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SMALE DIFFEOMORPHISMS OF SURFACES: A CLASSIFICATION ALGORITHM

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Abstract. We are concerned here with Smale (*i.e.* C^1-structurally stable) diffeomorphisms of compact surfaces. Bonatti and Langevin have produced some combinatorial descriptions of the dynamics of any such diffeomorphism ([2]). Actually, each diffeomorphism admits infinitely many different combinatorial descriptions. The aim of the present article is to describe an algorithm which decides whether two combinatorial descriptions correspond to the same diffeomorphism or not. This provides an algorithmic way to classify Smale diffeomorphisms of surfaces up to topological conjugacy (on canonical neighbourhoods of the basic pieces).

1. **Introduction.** A C^1-diffeomorphism f of a manifold M is said to be C^1*-structurally stable* if there exists a neighbourhood U of f in $\text{Diff}^1(M)$, such that every diffeomorphism $g \in U$ is topologically conjugate to f. In tribute to the pioneering work of S. Smale, C^1-structurally stable diffeomorphisms of compact manifolds are often called *Smale diffeomorphisms*.

We are concerned here with the classification of Smale diffeomorphisms of compact surfaces up to topological conjugacy.

1.1. **Combinatorial descriptions of Smale diffeomorphisms of surfaces.** In [2], C. Bonatti and R. Langevin have produced some combinatorial descriptions of the global dynamics of every Smale diffeomorphism of a compact surface ; we will briefly summarize the work of these authors. For sake of simplicity, all the surfaces that we consider are oriented, and all the diffeomorphisms that we consider are orientation-preserving.

It is well-known that the non-wandering set of a Smale diffeomorphism f can be decomposed as finite union of disjoined compact invariant topologically transitive sets: the *basic pieces* of f (see [8, chapter 8]). Then, the main difficulty in the investigation of the dynamics of a Smale diffeomorphism is to understand the dynamics in the neighbourhood of each basic piece.

The classification of Smale diffeomorphisms of surfaces can be reduced to the classification of Smale diffeomorphisms of surfaces whose non-wandering set is a totally discontinuous (see [2, section 2.3]). Moreover, the dynamics of a Smale diffeomorphism in the neighbourhood of an isolated periodic orbit is completely trivial. These are the reasons why we will focus our attention on totally discontinuous basic pieces which are not isolated periodic orbits. We call such a basic piece a *non-trivial saddle basic piece*.

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Let f be a Smale diffeomorphism of a compact surface. To study the dynamics of f in the neighbourhood of a non-trivial saddle basic piece K, we use some *geometrized Markov partitions* of K. Roughly speaking, a *Markov partition* of K is a finite collection $\mathcal{R} = \{R_1, \dots, R_n\}$ of rectangles embedded in the surface S, such that K is included $R_1 \cup \dots \cup R_n$, and such that each connected component of $f(R_i) \cap R_j$ is a subrectangle which crosses R_j "vertically" and crosses $f(R_i)$ "horizontally" (see subsection 3.1). A *geometrized Markov partition* is a Markov partition, whose rectangles are ordered and endowed with some choices of orientations.

For every geometrized Markov partition $\mathcal{R}$ of a non-trivial saddle basic piece of f, Bonatti and Langevin have defined a finite combinatorial object, called *the geometrical type* of the Markov partition $\mathcal{R}$. This combinatorial object describes how the rectangles of $\mathcal{R}$ are intersected by their images (number of connected components in the intersection, relative positions, and orientations of these connected components).

Besides, for every non-trivial saddle basic piece K of f, Bonatti and Langevin have defined an invariant neighbourhood $\Delta(f, K)$ of K, which they call the *domain* of K. The rectangles of every Markov partition of K are included in the domain $\Delta(f, K)$. The work of Bonatti and Langevin culminates in the following theorem:

Theorem 1 (Bonatti, Langevin [2]). *Let f_1, f_2 be Smale diffeomorphisms of compact surfaces S_1, S_2, let K_1, K_2 be non-trivial saddle basic pieces of f_1, f_2, and let $\mathcal{R}_1, \mathcal{R}_2$ be geometrized Markov partitions of K_1, K_2. Assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same geometrical type. Then, there exists an homeomorphism $h : \Delta(f_1, K_1) \to \Delta(f_2, K_2)$ which conjugates the restrictions of f_1 and f_2.*

