta > 0$. Then*
i) Assume $p_0 + p_1 = 0$. For all $b > 0$, $\mathbb{E}(Y^{-b}) < \infty$ and more precisely
$$\mathbb{E}(Y^{-b}) \leq \frac{1}{b\Gamma(b)} + \frac{1}{\Gamma(b)} \int_1^{\infty} t^{b-1} e^{-\frac{|\log \psi(1)|}{m} t^{\frac{\log m}{\log \frac{1}{\delta}}}}\, dt.$$
ii) Assume $p_0 + p_1 > 0$. For all $b \in]0, \frac{\log s}{\log \delta}[$, $\mathbb{E}(\mathbb{1}_{\{Y>0\}}\, Y^{-b}) < \infty$.
Moreover, if $t_0 > 0$ is large enough to have $s(t_0) = s + C(p)\, p(t_0) < 1$ then for all $b \in]0, \frac{\log s(t_0)}{\log \delta}[$, $$\mathbb{E}(\mathbb{1}_{\{Y>0\}}\, Y^{-b}) \leq \frac{(t_0/\delta)^b}{b\Gamma(b)} + \frac{s(t_0)\psi(t_0)(t_0/\delta)^b}{(\frac{\log s(t_0)}{\log \delta} - b)\Gamma(b)}.$$
2) Suppose $\delta = 0$. Assume moreover that there exists $a > 0$ such that for $j \in \mathcal{C}$, if $P(W_j > 0) > 0$ then $\mathbb{E}(\mathbb{1}_{\{W_j>0\}}\, W_j^{-a}) < \infty$. Then $m_a = \mathbb{E}(\mathbb{1}_{\{A>0\}}\, A^{-a}) < \infty$ and
i) Assume $p_0 + p_1 = 0$. For all $b \in]0, a[$, $\mathbb{E}(Y^{-b}) < \infty$. Moreover if $\alpha > 1$ is given and $t_0 > 0$ is large enough to have $p(t_0) < 1$ and $p(t_0)^{\frac{\alpha}{\alpha^*}}\, m_a < 1$ then for all $b \in]0, \frac{a}{\alpha}[$, $$\mathbb{E}(Y^{-b}) \leq \frac{t_0^b}{b\Gamma(b)}\Big(1 + \frac{b}{(\frac{a}{\alpha} - b)}\, \frac{p(t_0)^{\frac{1}{\alpha^*}}\, m_a^{\frac{1}{\alpha}}}{(1 - p(t_0)^{\frac{\alpha}{\alpha^*}}\, m_a)^{\frac{1}{\alpha}}}\Big).$$
ii) Assume $p_0 + p_1 > 0$. For $\beta \geq 1$, define $s(\beta) = \| \sum_{j=1}^{c} k p^{k-1} \mathbb{1}_{\{\tilde{c}=k\}} \|_\beta$.
a) If $p_0 = 0$ then for all $\beta \geq 1$ $s(\beta) = p_1 < 1$. If $p_0 > 0$ and β is close enough to 1 then $s(\beta) < 1$.

b) Fix $\alpha > 1$ such that $s(\alpha^*) < 1$. Let $b_a(\alpha)$ stand either for $\frac{a}{\alpha}$ if $s(\alpha^*)^\alpha\, m_a < 1$ or for $\frac{a}{\alpha}\frac{\log \frac{1}{s(\alpha^*)^\alpha}}{\log m_a}$ if $s(\alpha^*)^\alpha\, m_a \geq 1$. For all $b \in]0, b_a(\alpha)[$, $\mathbb{E}(\mathbb{1}_{\{Y>0\}}\, Y^{-b}) < \infty$. For $t > 0$ define $s(\alpha^*, t) = s(\alpha^*) + C(p)p(t)^{\frac{1}{\alpha^*}}$. If $t_0 > 0$ is large enough to have $s(\alpha^*, t_0) < 1$, let $b_a(\alpha, t_0)$ stand either for $\frac{a}{\alpha}$ if $s(\alpha^*, t_0)^\alpha\, m_a < 1$ or for $\frac{a}{\alpha}\frac{\log \frac{1}{s(\alpha^*,t_0)^\alpha}}{\log m_a}$ if $s(\alpha^*, t_0)^\alpha\, m_a \geq 1$. Then for $b \in]0, b_a(\alpha, t_0)[$,

$$\mathbb{E}(\mathbb{1}_{\{Y>0\}}Y^{-b}) \leq \frac{t_0^b}{b\Gamma(b)}(1+\frac{b}{b_a(\alpha,t_0)-b}\,\frac{s(\alpha^*,t_0)\,\mathbb{E}(\mathbb{1}_{\{A>0\}}\,A^{-\alpha b_a(\alpha,t_0)})^{\frac{1}{\alpha}}}{(1-s(\alpha^*,t_0)^\alpha\,\mathbb{E}(\mathbb{1}_{\{A>0\}}\,A^{-\alpha b_a(\alpha,t_0)}))^{\frac{1}{\alpha}}}).$$

Remark 2.1. 1) In [19] and [28], the problem of the existence of finite moments of negative orders is solved when $W \in (\mathbb{R}_+^*)^c$. In [3], we find nearly the same conclusions as in [19] and [28], and give good general conditions to solve it when $p_0 + p_1 = 0$ or $p_0 + p_1 > 0$ and $\delta > 0$; when $p_0 + p_1 > 0$ and $\delta = 0$, we solve it under some restriction on the probability distribution of W. These three works was realized independently. In order to complete the statement of Theorem 2.3., we give one of their conclusions: when the $W_j's$ are i.i.d. and positive, if there exists $a > 0$ such that $\mathbb{E}(W_0^{-a}) < \infty$ then $\mathbb{E}(Y^{-ac}) < \infty$ and if $\mathbb{E}(W_0^{-a}) = \infty$ then for all $b > ac$, $\mathbb{E}(Y^{-b}) = \infty$. In [24] and [25], where the previous studies are improved, the case $p_0 = 0$, $p_1 > 0$ and $\delta = 0$ is treated under good conditions and when $p_0 > 0$ and $\delta = 0$, the problem is solved only when the $W_j's$ are i.i.d., but in the satisfying following way : if $s\,\mathbb{E}(\mathbb{1}_{\{W_0>0\}}\,W_0^{-a}) < 1$ then for all $b \in]0, a[$, $\mathbb{E}(\mathbb{1}_{\{Y>0\}}\,Y^{-b}) < \infty$ and if $s\,\mathbb{E}(\mathbb{1}_{\{W_0>0\}}\,W_0^{-a}) \geq 1$ then for all $b \geq a$, $\mathbb{E}(\mathbb{1}_{\{Y>0\}}\,Y^{-b}) = \infty$.
This result leads to the following general remark already formulated in [3]: for any W, if $p_0+p_1 > 0$ then there exists $a_0 > 0$ such that for all $a > a_0$, $\mathbb{E}(\mathbb{1}_{\{Y>0\}}\,Y^{-a}) = \infty$.

2) [2], [24] and [25] study also the case where Y is constructed with c a random integer and ∂T a branching set in $\mathbb{N}^{\mathbb{N}}$.

The proof of Theorem 2.3. requires two lemmas. The first one can be deduced from Lemma 2.3 of [24] or [25].

Lemma 2.1 (Liu). *Let $f : [0, \infty[\to]0, 1]$ be a positive non increasing function. Let B be a non negative r. v. which is not almost surely a constant on $\{B > 0\}$. Assume that for some $0 < \lambda < 1$ and $t_0 > 0$ one has for all $t \geq t_0$, $f(t) \leq \lambda\,\mathbb{E}(\mathbb{1}_{\{B>0\}}\,f(Bt))$. If $m_a = \mathbb{E}(\mathbb{1}_{\{B>0\}}\,B^{-a}) < \infty$ for some $a > 0$ then*

i) either $\lambda\, m_a < 1$ and for all $t \geq t_0$ $f(t) \leq \frac{\lambda\, t_0^a\, m_a}{1-\lambda\, m_a} t^{-a}$

ii) or $\lambda\, m_a \geq 1$ and for all $t \geq t_0$ $f(t) \leq \frac{\lambda\, t_0^{a'}\, m_{a'}}{1-\lambda\, m_{a'}} t^{-a'}$, where $a' = a\,\frac{\log \frac{1}{\lambda}}{\log m_a}$.

Lemma 2.2. *For $t > 0$ and $\beta \geq 1$, one has $\mathbb{E}(\mathbb{1}_{\{\tilde{c}\geq 2\}}\,\psi^\beta(At)) \leq p(t)$. Moreover* $\lim_{t\to\infty} p(t) = 0$ *($p(t)$ is defined just before the statement of Theorem 2.3).*

Proof. Left to the reader.

Proof of Theorem 2.3. We prove 2)ii) and leave the other cases to the reader (the proof of 2)i) is similar to the one of 2)ii) and those of 1)i) and 1)ii) result from standard sequences of iterations).
By (E_ϵ) (Th. 2.1), for all $t \geq 0$, $\mathbb{E}(e^{-tY}) = p + \psi(t) = \mathbb{E}(\prod_{j=0}^{c-1}(p + \psi(tW_j)))$. So as ψ is decreasing $p + \psi(t) \leq p_0 + \mathbb{E}(\mathbb{1}_{\{\tilde{c}\geq 1\}}\,(p + \psi(tA))^{\tilde{c}})$

$= p_0 + \sum_{k=1}^{c} p_k p^k + \sum_{k=1}^{c} \sum_{j=1}^{k} \binom{k}{j} p^{k-j}\, \mathbb{E}(\mathbb{1}_{\{\tilde{c}=k\}}\, \psi^j(tA))$. Since $\psi \leq 1$ and by Theorem 2.1. $p = \sum_{k=0}^{c} p_k p^k$, it follows that for $t \geq 0$,

(2.1) $\psi(t) \leq \sum_{k=1}^{c} k p^{k-1} \mathbb{E}(\mathbb{1}_{\{\tilde{c}=k\}}\, \psi(tA)) + C(p)\mathbb{E}(\mathbb{1}_{\{\tilde{c}\geq 2\}}\, \psi^2(tA))$.

(2.2) We recall that for $b > 0$, $\mathbb{E}(\mathbb{1}_{\{Y>0\}}\, Y^{-b}) = \dfrac{1}{\Gamma(b)} \displaystyle\int_0^\infty t^{b-1}\, \psi(t)\, dt$.

We use the same approach as Liu in [23] and [24] where $p_0 = 0$ (excepted for the i.i.d. case, see remark 2.1.).

2)ii)a) By (H_0), f (see Th. 2.1) is strictly convex on $[0,1]$. Thus, as $f(1) = 1$, $f(p) = p$, and $p < 1$, $s = f'(p) < 1$. Then, as $s(1) = s$, the conclusion follows from the continuity of $\beta \mapsto s(\beta)$.

b) Fix $\alpha > 1$ such that $s(\alpha^*) < 1$ and choose $t_0 > 0$ such that $s < s(\alpha, t_0) < 1$. By (2.1), lemma 2.2., and the Hölder inequality, for $t \geq t_0$

$$\begin{aligned}
\psi^\alpha(t) &\leq \big(s(\alpha^*) + C(p)(\mathbb{E}(\mathbb{1}_{\{\tilde{c}\geq 2\}}\, \psi^{\alpha^*}(tA))^{\frac{1}{\alpha^*}}\big)^\alpha \mathbb{E}(\mathbb{1}_{\{A>0\}}\, \psi^\alpha(tA)) \\
&\leq (s(\alpha^*) + C(p)p(t)^{\frac{1}{\alpha^*}})^\alpha\, \mathbb{E}(\mathbb{1}_{\{A>0\}}\, \psi^\alpha(tA)) \\
&\leq s(\alpha^*, t_0)^\alpha \mathbb{E}(\mathbb{1}_{\{A>0\}}\, \psi^\alpha(tA)). \text{ So by Lemma 2.1., for } t \geq t_0
\end{aligned}$$

$\psi(t) \leq \frac{s(\alpha^*,t_0)(\mathbb{E}(\mathbb{1}_{\{A>0\}}\, A^{-\alpha b_a(\alpha,t_0)}))^{\frac{1}{\alpha}} t_0^{b_a(\alpha,t_0)}}{(1-s(\alpha^*,t_0)^\alpha \mathbb{E}(\mathbb{1}_{\{A>0\}}\, A^{-\alpha b_a(\alpha,t_0)}))^{\frac{1}{\alpha}}}\, t^{-b_a(\alpha,t_0)}$. We conclude by using (2.2) and the fact that $\lim_{t\to\infty} b_a(\alpha, t) = b_a(\alpha)$.

