Pseudo-rotations of the closed annulus: variation on a theorem of J. Kwapisz

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Abstract

Consider a homeomorphism h of the closed annulus $\mathbb{S}^1 \times [0,1]$, isotopic to the identity, such that the rotation set of h is reduced to a single irrational number α (we say that h is an *irrational pseudo-rotation*). For every positive integer n, we prove that there exists a simple arc γ joining one of the boundary component of the annulus to the other one, such that γ is disjoint from its n first iterates under h. As a corollary, we obtain that the rigid rotation of angle α can be approximated by homeomorphisms which are conjugate to h. The first result stated above is an analog of a theorem of J. Kwapisz dealing with diffeomorphisms of the two-torus; we give some new, purely two-dimensional, proofs, that work both for the annulus and for the torus case.

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1 Introduction

The concept of *rotation number* was introduced by H. Poincaré to study the dynamics of circle homeomorphisms (in the context of torus flows, see [16], chapitre XV). More precisely, for every orientation preserving homeomorphism h of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, Poincaré defined an element of $\mathbb{S}^1$, measuring the "asymptotic speed of rotation of the orbits of h around the circle": the so-called *rotation number of* h. The central question in this theory is: how much does the dynamics of an orientation preserving homeomorphism of the circle of rotation number α look like the dynamics of a rigid rotation R_α? The classical results obtained by Poincaré and A. Denjoy ([3]) provide a quite comprehensive list of answers to this question:

• If $\alpha = p/q$ (where p, q are relatively prime integers), then h has at least one periodic orbit, all the periodic orbits of h have prime period q, and the cyclic order of the points of any periodic orbit of h is the same as the cyclic order of the points of an orbit of the rotation R_α. If α is irrational, then h does not have any periodic orbit, and the cyclic order of the points of any orbit of h is the same as the cyclic order of the points of an orbit of the rotation R_α.

• If α is irrational, then h is semi-conjugate to the rotation R_α; moreover, h is in the closure of the conjugacy class of the rotation R_α, and R_α is in the closure of the conjugacy

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class of h (*i.e.* h can be conjugated to a homeomorphism arbitrarily close to the rotation R_α, and the rotation R_α can be conjugated to a homeomorphism arbitrarily close to h).
• If α is irrational and h is a C^2-diffeomorphism, then h is conjugate to the rotation R_α.

Poincaré's construction of the rotation number can be generalized for homeomorphisms of the closed annulus $\mathbb{A} := \mathbb{S}^1 \times [0, 1]$. More precisely, for every homeomorphism h of the closed annulus $\mathbb{A}$ which is isotopic to the identity, the *rotation set of* h is a closed interval[1] of $\mathbb{R}$, defined up to the addition of an integer, which measures the asymptotic speeds of rotation of the orbits of h around the annulus (see section 2.2). In the present article, we focus on homeomorphisms whose rotation set is a "small" interval. In particular, we call *irrational pseudo-rotation* every homeomorphism of the closed annulus $\mathbb{A}$, isotopic to the identity, whose rotation set is reduced to a single irrational number α (and we say that α is the *angle* of the pseudo-rotation). In this context, the natural question is: how much does the dynamics of an irrational pseudo-rotation of angle α look like the dynamics of the rigid rotation of angle α? The aim of the present article is to give some partial answer to this question.

We define an *essential simple arc* in the annulus $\mathbb{A}$ as a simple arc in $\mathbb{A}$ joining one of the boundary components of $\mathbb{A}$ to the other one. We shall prove the following theorem (which is a variation on a result of J. Kwapisz, dealing with torus diffeomorphisms, see [11]):

Theorem 1.1 ("arc translation theorem")**.** *Let $h : \mathbb{A} \to \mathbb{A}$ be an irrational pseudo-rotation of angle α. Then, for every positive integer n, there exists an essential simple arc γ_n in $\mathbb{A}$, such that the arcs $\gamma_n, \dots, h^n(\gamma_n)$ are pairwise disjoint. Moreover, the cyclic order of these arcs is the same as the cyclic order of the n first iterates of a vertical segment $\{\theta\} \times [0, 1]$ under the rigid rotation of angle α.*

Actually, theorem 1.1 will appear as a corollary of a more technical statement, concerning the homeomorphisms of the annulus whose rotation set is a "small" interval (more precisely, a *Farey interval*, see theorem 2.3).

Theorem 1.1 can be considered as a two-dimensional version of the above mentioned result concerning the cyclic order for circle homeomorphisms. Nevertheless, the situation is quite more complicated than in the circle. For example, the techniques developped by D. Anosov and A. Katok in [1] allow the construction of C^∞ irrationnal pseudo-rotation of the closed annulus $\mathbb{A}$ for which the Lebesgue measure is invariant and ergodic. Moreover, it is should be noticed that the statement of theorem 1.1 is optimal in the sense that it is impossible to make the arc γ_n independent of n. More precisely, mixing the Anosov-Katok techniques with an idea of M. Handel, M. Herman has constructed a C^∞ irrational pseudo-rotation h with a minimal invariant set which is a "pseudo-circle" ; it is not difficult to see that no essential simple arc is disjoint of all its iterates under h (see [9] and [8]).

As we have already said, theorem 1.1 is a variation of an analog result of Kwapisz, dealing with diffeomorphisms of the torus $\mathbb{T}^2$. The true aim of our article is not to adapt Kwapisz's proof to the case of annulus homeomorphisms, but rather to provide some completely different and (in our opinion) more natural proofs of this result. Indeed, in his proof, Kwapisz introduces the suspension of the diffeomorphism under consideration, and uses some 3-dimensional topology techniques to find the wanted curve as the intersection

[1]Contrary to what happens in the case of the circle, two different orbits of a homeomorphism of the annulus might have different "asymptotic speeds of rotation".

of two cross-sections of this suspension. The two proofs of theorem 1.1 that we give in the present article are purely two-dimensional, and only involve some classical manipulations on arcs. By the way, our proofs also work in the torus case (see appendix B).

As a corollary of theorem 1.1, we obtain the following result[2]:

Corollary 1.2. *Let $h : \mathbb{A} \to \mathbb{A}$ be an irrational pseudo-rotation of angle α. Then the rigid rotation R_α of angle α is in the closure of the conjugacy class of h.*

In other words, any irrational pseudo-rotation can be conjugated to obtain a homeomorphism which is arbitrarily close to a rigid rotation. Note that the converse is true: if the rigid rotation R_α is in the closure of the conjugacy class of some homeomorphism h (for some irrationnal number α), then h is a pseudo-rotation of angle α. This follows from an easy semi-continuity property of the rotation set, see remark 2.1. Nevertheless, we point out that we were not able to solve the "symmetric" problem:

Question. *Is every area-preserving irrational pseudo-rotation of angle α in the closure of the conjugacy class of the rotation R_α ?*

Corollary 1.2 was motivated by the situation on the two-torus. Indeed, an analog of corollary 1.2 holds on the two-torus; it is actually a consequence of another theorem by Kwapisz, called the *tiling theorem* (see [12]). The tiling theorem asserts roughly that if the rotation set of a two-torus diffeomorphism h is reduced to a single irrational point, then for any n, one can find a finite tiling, which is almost invariant under h (there are only three tiles whose images do not fit in with the tiling), and such that the restriction of h to the 1-skeleton is conjugate to the restriction of the corresponding rigid rotation to the 1-skeleton of a similar tiling. In the case of the annulus, theorem 1.1 also provides a kind of almost invariant tiling of the annulus. Nevertheless, corollary 1.2 is a little more difficult to derive in the annulus case because, unlike what happens in the torus setting, the diameter of the tiles of the corresponding tiling for the rigid rotation does not go to zero when the number of tiles increase.

Finally, it is interesting to associate theorem 1.1 with some generalizations of the Poincaré-Birkhoff theorem obtained by J. Franks ([5], [6]) or C. Bonatti and L. Guillou ([7]). Let us recall the result of Bonatti and Guillou. It deals with homeomorphisms h of the closed annulus $\mathbb{A}$ that are isotopic to the identity, and claims that *if h is fixed point free, then there exists an essential simple arc in $\mathbb{A}$ that is disjoint from its image under h, or there exists a non-homotopically trivial simple closed curve in $\mathbb{A}$ which is disjoint from its image under h.* Putting the quoted result together with theorem 1.1, we obtain the following corollary:

Corollary 1.3. *Let h be a homeomorphism of the annulus $\mathbb{A}$, which is isotopic to the identity, and which does not have any periodic point. Then at least one of the following property holds:*

(i) *there exists a non-homotopically trivial simple closed curve in $\mathbb{A}$, which is disjoint from its image under h,*

(ii) *the homeomorphism h is an irrational pseudo-rotation, and, for every positive integer n, there exists an essential simple arc γ_n in $\mathbb{A}$, such that the arcs $\gamma_n, h(\gamma_n), \dots, h^n(\gamma_n)$ are pairwise disjoint.*

[2]It might be interesting to notice that corollary 1.2 is actually equivalent to theorem 1.1.

In particular, any area-preserving orientation-preserving periodic point free homeomorphism of the annulus is an irrational pseudo-rotation (this can also be deduced from the results of J. Franks, see [6], theorem 2.2).

In a forthcoming paper, we shall prove some analogs of theorem 1.1, corollaries 1.2 and 1.3 for homeomorphisms of the open annulus $\mathbb{S}^1 \times]0,1[$. Some completely different (and more sophisticated) proofs are needed. All the difficulty arise from the lack of compacity of the open annulus, which forces to change the definition of the rotation set (in particular, one has to restrict to measure-preserving homeomorphisms).

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2 Definitions and precise statement

2.1 Notations

In this paper, we denote by $\mathbb{A}$ the closed annulus $\mathbb{S}^1 \times [0,1]$, and by $\widetilde{\mathbb{A}} := \mathbb{R} \times [0,1]$ the universal covering of $\mathbb{A}$. We denote by π the canonical projection of $\widetilde{\mathbb{A}}$ onto $\mathbb{A}$, and by $p_1 : \widetilde{\mathbb{A}} = \mathbb{R} \times [0,1] \to \mathbb{R}$ the first coordinate projection. Given a subset E of $\mathbb{A}$ (resp. $\widetilde{\mathbb{A}}$), we denote by $\mathrm{Int}(E)$ the interior of E in $\mathbb{A}$ (resp. $\widetilde{\mathbb{A}}$).

We denote by $T : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ the translation defined by $T : (\theta, t) \to (\theta + 1, t)$. Of course, the translation T is a generator of the automorphism group of the projection $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$. We observe that if h is a homeomorphism of the annulus $\mathbb{A}$ which is isotopic to the identity, and if $\widetilde{h}$ is a lift of h to the band $\widetilde{\mathbb{A}}$, then $\widetilde{h}$ commutes with T. Conversely, every homeomorphism of $\widetilde{\mathbb{A}}$ which is isotopic to the identity and which commutes with T is the lift of a homeomorphism of the annulus $\mathbb{A}$ isotopic to the identity. For every $\alpha \in \mathbb{R}$, we denote by $R_\alpha : (\theta, t) \mapsto (\theta + \alpha, t)$ the rigid rotation of angle α in the annulus $\mathbb{A}$.

2.2 The rotation set of a homeomorphism of the annulus $\mathbb{A}$

Let h be a homeomorphism of the bounded annulus $\mathbb{A}$ which is isotopic to the identity, and $\widetilde{h} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h. We define the n^{th} *displacement set* of $\widetilde{h}$ to be the set

$$D_n(\widetilde{h}) = \left\{ \frac{p_1(\widetilde{h}^n(\widetilde{x})) - p_1(\widetilde{x})}{n} \mid \widetilde{x} \in \widetilde{\mathbb{A}} \right\}.$$

Then a real number v is called a *rotation vector* if it is the limit of a sequence $(v_{n_k})_{k \geq 0}$ such that each v_{n_k} belongs to the n_k^{th} displacement set of $\widetilde{h}$, where the sequence (n_k) tends to $+\infty$. The *rotation set of* $\widetilde{h}$ is the set $\mathrm{Rot}(\widetilde{h})$ of all rotation vectors. It is easy to see that the sets $D_n(\widetilde{h})$ and $\mathrm{Rot}(\widetilde{h})$ are compact intervals. This definition of the rotation set of $\widetilde{h}$ is the analog of a definition given by Misiurewicz and Zieman in [15] in the case of torus homeomorphisms.

Let us recall briefly an alternative definition that follows an idea of S. Schwartzman [18]. If μ is an invariant probability measure for $\widetilde{h}$, the *rotation vector of* μ is

$$\int_D \left(p_1(\widetilde{h}(\widetilde{x})) - p_1(\widetilde{x}) \right) d\mu(\widetilde{x})$$

where D is any fundamental domain of the covering $\widetilde{\mathbb{A}}$ (for example $D = [0,1[\times[0,1])$. If μ is ergodic, then the ergodic theorem implies that the rotation number v of μ is realized, in the following strong sense: there exists a point $\widetilde{x}$ such that

$$\lim_{n\to+\infty} \frac{p_1(\widetilde{h}^n(\widetilde{x})) - p_1(\widetilde{x})}{n} = v.$$

One can deduce from this that the rotation set of $\widetilde{h}$ coincides with the set of rotation vectors of all invariant measures, and that the endpoints of the interval $\mathrm{Rot}(\widetilde{h})$ are realized in the sense defined above.

Remark 2.1. *From this alternative definition it follows that the rotation set is upper semi-continuous. Indeed, let (h_k) be a sequence of homeomorphisms converging to h, and let (μ_k) be a sequence of invariant probability measures. Taking a subsequence, one can assume that (μ_k) converges to a measure μ. Then the rotation vector of μ for h is the limit of the sequence of the rotation vectors of μ_k for h_k.*

For any integers p, q, the map $\widetilde{h}^q \circ T^{-p}$ is a lift of h^q. Using the fact that $\widetilde{h}$ and T commute, we have the following easy property:

Lemma 2.2. *For any couple (p, q) of integers, the rotation set of $\widetilde{h}^q \circ T^{-p}$ is given by*

$$\mathrm{Rot}(\widetilde{h}^q \circ T^{-p}) = q \times \mathrm{Rot}(\widetilde{h}) - p.$$

In particular, the rotation sets of two different lifts differ by an integer, so that the rotation set of h is well defined as an interval of $\mathbb{R}$, modulo $\mathbb{Z}$. The rotation set is invariant with respect to conjugacy by homeomorphisms of $\mathbb{A}$ which are isotopic to the identity.