A couple of remarks about the objects involved in theorem 1:
— The domain $\Delta(f, K)$ of a basic piece K is a very "nice" neighbourhood of K: it is homeomorphic to a compact surface with boundary (see [2]).
— If $\mathcal{R}$ is a geometrized Markov partition of a basic piece K of a Smale diffeomorphism f of a compact surface, then the information captured by the geometrical type of $\mathcal{R}$ contains the information captured by the classical *incidence matrix* of $\mathcal{R}$. The incidence matrix of $\mathcal{R}$ characterizes the restriction of f to the basic piece K up to topological conjugacy (see [8, chapter 10]), whereas the geometrical type of $\mathcal{R}$ characterizes the restriction of f to the domain $\Delta(f, K)$.

1.2. **Statement of the main result.** According to theorem 1, the restriction of a Smale diffeomorphism f to the domain of a non-trivial saddle basic piece K is characterized (up to topological conjugacy) by the geometrical type of any geometrized Markov partition of K. Nevertheless, such a non-trivial saddle basic piece K admits infinitely many Markov partitions. This suggests the following definitions:

Definition (realizability of a geometrical type). *A geometrical type T is said to be* realizable *if there exists a Smale diffeomorphism f on a compact surface and a non-trivial saddle basic piece K of f, such that K admits a geometrized Markov partition of geometrical type T.*

Definition (strong equivalence of geometrical types). *Two realizable geometrical types T_1, T_2 are said to be* strongly equivalent *if there exists a Smale diffeomorphism f of a compact surface and a non-trivial saddle basic piece K of f, such that K admits a geometrized Markov partition of geometrical type T_1, and a geometrized Markov partition of geometrical type T_2.*

The aim of the present article is to prove the following theorem:

Theorem 2. *There exists a finite algorithm which takes two realizable geometrical types, and decides whether these geometrical types are strongly equivalent or not.*

Theorem 2 provides an algorithmic way to classify Smale diffeomorphisms of surfaces in restriction to the domains of non-trivial saddle basic pieces. A few remarks about this theorem:
— There exist effective criterions to decide whether a given geometrical type is realizable or not (see [1] and [4]).
— Unfortunately, the complexity of the algorithm announced in theorem 2 is exponential (as a function of the number of rectangle of the geometrical types).
— The restriction of a Smale diffeomorphism to a non-trivial saddle basic piece K is topologically conjugate to a *subshift of finite type* (SSFT), and is characterized by the incidence matrix of any Markov partition of K. R. F. Williams has described a algebraic citerion to decide whether two incidence matrices correspond to the same SSFT or not (see [10]). Nevertheless, it is not known whether Williams criterion is algorithmic or not (see [11]). As a consequence, the classification of Smale diffeomorphisms of surfaces in restriction to the domains of the basic pieces is better understood than the classification of these diffeomorphisms in restriction to the basic pieces themselves.
— A. Y. Zhirov has studied the dynamics of Smale diffeomorphism of surfaces in the neighbourhood of 1-dimensional attractors, and has proved a analog of theorem 2 in this context (see [9]). Using some classical operations, the dynamics of Smale diffeomorphism in the neighbourhood of 1-dimensional attractors can be seen a particular case of the dynamics of Smale diffeomorphism in the neighbourhood non-trivial saddle basic pieces (see [2, section 2.3]).

1.3. **Organization of the article.** In sections 2 and 3, we recall the definitions and basic properties of Smale diffeomorphisms, Markov partitions and their geometrical types. In sections 4 and 5, we consider a Smale diffeomorphism f of a compact surface, and a non-trivial saddle basic piece K of f. We define a positive integer $p_{min}(f, K)$. Then, for every integer $p \geq p_{min}(f, K)$, we construct a set $\mathcal{R}(f, K, p)$ of particular Markov partitions of the basic piece K. The set made of the geometrical types of these Markov partitions is denoted by $\mathcal{T}(f, K, p)$; it satisfies the following important properties:
(i) For every $p \geq p_{min}(f, K)$, the set $\mathcal{T}(f, K, p)$ is made of a finite number of geometrical types.
(ii) For every $p \geq p_{min}(f, K)$, the set $\mathcal{T}(f, K, p)$ is a complete invariant for topological conjugacy (see proposition 7 for a precise statement).