3. TWO TYPES OF STATISTICALLY SELF-SIMILAR MEASURES.

Let W be in $\mathcal{E}_1 \cup \mathcal{E}_2$ and let $(W(a))_{a\in T}$ be a sequence of independent copies of W. From now, for $a \in T$, we denote by $Y_W(a)$ the r. v. $Y(a)$ constructed in the part 2.

Definition 3.1. *The relations $((E_a))_{a\in T}$ (see Theorem 2.1.) make possible to define an unique random statistically self-similar measure μ_W on $(\partial T, \mathcal{T})$ given on the I_a's by the following relations:*
with probability one, for all $a \in T$, $\mu_W(I_a) = w(a)\, Y_W(a)$.
We say that μ_W is of type 1 (resp. 2) if $W \in \mathcal{E}_1$ (resp. $\mathcal{E}_2$).
We denote by ∂T_W the support of μ_W and $\mathcal{T}_W$ the $\sigma-$algebra generated in ∂T_W by the $I_a \cap \partial T_W$'s, $a \in T$.

Proposition 3.1. *Define $L = \lim_{n\to\infty} \max_{a\in T_n} \mu_W(I_a)$ and A_W the set of the atoms of μ_W.*

i) One has either $P(L = 0) = P(A_W = \emptyset) = 1$ or for P-almost every ω in $\{\mu_W \neq 0\}$, A_W is countable, dense in ∂T_W equipped with the standard ultrametric distance, and it is, up to a finite set if $W \in \mathcal{E}_2$, contained in $\partial T(\mathbb{E}(W_0), \dots, \mathbb{E}(W_{c-1}))$.

ii) If $P(A_W = \emptyset) = 1$ then for P-almost every ω in $\{\mu_W \neq 0\}$, the mapping $d_W : (x, y) \mapsto \mu_W(I_{x\wedge y})$ is an ultrametric distance on ∂T_W.

iii) If $W \in \mathcal{E}_1$ and $\mathbb{E}(Y_W\,|\log Y_W|) < \infty$, then $P(A_W = \emptyset) = 1$.

Remark 3.1. 1) If $W \in \mathcal{E}_1$ then by Theorem 2.2, $\mathbb{E}(Y_W\,|\log Y_W|) < \infty$ as soon as $\mathbb{E}(\sum_{j=0}^{c-1} W_j^h) < 1$ for some $h > 1$. If moreover the W_j's are i.i.d., Kahane shows in [18] that the condition $W \in \mathcal{E}_1$ is sufficient for μ_W having no atoms.

2) We do not know sufficient condition on $W \in \mathcal{E}_2$ for having one of the alternatives concerning A_W.

Now we choose $(W, W') \in (\mathcal{E}_1 \cup \mathcal{E}_2)^2$ and $((W(a), W'(a)))_{a \in T}$ a sequence of independent copies of (W, W'). So we obtain two measures, μ_W and $\mu_{W'}$, simultaneously.

Proposition 3.2. *i) One has with probability one $\partial T_{W'} \subset \partial T_W$ if and only if $\sum_{j=0}^{c-1} P(W_j = 0, W'_j > 0) = 0$.*
ii) Assume that $\mu_{W'}$ has no atoms. One has with probability one $\mu_{W'}(\partial T_W) = 0$ if and only if there exists $j \in \mathcal{C}$ such that $P(W_j = 0, W'_j > 0) > 0$.

Proof of Proposition 3.1. i) First note that $L > 0$ if and only if μ_W has an atom. For $n \geq 1$, define $m_n = \max_{a \in T_n} \mu_W(I_a)$. By (E_ϵ), $m_{n+1} = \max_{j \in \mathcal{C}} W_j\, m_{n,j}$ where the $m_{n,j}$'s are independent copies of m_n, which are also independent of W (see Th 2.1). For $j \in \mathcal{C}$, define $L_j = \lim_{n \to \infty} m_{n,j}$. With probability one, $L = \max_{j \in \mathcal{C}} W_j\, L_j$ and the inequality $m_n \leq Y_W$ yields $L < \infty$. By (E_ϵ) again, $P(L = 0)$ and $P(Y_W = 0)$ are fixed points of f. Moreover, $\{Y_W = 0\} \subset \{L = 0\}$. So, either $P(L = 0) = 1$ or $P(L = 0, Y_W > 0) = 0$. If the last situation arises, for P-almost every $\omega \in \{\mu_W \neq 0\}$, $L > 0$ and μ_W has an atom in ∂T_W. Then, by the self-similarity of the construction, μ_W has an atom in every I_a such that $I_a \cap \partial T_W \neq \emptyset$. So A_W is dense and countable since $\|\mu_W\| < \infty$.
The end of the proof will be given in 4.2.
ii) Left to the reader.
iii) It is the Corollary IV.*a.ii*) of [3].

Proof of Proposition 3.2. i) It is a simple improvement of [3] Proposition IV.
ii) With probability one $\partial T_W = \cap_{n \to \infty} \downarrow \cup_{a \in T_n,\ \mu_W(I_a) > 0} I_a$. So if $h \in]0, 1[$, using Proposition 3.1. we obtain with probability one
$\mu_{W'}(\partial T_W) \leq \liminf_{n \to \infty} \sum_{a \in T_n} \mathbb{1}_{\{\mu_W(I_a) > 0\}}\, \mu_{W'}^h(I_a)$. Taking the mean and using Fatou's Lemma,
$\mathbb{E}(\mu_{W'}(\partial T_W)) \leq \liminf_{n \to \infty} \big(\mathbb{E}\big(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j > 0\}}\, {W'_j}^h\big)\big)^n\, \mathbb{E}(\mathbb{1}_{\{Y_W > 0\}}\, Y_{W'}^h)$. By Theorem 2.2., $\mathbb{E}(\mathbb{1}_{\{Y_W > 0\}}\, Y_{W'}^h)$ is always finite. Moreover, if for some $j \in \mathcal{C}$, $P(W_j = 0, W'_j > 0) > 0$ then for h close enough to 1, $\mathbb{E}(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j > 0\}}\, {W'_j}^h) < 1$; so $\mathbb{E}(\mu_{W'}(\partial T_W)) = 0$.
The converse is obvious by *i*).

4. LOCAL HÖLDER EXPONENTS.

Let (W, W', L) be in $(\mathcal{E}_1 \cup \mathcal{E}_2)^3$ and $((W(a), W'(a), L(a)))_{a \in T}$ a sequence of independent copies of (W, W', L). We want to study for P-almost every $\omega \in \{\mu_{W'} \neq 0\}$, for $\mu_{W'}$-almost every x of a subset of ∂T_L of positive $\mu_{W'}$-measure, the existence of local Hölder exponents for μ_W on ∂T_L equipped with the distance d_L defined in Proposition 3.1. To do that, we assume the following hypotheses:

(H_1) For $(V, V') \in \{W, W'\} \times \{W, L\}$, $\sum_{j=0}^{c-1} P(\{V'_j = 0, V_j > 0\}) = 0$; μ_L has no atoms.

Then, by Propositions 3.1 and 3.2, with probability one $\partial T_{W'} \subset \partial T_W \subset \partial T_L$ and ∂T_L is equipped with d_L. We denote by $|B|$ the diameter of a subset B of ∂T_L with respect to d_L.

Remark 4.1. 1) For P-almost every $\omega \in \{Y_L > 0\}$, for all $x \in \partial T_L$ and $n \geq 1$: if $V \in \{W, W'\}$, $\mu_V(I_n(x)) = \mu_V(I_n(x) \cap \partial T_L)$ and $\mu_L(I_n(x)) = |I_n(x) \cap \partial T_L|$.
2) For $V \in \{W, L\}$, $P(Y_{W'} > 0, Y_V = 0) = 0$.

4.1. The case $W' \in \mathcal{E}_1$.

The following results improve those of [3] IV., because here μ_W can be of type 2 and ∂T_L may be different of ∂T.

Theorem 4.1. *Assume that for* $V \in \{W, W', L\}$, $\mathbb{E}(\sum_{j=0}^{c-1} W'_j\,|\log V_j|) < \infty$ *and* $\mathbb{E}(Y_{W'}\,|\log Y_V|) < \infty$. *Then, for* P*-almost every* ω *in* $\{\mu_{W'} \neq 0\}$, *for* $\mu_{W'}$*-almost every* $x \in \partial T_L$, *for all* $V \in \{W, W', L\}$

$$\lim_{n\to\infty} \frac{\log\, \mu_V(I_n(x) \cap \partial T_L)}{n} = \mathbb{E}(\sum_{j=0}^{c-1} W'_j \,\log\, V_j) = -D_{W',V}.$$

Corollary 4.1. *Assume the hypotheses of Theorem 4.1. and* $D_{W',L} \neq 0$. *Then, for* P*-almost every* ω *in* $\{\mu_{W'} \neq 0\}$, *for* $\mu_{W'}$*-almost every* $x \in \partial T_L$,
$$\lim_{n\to\infty} \frac{\log\, \mu_W(I_n(x) \cap \partial T_L)}{\log\, |I_n(x) \cap \partial T_L|} = \frac{D_{W',W}}{D_{W',L}}.$$
If $W = W'$ *then for* P*-almost every* ω *in* $\{\mu_W \neq 0\}$, μ_W *is carried by the set* $E_D = \{x \in \partial T_W;\ \lim_{n\to\infty} \frac{\log\, \mu_W(I_n(x) \cap \partial T_L)}{\log\, |I_n(x) \cap \partial T_L|} = D = \frac{D_{W,W}}{D_{W,L}}\}$ *of Hausdorff dimension* D *(with respect to* d_L*), and an element of* $\mathcal{T}_L$ *of Hausdorff dimension less than* D *is of* μ_W*-measure 0.*

Proof. Once the remark 4.1 is made, it is the same proof as in [3] IV.

4.2. The case $W' \in \mathcal{E}_2$.

By Theorem 2.2., $\mathbb{E}(Y_{W'}) = \infty$. Thus Peyrière's probability on $\Omega \times \partial T$ (see [3] IV.), which is essential to obtain Theorem 4.1., fails to be defined. Our approach is most elementary and typic of this kind of problems when the situation is deterministic ([7], [8], [12]). A such approach was developped in [14] to study the multifractal spectrum of a class of measures of type 1.

For $V \in \{W, W', L\}$ and $(x, y) \in \mathbb{R}^2$, let $\varphi_{W',V}(x, y) = \mathbb{E}(\sum_{j=0}^{c-1} \mathbb{1}_{\{W'_j>0\}}\, W'^{x}_j\, V_j^y)$ and $\psi_{W',V}(x, y) = \mathbb{E}(\mathbb{1}_{\{Y_{W'}>0\}}\, Y_{W'}^x\, Y_V^y)$.