2.3 Cyclic order on the circle and the annulus

The natural orientation of $\mathbb{R}$ induces an orientation on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Given three distinct points p_1, p_2, p_3 on $\mathbb{S}^1$, we will say that the triplet (p_1, p_2, p_3) is *positive* if the point p_2 is crossed when going from p_1 to p_3 in the positive way.

A simple arc $\gamma : [0,1] \to \mathbb{A}$ is called an *essential simple arc* if γ joins one of the boundary components of $\mathbb{A}$ to the other, and if $\gamma(]0,1[)$ is included in the interior of $\mathbb{A}$. We define similarly the notion of *essential simple arc* in the band $\widetilde{\mathbb{A}}$. Similarly to the cyclic order for distinct points on the circle, we define a cyclic order on triplets of essential simple arcs in $\mathbb{A}$ that are pairwise disjoint. Note that this can be done simply by considering the endpoints of the three arcs on one of the boundary components. We will also use the (related) total order on sets of pairwise disjoint essential simple arcs in $\widetilde{\mathbb{A}}$.

2.4 Farey intervals

In this article, all the rational numbers $\frac{p}{q}$ (with $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$) will be written as irreducible fractions. A *Farey interval* is an interval $]\frac{p}{q}, \frac{p'}{q'}[$ of $\mathbb{R}$ with rational endpoints, such that $p'q - pq' = 1$ (which amounts to saying that the length of the interval is $\frac{1}{qq'}$). Some elementary properties of Farey intervals are proved in appendix A and used in sections 3 and 4.

2.5 Precise statement of the "arc translation theorem"

Here is the precise statement that we shall prove:

Theorem 2.3 ("arc translation theorem"). *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity, and let $\widetilde{h} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h. Assume that the rotation set of $\widetilde{h}$ is included in a Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$. Then there exists an essential simple arc γ in the annulus $\mathbb{A}$ such that the arcs $\gamma, h(\gamma), \dots, h^{q+q'-1}(\gamma)$ are pairwise disjoint. Moreover the cyclic order of these arcs in $\mathbb{A}$ is the same as the cyclic order of the first iterates of a vertical segment under any rigid rotation of angle $\alpha \in]\frac{p}{q}, \frac{p'}{q'}[$.*

The statement on cyclic order means that for any $k_1, k_2, k_3 \in \{0, \dots, q+q'-1\}$, one has the equivalence

$$(h^{k_1}(\gamma), h^{k_2}(\gamma), h^{k_3}(\gamma)) \text{ is positive } \Leftrightarrow (k_1\alpha, k_2\alpha, k_3\alpha) \text{ is positive}$$

where the numbers $k_i\alpha$ are considered as elements of the circle $\mathbb{R}/\mathbb{Z}$.

The result announced in the introduction (theorem 1.1) follows directly from this new one, by noting that given any irrational number α, one can find a Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$ containing α with $q + q'$ arbitrarily big.

Remark 2.4. *Theorem 2.3 is optimal: for any Farey interval $I =]\frac{p}{q}, \frac{p'}{q'}[$ there exists a homeomorphism h with the rotation set of $\widetilde{h}$ is included in I, for which there is no essential simple arc γ such that γ is disjoint from $h^{q+q'}(\gamma)$. For example, the rotation of angle $\frac{p+p'}{q+q'}$ is suitable.*

2.6 A basic property

Lemma 2.5. *Let h be a homeomorphism of the bounded annulus $\mathbb{A}$ which is isotopic to the identity, and $\widetilde{h} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h. Suppose that the rotation set of $\widetilde{h}$ is included in $]0, +\infty[$, and choose a real number ρ such that $0 < \rho < \inf(\mathrm{Rot}(\widetilde{h}))$. Then there exists a real number s such that for every $\widetilde{x}$ in $\widetilde{\mathbb{A}}$, for every positive integer n,*

$$p_1(\widetilde{h}^n(\widetilde{x})) \geq p_1(\widetilde{x}) + \rho n - s. \tag{1}$$

The proof is easy, and left to the reader. We shall use the following consequence of this lemma: *under the hypotheses of lemma 2.5, for every compact subset K of $\widetilde{A}$ and every $s_0 > 0$, there exists $n_0 > 0$ such that for all $n \geq n_0$, $\widetilde{h}^n(K) \subset [s_0, +\infty[\times [0, 1]$.*

Remark 2.6. *Another consequence is that, under the hypotheses of lemma 2.5, the quotient space $\widetilde{\mathbb{A}}/\widetilde{h}$ is separated. Using the classification of surfaces, one can see that $\widetilde{\mathbb{A}}/\widetilde{h}$ is necessarily homeomorphic to a closed annulus, which implies that $\widetilde{h}$ is conjugate to a translation. We shall not need this fact.*

3 First proof of the main theorem

Two proofs of the arc translation theorem 2.3 will be given, the first one in this section and the second one in the following section. These sections can be read in any order.

The first proof uses two kinds of ingredients: some elementary arithmetical properties of Farey intervals, and some (classical) operations on essential simple arcs in the band $\widetilde{\mathbb{A}}$.

3.1 Some more notations

For every essential simple arc Γ in the band $\widetilde{\mathbb{A}}$, we denote by $R(\Gamma)$ the closure of the connected component of $\widetilde{\mathbb{A}} \setminus \Gamma$ which is "on the right" of the arc Γ.

Given an essential simple arc Γ in $\widetilde{\mathbb{A}}$ and a homeomorphism $\Psi : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$, we say that the set $R(\Gamma)$ is an *attractor* (resp. a *strict attractor*) for Ψ if the image of $R(\Gamma)$ under Ψ is included in $R(\Gamma)$ (resp. in the interior of $R(\Gamma)$). Observe that if Ψ is isotopic to the identity, we have $R(\Psi(\Gamma)) = \Psi(R(\Gamma))$, so that the set $R(\Gamma)$ is a (strict) attractor for Ψ if and only if the image of the arc Γ under Ψ is included in (the interior of) $R(\Gamma)$.

Given two essential simple arcs Γ_1 and Γ_2, we write $\Gamma_1 < \Gamma_2$ if $\Gamma_2 \subset \mathrm{Int}(R(\Gamma_1))$, and we write $\Gamma_1 \leq \Gamma_2$ if $\Gamma_2 \subset R(\Gamma_1)$.

3.2 Attractors for families of commuting homeomorphisms

Theorem 2.3 will follow from elementary arithmetical properties of Farey intervals, and from the following proposition:

Proposition 3.1. *Let $\Psi_0, \dots, \Psi_p$ be some homeomorphisms of the band $\widetilde{\mathbb{A}}$ isotopic to the identity, pairwise commuting, and commuting with the translation T. Assume that the rotation set of each of these homeomorphisms is included in $]0, +\infty[$. Then there exists an essential simple arc Γ in $\widetilde{\mathbb{A}}$, such that the set $R(\Gamma)$ is a strict attractor for each of the homeomorphisms $\Psi_1, \dots, \Psi_p$.*

We begin with a technical point which consists in describing an operation on essential simple arcs. This operation will be used intensively to construct the simple arc demanded by proposition 3.1. The proof of the following lemma is postponed to section 3.4.

Lemma and notation 3.2. *Let Γ_1 and Γ_2 be two essential simple arcs in $\widetilde{\mathbb{A}}$, and let U be the unique non-bounded connected component of the set $(\widetilde{\mathbb{A}} \setminus R(\Gamma_1)) \cap (\widetilde{\mathbb{A}} \setminus R(\Gamma_2))$. The boundary (in $\widetilde{\mathbb{A}}$) of U is an essential simple arc, that we denote by $\Gamma_1 \vee \Gamma_2$ (see figure 1).*

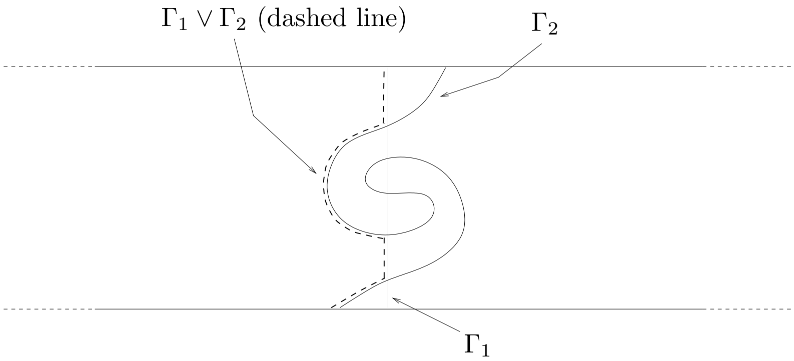

Figure 1: Two essential simple arcs Γ_1 and Γ_2, and the arc $\Gamma_1 \vee \Gamma_2$

Remark 3.3. *Let Γ_1, Γ_2, Γ_3 be three essential simple arcs in $\widetilde{\mathbb{A}}$. The following properties are immediate consequences of the definition of the arc $\Gamma_1 \vee \Gamma_2$:*

(i) *The arc $\Gamma_1 \vee \Gamma_2$ is included in the union of the arcs Γ_1 and Γ_2. Hence, if $\Gamma_3 < \Gamma_1$ and $\Gamma_3 < \Gamma_2$, then $\Gamma_3 < \Gamma_1 \vee \Gamma_2$.*

(ii) *The sets $R(\Gamma_1)$ and $R(\Gamma_2)$ are included in the set $R(\Gamma_1 \vee \Gamma_2)$. In other words, we have $\Gamma_1 \vee \Gamma_2 \leq \Gamma_1$ and $\Gamma_1 \vee \Gamma_2 \leq \Gamma_2$.*

Remark 3.4. *The operation which maps two essential simple arcs Γ_1, Γ_2 to the essential simple arc $\Gamma_1 \vee \Gamma_2$ is associative (and commutative). Therefore, given any finite number of essential simple arcs $\Gamma_1, \dots, \Gamma_n$, the essential simple arc $\Gamma_1 \vee \dots \vee \Gamma_n$ is well-defined.*

Now we are in a position to prove proposition 3.1.

Proof of proposition 3.1. We proceed by induction. For every $k \in \{1, \dots, p\}$, we shall construct an essential simple arc Γ_k such that the set $R(\Gamma_k)$ is a strict attractor for the homeomorphisms $\Psi_1, \dots, \Psi_k$.

Construction of the arc Γ_1. Let Γ_0 be an essential simple arc in $\widetilde{\mathbb{A}}$. According to lemma 2.5, there exists an integer N such that $\Gamma_0 < \Psi_1^N(\Gamma_0)$. We consider some essential simple arcs $\Lambda_0, \dots, \Lambda_{N-1}$ such that

$$\Psi_1^{-N}(\Gamma_0) < \Lambda_{N-1} < \Lambda_{N-2} < \dots < \Lambda_1 < \Lambda_0 = \Gamma_0,$$

and define

$$\Gamma_1 := \Lambda_0 \vee \Psi_1(\Lambda_1) \vee \dots \vee \Psi_1^{N-1}(\Lambda_{N-1}) = \bigvee_{i=0}^{N-1} \Psi_1^i(\Lambda_i).$$

For every $i \in \{0, \dots, N-2\}$, one has $\Psi_1^{i+1}(\Lambda_{i+1}) < \Psi_1^{i+1}(\Lambda_i)$ (by definition of the Λ_k's) and $\Gamma_1 \leq \Psi_1^{i+1}(\Lambda_{i+1})$ (by item (ii) remark 3.3). Hence, for every $i \in \{0, \dots, N-2\}$, one has

$$\Gamma_1 < \Psi_1^{i+1}(\Lambda_i).$$

Moreover, one has $\Lambda_0 < \Psi_1^N(\Lambda_{N-1})$ and $\Gamma_1 \leq \Lambda_0$. Hence,

$$\Gamma_1 < \Psi_1^N(\Lambda_{N-1}).$$

Finally, using item (i) of remark 3.3, we get

$$\Gamma_1 < \bigvee_{i=0}^{N-1} \Psi_1^{i+1}(\Lambda_i) = \Psi_1(\Gamma_1).$$

In other words, $R(\Gamma_1)$ is a strict attractor for Ψ_1.

Induction step. Let $k \in \{1, \dots, p-1\}$. We assume that we have constructed an essential simple arc Γ_k such that the set $R(\Gamma_k)$ is a strict attractor for the homeomorphisms $\Psi_1, \dots, \Psi_k$, *i.e.* such that $\Gamma_k < \Psi_j(\Gamma_k)$ for every $j \leq k$. The construction of the arc Γ_{k+1} is somewhat similar to the construction of the arc Γ_1. By lemma 2.5, there exists an integer N such that $\Gamma_k < \Psi_{k+1}^N(\Gamma_k)$. We consider some essential simple arcs $\Lambda_0, \dots, \Lambda_{N-1}$ such that

$$\Psi_{k+1}^{-N}(\Gamma_k) < \Lambda_{N-1} < \Lambda_{N-2} < \dots < \Lambda_1 < \Lambda_0 = \Gamma_k,$$

and define

$$\Gamma_{k+1} := \Lambda_0 \vee \Psi_{k+1}(\Lambda_1) \vee \dots \vee \Psi_{k+1}^{N-1}(\Lambda_{N-1}) = \bigvee_{i=0}^{N-1} \Psi_{k+1}^i(\Lambda_i).$$

The same argument as in the construction of the arc Γ_1 shows that $R(\Gamma_{k+1})$ is a strict attractor for Ψ_{k+1}.