Then, for every realizable geometrical type T, we choose a Smale diffeomorphism f_T on a compact surface, and a non-trivial saddle basic piece K_T of f_T, such that the basic piece K_T admits a geometrized Markov partition of geometrical type T. We consider the integer $p_{min}(T) := p_{min}(f_T, K_T)$, and, for every integer $p \geq p_{min}(T)$, we consider the set of geometrical types $\mathcal{T}(T, p) := \mathcal{T}(f_T, K_T, p)$. This set satisfies the following important properties:
(i') For every $p \geq p_{min}(T)$, $\mathcal{T}(T, p)$ is a finite set of geometrical types.
(ii') For every $p \geq p_{min}(T)$, the set $\mathcal{T}(T, p)$ is a complete invariant for strong equivalence of geometrical types (see proposition 8 for a precise statement).

At that stage, we have found a complete invariant for strong equivalence of geometrical types, and we are left to prove that this complete invariant can be computed by a finite algorithm.

Remark 1. *The definition of the set of geometrical types $\mathcal{T}(T,p)$ uses the process of construction of Markov partitions described in section 5. This process of construction is completely geometric. Thus, it is not clear* a priori *that one can deduce an algorithm from this process of construction.*

Section 7, 8, 9 and 10 are devoted to the proof of the two following propositions:

Proposition 1. *For every realizable geometrical type T, the integer $p_{min}(T)$ is smaller than $142n_T^2$, where n_T is the number of rectangles of any Markov partition of geometrical type T.*

Proposition 2. *There exists an algorithm, which takes a realizable geometrical type T and an integer $p \geq p_{min}(T)$ as input, and gives back the finite set of geometrical types $\mathcal{T}(T,p)$.*

In section 7, we consider a Smale diffeomorphism f on a compact surface, a non-trivial saddle basic piece K of f, and a Markov partition $\mathcal{R}$ of K. For every positive integer N, we define a *finite* set of segments included in the images under f^N (resp. f^{-N}) of the sides of the rectangles of $\mathcal{R}$. These segments are called N*-segments of* $(f,K,\mathcal{R})$. Roughly speaking, the aim of the section is to find explicitly an integer N, such that every segment involved in the construction of the set of Markov partitions $\mathcal{R}(f,K,p)$ is a N-segment of $(f,K,\mathcal{R})$. In particular, this implies that every segment involved in the construction of the set of Markov partitions $\mathcal{R}(f,K,p)$ belongs to a finite set of segments that we know *a priori*. This is certainly the most important step in the proof of proposition 2. We also prove proposition 1 in section 7.

Sections 8, 9 and 10 are devoted to the description of the algorithm announced in proposition 2. In section 8, we introduce the elementary combinatorial objects that the algorithm will manipulate. Roughly speaking, these elementary combinatorial objects are 6-uples of integers and binary symbols, which encode the positions of the N-segments of $(f,K,\mathcal{R})$. In section 9, we define some elementary operations on these elementary combinatorial objects, and we prove that these elementary operations are algorithmic. Lastly, in section 10, we describe each step of the algorithm. Theorem 2 follows from properties (ii'), proposition 1 and 2.

2. **Smale diffeomorphisms of surfaces: basic properties.**

Recall that a C^1-diffeomorphism f of a manifold M is called a *Smale diffeomorphism* if there exists a neighbourhood U of f in $\mathrm{Diff}^1(M)$, such that every $g \in U$ is topologically conjugate to f. In this section, we will state some classical properties of Smale diffeomorphisms of compact surfaces. These basic properties will be used (sometimes tacitely) all along the article.

Let f be a C^1-diffeomorphism of a manifold M. Recall that the *non-wanderring set* $\Omega(f)$ is made of the points $x \in M$ such that, for every neighbourhood U of x, there exists an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The non-wanderring set $\Omega(f)$ is said to be *hyperbolic* if there exists a splitting $T_{\Omega(f)}M = E^s \oplus E^u$, a constant $\lambda > 1$, and a riemannian metric $\|.\|$ on M such that:
— E^s and E^u are continuous subbundles of $T_{\Omega(f)}M$ which are invariant under df,
— for every $x \in \Omega(f)$, and every $v \in E^s_x$, we have $\|df_x.v\| \leq \lambda\|v\|$,
— for every $x \in \Omega(f)$, and every $v \in E^u_x$, we have $\|df_x^{-1}.v\| \leq \lambda\|v\|$.
If the non-wanderring set $\Omega(f)$ is hyperbolic, then, for every point x in $\Omega(f)$, the set $W^s(x) := \{y \in M \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\}$ and the set $W^u(x) := \{y \in S \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\}$ are C^1-manifolds, injectively immersed in M, called respectively *stable* and *unstable manifold of x*.