Theorem 4.2. *1) Assume that for* $V \in \{W, L\} \setminus \{W'\}$, $\varphi_{W',V}(1, .)$ *is finite in a neighbourhood of 0 and there exists* $\delta > 0$ *such that* $\psi_{W',V}(0, -\delta) < \infty$. *Then, for* P*-almost every* ω *in* $\{\mu_{W'} \neq 0\}$, *there exists a subset* E *of* $\partial T_{W'}$ *such that* $\mu_{W'}(E) > 0$ *and for* $\mu_{W'}$*-almost every* $x \in E$, *for all* $V \in \{W, W', L\}$

$$\lim_{n\to\infty} \frac{\log\, \mu_V(I_n(x) \cap \partial T_L)}{n} = \mathbb{E}(\sum_{j=0}^{c-1} W'_j \,\log\, V_j) = -D_{W',V}.$$

If $V = W'$, *the set* E *can be choosen equal to* $\partial T_{W'}$.

Corollary 4.2. *Assume the hypotheses of Theorem 4.2. and* $D_{W',L} \neq 0$. *Then, for P-almost every ω in $\{\mu_{W'} \neq 0\}$, there exists $E \subset \partial T_{W'}$ such that $\mu_{W'}(E) > 0$ and for $\mu_{W'}$-almost every $x \in E$,* $\lim_{n\to\infty} \dfrac{\log\, \mu_W(I_n(x) \cap \partial T_L)}{\log\, |I_n(x) \cap \partial T_L|} = \dfrac{D_{W',W}}{D_{W',L}}$.

If $W = W'$ then for P-almost every ω in $\{\mu_W \neq 0\}$, the set $E_0 = \{x \in \partial T_W;\ \lim_{n\to\infty} \dfrac{\log\, \mu_W(I_n(x) \cap \partial T_L)}{\log\, |I_n(x) \cap \partial T_L|} = 0\}$ *is of positive μ_W-measure and its largest subset carrying μ_W is of Hausdorff dimension 0 (with respect to d_L) .*

We shall need the following lemma

Lemma 4.2. *Assume the hypotheses of Theorem 4.2. Let $V \in \{W, W', L\}$ and $\varepsilon > 0$. For $\eta > 0$ close enough to 0 and $\gamma \in]0,1[$ close enough to 1, one has* $\varphi_{W',V}(\gamma,-\eta)\, e^{-(D_{W',V}+\varepsilon)\,\eta} < 1$ *and* $\varphi_{W',V}(\gamma,\eta)\, e^{(D_{W',V}-\varepsilon)\,\eta} < 1$.

Proof. Simple study of function using the log-convexity of $\varphi_{W',V}$.

Proof of Theorem 4.2. During this proof we keep in mind remark 4.1.

By Proposition 3.1 and the fact that $\|\mu_{W'}\| < \infty$, for P-almost every ω in $\{\mu_{W'} \neq 0\}$, there exists a smallest integer $n(\omega)$ such that the set $E_\omega = \{x \in \partial T_{W'};\ \forall\, n \geq n(\omega),\ \mu_{W'}(I_n(x)) \leq 1\}$ is of positive $\mu_{W'}$-measure. We claim that if $V \in \{W, W', L\}$, then for P-almost every ω in $\{\mu_{W'} \neq 0\}$, for $\mu_{W'}$-almost every $x \in E_\omega$, $\limsup_{n\to\infty} \dfrac{\log\, \mu_V(I_n(x) \cap \partial T_L)}{-n} \leq D_{W',V}$:
Fix $\varepsilon > 0$, $\gamma \in]0,1[$ and $\eta \in]0,\gamma[$. For P-almost every ω in $\{\mu_{W'} \neq 0\}$ and $n \geq n(\omega)$, define $E_{n,\varepsilon} = \{x \in E_\omega;\ \frac{\log\, \mu_V(I_n(x))}{-n} \geq D_{W',V} + \varepsilon\}$. Then

$$\sum_{n\geq n(\omega)} \mu_{W'}(E_{n,\varepsilon}) \leq \sum_{n\geq n(\omega)} \ \sum_{a\in T_n,\ I_a\cap E_\omega \subset E_{n,\varepsilon}} \mu_{W'}^{\gamma}(I_a)$$
$$\leq \sum_{n\geq n(\omega)} \sum_{a\in T_n} {\mu_{W'}}^{\gamma}(I_a)\, {\mu_V}^{-\eta}(I_a)\, e^{-n(D_{W',V}+\varepsilon)\,\eta}$$
$$\leq \sum_{n\geq 1} \sum_{a\in T_n} w'^{\gamma}(a)\, v^{-\eta}(a)\, e^{-n(D_{W',V}+\varepsilon)\,\eta}\, Y_{W'}^{\gamma}(a)\, Y_V^{-\eta}(a).$$

Moreover, the mean of the right hand side of the last inequality is
$M = \sum_{n\geq 1} \left(\varphi_{W',V}(\gamma,-\eta)\, e^{-(D_{W',V}+\varepsilon)\,\eta}\right)^n \psi_{W',V}(\gamma,-\eta)$.

As for all $h \in]0,1[$, $\mathbb{E}(Y_{W'}^h) < \infty$, we can choose γ and η so as to satisfy the conclusions of Lemma 4.2. and such that $\psi_{W',V}(\gamma,-\eta) < \infty$ (by an Hölder's inequality). With this choice $M < \infty$ and for P-almost every ω in $\{\mu_{W'} \neq 0\}$, $\sum_{n\geq n(\omega)} \mu_{W'}(E_{n,\varepsilon}) < \infty$.
So, if D is a dense countable subset of $\mathbb{R}_+^*$, as it is countable, for P-almost every ω in $\{\mu_{W'} \neq 0\}$, for all $\varepsilon \in D$, $\sum_{n\geq n(\omega)} \mu_{W'}(E_{n,\varepsilon}) < \infty$, and the conclusion comes from the density of D and the Borel-Cantelli Lemma.
If $V = W'$, one can choose $\partial T_{W'}$ instead of E_ω because in the previous calculus, γ can be replaced by 1.

The study of $\liminf_{n\to\infty} \frac{\log\, \mu_V(I_n(x)\cap\partial T_L)}{-n}$ is similar for $V \in \{W, L\} \setminus \{W'\}$.

End of the proof of Proposition 3.1.i). We are under the notations of Proposition 3.1. By Theorem 4.1. and the proof of Theorem 4.2., if $V = (p_0, \ldots, p_{c-1})$ is a non negative real vector such that $\sum_{j=0}^{c-1} p_j = 1$ and $\sum_{j=0}^{c-1} P(W_j > 0, p_j = 0) = 0$, by considering the Bernouilli measure on ∂T generated by V, we obtain: on $\{\mu_W \neq 0\}$, for every atom x of μ_W if $W \in \mathcal{E}_1$ or for every atom x of μ_W such that $\mu_W(\{x\}) \leq 1$ if $W \in \mathcal{E}_2$, $\lim_{n\to\infty} \frac{\sum_{k=1}^n \log p_{x_k}}{n} = \sum_{j=0}^{c-1} \mathbb{E}(W_j) \log p_j$. So, as V can be choosen arbitrarily, we have the desired conclusion.

Proof of corollary 4.2. The part concerning the limits is clear. The part on the Hausdorff dimension of a set carriyng μ_W is, as in corollary 4.1. a consequence of the Billingsley Theorem ([6], p 136-145).

5. MULTIFRACTAL ANALYSIS.

In this paragraph, we are given the same measures μ_W and μ_L as in paragraph 4. In particular, we are under (H_1) and with probability one $\partial T_W \subset \partial T_L$. Our purpose is, for P-almost every $\omega \in \{\mu_W \neq 0\}$ to calculate for all α of a largest as possible subinterval of $\mathbb{R}_+$ the Hausdorff dimension of the set

$$E_\alpha = \{x \in \partial T_W;\ \lim_{n\to\infty} \frac{\log\, \mu_W(I_n(x) \cap \partial T_L)}{\log\, |I_n(x) \cap \partial T_L|} = \alpha\}.$$

Define $\mathcal{C}_1 = \{(q,t); \varphi_{W,L}(q,t) < \infty\}$ and $\mathcal{C}_2 = \{(q,t); \psi_{W,L}(q,t) < \infty\}$.

Our hypotheses are the following:

(H) $(0,0) \in \mathrm{Int}(\mathcal{C}_1)$ and if $W \in \mathcal{E}_1$ then $(1,0) \in \mathrm{Int}(\mathcal{C}_1)$. Moreover $D_{W,L} \neq 0$.

Then, by Theorem 2.2 and 2.3.: $(0,0) \in \mathrm{Int}(\mathcal{C}_2)$ and if $W \in \mathcal{E}_1$ then $(1,0) \in \mathrm{Int}(\mathcal{C}_2)$. We have to distinguish the two following alternatives:

(I) $\quad \nexists \delta > 0 \ \ \forall \ \ 0 \leq j \leq c-1 \quad \mathbb{1}_{\{W_j > 0\}}\, L_j = \mathbb{1}_{\{W_j>0\}}\, W_j^{\frac{1}{\delta}}$,

(II) $\quad \exists \delta > 0 \ \ \forall \ \ 0 \leq j \leq c-1 \quad \mathbb{1}_{\{W_j > 0\}}\, L_j = \mathbb{1}_{\{W_j>0\}}\, W_j^{\frac{1}{\delta}}$.

We shall need the following result, which gives an a priori upper bound for the dim E_α's associated to some positive measure on a subset of ∂T:

Proposition 5.1 (Multifractal formalism). *Let $B \neq \emptyset$ be a closed subset of ∂T for the standard topology, d a metric distance on B, and μ a positive measure on (B,d). For $\alpha \geq 0$, define*
$E_\alpha = \{x \in supp(\mu);\ \lim_{n\to\infty} \frac{\log\, \mu(I_n(x) \cap B)}{\log\, |I_n(x) \cap B|} = \alpha\}$, *where $|\ |$ denotes the diameter with respect to d.*
For $(q,t) \in \mathbb{R}^2$, define $C(q,t) = \limsup_{n\to\infty} \sum_{a \in T_n, I_a \cap \mathrm{supp}(\mu) \neq \emptyset} \mu^q(I_a \cap B)\, |I_a \cap B|^t$.
Then for $q \in \mathbb{R}$, $\tau(q) = \sup\{t \in \mathbb{R},\ C(q,t) = \infty\} = \inf\{t \in \mathbb{R},\ C(q,t) = 0\}$ is defined and
i) τ is convex, non increasing and $\tau(1) = 0$.
Recall that τ^* is the Legendre transform of τ (see 1.1). If $\alpha \in \mathbb{R}_+$ then
ii) If $\tau^*(\alpha) < 0$ then $E_\alpha = \emptyset$.
iii) If $\tau^*(\alpha) \geq 0$ then $\dim\, E_\alpha \leq \tau^*(\alpha)$.
If $\dim\, E_\alpha = \tau^*(\alpha)$, *say that the multifractal formalism holds at α.*

The proof of this proposition is left to the reader which can find such upper bound a priori in [7] or [30]. We shall use this proposition with $(B,d,\mu) = (\partial T_L, d_L, \mu_W)$.

5.1. Multifractal analysis in the case (I).

From now, we denote $\varphi_{W,L}$ by φ and $\psi_{W,L}$ by ψ. Before stating our result, we need to determine a good domain on which calculate the multifractal spectrum. This is the object of the following proposition.