Moreover, we may assume that the Λ_i's are choosen such that, for every $j \in \{1, \dots, k\}$,

$$\Psi_j^{-1}(\Gamma_k) < \Lambda_{N-1}, \dots, \Lambda_0 \leq \Gamma_k.$$

Let us fix $j \in \{1, \dots, k\}$; we will check that the set $R(\Gamma_{k+1})$ is still a strict attractor for Ψ_j (this will essentially follow from the fact that Ψ_j commutes with Ψ_k). For each $i \in \{0, \dots, N-1\}$, one has

$$\Psi_j^{-1}(\Lambda_i) \leq \Psi_j^{-1}(\Gamma_k) < \Lambda_i,$$

and thus, since Ψ_j commutes with Ψ_{k+1},

$$\Psi_j^{-1}(\Psi_{k+1}^i(\Lambda_i)) < \Psi_{k+1}^i(\Lambda_i).$$

Moreover, by item (ii) of remark 3.3, we have

$$\Psi_j^{-1}(\Gamma_{k+1}) \leq \Psi_j^{-1}(\Psi_{k+1}^i(\Lambda_i)).$$

Hence, for every $i \in \{0, \dots, N-1\}$, we get

$$\Psi_j^{-1}(\Gamma_{k+1}) < \Psi_{k+1}^i(\Lambda_i).$$

Finally, using item (i) of remark 3.3), we get

$$\Psi_j^{-1}(\Gamma_{k+1}) < \bigvee_{i=0}^{N-1} \Psi_j^{-1}(\Psi_{k+1}^i(\Lambda_i)) = \Gamma_{k+1}.$$

In other words, $R(\Gamma_{k+1})$ is a strict attractor for Ψ_j. □

3.3 Proof of the theorem

We now turn to the proof of theorem 2.3. It consists in applying proposition 3.1 to a well-chosen family of homeomorphisms $\Psi_1, \dots, \Psi_{q+q'}$. Each of these homeomorphisms will be obtained as the composition of a power of $\widetilde{h}$ and a power of T.

Proof of theorem 2.3. Let ρ be any number in the Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$. Elementary properties of Farey intervals (see section A.1 of appendix A) imply that:

- for every $k \in \{1, \dots, q+q'-1\}$, the number $k.\rho$ is not an integer, so we may define the number $\alpha_k \in]0,1[$ and the integer n_k such that $k.\rho = n_k + \alpha_k$;
- the numbers $\alpha_1, \dots, \alpha_{q+q'-1}$ are distinct, so we may consider the permutation σ of the set $\{1, \dots, q+q'-1\}$, such that

$$0 < \sigma(1).\rho - n_{\sigma(1)} < \sigma(2).\rho - n_{\sigma(2)} < \cdots < \sigma(q+q'-1).\rho - n_{\sigma(q+q'-1)} < 1;$$

- the integers $n_1, \dots, n_{q+q'-1}$ and the permutation σ actually do not depend on the choice of the number ρ in $]\frac{p}{q}, \frac{p'}{q'}[$.

Then, for every $k \in \{1, \dots, q+q'-1\}$, we consider the homeomorphism $\Phi_k := \widetilde{h}^{\sigma(k)} \circ T^{-n_{\sigma(k)}}$. Moreover, we set $\Phi_0 := \mathrm{Id}$ and $\Phi_{q+q'} := T$. Finally, for every $k \in \{1, \dots, q+q'\}$, we consider the homeomorphism $\Psi_k := \Phi_k \circ \Phi_{k-1}^{-1}$ (see figure **??**). It is clear that the homeomorphisms $\Psi_1, \dots, \Psi_{q+q'}$ commute with the translation T, and that these homeomorphisms are pairwise commuting.

Let $k \in \{2, \dots, q+q'-1\}$. We have

$$\Psi_k = \widetilde{h}^{\sigma(k)} \circ T^{-n_{\sigma(k)}} \circ \widetilde{h}^{-\sigma(k-1)} \circ T^{n_{\sigma(k-1)}} = \widetilde{h}^{\sigma(k)-\sigma(k-1)} \circ T^{-n_{\sigma(k)}+n_{\sigma(k-1)}}.$$

Hence, according to lemma 2.2, the rotation set of the homeomorphism Ψ_k is:

$$\mathrm{Rot}(\Psi_k) = \left\{ \sigma(k).\rho - n_{\sigma(k)} - \left(\sigma(k-1).\rho - n_{\sigma(k-1)}\right) \quad | \quad \rho \in \mathrm{Rot}(\widetilde{h}) \right\}.$$

Since by assumption the set $\mathrm{Rot}(\widetilde{h})$ is included in $]\frac{p}{q}, \frac{p'}{q'}[$, the definition of σ implies that $\mathrm{Rot}(\Psi_k)$ is included in $]0, +\infty[$. Similarly, we have:

$$\mathrm{Rot}(\Psi_1) = \left\{ \sigma(1).\rho - n_{\sigma(1)} \quad | \quad \rho \in \mathrm{Rot}(\widetilde{h}) \right\},$$

$$\mathrm{Rot}(\Psi_{q+q'}) = \left\{ 1 - \left(\sigma(q+q'-1).\rho - n_{\sigma(q+q'-1)}\right) \quad | \quad \rho \in \mathrm{Rot}(\widetilde{h}) \right\},$$

and these sets are also included in $]0, +\infty[$. Consequently, the homeomorphisms $\Psi_1, \dots, \Psi_{q+q'}$ satisfy the hypotheses of proposition 3.1. So this proposition provides us with an essential simple arc Γ in the band $\widetilde{\mathbb{A}}$ such that $R(\Gamma)$ is a strict attractor for each of the homeomorphisms $\Psi_1, \dots, \Psi_{q+q'}$. Let γ be the projection in the annulus $\mathbb{A}$ of the arc Γ. It is an essential arc in the annulus $\mathbb{A}$; let us prove that it is simple. To see this, we observe that the translation T is equal to the telescopic product $\Psi_{q+q'} \circ \dots \circ \Psi_1$. As a consequence, $R(\Gamma)$ is a strict attractor for T. In particular, the arc Γ is disjoint from its image under T, and γ is a simple arc.

We are left to prove that the arcs $\gamma, h(\gamma), \dots, h^{q+q'-1}(\gamma)$ are pairwise disjoint. By construction of the arc Γ, we know that the set $R(\Gamma)$ is a strict attractor for the homeomorphism $\Phi_1 = \Psi_1$; in other words, the arc $\Phi_1(\Gamma)$ is strictly on the right of the arc Γ. For every $k \in \{2, \dots, q+q'-1\}$, we know that the set $R(\Gamma)$ is a strict attractor for the homeomorphism Ψ_k; since Ψ_k and Φ_{k-1} commute, this implies that the set $\Phi_{k-1}(R(\Gamma))$ is a strict attractor for the homeomorphism Ψ_k; in other words, the arc $\Psi_k \circ \Phi_{k-1}(\Gamma)) = \Phi_k(\Gamma)$ is strictly on the right of the arc $\Phi_{k-1}(\Gamma)$. Similarly, the arc $\Psi_{q+q'} \circ \Phi_{q+q'-1}(\Gamma)) = T(\Gamma)$ is strictly on the right of the arc $\Phi_{q+q'-1}(\Gamma)$ (see figure **??**). So we have proved that the arcs $\Phi_1(\Gamma), \dots, \Phi_{q+q'-1}(\Gamma)$ are pairwise disjoint, and that all these arcs are strictly on the right of Γ and strictly on the left of $T(\Gamma)$, *i.e.* are included in the set $D := \mathrm{Int}(R(\Gamma)) \setminus R(T(\Gamma))$. Since the set D is a fundamental domain for the covering map $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$, this implies that the projections of the arcs $\Gamma, \Phi_1(\Gamma), \dots, \Phi_{q+q'-1}(\Gamma)$ are pairwise disjoint in the annulus $\mathbb{A}$. Now we observe that, for every k, the homeomorphism Φ_k is, by definition, a lift of the homeomorphism $h^{\sigma(k)}$; in particular, the projection of the arc $\Phi_k(\Gamma)$ is the arc $h^{\sigma(k)}(\gamma)$. Hence, we have proved that the arcs $\gamma, h^{\sigma(1)}(\gamma), \dots, h^{\sigma(q+q'-1)}(\gamma)$ are pairwise disjoint. Since σ is a permutation, this is equivalent to the fact that the arcs $\gamma, h(\gamma), \dots, h^{q+q'-1}(\gamma)$ are pairwise disjoint.

Moreover, from the discussion above, we see that the cyclic order of the arcs $\gamma, h(\gamma), \dots, h^{q+q'-1}(\gamma)$ in the annulus $\mathbb{A}$ is given by the permutation σ. More precisely, given two distincts integers $k_1, k_2 \in \{1, \dots, q+q'-1\}$, the triple $(\gamma, h^{k_1}(\gamma), h^{k_2}(\gamma))$ is positive if and only if $\sigma(k_1) < \sigma(k_2)$. Using the definition of σ, this immediately implies that the cyclic order of the arcs $\gamma, h(\gamma), \dots, h^{q+q'-1}(\gamma)$ is the same as the cyclic order of the first iterates of a vertical segment under any rigid rotation of angle $\rho \in]p/q, p'/q'[$. $\square$

3.4 Proof of lemma 3.2

The proof of lemma 3.2 relies heavily on the following classical result of Keréjártó (see [10], or [14, page 246] for a modern proof): *let U_1 and U_2 be two Jordan domains in the two-sphere, that is, connected open sets whose boundary is homeomorphic to the circle ; then each connected component of $U_1 \cap U_2$ is also a Jordan domain.*

Proof of lemma 3.2. Under the hypotheses of the lemma, let Γ denote the boundary of U in $\widetilde{\mathbb{A}}$. We have to prove that Γ is an essential simple arc. For that purpose, we see the band $\widetilde{\mathbb{A}} = \mathbb{R} \times [0,1]$ as a subset $\mathbb{R}^2$, and we see the two-sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ as the one-point compactification of $\mathbb{R}^2$. Then Keréjártó's result stated above implies that U is a Jordan domain in $\mathbb{S}^2$. In particular, Γ is included in a Jordan curve of $\mathbb{S}^2$ (passing through the point ∞). From this it follows easily that Γ is an essential simple arc. $\square$

4 Alternative proof

This section is devoted to a second proof of the "arc translation theorem" 2.3. It can be considered as a variation on the first proof given in the previous section. We use two independent arguments. The first one is a purely arithmetic argument, and tells that it is enough to find an essential simple arc in $\widetilde{\mathbb{A}}$ which is disjoint from its images under the two "first-return maps" $\Phi_1 = T^{-p} \circ \widetilde{h}^q$ and $\Phi_2 = T^{p'} \circ \widetilde{h}^{-q'}$. The second argument goes the following way. Suppose we are given a family of k pairwise commuting maps of $\widetilde{\mathbb{A}}$, and consider sequences obtained by starting with any point in the closed band $\widetilde{\mathbb{A}}$ and iterating each time by one of the maps of the family (that is, we are considering a positive orbit of the $\mathbb{Z}^k$-action generated by the family). We prove that if the rotation sets of the k maps are all positive, then all the sequences obtained this way have a universally bounded leftward displacement (actually, the proof is given only for $k = 2$, since we have the arithmetic argument in mind). Moreover, by continuity, this remains true if we consider pseudo-orbits instead of orbits, *i. e.* if we allow a little "jump" (or "error") takes place at each step. Then we construct the essential simple arc Γ using a *brick decomposition*. This is a sort of triangulation which produces attractors in an automatic way, as far as the behaviour of pseudo-orbits is controlled. Brick decompositions were introduced by Flucher ([4]). They have been used by P. Le Calvez and A. Sauzet ([13]) in order to prove the existence of attractors for Brouwer homeomorphisms. In this text, we only need the easy version, without the maximality property introduced by A. Sauzet ([17]).

4.1 Structure of the proof

In the appendix A, using arithmetic properties of Farey intervals, we shall prove the following result:

Proposition 4.1. *Let $\widetilde{h}$ be the lift to $\widetilde{\mathbb{A}}$ of some annulus homeomorphism h which is isotopic to the identity and let $\Gamma \subset \widetilde{\mathbb{A}}$ be some essential simple arc in $\widetilde{\mathbb{A}}$. Assume that there exists a Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$ such that (using notations of subsection 3.1)*

$$T^{-p'} \circ \widetilde{h}^{q'}(\Gamma) < \Gamma < T^{-p} \circ \widetilde{h}^{q}(\Gamma),$$

Then, the following properties hold:

1. *the arcs $T^{-\ell} \circ \widetilde{h}^k(\Gamma)$, with $k \in \{0, \dots, q + q' - 1\}$ and $\ell \in \mathbb{Z}$, are pairwise disjoint;*

2. these arcs are ordered in $\widetilde{\mathbb{A}}$ as the lifts of the $q+q'-1$ first iterates of a vertical segment in $\mathbb{A}$ under the rotation R_α for any $\alpha \in]\frac{p}{q}, \frac{p'}{q'}[$. More precisely: given two pairs of integers (k,ℓ) and (k',ℓ') in $\{0,\ldots,q+q'-1\}\times\mathbb{Z}$, we have

$$T^{-\ell}\circ \widetilde{h}^k(\Gamma) < T^{-\ell'}\circ \widetilde{h}^{k'}(\Gamma) \Longleftrightarrow k\alpha-\ell < k'\alpha-\ell'.$$

Remark 4.2. *In particular, the arc Γ is disjoint from the arc $T^\ell(\Gamma)$ for any $\ell\in\mathbb{Z}$. Thus $\gamma=\pi(\Gamma)$ is an essential simple arc in $\mathbb{A}$, disjoint from its $q+q'-1$ first iterates under h. Moreover, the cyclic order of the arcs $\gamma, h(\gamma),\ldots,h^{q+q'-1}(\gamma)$ is the same as the cyclic order of the iterates of a vertical segment under the rotation of angle α, for any α in the Farey interval $I=\left]\frac{p}{q},\frac{p'}{q'}\right[$.*

The above proposition will be combined with the following one.

Proposition 4.3. *Let Φ_1, Φ_2 be two homeomorphisms of $\widetilde{\mathbb{A}}$, isotopic to the identity, which commute and commute with the translation T, and whose rotation sets are included in $]0,+\infty[$. Then there exists an essential simple arc Γ which is disjoint from its images $\Phi_1(\Gamma)$ and $\Phi_2(\Gamma)$.*

Now we explain how theorem 2.3 follows from these propositions. Then the remaining of the section will be devoted to the proof of proposition 4.3.