The study of Smale diffeomorphisms relies on the following characterization:

Theorem 3 (Robbin, Robinson, Mañé, [7], [5])**.** *A C^1-diffeomorphism f of a compact manifold M is a Smale diffeomorphism if and only if:*
— the non-wanderring set $\Omega(f)$ is hyperbolic, the periodic points are dense in $\Omega(f)$,
— for every $x, y \in \Omega(f)$, $W^s(x)$ is transverse to $W^u(y)$.

From now on, we consider a Smale diffeomorphism f of a compact surface S. We recall that the surface S is assumed to be orientable, and that the diffeomorphism f is assumed to be orientation-preserving.

A *basic piece* of f is a maximal compact invariant transitive subset of the non-wanderring set $\Omega(f)$. It was proved by S. Smale that f has finitely many basic pieces, that these basic pieces are pairwise disjoined, and that $\Omega(f)$ is the union of these basic pieces (see [8]).

Remark 2. *Using some classical techniques, one can prove that the classification of Smale diffeomorphisms of compact surfaces is equivalent to the classification of Smale diffeomorphism of compact surfaces whose non-wanderring set is totally discontinuous (see* [2, section 2.3]*). This is the reason why we will focus our attention on totally discontinuous basic pieces of f.*

Now, we consider a *non-trivial saddle basic piece* K of f, *that is* a totally discontinuous basic piece of f which is not a periodic orbit. The properties below are all classical:
1. Since S is a surface, and since K is neither a sink nor a source, the stable direction E^s_x and the unstable direction E^u_x are one-dimensional for every $x \in K$.
2. K is homeomorphic to a Cantor set.
3. Let $x \in K$. The *stable manifold* $W^s(x)$ is a one-dimensional manifold, diffeomorphic to the real line, injectively immersed in the surface S. Similarly, the *unstable manifold* $W^u(x)$ is a one-dimensional manifold, diffeomorphic to the real line, injectively immersed in S, and transversal to $W^s(x)$. Moreover, we have $W^s(f(x)) = f(W^s(x))$ and $W^u(f(x)) = f(W^u(x))$.
4. The set $W^s(K) = \bigcup_{x\in K} W^s(x)$ is a one-dimensional lamination. Similarly, the set $W^u(K) = \bigcup_{x\in K} W^u(x)$ is a one-dimensional lamination, transversal to $W^s(K)$. Clearly, we have $f(W^s(K)) = W^s(K)$ and $f(W^u(K)) = W^u(K)$. Moreover, we have $K = W^s(K) \cap W^u(K)$.
5. Let $x \in K$. A *stable separatrix* of x is a connected component of $W^s(x) \setminus \{x\}$. A stable separatrix of x is said to be *free* if it does not intersect lamination $W^u(K)$. If W^s is a stable separatrix of x which is not free, then it is easy to prove that the orbit of W^s under f is dense in the lamination $W^s(K)$ (see [2, lemme 6.1.2]).
Similarly, an *unstable separatrix* of x is a connected component of $W^u(x) \setminus \{x\}$. An unstable separatrix of x is said to be *free* if it does not intersect $W^s(K)$. If W^u is an unstable separatrix of x which is not free, then the orbit of W^u is dense in the lamination $W^u(K)$.
6. A *stable interval* (for the basic piece K) is a connected subset of a leaf of the lamination $W^s(K)$. A *stable segment* is a compact stable interval. If the points a and b lie on the same leaf of the lamination $W^s(K)$, then we denote by $[a, b]^s$ the stable segment whose ends are a and b. The notions of *unstable intervals* and *unstable segments*, and the notation $[a, b]^u$ are defined similarly.
The hyperbolicity of the non-wandering set $\Omega(f)$ implies that we can find a riemanian metric on S such that:
– for every stable interval I, we have $\text{length}(f(I)) \leq \lambda.\text{length}(I)$,
– for every unstable interval J, we have $\text{length}(f(J)) \geq \lambda^{-1}.\text{length}(J)$.