Proposition-definition 5.2. *1) There exits a largest interval J_1 containing $]0,1]$ such that*
i) If $W \in \mathcal{E}_1$ then $]0,1] \subset Int(J_1)$ and $\tilde{\tau}(q) = \inf\{t \in \mathbb{R},\ \varphi(q,t) < 1\}$ is defined for all $q \in J_1$ and satisfy $\varphi(q,\tilde{\tau}(q)) = 1$.
i′) If $W \in \mathcal{E}_2$ then $1 = \max(J_1)$ and $\tilde{\tau}(q) = \inf\{t \in \mathbb{R},\ \varphi(q,t) < 1\}$ is defined for all $q \in J_1 \setminus \{1\}$ and satisfy $\varphi(q,\tilde{\tau}(q)) = 1$. Define $\tilde{\tau}(1) = 0$.

ii) The function $q \mapsto \tilde{\tau}(q)$ is C^∞, strictly convex and decreasing on J_1.

iii) $0 \notin J_1$ if and only if ($L \in \mathcal{E}_2$ and $\varphi(0,1) = 1$), which is also equivalent to say that $\lim_{q\to 0^+} \tilde{\tau}'(q) = -\infty$. Then defining $\tilde{\tau}(0) = 1$ makes $\tilde{\tau}$ continuous at 0.

If $q \notin J_1$ then define $\tilde{\tau}(q) = \infty$.

Define $J = \{q \in J_1;\ \tilde{\tau}^*(-\tilde{\tau}'(q) \geq 0\}$, $I = -\tilde{\tau}'(J)$, $\alpha_{\inf} = \inf(I)$ and $\alpha_{\sup} = \sup(I)$. If $\alpha \geq 0$ is such that $\tilde{\tau}^*(\alpha) = 0$ then $\alpha \in \{\alpha_{\inf}, \alpha_{\sup}\}$.

For $q \in J_1$, define $W_q = (\mathbb{1}_{\{W_j>0\}} W_j^q L_j^{\tilde{\tau}(q)})_{j\in\mathcal{C}}$.

Define $F_1 = \{q \in J_1;\ \tilde{\tau}^*(-\tilde{\tau}'(q)) > 0,\ \exists\ V \in \{W,L\},\ \mathbb{E}(Y_{W_q} |\log Y_V|) = \infty\}$, $F_2 = \{q \in J_1;\ \tilde{\tau}^*(-\tilde{\tau}'(q)) = 0,\ W_q \notin \mathcal{E}_2\}$ and $S = -\tilde{\tau}'(F_1 \cup F_2)$; $S \subset \{\alpha_{\inf}, \alpha_{\sup}\}$.

Finally define $J' = \{q \in J_1;\ \exists \varepsilon > 0,\ \{q\} \times [\tilde{\tau}(q), \tilde{\tau}(q)+\varepsilon] \subset \mathcal{C}_2\}$ and $I' = -\tilde{\tau}'(J')$.

iv) If $W \in \mathcal{E}_1$ then J and J' are intervals which contain $]0,1]$ in their interior; $I \subset \mathbb{R}_+$ and $I' \subset \mathbb{R}_+$.
iv′) If $W \in \mathcal{E}_2$ then J and J' are intervals which contain $]0,1[$; $1 \in J \setminus (F_1 \cup F_2)$; $0 \in I \setminus S \subset \mathbb{R}_+$ and $I' \subset \mathbb{R}_+$.

Remark 5.1. 1) Remark 2.1. suggests that J and J', and so I and I' might be different, in particular $I' \not\subset I$ or $I \not\subset I'$, and there are effectively examples of such situations.
2) When $W \in \mathcal{E}_1$, in (H) we assume that $(0,1) \in \mathrm{Int}(C_1)$ in view to have $\dim E_D = D$ by Corollary 4.1.
3) $F_1 \cup F_2 \subset \{\inf(J), \sup(J)\}$ and $J \setminus F_1 \cup F_2$ is the set of the elements q of J such that one can apply Corollary 4.1 or 4.2 with (W, W_q, L).
4) $\tilde{\tau}(0) \leq 1$ and $\tilde{\tau}(0) = 1$ if and only if $P(\partial T_W = \partial T_L) = 1$.

Recall that for $q \in J$, $D_{W_q,L} = \mathbb{E}(\sum_{j=0}^{c-1} W_{q,j} \log L_j)$. Now, remark 3) made in introduction on the Legendre transform gives a better understanding of the following

Theorem 5.1. *For P-almost every $\omega \in \{\mu_W \neq 0\}$, one has:*

i) Study of E_α when $\alpha \in]\alpha_{\inf}, \alpha_{\sup}[$ (main result).
a) $\dim E_\alpha \geq \tilde{\tau}^*(\alpha) > 0$ *for all* $\alpha \in]\alpha_{\inf}, \alpha_{\sup}[$.
b) $\dim E_\alpha = \tau^*(\alpha) = \tilde{\tau}^*(\alpha)$ *for all* $\alpha \in]\alpha_{\inf}, \alpha_{\sup}[\cap Cl(I')$.
The multifractal formalism holds on $]\alpha_{\inf}, \alpha_{\sup}[\cap Cl(I')$.

ii) Study of $E_{\alpha_{\inf}}$ and $E_{\alpha_{\sup}}$.

a) If $\mathbb{R}_+^* \subset J$, $D_{W_q,L} \not\to 0$ *as* $q \to \infty$ *and* $\tilde{\tau}^*(\alpha_{\inf}) > 0$, *or* $\mathbb{R}_+^* \not\subset J$ *and* $\alpha_{\inf} \in I \setminus S$, *then* $E_{\alpha_{\inf}} \neq \emptyset$ *and* $\dim E_{\alpha_{\inf}} \geq \tilde{\tau}^*(\alpha_{\inf})$.
b) If $\alpha_{\sup} < \infty$ *then: if* $\mathbb{R}_- \subset J$, $D_{W_q,L} \not\to 0$ *as* $q \to -\infty$ *and* $\tilde{\tau}^*(\alpha_{\sup}) > 0$, *or* $\mathbb{R}_- \not\subset J$ *and* $\alpha_{\sup} \in I \setminus S$, *then* $E_{\alpha_{\sup}} \neq \emptyset$ *and* $\dim E_{\alpha_{\sup}} \geq \tilde{\tau}^*(\alpha_{\sup})$.

Moreover, in the cases a) and b), if $\alpha \in \{\alpha_{\inf}, \alpha_{\sup}\}$ *is in* $Cl(I')$, *then* $\dim E_\alpha = \tilde{\tau}^*(\alpha) = \tau^*(\alpha)$, *the multifractal formalism holds at* α.

iii) Study of E_α when $\alpha \notin [\alpha_{\inf}, \alpha_{\sup}]$.
a) If $\alpha_{\inf} \in Cl(I')$ *then, if* $\mathbb{R}_+^* \subset J$ *or* $\mathbb{R}_+^* \not\subset J$ *and* $\tilde{\tau}^*(\alpha_{\inf}) = 0$ *then for all* $\alpha \in [0, \alpha_{\inf}[$, $E_\alpha = \emptyset$.
b) If $\alpha_{\sup} \in Cl(I')$ *then, if* $\mathbb{R}_- \subset J$ *or* $\mathbb{R}_- \not\subset J$, $\alpha_{\sup} < \infty$ *and* $\tilde{\tau}^*(\alpha_{\sup}) = 0$ *then for all* $\alpha \in]\alpha_{\sup}, \infty[$, $E_\alpha = \emptyset$.

iv) Comparison of τ with $\tilde{\tau}$.
a) $\tau(q) \leq \tilde{\tau}(q)$ *for all* $q \in Cl^+(J')$.
b) $\tau(q) = \tilde{\tau}(q)$ *for all* $q \in Int(J) \cap Cl^+(J')$; $\tau(1) = \tilde{\tau}(1) = 0$. *If* $W \in \mathcal{E}_2$ *then for all* $q \geq 1$, $\tau(q) = 0$.

The following corollary gives examples where $L \in \mathcal{E}_1$, the case for which we know a sufficient condition for μ_L defining a distance d_L (see Prop. 3.1).

Corollary 5.1. (Examples). *Assume* $L \in \mathcal{E}_1$ *and* $\mathbb{E}(Y_L\,|\log Y_L|) < \infty$. *For* $V \in \{W, L\}$, *define* $h_V = \sup\{h \geq 1; \mathbb{E}(\sum_{j=0}^{c-1} V_j^h) = 1\}$. *For* $\omega \in \{\mu_W \neq 0\}$, *let* $\mathcal{I}$ *be the largest interval such that for all* $\alpha \in \mathcal{I}$, $E_\alpha \neq \emptyset$ *and* $\dim E_\alpha = \tilde{\tau}^*(\alpha) = \tau^*(\alpha)$. *Then for P-almost every ω in* $\{\mu_W \neq 0\}$,
i) if $W \in \mathcal{E}_1$, $P(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j > 0\}} \geq 2) = 1$ *and* $\mathcal{C}_1 = \mathbb{R}^2$, *then*
a) if $h_W = h_L = 1$ *then* $]\alpha_{\inf}, \alpha_{\sup}[\subset \mathcal{I} \subset [\alpha_{\inf}, \alpha_{\sup}]$;
b) if $h_W = 1$, $h_L > 1$ *and* $\tilde{\tau}(\inf(J)) \leq h_L$, *then* $]\alpha_{\inf}, \alpha_{\sup}] \subset \mathcal{I} \subset [\alpha_{\inf}, \alpha_{\sup}]$;
c) if $h_W > 1$ *and* $h_L = 1$ *then* $[\alpha_{\inf}, \alpha_{\sup}[\subset \mathcal{I} \subset [\alpha_{\inf}, \alpha_{\sup}]$;
d) if $h_W > 1$, $h_L > 1$, $\sup(J) \leq h_W$ *and* $\tilde{\tau}(\inf(J)) \leq h_L$, *then* $\mathcal{I} = [\alpha_{\inf}, \alpha_{\sup}]$.

ii) If $W \in \mathcal{E}_2$, $P(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j > 0\}} \geq 2) = 1$, *for all* $q < 0$, $\varphi(q, 0) < \infty$ *and for all* $t \geq 1$, $\varphi(0, t) < \infty$, *then*
a) if $h_L = 1$ *then* $[0, \alpha_{\sup}[\subset \mathcal{I} \subset [0, \alpha_{\sup}]$;
b) if $h_L > 1$ *and* $\tilde{\tau}(\inf(J)) \leq h_L$ *then* $\mathcal{I} = [0, \alpha_{\sup}]$.

Moreover, in i)a)b) (resp. i)a)c) and ii)a)), if $\mathbb{R}_+$ *(resp.* $\mathbb{R}_-$*)* $\subset J$ *and neither* $\tilde{\tau}^*(\alpha_{\inf}) = 0$ *nor* $\lim_{q\to\infty} D_{W_q,L} = 0$ *(resp. neither* $\alpha_{\sup} = \infty$ *nor* $\alpha_{\sup} < \infty$ *and* $\tilde{\tau}^*(\alpha_{\sup}) = 0$ *or* $\lim_{q\to-\infty} D_{W_q,L} = 0$*) arises, then* $\alpha_{\inf}$ *(resp.* $\alpha_{\sup}$*)* $\in \mathcal{I}$; *in i) and ii), for all* $\alpha \notin [\alpha_{\inf}, \alpha_{\sup}]$, $E_\alpha = \emptyset$.

Remark 5.2. 1) Assertion *ii)* of Theorem 5.1. is a first improvement of the other works on the measures of type 1, which tell nothing about E_α when α is an end point of $[\alpha_{\inf}, \alpha_{\sup}]$. When $\alpha_{\inf} \in I \setminus S$ and $\tilde{\tau}^*(\alpha_{\inf}) = 0$, the proof will show the fundamental role of the existence of the measures of type 2.
Nevertheless this point does not conclude when $\mathbb{R}_+ \subset J$ and $\tau^*(\alpha_{\inf}) = 0$ or $\alpha_{\inf} \in S$ and $\sup(J) < \infty$ (and in the symetric cases on $\mathbb{R}_-$).
Taking $L_j = \frac{1}{c}$ and the W_j's i.i.d. with distribution of the form $\lambda\delta_a + (1-\lambda)\delta_b$ with $\lambda \in]0,1[$, $0 < a < b < 1$ and $\lambda a + (1-\lambda)b = \frac{1}{c}$, provides examples of the alternatives $\mathbb{R}_+ \subset J$ and $\tau^*(\alpha_{\inf}) > 0$ and $\mathbb{R}_+ \subset J$ and $\tau^*(\alpha_{\inf}) = 0$.