Alternative proof of theorem 2.3 assuming propositions 4.1 and 4.3. Let $\widetilde{h}$ be as in theorem 2.3. We consider the two "return maps" $\Phi_1 := T^{-p}\circ\widetilde{h}^q$ and $\Phi_2 := T^{p'}\circ\widetilde{h}^{-q'}$. According to lemma 2.2, the rotation sets of both maps are included in $]0,+\infty[$. So we can apply proposition 4.3, and we get a curve Γ which does not meet its images $\Phi_1(\Gamma)$ and $\Phi_2(\Gamma)$. Then, since the rotation sets are positive, the order of the curves must be such that $\Phi_2^{-1}(\Gamma)<\Gamma<\Phi_1(\Gamma)$ (this also follows from the proof of proposition 4.3). Now we can apply proposition 4.1. Letting γ be the projection of Γ to the annulus $\mathbb{A}$, it follows that the arcs γ, $h(\gamma)$, …, $h^{q+q'-1}(\gamma)$ are pairwise disjoint. By remark 4.2 the cyclic order of these arcs is the same as the cyclic order of the iterates of a vertical segment under the rigid rotation. This concludes the proof of the theorem. □

The remainder of the section is devoted to the proof of proposition 4.3.

4.2 Pseudo-orbits for commuting homeomorphisms with positive rotation sets

During the whole section, we consider two homeomorphisms Φ_1, Φ_2 of $\widetilde{\mathbb{A}}$ which commute and commute with the translation T, and we make the assumption that both rotation sets of Φ_1 and Φ_2 are included in $]0,+\infty[$.

A sequence $(x_n)_{n\geq 0}$ of points in $\widetilde{\mathbb{A}}$ is called a (Φ_1,Φ_2)-*orbit* if for all n, we have $x_{n+1}=\Phi_1(x_n)$ or $\Phi_2(x_n)$. Let d denote the Euclidean distance on $\widetilde{\mathbb{A}}=\mathbb{R}\times[0,1]$ and ε a positive real number. An ε-(Φ_1,Φ_2)-*pseudo-orbit* is a sequence $(x_n)_{n\geq 0}$ of points in $\widetilde{\mathbb{A}}$ such that for all n, we have $d(\Phi_1(x_n),x_{n+1})<\varepsilon$ or $d(\Phi_2(x_n),x_{n+1})<\varepsilon$. The main result that makes this definition useful is that we can choose $\varepsilon>0$ such that the leftward displacement of any ε-(Φ_1,Φ_2)-pseudo-orbit is universally bounded:

Proposition 4.4. *There exist $\varepsilon>0$ and $M>0$ such that for any ε-(Φ_1,Φ_2)-pseudo-orbit $(x_n)_{n\geq 0}$, for any $n\geq 0$,*

$$p_1(x_n)\geq p_1(x_0)-M.$$

To prove this proposition, we use lemma 4.6 below which bounds the leftward displacement of the ε-(Φ_1, Φ_2)-pseudo-orbits over long periods. First, we prove a version of this lemma for (Φ_1, Φ_2)-orbits:

Lemma 4.5. *There exists an integer $N > 0$ with the following property. For each couple of non-negative integers (N_1, N_2) such that $N_1 + N_2 \geq N$, for every point x in $\widetilde{\mathbb{A}}$,*

$$p_1(\Phi_1^{N_1}\Phi_2^{N_2}(x)) \geq p_1(x) + 2.$$

Proof of lemma 4.5. Applying lemma 2.5 twice, we find numbers ρ, s satisfying the inequality (1) of this lemma for both Φ_1 and Φ_2 (and for every point x and every positive integer n). Take a couple of non-negative integers (N_1, N_2). Then writing $p_1(\Phi_1^{N_1}\Phi_2^{N_2}(x)) - p_1(x)$ as the sum

$$\left[p_1(\Phi_1^{N_1}(\Phi_2^{N_2}(x))) - p_1(\Phi_2^{N_2}(x))\right] + \left[p_1(\Phi_2^{N_2}(x)) - p_1(x)\right],$$

we see that this quantity is greater than $\rho(N_1 + N_2) - 2s$. We conclude that any integer N such that $\rho N - 2s \geq 2$ will satisfy the conclusion of the lemma. $\square$

Let us tackle the case of ε-(Φ_1, Φ_2)-pseudo-orbits:

Lemma 4.6. *There exist an integer $N > 0$ and a constant $\varepsilon > 0$ with the following property. For every ε-(Φ_1, Φ_2)-pseudo-orbit $(x_0, \dots, x_N)$ of length N,*

$$p_1(x_N) \geq p_1(x_0) + 1.$$

Proof of lemma 4.6. Let N be the integer given by lemma 4.5. We shall say that an ε-(Φ_1, Φ_2)-pseudo-orbit $(x_0, \dots, x_N)$ of length N is *of type σ*, where $\sigma \in \{1, 2\}^N$, if for every $n \in \{0, \dots, N-1\}$ we have $d(x_{n+1}, \Phi_{\sigma_{n+1}}(x_n)) < \varepsilon$. Since the set $\{1, 2\}^N$ is finite, it is sufficient to prove the lemma for each type σ. In the remainder of the proof, the type σ of the pseudo-orbits is fixed.

To prove the lemma we identify the tangent spaces $T_x\widetilde{\mathbb{A}}$ with the plane $\mathbb{R}^2$. Given a point x_0 in $\widetilde{\mathbb{A}}$ and a finite sequence of vectors of the plane $\mathbf{v} = (\vec{v}_1, \dots, \vec{v}_N)$, we define recursively

$$x_1 := \Phi_{\sigma_1}(x_0) + \vec{v}_1 \ , \ \dots \ , \ x_N := \Phi_{\sigma_N}(x_{N-1}) + \vec{v}_N.$$

The vectors $\vec{v}_i$ will be chosen in the compact unitary ball $\mathbb{D}$ of $\mathbb{R}^2$. Then we can consider the map

$$\begin{array}{rcl} \mathcal{F} : \widetilde{\mathbb{A}} \times (\mathbb{D})^N & \longrightarrow & \mathbb{R}^2 \\ (x_0, \mathbf{v}) & \longmapsto & x_N. \end{array}$$

Clearly, the map $\mathcal{F}$ is continuous. Since $\mathcal{F}$ commutes with the deck transformation T (meaning that $\mathcal{F}(T(x_0), \mathbf{v}) = T(\mathcal{F}(x_0, \mathbf{v}))$, and since the quotient annulus $\mathbb{A}$ is compact, $\mathcal{F}$ is uniformly continuous. Therefore there exists $\varepsilon \in]0, 1[$ such that for every x_0 in $\widetilde{\mathbb{A}}$, for every sequence $\mathbf{v} = (\vec{v}_1, \dots, \vec{v}_N)$ of vectors whose Euclidean norms are less than ε, we have $d(\mathcal{F}(x_0, \mathbf{v}), \mathcal{F}(x_0, (\vec{0}))) < 1$. We observe that :

- for every ε-(Φ_1, Φ_2)-pseudo-orbit $(x_0, \dots, x_N)$ of type σ, x_N can be expressed as $\mathcal{F}(x_0, \mathbf{v})$ for some sequence $\mathbf{v} = (\vec{v}_0, \dots, \vec{v}_N)$ with $\|\vec{v}_i\| < \varepsilon$;
- we have the equality $\mathcal{F}(x_0, (\vec{0})) = \Phi_{\sigma_N} \circ \dots \circ \Phi_{\sigma_1}(x_0)$.

As a consequence, for every ε-(Φ_1, Φ_2)-pseudo-orbit $(x_0, \ldots, x_N)$ of length N and type σ, we have the following inequalities:

$$p_1(x_N) - p_1(x_0) \geq p_1(\Phi_{\sigma_N} \circ \cdots \circ \Phi_{\sigma_1}(x_0)) - p_1(x_0) - 1 \geq 2 - 1 = 1.$$

(the latest inequality is a consequence of lemma 4.5 applied to the (Φ_1, Φ_2)-orbit $(x_0\,, \Phi_{\sigma_1}(x_0), \ldots, \Phi_{\sigma_N} \circ \cdots \circ \Phi_{\sigma_1}(x_0))$; observe that we use here the fact that Φ_1 and Φ_2 commute). This gives the lemma. □

We end with the proof of proposition 4.4.

Proof of proposition 4.4. Let N and $\varepsilon > 0$ be the integer and the constant given by lemma 4.6. Using repeatedly lemma 2.5, we obtain that, for any ε-(Φ_1, Φ_2)-pseudo-orbit $(x_0, \ldots, x_\ell)$ of length $\ell < N$ we have,

$$\begin{array}{rcl} p_1(x_\ell) & \geq & p_1(x_{l-1}) - s - \varepsilon \\ & \geq & p_1(x_{l-2}) - 2s - 2\varepsilon \\ & \cdots & \\ & \geq & p_1(x_0) - N(\varepsilon + s), \end{array}$$

where $s > 0$ is the bound given by lemma 2.5.

Let us now consider any positive integer n and any ε-(Φ_1, Φ_2)-pseudo-orbit $(x_0, \ldots, x_n)$ of length n. We decompose n as $kN + \ell$ with $k \geq 0$ and $\ell \in \{0 \ldots, N-1\}$. On the one hand, lemma 4.6 implies

$$p_1(x_{kN}) \geq p_1(x_0) + k.$$

On the other hand, from the inequality above, we get

$$p_1(x_\ell) \geq p_1(x_{kN}) - N(\varepsilon + s).$$

Putting all these inequalities together, one deduces

$$\begin{array}{rcl} p_1(x_n) - p_1(x_0) & = & [p_1(x_n) - p_1(x_{kN})] + [p_1(x_{kN}) - p_1(x_0)] \\ & \geq & k - N(\varepsilon + s) \geq -N(\varepsilon + s) \end{array}$$

Hence, the proposition is proved for the constant $M = N(\varepsilon + s)$. □

4.3 Brick decomposition

We now turn to the proof of proposition 4.3 itself. We consider a *brick decomposition* of $\mathbb{A}$, as shown on figure 2. Essentially, this amounts to taking an embedded triadic graph F in $\widetilde{\mathbb{A}}$ (*triadic* meaning that each vertex belongs to exactly three edges). We demand that F contains the boundary of $\widetilde{\mathbb{A}}$. A *brick* is defined to be the closure of a complementary domain of F in $\mathbb{A}$; it is a topological closed disk. The last requirement in the definition of F is the following key feature: every brick is of diameter less than the number ε given by proposition 4.4 (for the Euclidean metric on $\mathbb{A} = \mathbb{S}^1 \times [0, 1]$).

Remark 4.7. *Since F is triadic, the topological boundary of the union of any family of bricks is a 1-submanifold in $\widetilde{\mathbb{A}}$, with boundary included in the boundary of $\widetilde{\mathbb{A}}$.*

Remark 4.8. *Any subset of $\widetilde{\mathbb{A}}$ is included in the* interior *of the union of the bricks that it meets (the bricks are topological closed disks).*

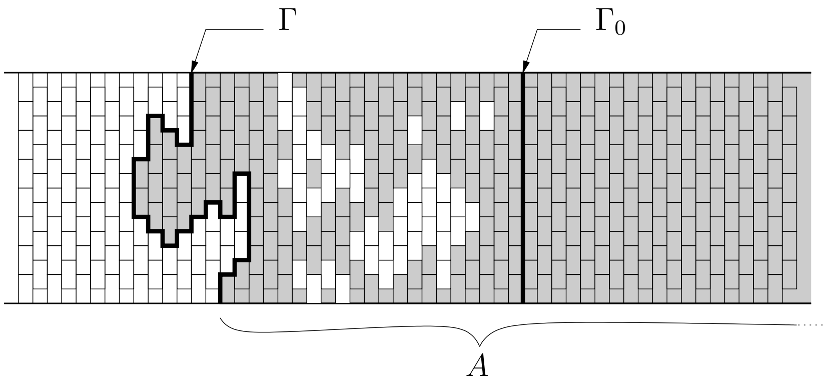

Figure 2: A brick decomposition

A *brick chain (from the brick D_0 to the brick D_i)* is a sequence $(D_0, \dots D_i)$ of bricks in $\widetilde{\mathbb{A}}$ such that $\Phi_1(D_0) \cup \Phi_2(D_0)$ meets $D_1, \dots, \Phi_1(D_{i-1}) \cup \Phi_2(D_{i-1})$ meets D_i.

Take $\Gamma_0 = \{0\} \times [0, 1]$; we can suppose that Γ_0 is included in F (as on figure 2). We define a subset A of $\widetilde{\mathbb{A}}$ in the following way:
– to any brick D_0, we associate the union $\mathcal{D}(D_0)$ of all the bricks D of the decomposition such that there exists a brick chain from D_0 to D;
– the set A is the union of all the sets $\mathcal{D}(D_0)$, where D_0 ranges over the set of all the bricks lying on the right of the arc Γ_0 (the brick D_0 may meet Γ_0).

Fact 4.9. *The set A contains all the bricks on the right of Γ_0 and is bounded on the left: there exists a constant M such that A is included in $[-M, +\infty[\times[0, 1]$.*

Proof. Indeed, if $(D_0, \dots D_i)$ is a brick chain, and x is any point in D_i, then there exists an ε-(Φ_1,Φ_2)- pseudo-orbit (x_n) such that x_0 is in D_0 and $x_i = x$. Remember that ε is given by proposition 4.4; let M be the other constant given by this proposition. Then we have $p_1(x_i) \geq p_1(x_0) - M$, so that if D_0 is on the right of Γ_0, then $p_1(x_0) \geq 0$, and consequently D_i is included in $[-M, +\infty[\times[0, 1]$. We conclude that A is included in $[-M, +\infty[\times[0, 1]$. The fact that A contains all the bricks on the right of Γ_0 follows from the definition of A (considering chains made of only one brick). □

Fact 4.10. *The set A is a strict attractor for Φ_1 and Φ_2, i. e.*

$$\Phi_1(A) \subset \mathrm{Int}(A) \textit{ and } \Phi_2(A) \subset \mathrm{Int}(A).$$

Proof. Indeed, let D be included in A. By definition, there exists a brick chain $(D_0, \dots D)$ with D_0 on the right of Γ_0. Then for any brick D' meeting $\Phi_1(D)$, the sequence $(D_0, \dots D, D')$ is again a brick chain, so D' is also included in A. Then the fact follows from remark 4.8: the set $\Phi_1(D)$ is included in the *interior* of the union of the bricks that it meets[3]. Of course, the same argument can be applied to the homeomorphism Φ_2. □

Consider the essential arc Γ "bounding A on the left" (see figure 2); more precisely, using fact 4.9 and remark 4.7, this can be defined as the boundary of the connected component of $\widetilde{\mathbb{A}} \setminus A$ containing $]-\infty, -M[\times[0, 1]$. From fact 4.10 it follows that Γ is disjoint from its images $\Phi_1(\Gamma)$ and $\Phi_2(\Gamma)$. This ends the proof of proposition 4.3.