7. Consider a point $x \in K$. Since K is homeomorphic to a Cantor set (see property 2), the point x is not isolated in K. Moreover, for every stable interval I such that $x \in \text{int}(I)$, the point x is not isolated in $I \cap K$. Similarly, for every unstable interval J such that $x \in \text{int}(J)$, the point x is not isolated in $J \cap K$. In particular, if $x \in K$, then at least one of the two stable (resp. unstable) separatrices of x is not free.

3. Markov partitions and geometrical types.

The aim of this section is to define the notion of *geometrized Markov partition* of a basic piece, and the notion of *geometrical type* of a geometrized Markov partition.

3.1. Markov partitions.

We consider a Smale diffeomorphism f of a compact surface S, and a non-trivial saddle basic piece K of f.

Definition (rectangle). *Let Q be a subset of the surface S. We say that Q is a* rectangle *(for the basic piece K) if there exists an homeomorphism $h : [0,1]^2 \to Q$ such that (see figure 1):*
– $Q \cap W^s(K) = h([0,1] \times F^s)$, where F^s is a closed subset of $[0,1]$ and $\{0,1\} \subset F^s$,
– $Q \cap W^u(K) = h(F^u \times [0,1])$, where F^u is a closed subset of $[0,1]$ and $\{0,1\} \subset F^u$.

Let Q be a rectangle (for the basic piece K). The *stable sides* of Q are the connected components of $\partial Q \cap W^s(K)$. The *unstable sides* of Q are the connected components of $\partial Q \cap W^u(K)$. The union of the two stable sides of Q is denoted by $\partial^s Q$. The union of the two unstable sides of Q is denoted by $\partial^u Q$.

A *stable cross bar* of the rectangle Q is a connected component of $W^s(K) \cap Q$. An *unstable cross bar* of Q is a connected component of $W^u(K) \cap Q$. Observe that the ends of every stable (resp. unstable) cross bar of Q lie in the unstable (resp. stable) sides of Q.

An *horizontal subrectangle* of Q is a rectangle H, included in Q, such that the unstable sides of H are included in the unstable sides of Q. A *vertical subrectangle* of Q is a rectangle V, included in Q, such that the stable sides of V are included in the stable sides of Q (see figure 1)

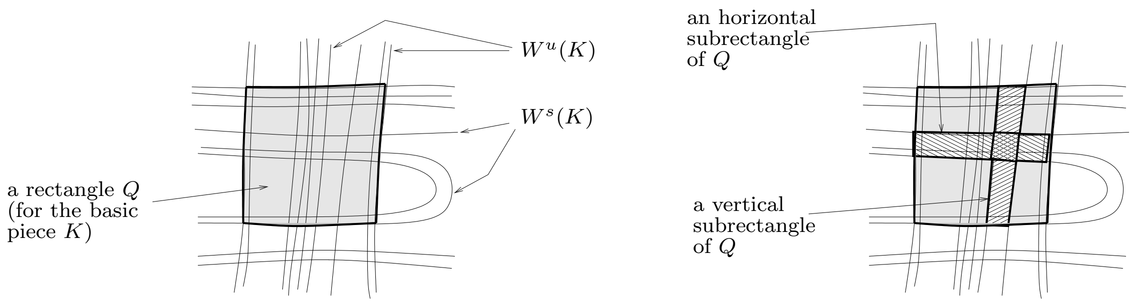

FIGURE 1. A rectangle for the basic piece K

Definition (Markov partition). *A* Markov partition $\mathcal{R}$ *of the basic piece K is a finite collection of pairwise disjoined rectangles such that (see figure 2):*
— K is included in the union R of the rectangles of $\mathcal{R}$,
— $f(\partial^s R) \subset \partial^s R$ and $f(\partial^u R) \supset \partial^u R$, where $\partial^s R$ (resp. $\partial^u R$) is the union of the stable (resp. unstable) sides of all the rectangles of $\mathcal{R}$,
— each connected component of $f(R) \cap R$ has non-empty interior.

Definition (geometrized Markov partition)**.** *A* geometrized Markov partition *is a Markov partition $\mathcal{R}$ endowed with:*
— an order of the rectangles of $\mathcal{R}$,
— an orientation of the unstable cross bars of each rectangle of $\mathcal{R}$.