Under our hypothesis $D_{W,L} \neq 0$ one has $D_{W_q,L} \not\to 0$ as $q \to \infty$ or $D_{W_q,L} \not\to 0$ as $q \to -\infty$ for example as soon as the L_j's are bounded by 1.

2) Assertion *i*) of Theorem 5.1. is the main result announced in the introduction. It claims that for the measures of type 1 and 2, it is possible to permute the "for all α" and the "with probability one" usually met in the results on the multifractal analysis of the measures of type 1 ([16], [19], [14], [29], [28], [1], [3]).

3) If $S' =]\alpha_{\inf}, \alpha_{\sup}[\setminus \mathrm{Cl}(I') \neq \emptyset$ and $\alpha \in S'$, we do not know how to establish the inequality $\dim E_\alpha \leq \tilde{\tau}^*(\alpha)$.

4) The ultrametric distance is given by a measure, so the multifractal analysis made here may be compared with the one made in an other context in [9]. Moreover, if we suppress the hypothesis that μ_L has no atoms, we have nevertheless the

Theorem 5.2. *Assume the hypotheses of Theorem 5.1, excepted the fact that μ_L has no atoms. For $\alpha \in \mathbb{R}_+$ define $F_\alpha = \{x \in \partial T_W;\ \lim_{n\to\infty} \frac{\log \mu_W(I_n(x) \cap \partial T_L)}{\log \mu_L(I_n(x) \cap \partial T_L)} = \alpha\}$. Then, for P-almost every $\omega \in \{\mu_W \neq 0\}$,*

i) $F_\alpha \neq \emptyset$ for all $\alpha \in]\alpha_{\inf}, \alpha_{\sup}[$.

ii) a) If $\mathbb{R}_+^* \subset J$, $D_{W_q,L} \not\to 0$ as $q \to \infty$ and $\tilde{\tau}^*(\alpha_{\inf}) > 0$, or $\mathbb{R}_+^* \not\subset J$ and $\alpha_{\inf} \in I \setminus S$, then $F_{\alpha_{\inf}} \neq \emptyset$.
b) If $\alpha_{\sup} < \infty$ then: if $\mathbb{R}_- \subset J$, $D_{W_q,L} \not\to 0$ as $q \to -\infty$ and $\tilde{\tau}^*(\alpha_{\sup}) > 0$, or $\mathbb{R}_- \not\subset J$ and $\alpha_{\sup} \in I \setminus S$, then $F_{\alpha_{\sup}} \neq \emptyset$.
As an example, if $W \in \mathcal{E}_2$, $L \in \mathcal{E}_2$ and $P(\partial T_W = \partial T_L) = 1$, then for all $\alpha \in [0, \infty[$, $F_\alpha \neq \emptyset$.

The proof of Proposition 5.2 is left to the reader. The ideas are the same as in the proof of Proposition VI.A.a. of [3], and based on the strict log-convexity of φ under (I). To prove Theorem 5.1., we need a serie of lemmas.

Lemma 5.1. *For all $q \in Int(J)$, there exists a neighbourhood V_q of q in $\mathbb{C}$ such that*
i) for all $z \in V_q$, there exists $\tilde{\tau}_q(z) \in \mathbb{C}$ such that $\mathbb{E}(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j>0\}}\, W_j^z\, L_j^{\tilde{\tau}_q(z)}) = 1$ and the function $z \mapsto \tilde{\tau}_q(z)$ is an analytic extension of $\tilde{\tau}_{|\mathbb{R}\cap V_q}$.
For $a \in T$ and $z \in V_q$ let $W_z(a)$ denote the r. v. $(\mathbb{1}_{\{W_j(a)>0\}}\, W_j^z(a)\, L_j^{\tilde{\tau}_q(z)}(a))_{j\in\mathcal{C}}$.
ii) With probability one, for all $a \in T$, the sequence
$(z \mapsto Y_{z,n}(a) = \sum_{b\in T_n} W_{z,b_1}(a)\, W_{z,b_2}(ab_1) \ldots W_{z,b_n}(ab_1 \ldots b_{n-1}))_{n\geq 1}$ of analytic functions on V_q converges, uniformly on the compact sets, to an analytic function $z \mapsto Y_z(a)$.

Proof. i) Let q be in $\mathrm{Int}(J)$. In a neighbourhood O of $(q, \tilde{\tau}(q))$ in $\mathbb{C}^2$, $(z, z') \mapsto \mathbb{E}(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j>0\}} \mid W_j^z \mid \mid L_j^{z'} \mid)$ is finite and $\tilde{\varphi} : (z, z') \mapsto \mathbb{E}(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j>0\}}\, W_j^z\, L_j^{z'})$ is analytic in z and z'. Since $q \in \mathrm{Int}(J)$, as in [3] Lemma VI.A., the strict log convexity of φ yields $h > 1$ such that $\varphi(hq, h\tilde{\tau}(q)) < 1$. So, one can choose h and O to have
$\sup_{(z,z')\in O} \mathbb{E}(\sum_{j=0}^{c-1} \mathbb{1}_{\{W_j>0\}}\, |W_j^z|^h\, |L_j^{z'}|^h) < 1$. Moreover, as in [3] (proof of Prop. VI.A.a.ii)), $\frac{\partial}{\partial t}\varphi(q, \tilde{\tau}(q)) \neq 0$ and so $\frac{\partial}{\partial z'}\tilde{\varphi}(q, \tilde{\tau}(q)) = \frac{\partial}{\partial t}\varphi(q, \tilde{\tau}(q)) \neq 0$. So, there exist a neighbourhood V_1 (resp. V_2) of q (resp. $\tilde{\tau}(q)$) in $\mathbb{C}$ such that $V_1 \times V_2 \subset O$ and an

analytic function $\tilde{\tau}_q : V_1 \to V_2$ such that for all $z \in V_1$, $\tilde{\varphi}(z, \tilde{\tau}_q(z)) = \varphi(q, \tilde{\tau}(q)) = 1$. Choose $V_q = V_1$.
ii) For $a = \epsilon$, the proof follows the same lines as the one of the corresponding point in Theorem 1.2. Then the result comes from the fact that for all $a \in T$ and $n \geq 1$, $z \mapsto Y_{z,n}(a) \sim z \mapsto Y_{z,n}(\epsilon)$ and T is countable.

Corollary 5.2. (Simultaneous auxiliary measures). *For $a \in T$ and $q \in Int(J)$ let $W_q(a)$ denote the random vector $(\mathbb{1}_{\{W_j(a)>0\}} W_j^q(a) L_j^{\tilde{\tau}(q)}(a))_{0 \leq j \leq c-1}$. For a fixed q, the $W_q(a)$'s, $a \in T$, are i.i.d., in $\mathcal{E}_1$, and they allow to define a measure μ_{W_q} as in 3. In fact these measures μ_{W_q} are defined simultaneously:*
i) With probability one, for all $a \in T$ and $q \in Int(J)$, the sequence $\big(Y_{W_q,n} = \sum_{b \in T_n} W_{q,b_1}(a) W_{q,b_2}(ab_1) \ldots W_{q,b_n}(ab_1 \ldots b_{n-1})\big)_{n \geq 1}$ converges to a finite limit $Y_q(a)$, ($Y_q(\epsilon) = Y_q$).
Moreover, with probability one, for all $a \in T$:
α) the function $q \mapsto Y_q(a)$ is analytic on $Int(J)$.
β) For all $q \in Int(J)$, $Y_q(a) = \sum_{j=0}^{c-1} W_{q,j}(a) Y_q(aj)$.

ii) Let a be in T. α) For P-almost every ω in $\{Y_W(a) = 0\}$, $q \mapsto Y_q(a)$ is identically equal to zero.
β) For P-almost every ω in $\{Y_W(a) \neq 0\}$, $q \mapsto Y_q(a)$ has no zero in $Int(J)$.

iii) For P-almost every ω in $\{Y_W \neq 0\}$, for all $q \in Int(J)$, the following relations define an unique positive measure μ_q on $(\partial T, \mathcal{T})$: for all $a \in T$, $\mu_q(I_a) = w_q(a) Y_q(a)$. Moreover, for P-almost every $\omega \in \{\mu_W \neq 0\}$, for all $q \in Int(J)$, $supp(\mu_q) = supp(\mu_W) \subset \partial T_L$.

Proof of Corollary 5.2. We saw in the proof of Lemma 5.1 that for all q in $\mathrm{Int}(J)$, there exists $h > 1$ such that $\varphi(hq, h\tilde{\tau}(q)) = \mathbb{E}(\sum_{j=0}^{c-1} W_{q,j}^h) < 1$, so $W_q \in \mathcal{E}_1$ by construction.
$i)\alpha)\beta)$ Simple consequences of Lemma 5.1. and the construction of the $Y_q(a)$'s.
ii) Consequence of the continuity of $q \mapsto Y_q$ and Proposition 3.2. applied simultameously for all the W_q's, $q \in \mathrm{Int}(J) \cap \mathbb{Q}$.

β) With probability one, $q \mapsto Y_q$ is continuous and non negative, so if $\mathcal{I}$ is a subinterval of $\mathrm{Int}(J)$, $S_{\mathcal{I}} = \{\exists\, q \in \mathcal{I},\ Y_q = 0\}$ is an event.

Let K be a compact subinterval of $\mathrm{Int}(J)$. For $\mathcal{J} \subset \mathcal{C}$, define $A_{\mathcal{J}} = \{\forall\, j \in \mathcal{J},\ W_j > 0;\ \forall\, j \in \mathcal{J}^c,\ W_j = 0\}$, $B_{\mathcal{J}} = \{\exists\, q \in K,\ \forall\, j \in \mathcal{J},\ Y_q(j) = 0\}$ and $\tilde{B}_{\mathcal{J}} = \cap_{j \in \mathcal{J}} B_{\{j\}}$. They are also events. As the processes $q \mapsto Y_q(j)$, $j \in \mathcal{C}$, are i.i.d. with $q \mapsto Y_q$, for all $j \in \mathcal{C}$, $P(B_{\{j\}}) = P(S_K)$ and for $\mathcal{J} \subset \mathcal{C}$, $P(\tilde{B}_{\mathcal{J}}) = (P(S_K))^{\#\mathcal{J}}$. By (E_ϵ) (Th. 2.1),
$S_K = \{\exists\, q \in K,\ \forall\, j \in \mathcal{C},\ W_{q,j} Y_q(j) = 0\} = \{\exists\, q \in K,\ \forall\, j \in \mathcal{C},\ W_j Y_q(j) = 0\}$. So

$P(S_K) = \sum_{\mathcal{J} \subset \mathcal{C}} P(A_{\mathcal{J}}) P(B_{\mathcal{J}}) \leq \sum_{\mathcal{J} \subset \mathcal{C}} P(A_{\mathcal{J}}) P(\tilde{B}_{\mathcal{J}}) = \sum_{\mathcal{J} \subset \mathcal{C}} P(A_{\mathcal{J}}) (P(S_K))^{\#\mathcal{J}}$ $= f(P(S_K))$ (see Th. 2.1). So as f is strictly convex on $[0,1]$ (by (H_0)) and its fixed points are $P(\{Y_W = 0\})$ and 1 (see Th 2.1.), one has either $0 \leq P(S_K) \leq P(\{Y_W = 0\})$ or $P(S_K) = 1$.
If $P(S_K) = 1$ then there are only equalities in the previous calculus, and by (H_0) there exists $\mathcal{J} \subset \mathcal{C}$, with $\#\mathcal{J} \geq 2$, such that $P(B_{\mathcal{J}}) = 1$. Let (j_1, j_2), $j_1 \neq j_2$, be in $\mathcal{J}^2$; $P(B_{\{j_1,j_2\}}) = 1$ and for all $\mathcal{I}_1$ and $\mathcal{I}_2$ subintervals of K, $S^i_{\mathcal{I}_i} = \{\exists\, q \in$