[3]Note that, to get a *strict* attractor, it is crucial that the bricks are defined to be *closed*.

5 Closure of the conjugacy class of a pseudo-rotation: proof of corollary 1.2

Using Poincaré's classical results (see the introduction), one can easily prove that, for any orientation-preserving circle homeomorphism h of rotation number α, the rigid rotation of angle α is in the closure of the conjugacy class of h. In this section, we extend this result to irrational pseudo-rotations of the annulus, *i.e.* we prove corollary 1.2.

For every $\alpha \in \mathbb{R}$, we denote by T_α the rigid translation in the band $\widetilde{\mathbb{A}}$ given by $(\theta, t) \mapsto (\theta + \alpha, t)$. Of course, T_α is a lift of the rotation R_α.

Corollary 1.2 is an immediate consequence of the following proposition:

Proposition 5.1. *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism that is isotopic to the identity and $\widetilde{h} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h. Suppose that the rotation set $\mathrm{Rot}(\widetilde{h})$ is contained in some Farey interval $]\frac{p}{q}, \frac{p'}{q'}[\subset \mathbb{R}$. Then for any (rational or irrational) number $\alpha \in]\frac{p}{q}, \frac{p'}{q'}[$, there exists a homeomorphism σ of $\mathbb{A}$, isotopic to the identity, such that for any lift $\widetilde{\sigma}$ of σ to $\widetilde{\mathbb{A}}$, we have*

$$\mathrm{d}(\widetilde{\sigma} \circ \widetilde{h} \circ \widetilde{\sigma}^{-1}, T_\alpha) < \frac{40}{\min(q, q')}.$$

Proof of corollary 1.2 assuming proposition 5.1. Let us consider an irrational pseudo-rotation h of angle α. Since α is irrational, it belongs to some Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$ with q and q' arbitrarily large. Hence, by proposition 5.1, h is conjugate to some homeomorphism $\sigma \circ h \circ \sigma^{-1}$ arbitrarily close to the rigid rotation R_α. $\square$

The remainder of section 5 is devoted to the proof of proposition 5.1. Here is the idea of the proof. We begin by applying our main theorem 2.3, thus finding an essential simple arc γ in $\mathbb{A}$ which is disjoint from its first $q + q' - 1$ iterates under h. Let γ_0 be the vertical segment $\{0\} \times [0, 1]$ in $\mathbb{A}$ (then γ_0 is disjoint from all its iterates by the rotation R_α). Since the cyclic order of the first iterates of γ under h is the same as the cyclic order of the first iterates of γ_0 under R_α, one can perform a first conjugacy, by a homeomorphism σ_a sending γ on γ_0, so that $h_a := \sigma_a \circ h \circ \sigma_a^{-1}$ coincides with R_α on the iterates $\gamma_0, R_\alpha(\gamma_0), \ldots, R_\alpha^{q+q'-2}(\gamma_0)$ (first step of the proof).

In the second step, we use the dynamical tiling generated by these arcs. More precisely, let us call D and D' the two tiles adjacent to the arc γ_0 in this tiling (see figure 3). Since the $q' - 1$ first iterates of D under h_a have mutually disjoint interiors, and since they coincide with the iterates under R_α, we can conjugate h_a by a homeomorphism supported by the union of these discs so that the conjugated homeomorphism coincides with R_α on all these discs but the last one. We do the same on the iterates of D'. Now we notice that the $q' - 1$ first iterates of D, together with the $q - 1$ first iterates of D', cover the whole annulus. Thus the second step of the proof provides us with a homeomorphism σ_b so that $h_b := \sigma_b \circ h_a \circ \sigma_b^{-1}$ coincides with R_α on the whole annulus except on the set $R_\alpha^{q'-1}(D) \cup R_\alpha^{q-1}(D')$, which happens to be the topological disc $O' := R_\alpha^{-1}(D \cup D')$ (see figure 3). Note that the interior of this disc O' is disjoint from its first $s-1$ iterates, where $s := \min(q, q')$.

For the last step, we consider the difference homeomorphism $g := R_\alpha^{-1} \circ h_b$ on the topological disc O'. A key lemma, dealing with disc homeomorphisms, allows us to write g as the composition of N homeomorphisms $g_N, \cdots, g_1$ of the disc O' which are ε-close to the identity, *the integer N depending on ε but not on g*. We choose ε so that $N < s$, hence the disc O' is disjoint from its first $N-1$ iterates, and we can make a last conjugacy

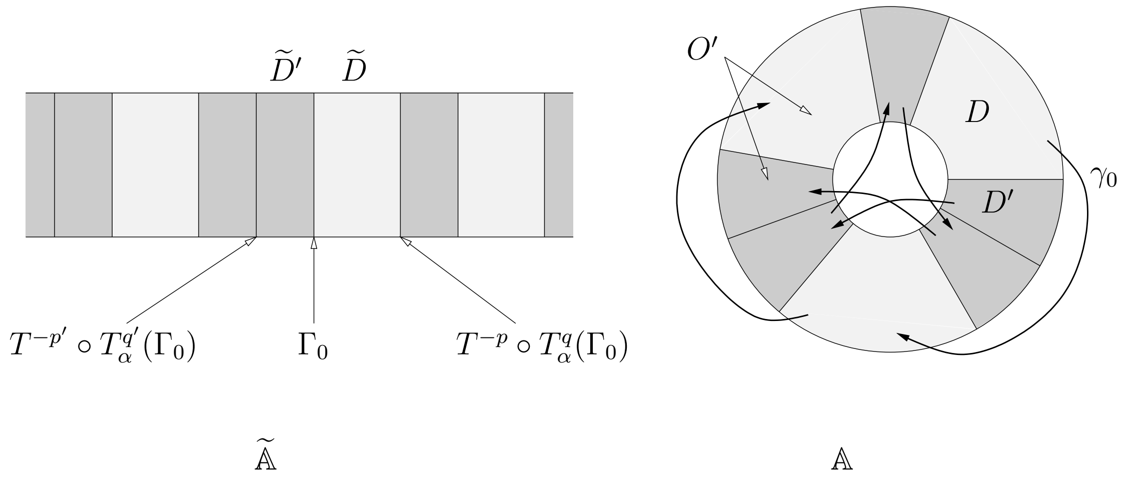

Figure 3: The dynamical tilings in $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ (here $]\frac{p}{q}, \frac{p'}{q'}[=]3/5, 2/3[$)

σ_c that distributes the difference g on these iterates. Thus we get a homeomorphism $h_c := \sigma_c \circ h_b \circ \sigma_c^{-1}$ such that on O' and its first $N-1$ iterates, h_c coincides with the rotation R_α up to one of the homeomorphisms g_k (and consequently is ε-close to R_α), and such that h_c still exactly coincides with R_α everywhere else. Hence h_c satisfies the conclusion of proposition 5.1.

5.1 Preliminaries: decomposition of disc homeomorphisms

We denote by $\mathbb{D}$ the unitary closed disc for the Euclidean metric of $\mathbb{R}^2$. We denote by $\mathrm{Homeo}^+(\mathbb{D})$ the set of homeomorphisms of the disc $\mathbb{D}$ isotopic to the identity, and by $\mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$ the set of those that coincide with the identity on the boundary of $\mathbb{D}$. We consider the usual distance $d(h, h') = \sup\{d(h(x), h'(x)), x \in \mathbb{D}\}$ on these sets. We will say that two homeomorphisms h, h' are *ε-close* if $d(h, h') < \varepsilon$. The aim of this subsection is to prove the following lemma and corollary:

Lemma 5.2. *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that every homeomorphism $h \in \mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$ can be written as a product $h = h_N \circ \cdots \circ h_1$ of N homeomorphisms in $\mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$ which are ε-close to the identity. Moreover, we can choose N less than $\frac{4}{\varepsilon} + 4$.*

Let α_1, α_2 be two disjoint closed arcs included in the boundary $\partial\mathbb{D}$. We denote by $\mathrm{Homeo}^+(\mathbb{D}, \alpha_1 \cup \alpha_2)$ the set of homeomorphisms of the disc $\mathbb{D}$, isotopic to the identity, that coincide with the identity on $\alpha_1 \cup \alpha_2$ (note that the arc α_1 or/and the arc α_2 may possibly be reduced two a single point).

Corollary 5.3. *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that every homeomorphism $h \in \mathrm{Homeo}^+(\mathbb{D}, \alpha_1 \cup \alpha_2)$ can be written as a product $h = h_N \circ \cdots \circ h_1$ of N homeomorphisms in $\mathrm{Homeo}^+(\mathbb{D}, \alpha_1 \cup \alpha_2)$ which are ε-close to the identity. Moreover, we can choose N less than $\frac{4+2\pi}{\varepsilon} + 5$.*

Remark 5.4. *Note that, under the conclusion of the lemma, for every $k \leq N$, the homeomorphism $h_k \circ h_{k-1} \circ \cdots \circ h_1$ is ε-close to the homeomorphism $h_{k-1} \circ \cdots \circ h_1$. On the contrary, it is not true in general that the homeomorphism $h_N \circ \cdots \circ h_k$ is ε-close to the homeomorphism $h_N \circ \cdots \circ h_{k+1}$.*

Proof of lemma 5.2. We fix a number ε with $1/2 > \varepsilon > 0$, and we consider a homeomorphism $h \in \mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$.

Step 1. We use Alexander's trick to prove that h can be written as a product $h = h' \circ h_0$ of two homeomorphisms $h_0, h' \in$ $\mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$, where h_0 is ε-close to the identity, and where h' coincides with the identity on a neighbourhood of $\partial\mathbb{D}$.
First, we extend h on the whole plane $\mathbb{R}^2$ by the identity on $\mathbb{R}^2 \setminus \mathbb{D}$. Then for $t \in]0,1]$, we consider the homeomorphism $A_t \in \mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$ defined by $A_t(x) = t.h(x/t)$ for every $x \in \mathbb{D}$. Now we set $h' := A_{t_0}$ and $h_0 := A_{t_0}^{-1} \circ h$. It is easy to check that if t_0 is close enough to 1 the desired properties are satisfied.

Step 2. It remains to prove that the homeomorphism h' can be written as a product of n elements of $\mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$ *which are ε-close to the identity, with $n \leq \frac{4}{\varepsilon} + 3$.*
Let δ be a number such that $1/2 > \delta > 0$, and such that the homeomorphism h' coincides with the identity outside the Euclidean ball $B(0, 1-\delta)$. Let us consider now the radial homeomorphism $g_{\delta,\varepsilon}$ in $\mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$ given by

$$g_{\delta,\varepsilon}(x) = \varphi(\|x\|).\frac{x}{\|x\|} \qquad \text{and} \qquad g_{\delta,\varepsilon}(0) = 0$$

where the homeomorphism φ of $[0,1]$ is defined on figure 4. The homeomorphism $g_{\delta,\varepsilon}$

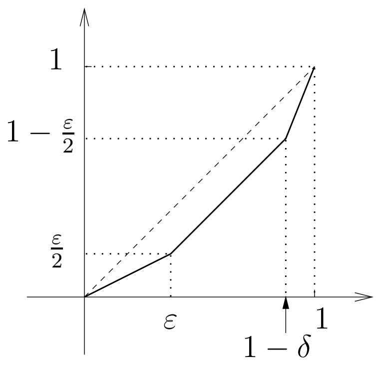

Figure 4: The function φ

is ε-close to the identity. Now, let m be the integer in $[\frac{2}{\varepsilon}, \frac{2}{\varepsilon} + 1[$. We observe that $\varphi^m(1-\delta) < \frac{\varepsilon}{2}$. Hence the homeomorphism $g_{\delta,\varepsilon}^m$ maps the ball $B(0, 1-\delta)$ inside the ball $B(0, \frac{\varepsilon}{2})$, and thus the homeomorphism $h_b = g_{\delta,\varepsilon}^m \circ h' \circ g_{\delta,\varepsilon}^{-m}$ coincides with the identity outside the ball $B(0, \frac{\varepsilon}{2})$ (of center 0 and radius $\frac{\varepsilon}{2}$). In particular, the homeomorphism h_b is ε-close to the identity. So, writing

$$h' = g_{\delta,\varepsilon}^{-m} \circ h_b \circ g_{\delta,\varepsilon}^m \ ,$$

we obtain that h' is the product of $n = 2m + 1$ elements of $\mathrm{Homeo}(\mathbb{D}, \partial\mathbb{D})$, all of which are ε-close to the identity. Note that $n \leq \frac{4}{\varepsilon} + 3$ as announced. $\square$

The proof of the corollary will make use of the following fact.

Fact 5.5. *Every increasing homeomorphism of the interval* $[0,1]$ *can be written as a product of* N *homeomorphisms that are* ε*-close to the identity, with* $N \leq \frac{1}{\varepsilon}+1$.