Remarks 1. *Let $\mathcal{R}$ be a geometrized Markov partition.*
(i) *Since the rectangles of $\mathcal{R}$ are ordered, we can denote them by $R_1, \dots, R_n$.*
(ii) *Let $i \leq n$. The orientation of the unstable cross bars of the rectangle R_i induces an orientation of the stable cross bars of the rectangle R_i (via the two-dimensional orientation of the surface S).*
(iii) *Let $i \leq n$. If Q is a subrectangle of the rectangle R_i, then every stable (resp. unstable) cross bars of Q is included in a stable (resp. unstable) cross bar of R_i. As a consequence, the orientation of the stable (resp. unstable) cross bars of the rectangle R_i, induces an orientation of the stable (resp. unstable) cross bars of every subrectangle of R_i.*

The following remark plays a fundamental role in the defintion of the geometrical type of a Markov partition:

Remark 3. *Let R_i and R_j be two rectangles of a Markov partition of the basic piece K. It follows immediately from definition that every connected component of $f(R_i) \cap R_j$ is:*
— a vertical subrectangle of the rectangle R_j,
— the image under f of an horizontal subrectangle of the rectangle R_i.

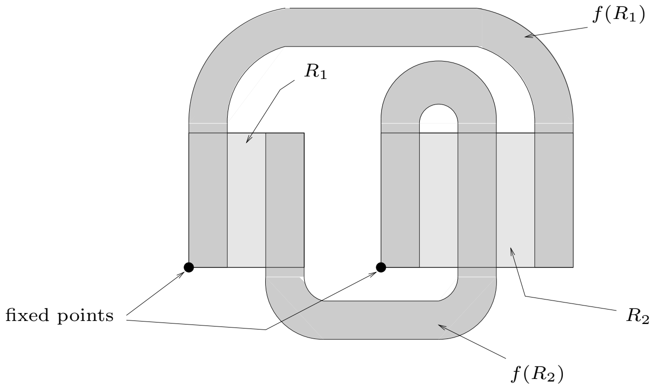

FIGURE 2. A Markov partition

3.2. Geometrical types.

Definition (geometrical type)**.** *A* geometrical type $T = (n,\, h,\, v,\, \Phi,\, \varepsilon)$ *is 5-uple made of:*
— a positive integer n,
— two n-uples of positive integers $h = (h_1, \dots, h_n)$ and $v = (v_1, \dots, v_n)$ which satisfy the equality $\sum h_i = \sum v_i$,
— a bijection $\Phi : \{(i,j) \mid i \leq n,\ j \leq h_i\} \to \{(k,l) \mid k \leq n,\ l \leq v_k\}$,
— a function $\varepsilon : \{(i,j) \mid i \leq n,\ j \leq h_i\} \to \{+,-\}$.

Now, we will associate a *geometrical type* to any geometrized Markov partition.

Definition (geometrical type of a Markov partition)**.** *Let $\mathcal{R} = \{R_1, \dots, R_n\}$ be a geometrized Markov partition of the non-trivial saddle basic piece K. The* geometrical type $T = (n, h, v, \Phi, \varepsilon)$ of $\mathcal{R}$ *is defined as follows:*
— n is the number of rectangles of $\mathcal{R}$.
— $h = (h_1, \dots, h_n)$ where h_i is the number of connected components of the intersection $(R_1 \cup \dots \cup R_n) \cap f(R_i)$. The connected components of $(R_1 \cup \dots \cup R_n) \cap f(R_i)$ are the images under f of some horizontal subrectangles of the rectangle R_i (remark 3) ; we denote these horizontal subrectangles by $H_i^1, \dots, H_i^{h_i}$, the order being induced by the orientation of the vertical cross bars of R_i.
— $v = (v_1, \dots, v_n)$ where v_k is the number of connected components of the intersection $R_k \cap f(R_1 \cup \dots \cup R_n)$. The connected components of $R_k \cap f(R_1 \cup \dots \cup R_n)$ are some vertical subrectangles of the rectangle R_k (remark 3) ; we denote these vertical subrectangles by $V_k^1, \cdots, V_k^{v_k}$, the order being induced by the orientation of the horizontal cross bars of R_k.
— $\Phi(i, j) = (k, l)$ if the diffeomorphism f maps the horizontal subrectangle H_i^j to the vertical subrectangle V_k^l.
- $\varepsilon(i, j) = +$ if and only if the diffeomorphism f maps the orientation of the unstable cross bars of H_i^j to the orientation of the unstable cross bars of V_k^l (by definition, the orientation of the unstable cross bars of the subrectangle H_i^j is the orientation of the unstable cross bars of the rectangle R_i, and the orientation of the unstable cross bars of the subrectangle V_k^l is the orientation of the unstable cross bars of the rectangle R_k; see item (iii) of remark 1)