$\mathcal{I}_i,\ Y_q(j_i)=0\}$, $i=1,2$, are independent events and $P(S^1_{\mathcal{I}_1})=P(S^2_{\mathcal{I}_1})$. We can suppose that $K=[0,1]$. If $P(S^1_{[0,\frac12[})<1$ and $P(S^1_{[\frac12,1]})<1$ then $P(\Omega\setminus S^1_{[0,\frac12[}\cap\Omega\setminus S^2_{[\frac12,1]})>0$, and $P(B_{\{j_1,j_2\}})<1$. There is a contradiction. So one constructs a decreasing sequence $(I_n)_{n\geq 0}$ of dyadic closed subintervals of $[0,1]$ such that for all $n\geq 0$, $P(S_{I_n})=1$ and $S_{I_{n+1}}\subset S_{I_n}$. Let q_0 be the unique point of $\cap_{n\geq 0}I_n$: $\{Y_{q_0}=0\}=S_{\{q_0\}}=\cap_{n\geq 0}S_{I_n}$, so $P(\{Y_{q_0}=0\})=1$, new contradiction since $W_{q_0}\in\mathcal{E}_1$.
Thus, $0\leq P(S_K)\leq P(\{Y_W=0\})$ and since $\{Y_W=0\}\subset S_K$, one has $P(\{Y_W\neq 0\}\cap S_K)=0$. The conclusion comes from the fact that $\mathrm{Int}(J)$ can be written as a countable union of compact subintervals.
iii) It is a simple consequence of the points *i*) and *ii*).

Lemma 5.2. *i*) *Recall that* $\tilde c=\sum_{j=0}^{c-1}\mathbb{I}_{\{W_j>0\}}$. *For* $q\in Int(J)$, *define* $A_q=\min_{0\leq j\leq c-1,\ W_{q,j}>0}W_{q,j}$ *on* $\{\tilde c\geq 1\}$ *and* $A_q=0$ *on* $\{\tilde c=0\}$. *Then define* $\delta(q)=$ *ess inf*$_{\{\tilde c\geq 1\}}$ A_q. *There exists a finite subset* F *of* $Int(J)$ *such that for all compact interval contained in* $Int(J)\setminus F$ *one has the following alternatives: either* $\inf_{q\in K}\delta(q)>0$ *or for all* $q\in K$, $\delta(q)=0$.

Fix such an F.
ii) *Let* K *be a compact interval contained in* $Int(J)\setminus F$. *There exists* $\eta>0$ *such that* $\sup_{q\in K}\mathbb{E}(\mathbb{I}_{\{Y_q>0\}}\,Y_q^{-\eta})<\infty$.

Proof. *i*) We leave the verification to the reader.

ii) Let $(q\mapsto\tilde Y_q)$ be a copie of $(q\mapsto Y_q)$ on K, which is independent of $(q\mapsto A_q)$ on K. For all $q\in K$ define $\psi_q\,:\,t\geq 0\mapsto\mathbb{E}(\mathbb{I}_{\{Y_q>0\}}e^{-tY_q})$. Then for $q\in K$ define $p_q\,:\,t>0\mapsto P(0<A_q\tilde Y_q\leq t^{-1/2})+e^{-t^{1/2}}$.
By using *i*), Theorem 2.3, (H), and the fact that for a fixed $t>0$, $q\mapsto p_q(t)$ and $q\mapsto\psi_q(t)$ are continuous on K, one obtains that for all $q\in K$, there exists a neighbourhod O_q of q and $\eta_q>0$ such that $\sup_{q'\in O_q}\mathbb{E}(\mathbb{I}_{\{Y_{q'}>0\}}\,Y_{q'}^{-\eta_q})<\infty$. One ends the proof by using a covering of K by a finite number of O_q's.

Lemma 5.3. *Let* K *be a compact subinterval of* $Int(J)$. *Fix* $\varepsilon>0$. *For* $\eta\in]0,1[$ *close enough to 0, one has for* $\delta\in\{-1,1\}$ *and* $V\in\{W,L\}$:
i)*a*) $\sup_{q\in K}\mathbb{E}(\sum_{j=0}^{c-1}W_{q,j}^{1-\delta\eta})\,e^{-\delta(D_{W_q,W_q}+\delta\varepsilon)\eta}<1$.
b) $\sup_{q\in K}\mathbb{E}(\sum_{j=0}^{c-1}\mathbb{I}_{\{W_{q,j}>0\}}\,|\frac{d}{dq}W_{q,j}|W_{q,j}^{-\delta\eta})<\infty$.
ii)*a*) $\sup_{q\in K}\mathbb{E}(\sum_{j=0}^{c-1}W_{q,j}\,V_j^{-\delta\eta})e^{-\delta(D_{W_q,V}+\delta\varepsilon)\eta}<1$.
b) $\sup_{q\in K}\mathbb{E}(\sum_{j=0}^{c-1}|\frac{d}{dq}W_{q,j}|V_j^{-\delta\eta}(q))<\infty$.
iii)*a*) $\sup_{q\in K}\mathbb{E}(Y_q^{1-\delta\eta})\ <\infty$.
b) *if* $K\cap F=\emptyset$ *then* $\sup_{q\in K}\mathbb{E}(\mathbb{I}_{\{Y_q>0\}}\,|\frac{d}{dq}Y_q|\,Y_q^{-\delta\eta})<\infty$.
iv) $\sup_{q\in K}\mathbb{E}(Y_q\,Y_V^{-\delta\eta})<\infty$ *and* $\sup_{q\in K}\mathbb{E}(|\frac{d}{dq}Y_q|\,Y_V^{-\delta\eta})<\infty$.

Proof. The most delicate point is *iii*). *iii*)*a*) follows from the fact that in lemma 5.1, the convergence is locally uniform in L^h norm for some $h>1$. The proof of *iii*)*b* is similar to the one of the corresponding point in the proof of Theorem 1.2. and uses Lemma 5.2.

Lemma 5.4. *Fix $\varepsilon > 0$ and let $\eta \in]0,1[$. Assume Corollary 5.2. Let Ω_1 denote the subset of $\{Y_W \neq 0\}$ on which $q \mapsto Y_q$ is positive and analytic on Int(J) and for all $q \in$ Int(J), supp$(\mu_q) =$ supp$(\mu_W) \subset \partial T_L$. For $\omega \in \Omega_1$, define for all $n \geq 1$, $\delta \in \{-1, 1\}$ and $V \in \{W, L\}$ the following functions on Int (J):*

$$g_{\delta,n} \, : \, q \mapsto \sum_{a \in T_n} w_q^{1-\delta\eta}(a) \, e^{-\delta n (D_{W_q,W_q}+\delta\varepsilon)\eta} \, Y_q^{1-\delta\eta}(a),$$
$$f_{\delta,V,n} \, : \, q \mapsto \sum_{a \in T_n} w_q(a) \, v^{-\delta\eta}(a) \, e^{-\delta n (D_{W_q,V}+\delta\varepsilon)\eta} \, Y_q(a) \, Y_V^{-\delta\eta}(a).$$

For $\omega \in \Omega \setminus \Omega_1$, define the same functions to be identically equal to 0.

Let K be a compact interval contained in Int$(J) \setminus F$. If $\eta > 0$ is small enough then for P-almost every ω in $\{Y_W \neq 0\}$, the series $\sum_{n\geq 1} g_{\delta,n}$ and $\sum_{n \geq 1} f_{\delta,V,n}$, $V \in \{W, L\}$, converge uniformly on K and they converge on F (F was defined in Lemma 5.2).

Proof. Write $K = [q_1, q_2]$ $(q_1 < q_2)$. One verifies easily that proving the following assertion gives the conclusion:

With probability one,
i) For all $q \in F \cup \{q_1\}$, $\sum_{n \geq 1} g_{\delta,n}(q) < \infty$ and for $V \in \{W, L\}$ $\sum_{n\geq 1} f_{\delta,V,n}(q) < \infty$.
ii) $\sum_{n\geq 1} \int_K |g'_{\delta,n}(x)| \, dx < \infty$ and for $V \in \{W, L\}$ $\sum_{n \geq 1} \int_K \left| f'_{\delta,V,n}(x) \right| \, dx < \infty$.

Choose $\eta \in]0,1[$ such that: all the conclusions of Lemma 5.3 hold for K; all the conclusions of Lemma 5.3, excepted the *iii)b)*, hold for each $\{q\}$, $q \in F$. Then one proceeds as in Corollary 1.1. to study $\sum_{n\geq 1} f_n$.

Lemma 5.5. *Assume Corollary 5.2. For P-almost every $\omega \in \{Y_W \neq 0\}$, for all $q \in$ Int(J), if $V \in \{W, W_q, L\}$, for μ_q-almost every $x \in \partial T_L$,*

$$\lim_{n \to \infty} \frac{\log \, \mu_V(I_n(x) \cap \partial T_L)}{n} = \mathbb{E}(\sum_{j=0}^{c-1} W_{q,j} \, \log \, V_j) = -D_{W_q,V}.$$

Proof. For $\omega \in \Omega_1$ (see Lemma 5.4), $\varepsilon > 0$, $q \in \mathrm{Int}(J)$, $V \in \{W, W_q, L\}$ and $n \geq 1$, define $E^{1,V}_{q,n,\varepsilon} = \{x \in \partial T_L; \; \frac{\log \, \mu_V(I_n(x))}{-n} \geq D_{W_q,V} + \varepsilon\}$ and
$E^{-1,V}_{q,n,\varepsilon} = \{x \in \partial T_L; \; \frac{\log \, \mu_V(I_n(x))}{-n} \leq D_{W_q,V} - \varepsilon\}$.
Fix $\varepsilon > 0$ and $\eta \in]0,1[$. For $\omega \in \Omega_1$, $q \in \mathrm{Int}(J)$ and $V \in \{W, W_q, V\}$, a similar calculus to the one made in the proofs of Corollary 1.1. and 4.2. yields
$\sum_{n\geq 1} \mu_q(E^{1,W_q}_{q,n,\varepsilon}) + \mu_q(E^{-1,W_q}_{q,n,\varepsilon}) \leq \sum_{n\geq 1} g_{1,n}(q) + g_{-1,n}(q)$ and if $V \in \{W, L\}$ $\sum_{n \geq 1} \mu_q(E^{1,V}_{q,n,\varepsilon}) + \mu_q(E^{-1,V}_{q,n,\varepsilon}) \leq \sum_{n\geq 1} f_{1,V,n}(q) + f_{-1,V,n}(q)$ (with the notations of Lemma 5.4.). One concludes by using lemma 5.4. and as in Corollary 1.1., the Borel-Cantelli Lemma.