Proof of the fact. Let h be an increasing homeomorphism of $[0,1]$ such that $d(h, Id) \geq \varepsilon$. We consider the homeomorphism

$$h_\varepsilon : x \mapsto \begin{cases} x+\varepsilon & \text{if } h(x) > x+\varepsilon \\ h(x) & \text{if } x-\varepsilon \leq h(x) \leq x+\varepsilon \\ x-\varepsilon & \text{if } h(x) < x-\varepsilon. \end{cases}$$

Then $d(h_\varepsilon, Id) = \varepsilon$, and $d(h \circ {h_\varepsilon}^{-1}, Id) = d(h, Id) - \varepsilon$. The fact follows. □

Proof of corollary 5.3. We fix a number $\varepsilon > 0$ and we consider a homeomorphism $h \in \mathrm{Homeo}^+(\mathbb{D}, \alpha_1 \cup \alpha_2)$.

Let $\mathrm{Homeo}^+(\partial\mathbb{D}, \alpha_1 \cup \alpha_2)$ be the space of orientation preserving homeomorphisms of the circle $\partial\mathbb{D}$ that are the identity on $\alpha_1 \cup \alpha_2$. We see $\partial\mathbb{D}$ as the boundary of the unit disk in the euclidian plane; instead of the usual metric on $\partial\mathbb{D}$, we use the *extrinsic* metric of the plane (the distance between x and y is not the length of the arc but the length of the segment $[xy]$). This metric induces a metric on $\mathrm{Homeo}^+(\partial\mathbb{D}, \alpha_1 \cup \alpha_2)$. For any arc α in $\partial\mathbb{D}$, there exists a 2π-Lipschitz homeomorphism from $[0,1]$ to α. So fact 5.5 implies that every element of $\mathrm{Homeo}^+(\partial\mathbb{D}, \alpha_1 \cup \alpha_2)$ can be decomposed as a product of N elements that are ε-close to the identity, allowing $N \leq \frac{2\pi}{\varepsilon}+1$. According to this we write

$$h|_{\partial\mathbb{D}} = H_N \circ \cdots \circ H_1.$$

Now consider the "circular extension mapping" Φ: using complex numbers notation,

$$\Phi : \begin{cases} \mathrm{Homeo}(\partial\mathbb{D}) & \longrightarrow & \mathrm{Homeo}(\mathbb{D}) \\ H & \longmapsto & \left(h : re^{i\theta} \mapsto rH(e^{i\theta})\right). \end{cases}$$

Note that the map Φ is an isometry (since we used the extrinsic metric on $\partial\mathbb{D}$). Let $h' := \Phi(h|_{\partial\mathbb{D}})$. Observe that $h' = \Phi(H_N \circ \cdots \circ H_1) = \Phi(H_N) \circ \cdots \circ \Phi(H_1)$, so that

$$h = \Phi(H_N) \circ \cdots \circ \Phi(H_1) \circ ({h'}^{-1} \circ h).$$

In this formula each homeomorphism $\Phi(H_k)$ is ε-close to the identity, and the homeomorphism ${h'}^{-1} \circ h$ is the identity on the boundary $\partial\mathbb{D}$. Now we complete the proof by applying the previous lemma 5.2 to the homeomorphism ${h'}^{-1} \circ h$. □

5.2 Proof of proposition 5.1

Let us fix α in $]\frac{p}{q}, \frac{p'}{q'}[$. We begin with some considerations on the rotation R_α.

We consider the essential simple arcs $\Gamma_0 = \{0\} \times [0,1]$ in $\widetilde{\mathbb{A}}$, and $\gamma_0 = \pi(\Gamma_0)$ in $\mathbb{A}$. We consider the topological closed discs (see figure 3)

$$\widetilde{D} := R(\Gamma_0) \setminus \mathrm{Int}(R(T^{-p} \circ T_\alpha^q(\Gamma_0))) \quad \text{and} \quad \widetilde{D}' := R(T^{-p'} \circ T_\alpha^{q'}(\Gamma_0)) \setminus \mathrm{Int}(R(\Gamma_0)).$$

We set $D = \pi(\widetilde{D})$ and $D' = \pi(\widetilde{D}')$, and we consider the family of topological closed discs

$$\mathcal{D} = \left\{ R_\alpha^k(D) \ , \ k = 0, \ldots, q'-1 \right\} \cup \left\{ R_\alpha^{k'}(D') \ , \ k' = 0, \ldots, q-1 \right\}.$$

Lemma 5.6. *The union of the discs in the family $\mathcal{D}$ cover the annulus $\mathbb{A}$, and their interiors are pairwise disjoint.*

Proof. By lemma A.1 of the appendix, no arc $R_\alpha^k(\gamma_0)$, with $k \in \{0, \dots, q+q'-1\}$, intersects the interior of D nor D'. Consequently, the discs of the family $\mathcal{D}$ have their interiors pairwise disjoint.

Similarly, the arcs $R_\alpha^k(\gamma_0)$, with $k \in \{0, \dots, q+q'-1\}$, are pairwise disjoint. We observe that each of these arcs is contained in the boundary of exactly two elements of the family $\mathcal{D}$. This implies that the family $\mathcal{D}$ covers the annulus $\mathbb{A}$. $\square$

We denote by O the topological closed disc $D' \cup D$. Observe that O is also equal to $R_\alpha^{q'}(D) \cup R_\alpha^q(D')$. Unless $q = q' = 1$ (and in this case the proposition will hold trivially from step 1 below), O is a topological closed disc. The interior of O is disjoint from its first $(\min(q, q') - 1)$ backward iterates under R_α. The boundary of O (as a topological manifold) is a simple closed curve C, and we have

$$C = R_\alpha^q(\gamma_0) \cup R_\alpha^{q'}(\gamma_0) \cup C^+ \cup C^-,$$

where $C^- = \pi([q'\alpha - p', q\alpha - p] \times \{0\})$ and $C^+ = \pi([q'\alpha - p', q\alpha - p] \times \{1\})$.

In order to compare the metrics on the topological disc O and the Euclidean disc $\mathbb{D}$, we introduce a homeomorphism $\psi : \mathbb{D} \to O$: first, we note that O is isometric to a Euclidean rectangle in $\mathbb{R}^2$, centered at $(0,0)$, with width $a = (q - q')\alpha - (p - p') \leq 1$ and height $b = 1$; then, we define ψ as follows:

$$\psi(x, y) = \frac{\sqrt{x^2 + y^2}}{\max(|x|, |y|)} \cdot \left(\frac{ax}{2}, \frac{by}{2}\right).$$

A rough estimate shows that

$$\forall z, z' \in \mathbb{D}, \quad \mathrm{d}_O(\psi(z), \psi(z')) \leq 2\, \mathrm{d}_{\mathbb{D}}(z, z'), \tag{2}$$

where d_O and $\mathrm{d}_{\mathbb{D}}$ are the usual Euclidean distances on O and $\mathbb{D}$. Therefore one can easily transpose the result of corollary 5.3, concerning the metric space $\mathrm{Homeo}^+(\mathbb{D})$, to the space $\mathrm{Homeo}^+(O)$ of homeomorphisms of O isotopic to the identity endowed with the metric induced by the Euclidean metric on O:

Remark 5.7. *The statement in corollary 5.3 still holds if we replace the metric space* $\mathrm{Homeo}^+(\mathbb{D}, \alpha_1 \cup \alpha_2)$ *by the metric space* $\mathrm{Homeo}^+(O, R_\alpha^q(\gamma_0) \cup R_\alpha^{q'}(\gamma_0))$ *of orientation preserving homeomorphisms of O which are the identity on $R_\alpha^q(\gamma_0) \cup R_\alpha^{q'}(\gamma_0)$, provided that we allow N to be less than $2(\frac{4+2\pi}{\varepsilon} + 5)$.*

Actually, we will apply this remark to the disc $O' = R_\alpha^{-1}(O)$ instead of O : note that this is all right since R_α is an isometry.

5.2.1 First step

In this first step, we build a homeomorphism $\sigma_a : \mathbb{A} \to \mathbb{A}$, isotopic to the identity, such that the homeomorphism $h_a = \sigma_a \circ h \circ \sigma_a^{-1}$ coincides with the rotation R_α on each arc $R_\alpha^k(\gamma_0)$ with $k \in \{0, \dots, q + q' - 2\}$.

Applying the arc translation theorem 2.3 to the homeomorphism h, we obtain an essential simple arc γ in $\mathbb{A}$ such that the arcs $\gamma, \dots, h^{q+q'-1}(\gamma)$ are pairwise disjoint. Now, we consider a homeomorphism $\sigma_a : \gamma \to \gamma_0$ which maps the endpoint of γ which is on $\pi(\mathbb{S}^1 \times \{1\})$ (resp. on $\pi(\mathbb{S}^1 \times \{0\})$) to the endpoint γ_0 which is on $\pi(\mathbb{S}^1 \times \{1\})$ (resp. on $\pi(\mathbb{S}^1 \times \{0\})$). Since the arcs $\gamma, \dots, h^{q+q'-1}(\gamma)$ are pairwise disjoint, we may extend σ_a to the union of these arcs: more precisely, on $h^k(\gamma)$, we let $\sigma_a := R_\alpha^k \circ \sigma_a|_\gamma \circ h^{-k}$. Now the key fact is that according to proposition 4.1, the cyclic order of the iterates of γ under h is the same as for the iterates of γ_0 under R_α. Consequently, thanks to a repeated use of Schoenflies theorem (see for example [2]), we may further extend σ_a to a homeomorphism of $\mathbb{A}$ (isotopic to the identity). For every $k \in \{0, \dots, q+q'-2\}$, the arc $h^k(\gamma)$ is mapped by σ_a on the arc $R_\alpha^k(\gamma_0)$, and the conjugate $h_a := \sigma_a \circ h \circ \sigma_a^{-1}$ of h coincides with the rotation R_α on the arc $R_\alpha^k(\gamma_0)$.

We note that, since the homeomorphisms $\widetilde{\sigma}_a \circ \widetilde{h} \circ \widetilde{\sigma}_a^{-1}$ and the translation T_α coincide on the arc $T^k(\Gamma_0)$ for every $k \in \mathbb{Z}$, we have

$$\mathrm{d}(\widetilde{\sigma}_a \circ \widetilde{h} \circ \widetilde{\sigma}_a^{-1}, T_\alpha) \leq 1.$$

Hence, proposition 5.1 already holds if $s = \min(q, q') \leq 40$. So, from now on, we shall assume that $s = \min(q, q')$ is bigger than 40.

5.2.2 Second step

In this second step, we build a homeomorphism $\sigma_b \in \mathrm{Homeo}(\mathbb{A})$ which is isotopic to the identity, and such that the conjugate $h_b := \sigma_b \circ h_a \circ \sigma_b^{-1}$ coincides with the rotation R_α everywhere except possibly on the topological disc $O' := R_\alpha^{-1}(O) = R_\alpha^{q'-1}(D) \cup R_\alpha^{q-1}(D')$.

According to the first step, for each $k \in \{0, \dots, q'-1\}$, we have $h_a^k(D) = R_\alpha^k(D)$; on this disc, we let $\sigma_b := R_\alpha^k \circ h_a^{-k}$. We use the same formula on the disc $h_a^k(D') = R_\alpha^k(D')$ for each $k \in \{0, \dots, q-1\}$. Note that the intersection of any two such discs, if not empty, is an arc $R_\alpha^k(\gamma_0)$ with $k \in \{0, \dots, q+q'-1\}$, and that on these arcs the map $R_\alpha^k \circ h_a^{-k}$ is the identity. Thus these formulae are coherent, and define a homeomorphism σ_b of $\mathbb{A}$ isotopic to the identity. Now one easily checks that the homeomorphism h_b defined as indicated above coincides with R_α on each disc $R_\alpha^k(D)$ with $k \in \{0, \dots, q'-2\}$ and $R_\alpha^k(D')$ with $k \in \{0, \dots, q-2\}$. Using lemma 5.6, we see that the union of all these discs cover the whole annulus but the set O', and thus the second step is complete.

5.2.3 Last step

We denote by s the minimum of q and q'. We consider the homeomorphism g of the topological disc O' defined by $g = R_\alpha^{-1} \circ h_b$. Note that g is the identity on the boundary arcs $R_\alpha^{q-1}(\gamma_0)$ and $R_\alpha^{q'-1}(\gamma_0)$. Now we apply the results on decomposition of disc homeomorphisms: according to corollary 5.3 and remark 5.7, we can write

$$g = g_N \circ \cdots \circ g_1$$

where each g_i is a homeomorphism of O' which is the identity on $R_\alpha^{q-1}(\gamma_0) \cup R_\alpha^{q'-1}(\gamma_0)$, and which is ε-close to the identity of O'. In applying this we choose $\varepsilon = \frac{2(4+2\pi)}{s-12}$, so that

$$N \leq 2(\frac{4+2\pi}{\varepsilon} + 5) \leq s - 2$$

and one can check that $\varepsilon \leq \frac{40}{s}$ (recall that $s = \min(q, q')$ is assumed to be larger than 40).

Now for each $k \in \{1, \dots, N\}$, we define σ_c on the disc $R_\alpha^k(O')$ by the formula

$$\sigma_c := R_\alpha^k \circ g_k \circ \dots \circ g_1 \circ h_b^{-k}.$$

We also let σ_c be equal to the identity on the remaining part of the annulus $\mathbb{A}$. It remains to check that these formulae are coherent, and that the homeomorphism $h_c := \sigma_c \circ h_b \circ \sigma_c^{-1}$ is ε-close to the identity.

Since $N \leq s-2$, the discs $R_\alpha(O'), \dots, R_\alpha^N(O')$ have their interiors pairwise disjoint; furthermore, the formulae defining σ_c gives the identity on the boundary arcs $R_\alpha^\ell(\gamma_0)$ of these discs. This proves that σ_c is a well-defined homeomorphism of O'.

Now observe first that σ_c is equal to $R_\alpha^N \circ g \circ h_b^{-N}$ on the disc $R_\alpha^N(O')$. Since $N < s-1$, we have $h_b^N = R_\alpha^N \circ g$ on O' (according to the second step). This shows that σ_c is the identity on $R_\alpha^N(O')$.