Example 1. *Consider the Markov partition of figure 2. Assume that the stable (resp. unstable) cross bars of the rectangles of this Markov partition are oriented from left to right (resp. from bottom to top). Then, the geometrical type of this Markov partition is $T = (n, h, v, \Phi, \varepsilon)$ where $n = 2$, $h = (2, 3)$, $v = (2, 3)$, and*

$\Phi(1,1) = (1,1)$	$\varepsilon(1,1) = +$	$\Phi(2,1) = (2,1)$	$\varepsilon(2,1) = +$
$\Phi(1,2) = (2,3)$	$\varepsilon(1,2) = -$	$\Phi(2,2) = (2,2)$	$\varepsilon(2,2) = -$
		$\Phi(2,3) = (1,2)$	$\varepsilon(2,3) = +$

4. **Boundary leaves and special points.**

In this section, we consider a Smale diffeomorphism f of a compact surface, and a non-trivial saddle basic piece K of f. The purpose of the section is to define some particular leaves of the laminations $W^s(K)$, $W^u(K)$, called *boundary leaves*, and some particular points of K, called *special points*.

4.1. **Boundary leaves, s and u-boundary points.**

Definition (boundary leaves, s and u-boundary points)**.** *A point $x \in K$ is said to be a* s-boundary point *if there exists a point $y \in W^u(x)$ such that $y \neq x$ and $]x, y[^u \cap W^s(K) = \emptyset$ (see figure 3). If x is a s-boundary point of K, then the stable manifold $W^s(x)$ is said to be a* boundary leaf of the lamination $W^s(K)$.

A point $x \in K$ is said to be a u-boundary point *if there exists a point $y \in W^s(x)$ such that $y \neq x$ and $]x, y[^s \cap W^u(K) = \emptyset$ (see figure 3). If x is a u-boundary point of K, then the unstable manifold $W^u(x)$ is said to be a* boundary leaf of $W^u(K)$.

It is quite easy to see that the stable (resp. unstable) sides of the rectangles of every Markov partition of K are included in the boundary leaves of the lamination $W^s(K)$ (resp. $W^u(K)$). This is the reason why these leaves will play a very important role in the sequel.

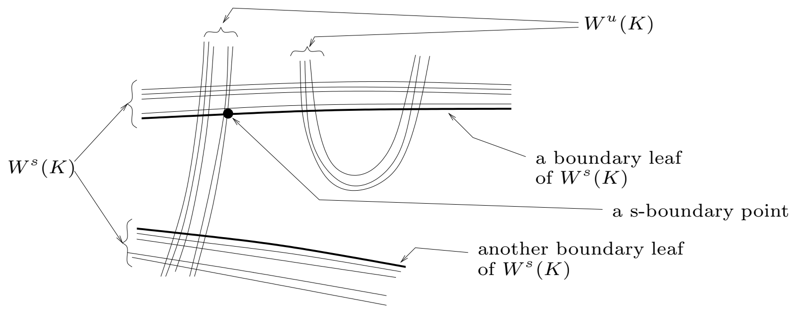

FIGURE 3.

The proposition below is essentially due to S. Newhouse and J. Palis (see [6]) ; a detailed proof of this proposition can be found in [2, propositon 2.1.1] (the proof relies on the so-called "local product structure of K").

Proposition 3 (Newhouse, Palis). **(i)** *If x is a s-boundary point and $x' \in W^s(x) \cap K$, then x' is a s-boundary point. In other words, every point of K which lies on a boundary leaf of the lamination $W^s(K)$ is a s-boundary point.*

(ii) *There exists at least one boundary leaf in the lamination $W^s(K)$.*

(iii) *Every boundary leaf of the lamination $W^s(K)$ is periodic under f,* i.e. *every boundary leaf of $W^s(K)$ is the stable manifold of a periodic s-boundary point of K.*

(iv) *If x is a periodic s-boundary point of K, then one of the two unstable separatrices of x is a free separatrix.*

(v) *There are only finitely many boundary leaves in the lamination $W^s(K)$. In other words, there are only finitely many periodic s-boundary points in K.*

Of course, there is an an analog of proposition 3 concerning u-boundary points of K, and boundary leaves of the lamination $W^u(K)$.