Lemma 5.6. *For P-almost every $\omega \in \{\mu_W \neq 0\}$:*
i) If $\mathbb{R}_+$(resp. $\mathbb{R}_-$) $\not\subset J$ and $\alpha = \alpha_{\inf}$(resp. $\alpha_{\sup}$) $\in I \setminus S$ then $E_\alpha \neq \emptyset$ and $\dim \, E_\alpha \geq \tilde{\tau}^*(\alpha)$.
ii) If $\mathbb{R}_+^* \subset J$, $D_{W_q,L} \not\to 0$ as $q \to \infty$ and $\tilde{\tau}^*(\alpha_{\inf}) > 0$, then $E_{\alpha_{\inf}} \neq \emptyset$ and $\dim \, E_{\alpha_{\inf}} \geq \tilde{\tau}^*(\alpha)$.
iii) If $\mathbb{R}_- \subset J$, $\alpha_{\sup} < \infty$, $D_{W_q,L} \not\to 0$ as $q \to -\infty$ and $\tilde{\tau}^*(\alpha_{\sup}) > 0$ then $E_{\alpha_{\sup}} \neq \emptyset$ and $\dim \, E_{\alpha_{\sup}} \geq \tilde{\tau}^*(\alpha)$.

Proof. First remark that with the notations of 4., for $\alpha \in I$, $\alpha = -\tilde{\tau}'(q_\alpha)$, (5.1) $\tilde{\tau}^*(\alpha) = \dfrac{D_{W_{q_\alpha},W_{q_\alpha}}}{D_{W_{q_\alpha},L}}$ and (5.2) $\alpha = \dfrac{D_{W_{q_\alpha},W}}{D_{W_{q_\alpha},L}}$.

i) By construction of J and by (H), if $\tilde{\tau}^*(\alpha) > 0$ (resp. $=0$) then by (5.1) $W(q_\alpha) \in \mathcal{E}_1$ (resp. $\mathcal{E}_2$) and we are under the hypotheses of Corollary 4.1 (resp. 4.2) with $(W, W(q_\alpha), L)$: for P-almost every $\omega \in \{\mu_{q_\alpha} \neq 0\}$, E_α carries a piece of $\mu_{W(q_\alpha)}$ and so $E_\alpha \neq \emptyset$ and $\dim E_\alpha \geq \tilde{\tau}^*(\alpha)$ by the Billingsley Theorem. But by Proposition 3.2., the event $\{\mu_{q_\alpha} \neq 0\}$ differs from $\{\mu_W \neq 0\}$ by a set of probability 0.

ii) By the geometrical remark made in the introduction on the Legendre transform, $\alpha_{\inf} > 0$ and $\tilde{\tau}^*(\alpha_{\inf}) = \lim_{q\to\infty} \tilde{\tau}^*(-\tilde{\tau}'(q))$.

We need

Lemma 5.7. *Assume that $D_{W_q,L} \not\to 0$ as $q \to \infty$. Then there exists a sequence $(q_n)_{n\geq 0} \in \mathbb{R}_+^{\mathbb{N}}$, $\lim_{n\to\infty} q_n = \infty$, such that as $n \to \infty$, (W, W_{q_n}, L) converges in distribution towards a random vector $(\tilde{W}, W', \tilde{L})$ and*
i) If $\tilde{\tau}^*(\alpha_{\inf}) > 0$ then $W' \in \mathcal{E}_1$, $\mathbb{E}(\sum_{j=0}^{c-1} W'_j|\log V_j|) < \infty$ for $V \in \{\tilde{W}, \tilde{L}\}$, there exists $\eta > 0$ such that $\mathbb{E}(\sum_{j=0}^{c-1} W_j'^{1+\eta}) < 1$ and :
$\frac{D_{W',\tilde{W}}}{D_{W',\tilde{L}}} = \alpha_{\inf}$ *and* $\frac{D_{W',W'}}{D_{W',\tilde{L}}} = \tilde{\tau}^*(\alpha_{\inf})$.
ii) If $\tilde{\tau}^*(\alpha_{\inf}) = 0$ then $P(\sum_{j=0}^{c-1} W'_j|1 - W'_j| = 0) = 1$ and $W' \notin \mathcal{E}_1 \cup \mathcal{E}_2$.

Assume lemma 5.7. Then, as $(\tilde{W}, \tilde{L}) \sim (W, L)$, in the initial construction we can replace the sequence $(W(a), L(a))_{a\in T}$ by a sequence $(\tilde{W}(a), W'(a), \tilde{L}(a))_{a\in T}$ of independent copies of $(\tilde{W}, W', \tilde{L})$. Thus we obtain $(\mu_{\tilde{W}}, \mu_{W'}, \mu_{\tilde{L}})$ with $(\mu_W, \mu_L) \sim (\mu_{\tilde{W}}, \mu_{\tilde{L}})$, we are under the conditions of Corollary 4.1. with $(\tilde{W}, W', \tilde{L})$ and we conclude as in *i*).
iii) the proof is similar to the one of *ii*).

Proof of Lemma 5.7. For $q \geq 0$, we denote by ν_q the probability distribution of (W, W_q, L) on $\mathbb{R}_+^{3c}$. For all $q \geq 0$, $\mathbb{E}(\sum_{j=0}^{c-1} W_j) = \mathbb{E}(\sum_{j=0}^{c-1} W_{q,j}) = \mathbb{E}(\sum_{j=0}^{c-1} L_j) = 1$. Thus, the $3c$ non negative components of (W, W_q, L) have a mean less than or equal to 1. So the family $(\nu_q)_{q\geq 0}$ is tight and the Prohorov's Theorem yields $(q_n)_{n\geq 0}$ converging towards ∞ such that as $n \to \infty$, (W, W_{q_n}, L) converges in distribution towards a random vector $(\tilde{W}, W', \tilde{L})$. Moreover, as by construction for all $q > 0$ $D_{W_q,L} < 0$, the hypothesis $D_{W_q,L} \not\to 0$ as $q \to \infty$ make possible to choose $(q_n)_{n\geq 0}$ in the set of the q's such that $D_{W_q,L} \leq -\varepsilon$ for some $\varepsilon > 0$.

There exists $\delta > 0$ such that for all $q \in [1, \infty[$, $\mathbb{E}(\sum_{j=0}^{c-1} W_{q,j}^{1+\delta}) \leq 1$:
As $\lim_{q\to\infty} -\tilde{\tau}'(q) = \alpha_{\inf}$ and as φ is log-convex, the set $\mathcal{D} \subset \{q \geq 0\}$ delimited by the graph of $\tilde{\tau}$ and the line $t = \tilde{\tau}(0) - \alpha_{\inf} q$ is contained in $\{\varphi \leq 1\}$. As $\alpha \to -\tilde{\tau}'(0^+)$, $\tilde{\tau}^*$ reaches the strict maximum $\tilde{\tau}(0)$. So $\tilde{\tau}(0) > \tilde{\tau}^*(-\tilde{\tau}'(1)) = -\tilde{\tau}'(1) > \alpha_{\inf}$. Let $\delta > 0$ be such that $\tilde{\tau}(0) - \alpha_{\inf}(1+\delta) > 0$. Since $q \mapsto \alpha_{\inf} q + \tilde{\tau}(q)$ is decreasing, for $q \geq 1$, the point $((1+\delta)q, (1+\delta)\tilde{\tau}(q))$ is under the line $t = \tilde{\tau}(0) - \alpha_{\inf} q$. Moreover $q \mapsto \frac{\tilde{\tau}(q)}{q}$ is strictly decreasing on $\mathbb{R}_+^*$. So $((1+\delta)q, (1+\delta)\tilde{\tau}(q))$ is above the graph of $\tilde{\tau}$. Thus for all $q \geq 1$,$((1+\delta)q, (1+\delta)\tilde{\tau}(q)) \in \mathcal{D}$ and so $\mathbb{E}(\sum_{j=0}^{c-1} W_{q,j}^{1+\delta}) \leq 1$.

Now, (H), the uniform inequality for all $q \in [1, \infty[$, $\mathbb{E}(\sum_{j=0}^{c-1} W_{q,j}^{1+\delta}) \leq 1$, the convergence in distribution of (W, W_{q_n}, L) towards $(\tilde{W}, W', \tilde{L})$, (5.1) and (5.2) imply that:

(5.3) $\mathbb{E}(\sum_{j=0}^{c-1} W'_j) = 1$, (5.4) $\mathbb{E}(\sum_{j=0}^{c-1} {W'_j}^{1+\delta'}) \leq 1$ for some $0 < \delta' \leq \delta$ and (5.5) $\mathbb{E}(\sum_{j=0}^{c-1} W'_j |\log V_j|) < \infty$ for $V \in \{\tilde{W}, \tilde{L}\}$, so $D_{W',\tilde{W}}$ and $D_{W',\tilde{L}}$ exist with $D_{W',\tilde{L}} \leq -\varepsilon$, and by construction $\frac{D_{W',\tilde{W}}}{D_{W',\tilde{L}}} = \alpha_{\inf}$ and $\frac{D_{W',W'}}{D_{W',\tilde{L}}} = \tilde{\tau}^*(\alpha_{\inf})$.
i) If $\tilde{\tau}^*(\alpha_{\inf}) > 0$ then by (5.3), (5.4) and (5.5) W' satisfies (H_0) and there exists $0 < \eta \leq \delta$ such that $\mathbb{E}(\sum_{j=0}^{c-1} {W'_j}^{1+\eta}) < 1$.
ii) If $\tilde{\tau}^*(\alpha_{\inf}) = 0$, the mapping $x \mapsto \mathbb{E}(\sum_{j=0}^{c-1} {W'_j}^{x})$ is convex, equal to 1 at 1 by (5.3), less or equal to 1 at some $x > 1$ and its derivate at 1 is equal to 0; so it is constant, from which comes *ii)*.

Proof of Theorem 5.1. iv)a) By using the same approach as in the proof of Th. VI.A.a. of [3], one obtains for P-almost every $\omega \in \{\mu_W \neq 0\}$, for all $q \in J'$, $\tau(q) \leq \tilde{\tau}(q)$. One can extend the inequality to $\mathrm{Cl}^+(J')$ because τ is always finite and non increasing on $[1, \infty[$.

ii)a), b) By Lemma 5.6, if $\alpha_0 \in \{\alpha_{\inf}, \alpha_{\sup}\}$, then for P-almost every $\omega \in \{\mu_W \neq 0\}$, $E_{\alpha_0} \neq \emptyset$ and $\dim E_{\alpha_0} \geq \tilde{\tau}^*(\alpha_0)$. By Proposition 5.1. this implies that $\dim E_{\alpha_0} \leq \tau^*(\alpha_0)$.
Moreover, by *iv)a)*, for P-almost every $\omega \in \{\mu_W \neq 0\}$, for all $\alpha \in I'$, $\alpha = -\tilde{\tau}'(q_\alpha)$, one has $\tau^*(\alpha) =$
$\inf_{q\in\mathbb{R}} \alpha q + \tau(q) \leq \inf_{q\in J'} \alpha q + \tau(q) \leq \inf_{q\in J'} \alpha q + \tilde{\tau}(q) = \alpha q_\alpha + \tilde{\tau}(q_\alpha) = \tilde{\tau}^*(\alpha)$.
The differentiability of $\tilde{\tau}$ where it is finite make possible to extend the inequality to $\mathrm{Cl}(I')$. So if $\alpha_0 \in \mathrm{Cl}(I')$ then for P-almost every $\omega \in \{\mu_W \neq 0\}$, $\dim E_{\alpha_0} = \tilde{\tau}^*(\alpha_0) = \tau^*(\alpha_0)$.

i)a) Lower bound comes from Lemma 5.5., relations (5.1) and (5.2), and the Billingsley Theorem. *i)b)* Proceeding as in *ii)*, one obtains that for P-almost every $\omega \in \{\mu_W \neq 0\}$, for all $\alpha \in \mathrm{Int}(I) \cap \mathrm{Cl}(I')$, $\dim E_\alpha \leq \tau^*(\alpha) \leq \tilde{\tau}^*(\alpha)$.