For $k \in \{0, \dots, N-1\}$, one can check that the homeomorphism h_c is equal to $R_\alpha \circ \left(R_\alpha^k \circ g_{k+1} \circ R_\alpha^{-k}\right)$ on the disc $R_\alpha^k(O')$. Since g_{k+1} is ε-close to the identity, and since R_α is an isometry, we get that h_c is ε-close to R_α on this disc. Moreover, the homeomorphisms h_b and h_c coincide on $E := \mathbb{A} \setminus \bigcup_{k=0}^{N-1} R_\alpha^k(O')$, since $\sigma_c = \mathrm{Id}$ on $R_\alpha^N(O')$. According to the second step, this proves that h_c is equal to R_α on the set E. Thus the proof of proposition 5.1 is complete.

6 Proof of corollary 1.3

Given theorem 1.1, corollary 1.3 reduces to the following fact: a homeomorphism which has the intersection property (any non-homotopically trivial simple closed curve meets its image) and no periodic point is an irrational pseudo-rotation. This fact seems to be well-known; however, as we were unable to find it in the literature, we provide a proof.

Let us first recall the statement of Bonatti-Guillou's result:

Theorem 6.1 ([7]). *Let h be a homeomorphism of the annulus $\mathbb{A}$, which is isotopic to the identity. Assume that h does not have any fixed point. Then at least one of the following property holds:*

(i) there exists a non-homotopically trivial simple closed curve in $\mathbb{A}$, which is disjoint from its image under h,

(ii) there exists an essential simple arc in $\mathbb{A}$ which is disjoint from its image under h.

To prove corollary 1.3, we also need two technical lemmas:

Lemma 6.2. *Let h be a homeomorphism of the annulus $\mathbb{A}$, which is isotopic to the identity. Assume that h does not have any fixed point. Assume that there exists a positive integer p, and a non-homotopically trivial simple closed curve σ in $\mathbb{A}$ which is disjoint from its image under h^p. Then there exists a non-homotopically trivial simple closed curve $\widehat{\sigma}$ in $\mathbb{A}$ which is disjoint from its image under h.*

Sketch of the proof. We use the same kind of arguments as in the beginning of section 3. For every non-homotopically trivial simple closed curve σ in $\mathbb{A}$, we denote by $B(\sigma)$ the closure of the connected component of $\mathbb{A} \setminus \sigma$ which is "below σ" (that is, which contains $\mathbb{S}^1 \times \{0\}$). Given two non-homotopically trivial simple closed curves σ_1 and σ_2 in $\mathbb{A}$, the boundary of the connected component of $(\mathbb{A} \setminus B(\sigma_1)) \cap (\mathbb{A} \setminus B(\sigma_2))$ containing $\mathbb{S}^1 \times \{1\}$ is a non-homotopically trivial simple closed curve, that we denote by $\sigma_1 \vee \sigma_2$ (the proof is the same as for lemma 3.2). We write $\sigma_1 < \sigma_2$ if $\sigma_2 \subset \mathrm{Int}(B(\sigma_1))$.

Now, we consider the integer p and the curve σ given in the hypothesis of lemma 6.2. We have either $h^p(\sigma) < \sigma$ or $\sigma < h^p(\sigma)$; assume for example that we are in the second case. Then, we consider some non-homotopically trivial simple closed curves $\tau_0, \dots, \tau_{p-1}$ such that

$$h^{-p}(\sigma) < \tau_{p-1} < \cdots < \tau_1 < \tau_0 = \sigma,$$

and we define

$$\widehat{\sigma} := \tau_0 \vee h(\tau_1) \vee \cdots \vee h^{p-1}(\tau_{p-1}).$$

The same arguments as in the beginning of the proof of proposition 3.1 show that $B(\widehat{\sigma})$ is a strict attractor for h (*i.e.* the image of $B(\widehat{\sigma})$ under h is included in the interior of $B(\widehat{\sigma})$). In particular, we obtain a non-homotopically trivial simple closed curve $\widehat{\sigma}$ which is disjoint from its image under h. □

Lemma 6.3. *Let h be a homeomorphism of the annulus $\mathbb{A}$, which is isotopic to the identity. Assume that h does not have any fixed point. Assume moreover that there exists an essential simple arc γ in $\mathbb{A}$ which is disjoint from its image under h. Then, for every lift $\widetilde{h}$ of h, the rotation set of $\widetilde{h}$ is disjoint from $\mathbb{Z}$.*

Proof. Let us consider a lift $\widetilde{h}$ of h and a lift Γ of γ in $\widetilde{\mathbb{A}}$. Since $T^k \circ \widetilde{h}$ is also a lift of h for every $k \in \mathbb{Z}$, and since $\mathrm{Rot}(T^k \circ \widetilde{h}) = \mathrm{Rot}(\widetilde{h}) + k$, it is sufficient to prove that 0 is not in the rotation set of $\widetilde{h}$.

Our assumption on γ implies that Γ is disjoint from its image under $\widetilde{h}$. We will suppose for instance that $\Gamma \cap R(\widetilde{h}(\Gamma)) = \emptyset$. This implies that $R(\Gamma)$ is a strict attractor for $\widetilde{h}$. The same is true for $R(T(\Gamma))$.

Recall that $\widetilde{\mathbb{A}}$ is the band $\mathbb{R} \times [0,1]$. We define a symmetry χ of $\mathbb{R}^2$ by $\chi(x,y) = (x,-y)$. One can then extend $\widetilde{h}$ to an orientation-preserving homeomorphism $\widetilde{h}'$ of $\mathbb{R}^2$ by setting for any $(x,y) \in \widetilde{\mathbb{A}}$ and $n \in \mathbb{Z}$:

- $\widetilde{h}'(x,-y) = \chi(\widetilde{h}(x,y))$;
- $\widetilde{h}'(x, y+2n) = \widetilde{h}'(x,y) + (0,2n)$.

Like h, the homeomorphisms $\widetilde{h}$ and $\widetilde{h}'$ do not have any fixed point. Brouwer's theory on homeomorphisms of the plane asserts that any orbit of $\widetilde{h}'$ goes to infinity in $\mathbb{R}^2$ (see [7]). Let us consider any point $z_0 \in R(\Gamma)$. The second coordinate of the orbit of z_0 by $\widetilde{h}$ remains in $[0,1]$. Since the orbit of z_0 is the same by $\widetilde{h}$ and by $\widetilde{h}'$, the modulus of the first coordinate along the positive orbit of z_0 by $\widetilde{h}$ must take arbitrarily large values. Using the fact that $R(\Gamma)$ is an attractor, we deduce that the first coordinate along the positive orbit of z_0 is bounded from below and is not bounded from above. Hence, there exists $n(z_0) \geq 0$ such that $\widetilde{h}^{n(z_0)}(z_0) \in R(T(\Gamma))$. Since $R(T(\Gamma))$ is an attractor, any iterate $\widetilde{h}^n(z_0)$ with $n \geq n(z_0)$ belongs to $R(T(\Gamma))$.

Note that since $R(T(\Gamma))$ is a strict attractor, for any z' close to z_0 and any $n > n(z_0)$, the iterate $\widetilde{h}^n(z')$ belongs to $R(T(\Gamma))$. By compacity of the square $Q = R(\Gamma) \setminus \mathrm{Int}(R(T(\Gamma)))$, there exists some integer $n_0 \geq 1$ such that for any point $z \in Q$ and any $n \geq n_0$, the iterate $\widetilde{h}^n(z)$ belongs to $R(T(\Gamma))$.

Since $\widetilde{h}$ and T commute and since $R(\Gamma) \setminus R(T(\Gamma)) \subset Q$ is a fundamental domain for the covering $\widetilde{\mathbb{A}} \to \mathbb{A}$, one deduces that the rotation set of $\widetilde{h}$ is included in $[\frac{1}{n_0}, +\infty[$ and does not contained 0. This concludes the proof. □

Now, we are ready to prove corollary 1.3:

Proof of corollary 1.3. Let h be a homeomorphism as in the statement of corollary 1.3, *i.e.* a homeomorphism of the annulus $\mathbb{A}$, which is isotopic to the identity, and which does not have periodic point. Let $\widetilde{h} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h.

First case. If there exists a non-homotopically trivial simple closed curve in $\mathbb{A}$ which is disjoint from its image under h, then we are done.

Second case. Now, we assume that there does not exist any non-homotopically trivial simple closed curve in $\mathbb{A}$ which is disjoint from its image under h. In order to apply theorem 1.1, we have to prove that h is an irrational pseudo-rotation. Let p be an integer. Since h does not have any periodic point, the homeomorphism h^p does not any fixed point. So we can apply theorem 6.1 to the homeomorphism h^p. Moreover, lemma 6.2 implies that there does not exist any non-homotopically simple closed curve in $\mathbb{A}$ which is disjoint from its image under h^p. Hence, theorem 6.1 provides us with an essential simple arc in $\mathbb{A}$ which is disjoint from its image under h^p. Then, using lemma 6.3, we obtain that the rotation set of the homeomorphism $\widetilde{h}^p$ is disjoint from $\mathbb{Z}$. Equivalently (see lemma 2.2), the rotation set of $\widetilde{h}$ is disjoint from $\frac{1}{p}.\mathbb{Z}$. So, we have proved that, for every $p \in \mathbb{Z}$, the rotation set of $\widetilde{h}$ is disjoint from $\frac{1}{p}.\mathbb{Z}$. Since the rotation set of $\widetilde{h}$ is an interval, this implies that it is a single irrational number; in other words, h is an irrational pseudo-rotation. Then, for every $n \in \mathbb{N}$, theorem 1.1 provides us with an essential simple arc γ_n, such that the arcs $\gamma_n, h(\gamma_n), \dots, h^n(\gamma_n)$ are pairwise disjoint. $\square$

A Some elementary properties of Farey intervals

A.1 Farey intervals and rotations

Let us fix a Farey interval $I = \left]\frac{p}{q}, \frac{p'}{q'}\right[$. We choose a number α in I. In this subsection, we prove that the cyclic order of the $q + q' - 1$ first iterates of any orbit under the circle rotation of angle α does not depend on the choice of α in I (proposition A.2 below).

Lemma A.1. *For every $k \in \{1, \dots, q + q' - 1\}$, the interval $\left]k.\frac{p}{q}, k.\frac{p'}{q'}\right[$ does not contain any integer.*

Proof. Let k be a positive integer such that the interval $]kp/q\,,\, kp'/q'[$ contains an integer ℓ. Then the rational number ℓ/k is in the Farey interval $]p/q, p'/q'[$. In particular, we have $p'/q' - p/q = (p'/q' - \ell/k) + (\ell/k - p/q)$. Moreover, we have $\ell/k - p/q = (\ell q - kp)/(kq) \geq 1/(kq)$ and $p'/q' - \ell/k = (p'k - q'\ell)/(kq') \geq 1/(kq')$. Besides, since $]p/q, p'/q'[$ is a Farey interval, we have $p'/q' - p/q = 1/(qq')$. Putting everything together, we obtain $1/(kq) + 1/(kq') \leq 1/(qq')$ which is equivalent to $k \geq q + q'$. $\square$

Lemma A.1 implies that, for every $k \in \{1, \dots, q + q' - 1\}$, there exists a unique integer $n_k \in \mathbb{Z}$, such that the interval $]\,k.\frac{p}{q} - n_k,\ k.\frac{p'}{q'} - n_k[$ is included in $]0, 1[$. Given a number α in the Farey interval I, and an integer $k \in \{1, \dots, q + q' - 1\}$, we set $\alpha_k := k\alpha - n_k$. We can now prove the announced result:

Proposition A.2. *The order of the numbers $\alpha_1, \dots, \alpha_{q+q'-1}$ in $]0, 1[$ does not depend on the choice of α in the Farey interval I.*

Proof. Let k_1 and k_2 be two integers in $\{1, \dots, q + q' - 1\}$, with $k_1 \neq k_2$. Then, using lemma A.1, we see that the difference $\alpha_{k_2} - \alpha_{k_1}$ is never null if α is in I. Since this quantity depends continuously on α, its sign does not depend on the choice of α in I. This completes the proof. $\square$

A.2 Farey intervals and rational approximations

It is well-known that any Farey interval $I =]p/q, p'/q'[$ is associated with a finite sequence of rational numbers $\left(\frac{p_n}{q_n}\right)_{1\leq n\leq n_0}$ which satisfies the following properties:

- $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2} = \frac{p_1}{q_1} + 1$ are two consecutive integers;
- For any $2 \leq n \leq n_0 - 1$, there exists $a_{n+1} \in \mathbb{N} \setminus \{0\}$ such that

$$p_{n+1} = a_{n+1}p_n + p_{n-1} \quad \text{and} \quad q_{n+1} = a_{n+1}q_n + q_{n-1}; \tag{3}$$

- $\frac{p_{n_0-1}}{q_{n_0-1}}$ and $\frac{p_{n_0}}{q_{n_0}}$ are the two endpoints of the Farey interval I, that is

$$\left]\frac{p}{q}, \frac{p'}{q'}\right[= \left]\frac{p_{n_0-1}}{q_{n_0-1}}, \frac{p_{n_0}}{q_{n_0}}\right[\text{ or } \left]\frac{p}{q}, \frac{p'}{q'}\right[= \left]\frac{p_{n_0}}{q_{n_0}}, \frac{p_{n_0-1}}{q_{n_0-1}}\right[.$$

The rational numbers $\frac{p_1}{q_1}, \ldots, \frac{p_{n_0}}{q_{n_0}}$ are the *common Farey approximations* of the elements of the interval I.

Remark A.3. *It is convenient to add to the sequence $\{(p_n, q_n), 1 \leq n \leq n_0\}$ a first term $(p_0, q_0) = (1, 0)$ so that (3) holds also with $n = 1$ and $a_2 = 1$.*

A.3 Iterates of essential simple arcs: proof of proposition 4.1

In order to prove proposition 4.1, the main task is to show that all the iterates of the arc Γ involved in the statement are disjoint from Γ. To prove this point, we will also need to collect some information about their order (but not all the information): this programme is realized by lemmas A.5 and A.7. Actually, the whole result concerning the order will follow easily from the disjointness.

We use the relations "$\Gamma_1 < \Gamma_2$" and "$\Gamma_1 \leq \Gamma_2$" introduced in subsection 3.1. We assume that we are given a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ isotopic to the identity, a lift $\widetilde{h} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of h, an essential simple arc Γ in $\widetilde{\mathbb{A}}$, and a Farey interval $I =]p/q, p'/q'[$ such that $T^{-p'} \circ \widetilde{h}^{q'}(\Gamma) < \Gamma < T^{-p} \circ \widetilde{h}^{q}(\Gamma)$. We have to prove that the arcs $T^{-\ell} \circ \widetilde{h}^{k}(\Gamma)$ for $k \in \{0, \ldots, q + q' - 1\}$ and $\ell \in \mathbb{Z}$ are pairwise disjoint, and ordered as the lifts of the first iterates of a vertical segment under the rotation R_α for any $\alpha \in]p/q, p'/q'[$. The proof relies on some arithmetical properties of Farey intervals, and on the fact that T and $\widetilde{h}$ commute. Let us first introduce some notations:

- For every $(\ell, k) \in \mathbb{Z}^2$, we denote by $\Gamma(\ell, k)$ the curve $T^{-\ell} \circ \widetilde{h}^{k}(\Gamma)$.
- We denote by $\frac{p_1}{q_1}, \ldots, \frac{p_{n_0}}{q_{n_0}}$ (with $p_i \wedge q_i = 1$) the common Farey approximations of the elements of the interval I (see subsection A.2). Moreover, we set $p_0 := 1$ and $q_0 := 0$.
- For $0 \leq n \leq n_0$, we denote by $\widetilde{h}_n$ the map $T^{-p_n} \circ \widetilde{h}^{q_n}$.
- Given three essential simple arcs $\Gamma, \Gamma', \Gamma''$, we shall say that Γ *separates* Γ' *and* Γ'', if these three arcs are disjoint and satisfy either $\Gamma' < \Gamma < \Gamma''$ or $\Gamma'' < \Gamma < \Gamma'$.

We will use intensively the following remark:

Remark A.4. *If H is a homeomorphism of $\widetilde{\mathbb{A}}$ isotopic to the identity, then H "preserves inequalities"*, that is *if Γ_1 and Γ_2 are two disjoint essential simple arcs in $\widetilde{\mathbb{A}}$, then*

$$\Gamma_1 < \Gamma_2 \Longrightarrow H(\Gamma_1) < H(\Gamma_2). \tag{4}$$

This remark explains why the argument of the proof of the proposition will be essentially one-dimensional.

Lemma A.5. *For any $n \in \{0, \dots, n_0-1\}$, the arc Γ separates $\Gamma(p_n, q_n)$ and $\Gamma(p_{n+1}, q_{n+1})$. Furthermore, if $n \neq 0$, then $\Gamma(p_{n+1}, q_{n+1})$ separates Γ and $\Gamma(p_{n-1}, q_{n-1})$.*

Corollary A.6. *The order of the whole family of arcs involved in the lemma is the following (for simplicity, we assume that n_0 is even):*

$$T^{-1}(\Gamma) = \Gamma(p_0, q_0) < \Gamma(p_2, q_2) < \cdots < \Gamma(p_{n_0}, q_{n_0}) < \Gamma < \cdots$$

$$\cdots < \Gamma < \Gamma(p_{n_0-1}, q_{n_0-1}) < \cdots < \Gamma(p_1, q_1) < T(\Gamma)$$

Proof of lemma A.5. The proof is a decreasing induction on n. First, we observe that the arc Γ separates $\Gamma(p_{n_0-1}, q_{n_0-1})$ and $\Gamma(p_{n_0}, q_{n_0})$ by assumption (2) in proposition 4.1.

Now, suppose that we have proven that the arc Γ separates the arcs $\Gamma(p_n, q_n)$ and $\Gamma(p_{n+1}, q_{n+1})$ for some $n < n_0$. To fix ideas, we assume that

$$\widetilde{h}_n(\Gamma) < \Gamma < \widetilde{h}_{n+1}(\Gamma). \tag{5}$$

The left "inequality" implies, using (4) and the transitivity of "<", that $\Gamma < \widetilde{h}_n^{-a_{n+1}}(\Gamma)$. Since $\widetilde{h}_{n+1}$ and $\widetilde{h}_n$ commute, using again (4), we have

$$\widetilde{h}_{n+1}(\Gamma) < \widetilde{h}_n^{-a_{n+1}} \circ \widetilde{h}_{n+1}(\Gamma). \tag{6}$$

By (3), we have $\widetilde{h}_{n-1} = \widetilde{h}_n^{-a_{n+1}} \circ \widetilde{h}_{n+1}$. Hence, putting together (5) and (6), we get the desired inequalities:

$$\Gamma(p_n, q_n) = \widetilde{h}_n(\Gamma) \; < \; \Gamma \; < \; \widetilde{h}_{n+1}(\Gamma) = \Gamma(p_{n+1}, q_{n+1}) \; < \; \widetilde{h}_{n-1}(\Gamma) = \Gamma(p_{n-1}, q_{n-1}).$$

□

Proof of corollary A.6. The only inequality that is not contained in the lemma is the last one. But we have $\Gamma(p_2, q_2) < \Gamma$, and composing with T gives $T(\Gamma(p_2, q_2)) < T(\Gamma)$; it remains to note that $T(\Gamma(p_2, q_2)) = \Gamma(p_1, q_1)$ (see subsection A.2). □

Lemma A.7. *For every integer $n \in \{0, \dots, n_0 - 1\}$, and every pair of integers $(\ell, k) \neq (0, 0)$ with $\ell \in \mathbb{Z}$ and $k \in \{0, \dots, q_n + q_{n+1} - 1\}$, the arcs Γ and $\Gamma(\ell, k)$ are disjoint.*

Proof of lemma A.7. We will actually prove the following statement : *For n, ℓ, k as in the lemma, the arc $\Gamma(\ell, k)$ does not meet the open topological disc whose boundary in $\widetilde{\mathbb{A}}$ is $\Gamma(p_n, q_n) \cup \Gamma(p_{n+1}, q_{n+1})$.* This, together with lemma A.5, implies the lemma. We proceed by induction on n.

The case $n = 0$ comes from the inequalities $T^{-1}(\Gamma) < \Gamma < h_1(\Gamma) < T(\Gamma)$ extracted from remark A.6 (note that $q_0 + q_1 - 1 = 0$).

Now we assume that the statement holds for some integer $n-1 \geq 0$. By lemma A.5, we may assume for instance that $\Gamma(p_n, q_n) < \Gamma < \Gamma(p_{n-1}, q_{n-1})$. Note that this in turn implies $\Gamma(p_n, q_n) < \Gamma < \Gamma(p_{n+1}, q_{n+1})$ (using the same lemma). By the induction hypothesis, every arc $\Gamma(\ell', k')$ with $(\ell', k') \neq (0,0)$, $\ell' \in \mathbb{Z}$ and $k' \in \{0, \ldots, q_{n-1} + q_n - 1\}$ satisfies either $\Gamma(\ell', k') \leq \Gamma(p_n, q_n)$ or $\Gamma(p_{n-1}, q_{n-1}) \leq \Gamma(\ell', k')$.

Let us consider some arc $\Gamma(\ell, k)$ with $(\ell, k) \neq (0,0)$, $\ell \in \mathbb{Z}$ and $k \in \{0, \ldots, q_n + q_{n+1} - 1\}$. We have to prove that either $\Gamma(\ell, k) \leq \Gamma(p_n, q_n)$ or $\Gamma(p_{n+1}, q_{n+1}) \leq \Gamma(\ell, k)$. According to lemma A.5, we have $\Gamma(p_{n+1}, q_{n+1}) \leq \Gamma(p_{n-1}, q_{n-1})$; combining this with our induction hypothesis solves the case $k \leq q_{n-1} + q_n - 1$.

We now suppose $k \geq q_{n-1} + q_n$. Then we write $k = sq_n + k'$ with $s \in \{1, \ldots, a_{n+1}\}$ and $k' \in \{0, \ldots, q_{n-1} + q_n - 1\}$. Similarly, we write $\ell = sp_n + \ell'$. By the induction hypothesis, one of the three following cases holds:

- $(\ell', k') = (0,0)$. In this case, we have $\Gamma(\ell, k) = \Gamma(sp_n, sq_n) = \widetilde{h}_n^s(\Gamma) \leq \widetilde{h}_n(\Gamma) = \Gamma(p_n, q_n)$ (the inequality follows from the hypothesis $\widetilde{h}_n(\Gamma) < \Gamma$, from inequality (4), and from the transitivity of $<$).

- $\Gamma(\ell', k') < \Gamma$. In this case $\Gamma(\ell, k) = \widetilde{h}_n^s(\Gamma(\ell', k')) < \widetilde{h}_n^s(\Gamma)$ and we conclude using the previous case.

- $\Gamma < \Gamma(\ell', k')$. In this case, using our induction hypothesis, we obtain that $\Gamma(p_{n-1}, q_{n-1}) \leq \Gamma(\ell', k')$. It follows that

$$\begin{aligned} \Gamma(p_{n+1}, q_{n+1}) = \widetilde{h}_{n-1} \circ \widetilde{h}_n^{a_{n+1}}(\Gamma) & \underset{(a)}{\leq} \widetilde{h}_{n-1} \circ \widetilde{h}_n^s(\Gamma) \\ & = \widetilde{h}_n^s(\Gamma(p_{n-1}, q_{n-1})) \\ & \underset{(b)}{\leq} \widetilde{h}_n^s(\Gamma(\ell', k')) = \Gamma(\ell, k) \end{aligned}$$

  where we used for (a) that $\widetilde{h}_n(\Gamma) < \Gamma$ and $s \leq a_{n+1}$ and for (b) the above inequality.

This completes the proof of the lemma. $\square$

We now turn to the proof of proposition 4.1. Let us first prove that the arcs are disjoint (first item). For this it suffices to note that

$$\Gamma(\ell, k) \cap \Gamma(\ell', k') = T^{\ell'} \circ \widetilde{h}^{-k'}(\Gamma(\ell - \ell', k - k') \cap \Gamma)$$

and to apply the last lemma (for $n = n_0 - 1$).

Now, let us turn to the proof of the second item of the proposition. First, we note that the "inequalities" $\Gamma(\ell, k) < \Gamma(\ell', k')$ and $\Gamma(\ell - \ell', k - k') < \Gamma$ are equivalent. So, it is sufficient to show that, for any $k \in \{0, \ldots, q + q' - 1\}$ and for any ℓ,

$$\begin{aligned} T^{-\ell} \circ \widetilde{h}^k(\Gamma) < \Gamma \quad & \text{implies that} \quad k\frac{p'}{q'} - \ell \leq 0, \\ \Gamma < T^{-\ell} \circ \widetilde{h}^k(\Gamma) \quad & \text{implies that} \quad k\frac{p}{q} - \ell \geq 0. \end{aligned}$$

Let us prove, for instance, the second implication. We suppose that $\Gamma < T^{-\ell} \circ \widetilde{h}^k(\Gamma)$. First note that by assumption we have $T^{-p'} \circ \widetilde{h}^{q'}(\Gamma) < \Gamma$, so that $\frac{p'}{q'} \neq \frac{l}{k}$. This also implies that the rotation set of the map $T^{-p'} \circ \widetilde{h}^{q'}$ is included in $]-\infty, 0]$, so that the rotation set of

$\widetilde{h}$ is included in $]-\infty, \frac{p'}{q'}]$ by lemma 2.2. Similarly, from the inequality $\Gamma < T^{-\ell} \circ \widetilde{h}^k(\Gamma)$ we get that the rotation set of $\widetilde{h}$ is included in $[\frac{\ell}{k}, +\infty[$. From all this we conclude that $\frac{l}{k} \leq \frac{p'}{q'}$, *i.e.* that $k\frac{p'}{q'} - \ell \geq 0$. Since the integer k is in $\{0, \dots, q+q'-1\}$, we get from lemma A.1 that $k\frac{p}{q} - \ell \geq 0$. This completes the proof of proposition 4.1.

B The case of the torus

In a way similar to section 2.2, on can define the rotation set $\mathrm{Rot}(\widetilde{h})$ of a lift $\tilde{h}$ of a torus homeomorphism h that is isotopic to the identity. It is a convex compact set of $\mathbb{R}^2$.

We recall the statement of the theorem proved (for diffeomorphisms) by J. Kwapisz ([11]) on the torus, and explain how our methods work in this case.

Theorem B.1. *Let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism of the torus isotopic to the identity, and let $\widetilde{h} : \widetilde{\mathbb{T}^2} \to \widetilde{\mathbb{T}^2}$ be a lift of h. Assume that the rotation set of $\widetilde{h}$ is included in a Farey band $]\frac{p}{q}, \frac{p'}{q'}[\times\mathbb{R}$. Then there exists an essential simple closed curve γ in the torus $\mathbb{T}^2$ such that the curves $\gamma, h(\gamma), \dots, h^{q+q'-1}(\gamma)$ are pairwise disjoint.*

Here an *essential simple closed curve* is a topological circle in $\mathbb{T}^2$ homotopic to the circle $\{0\} \times \mathbb{T}^1$, and a *Farey band* is a band $]\frac{p}{q}, \frac{p'}{q'}[\times\mathbb{R}$ where $]\frac{p}{q}, \frac{p'}{q'}[$ is a Farey interval. The proofs given in sections 3 and 4 work in the torus setting with very few changes :

- the annulus $\mathbb{A}$ has to be replaced by the torus $\mathbb{T}^2$, and the band $\widetilde{\mathbb{A}}$ has to be replaced by the intermediate covering $\mathbb{R} \times \mathbb{S}^1$ of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$,
- the notion of essential simple arcs in $\mathbb{A}$ or $\widetilde{\mathbb{A}}$ has to be replaced by the notion of essential simple closed curve in $\mathbb{T}^2$ or $\mathbb{R} \times \mathbb{S}^1$,
- the rotation set $\mathrm{Rot}(\widetilde{h})$ of a lift $\widetilde{h}$ to $\widetilde{\mathbb{A}}$ of a homeomorphism h of the annulus $\mathbb{A}$ has to be replaced by the first projection $p_1(\mathrm{Rot}(\widetilde{h}))$ of the rotation set of a lift $\widetilde{h}$ to $\mathbb{R}^2$ of a homeomorphism h of the torus $\mathbb{T}^2$.

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