iii)a) If $\alpha_{\inf} \in \mathrm{Cl}(I')$, $\mathbb{R}_+ \not\subset J$ and $\tilde{\tau}^*(\alpha_{\inf}) = 0$ then by the proof of *ii)*, for P-almost every $\omega \in \{\mu_W \neq 0\}$, $\tau^*(\alpha_{\inf}) \leq \tilde{\tau}^*(\alpha_{\inf}) \leq 0$. Moreover τ^* is strictly concave at $\alpha_{\inf}$ (since by *i)* $\tau^* = \tilde{\tau}^*$ on $\alpha \in \mathrm{Int}(I) \cap \mathrm{Cl}(I')$) so for $0 < \alpha < \alpha_{\inf}$, $\tau^*(\alpha) < 0$ and $E_\alpha = \emptyset$ by Proposition 5.1.
If $\mathbb{R}_+ \subset J$ then $\alpha_{\inf} > 0$ and if $0 < \alpha < \alpha_{\inf}$, for all $q \geq 0$, $\alpha q + \tilde{\tau}(q) \leq (\alpha - \alpha_{\inf})q + \tilde{\tau}(0)$, so $\tilde{\tau}^*(\alpha) < 0$ and we conclude as above.
iii)b) Same discussion as in *iii)a)*.

iv)b) By *i)*, for P-almost every $\omega \in \{\mu_W \neq 0\}$, for all $\alpha \in \mathrm{Int}(I) \cap \mathrm{Cl}(I')$, $\tilde{\tau}^*(\alpha) = \tau^*(\alpha)$. Moreover by *iv)a)* for all $q \in \mathrm{Cl}^+(J')$, $\tau(q) \leq \tilde{\tau}(q)$. Proceeding as in the proof of [3] Theorem $VI.A.a.iii)$, we obtain that for P-almost every $\omega \in \{\mu_W \neq 0\}$, for all $q \in \mathrm{Int}(J) \cap \mathrm{Cl}^+(J')$, $\tau(q) = \tilde{\tau}(q)$. We cannot extend the inequality to $\mathrm{Cl}(J) \cap \mathrm{Cl}(J')$ because for example at $\inf(J)$ τ can be finite while $\tilde{\tau}$ is infinite.
As $\tau = \tilde{\tau}$ on $]0, 1]$, $\tau(1) = \tilde{\tau}(1) = 0$, $\tilde{\tau}'(1) = 0$ and τ is non increasing, for all $q \geq 1$, $\tau(q) = 0$.

Proof of Corollary 5.1 and Theorem 5.2. We leave these verifications to the reader.

5.2. Multifractal analysis in the case (II).

In this simpler situation we only give a statement.

Theorem 5.3. *There exists a largest interval J containing $]0,1]$, such that for P-almost every ω in $\{\mu_W \neq 0\}$,*
i) for all $q \in J$, $\tau(q) = \delta(1-q)$.
ii) $dim\, E_\delta = \delta$.
iii) For all $\alpha \geq 0$ such that $\inf_{q\in J} \alpha q + \tau(q) < 0$, $E_\alpha = \emptyset$. Moreover, if $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{R}^2$ then for all $\alpha \in \mathbb{R}_+ \setminus \{\delta\}$, $E_\alpha = \emptyset$.

6. GEOMETRICAL REALIZATIONS.

We are under the same hypotheses as in paragraph 5.

6.1. Statistically self-similar measures on a random interval.

We define on $\Omega_L = \{Y_L \neq 0\}$ a random family of intervals. Let $\omega \in \Omega_L$. For all $a \in T$ define $F_a = \{0 \leq j \leq c-1,\ \mu_L(I_{aj}) > 0\}$. Define $J_\epsilon = [0, Y_L[$. Then cut J_ϵ in $\#F_\epsilon$ semi-open to the right subintervals, the J_j's, $j \in F_\epsilon$, which follow one another like the j's in F_ϵ and satisfy $|J_j| = \mu_L(I_j)$. For all $j \in F_\epsilon$, one can repeat the operation in J_j and obtain the $J_{jj'}$'s, $j' \in F_j$, which follow one another like the j''s in F_j and satisfy $|J_{jj'}| = \mu_L(I_{jj'})$, and so on.
Now, assume that with probability one, μ_W has no atoms. One can project it on the J_a's in a measure μ_J as follows: for all $a \in T$ such that $\mu_L(I_a) > 0$, $\mu_J(J_a) = \mu_W(I_a)$. Then, if one substitutes μ_W for μ_J, the I_a's for the J_a's, and if one uses Lemma 4.3.2. of [33] instead of Billingsley's Theorem in the computation of the Hausdorff dimensions, Theorem 5.1., Corollary 5.1 and Theorem 5.3. are still valid, excepted Theorem 5.1.*ii*) in the case where for $\alpha = \alpha_{\inf}$ (resp. $\alpha_{\sup}$), $J \not\subset \mathbb{R}_+$ (resp. $\mathbb{R}_-$), $\tilde{\tau}^*(\alpha) = 0$, and $\mu_{W_{q_\alpha}}$, which is of type 2, is carried only by points of ∂T which encode the end points of the J_a's (because of Proposition 3.1, this exception does not arise for example in the situation considered in [3] where $L \in (\mathbb{R}_+^*)^c$).

6.2. Statistically self-similar measures on a random Cantor set in $\mathbb{R}^n$.

In our second geometrical application in [3], which generalizes the result of Falconer [14], one can now choose W in $(\mathcal{E}_1 \cup \mathcal{E}_2)$ instead of $\mathcal{E}_1$, and improve the result of [3] by subsituting Theorem 5.1., Corollary 5.1., and Theorem 5.3. for Theorem VI.A.a. and VI.B.
Moreover, an attentive reading of the Arbeiter and Patzschke paper [1] (where the strong separation condition of Falconer [14] is replaced by the strong open set condition) shows that it is possible to extend Theorem 5.1.*ii*)*iii*)*iv*) and results of [3] to their Theorem 4.10, with $W \in \mathcal{E}_1$ and weaker hypotheses on the distribution of W. To extend Theorem 5.1.*i*) to their study, it would be sufficient to know that, with probability one, for all $q \in \mathrm{Int}(J)$, $\mu_q(\partial K) = 0$ where, in [1], K is some compact fixed under the action of the random similarities which define the random self-similar set. We do not know prove that for instance.

ACKNOWLEDGMENTS.

The author thanks Jacques Peyrière for valuable discussions.

References

[1] Arbeiter, M., Patzschke N., (1996): Random self-similar multifractals, Math. Nachr., **181**, 5–42.

[2] Barral, J., (1997) : Continuité, moments d'ordres négatifs, et analyse multifractale des cascades multiplicatives de Mandelbrot, Thèse, Université Paris-Sud, N^o d'ordre 4704.

[3] Barral, J., (1999) : Moments, continuité et analyse multifractale des martingales de Mandelbrot, Probab. Theory Relat. Fields 113, 535-569.

[4] Ben Nasr, F., (1987) : Mesures aléatoires associées à des substitutions, C. R. Acad. Sci. Paris, **304**, 255–258.

[5] Biggins, J. D., (1992) : Uniform convergence of martingales in the branching random walk, Ann. Prob., **20**, 137-151.

[6] Billingsley, P., (1965) : Ergodic Theory and Information, Wiley, New York.

[7] Brown, G, Michon, G., Peyrière, J., (1992) : On the Multifractal Analysis of Measures, J. Stat. Phys., Vol *66*, Nos 3/4, 775–790.

[8] Cawley, R., Mauldin, R. D., (1992) : Multifractal Decomposition of Moran Fractals, Adv. Math. **92**, 196–236.

[9] Cole, J., (1998) : Relative Multifractal Analysis, preprint, University of St. Andrews.

[10] Collet, J., Koukiou, F., (1992) : Large deviations for multiplicative chaos, Commun. Math. Phys. **147**, 329-342.

[11] Durrett, R., Liggett, Th., (1983) : Fixed points of the smoothing transformation, Z. Wahrsch. verw. Gebiete **64**, 275–301.

[12] Edgar, G. A., Mauldin, R. D., (1992) : Multifractal decomposition of digraph recursive fractals, Proc. London Math. Soc., **65**, 604–628.

[13] Falconer, K. J., (1990) : Fractal Geometry - Mathematical foundations and application, J. Wiley.

[14] Falconer, K. J., (1994) : The multifractal spectrum of statistically self-similar measures, J. Theor. Prob. Vol **7**, 681–702.

[15] Guivarc'h, Y., (1990) : Sur une extension de la notion de loi semi-stable, Ann. Inst. H. Poincaré, Probab. et Statist. **26** 261–285.

[16] Holley, R. and Waymire, E. C., (1992) : Multifractal dimensions and scaling exponents for strongly bounded random cascades, Ann. Appl. Probab. **2**, 819–845.

[17] Kahane, J.-P., (1974) : Sur le modèle de turbulence de Benoît Mandelbrot, C. R. Acad. Sci. Paris **278**, 621–623.

[18] Kahane, J.-P., (1987) : Multiplications aléatoires et dimensions de Hausdorff, Ann. Inst. H. Poincaré, Sup. au N^o 2, Vol **23**, 289–296.

[19] Kahane, J.-P., (1991) : Produits de poids aléatoires et indépendants et applications, in Fractal Geometry and Analysis, J. Bélair and S. Dubuc (eds.), 277–324.

[20] Kahane, J.-P., Peyrière, J., (1976) : Sur certaines martingales de Benoît Mandelbrot, Adv. Math. **22**, 131–145.

[21] Liu, Q., (1997) : Sur une équation fonctionnelle et ses applications: une extension du théorème de Kesten-Stigum concernant des processus de branchement, Adv. Appl. Prob., **29**, Number 2, 353-373.

[22] Liu, Q., (1998) : Fixed points of a generalized smoothing transformation and applications to branching processes, Adv. Appl. Prob., **30**, Number 1, 85-112.

[23] Liu, Q., (1997) : Self-similar cascades and the branching random walk, preprint, Univ. Rennes 1.

[24] Liu, Q., (1997) : Multiplicative cascades and the branching random walk : asymptotic properties and absolute continuity, personnal communication.

[25] Liu, Q., (1998) : On a distributional equation arising in multiplicative cascades and branching random walks : asymptotic behavior and absolute continuity, preprint, Univ. Rennes1.

[26] Mandelbrot, B., (1974) : Intermittent turbulence in self- similar cascades : divergence of hight moments and dimension of the carrier, J. fluid. Mech. **62**, 331–358.

[27] Mandelbrot, B., (1974) : Multiplications aléatoires itérées et distributions invariantes par moyennes pondérées, C. R. Acad. Sci. Paris, **278**, 289–292 355–358.

[28] Molchan, G. M., (1996) : Scaling Exponents and Multifractal Dimensions for Independent Random Cascades, Commun. Math. Phys., **179**, 681–702.

[29] Olsen, L., (1994) : Random geometrically graph directed self-similar multifractals, Pitman Res. Notes Math. Ser., **307**.

[30] Olsen, L., (1995) : A multifractal formalism, Adv. Math., **116**, 92–195.

[31] Peyrière, J., (1974) : Turbulence et dimension de Hausdorff, C. R. Acad. Sci. Paris, **278**, 567–569.

[32] Peyrière, J., (1977) : Calculs de dimensions de Hausdorff, Duke Math. J., Vol. **44** N^o 3, 591–601.

[33] Peyrière, J., (1979) : A Singular Random Measure Generated by Spliting $[0, 1]$, Z. Wahrsch. verw. Gebiete **47**, 289–297.

[34] Von Bahr, B., Esseen, C.-G., (1960) : Inequalities for the rth absolute moment of sum of random variables, $1 \leq r \leq 2$, Ann. Math. Statist. **36**, 299–303.

Julien BARRAL
Université Paris-Sud
Equipe d'Analyse Harmonique (URA 757 du CNRS)
Mathématiques, bât. 425
91405 ORSAY CEDEX
FRANCE
E-mail address: Julien.Barral@math.u-psud.fr