4.2. **Special points of K.**

Definition (special point). *A point $z \in K$ is said to be a* special point of K *if there exists a periodic s-boundary point x of K and a periodic u-boundary point y of K, such that $z \in W^s(x) \cap W^u(y)$, and such that $]x, z]^s \cap]y, z]^u = \{z\}$.*

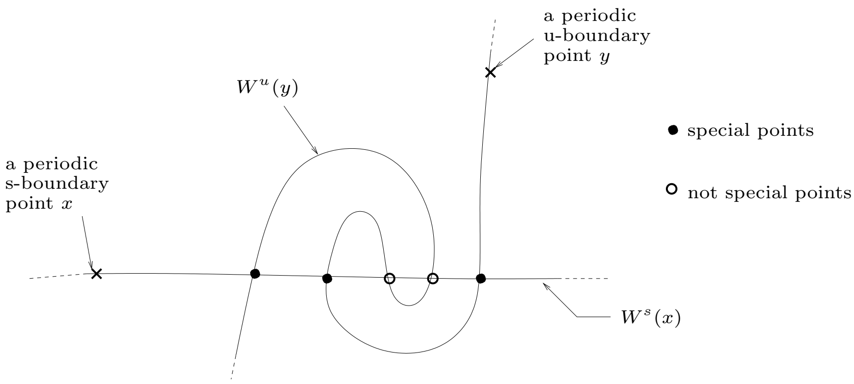

FIGURE 4. Special points and non-special points.

The special points of K will play a fundamental role in section 5. The main properties of these points are stated in propositions 4 and 5 below.

Proposition 4. *There are only finitely many orbits of special points in K.*

Lemma 1. *Let x be a periodic s-boundary point of K, and y be a periodic u-boundary point of K. Let W^s be a stable separatrix of x, and W^u be an unstable separatrix of y. Then, there are finitely many orbits of special points in $W^s \cap W^u$.*

Proof of lemma 1. We consider a positive integer q such that $f^q(W^s) = W^s$ and $f^q(W^u) = W^u$, and a point z_0 in $W^s \cap W^u$.

On the one hand, for every point $z \in W^s$, there exists an integer k such that $f^k(z) \in]f^q(z_0), z_0]^s$ (since the stable segment $[z_0, f^q(z_0)]^s$ is a fundamental domain of W^s for the action of f^q). On the other hand, if $f^k(z) \in]f^q(z_0), z_0]^s \cap W^u$ and $f^k(z) \notin]f^q(z_0), z_0]^s \cap [y, f^q(z_0)]^u$, then $f^k(z)$ is not a special point of K (since the point $f^q(z_0)$ lies in $]x, f^k(z)[^s \cap]y, f^k(z)[^u$). Thus, we have the following statement: for every special point $z \in W^s \cap W^u$, there exists an integer k such that $f^k(z) \in [f^q(z_0), z_0]^s \cap [y, f^q(z_0)]^u$.

Now, we observe that $[f^q(z_0), z_0]^s \cap [y, f^q(z_0)]^u$ is a finite set (since it is the intersection of two transversal segments). This completes the proof. △

Proof of proposition 4. The result follows immediately from lemma 1 and from item (v) of proposition 3. △

Proposition 5. *Let g be a Smale diffeomorphism of a compact surface and L be a non-trivial saddle basic piece of g. Recall that we denote by $\Delta(g, L)$ the canonical invaraint neighbourhood of L defined in* [2] *(see subsection 1.1). Assume that there exists an homeomorphism $h : \Delta(f, K) \to \Delta(g, L)$ such that $h \circ f = g \circ h$. Then, z is a special point of K if and only if $h(z)$ is a special point of L.*

Morally, proposition 5 tells us that the (finite) set of the special points of f is a conjugacy invariant of f ; this is a crucial point in the strategy of our proof.

Proof of proposition 5. Let $x \in K$ be a periodic s-boundary point, $y \in K$ be a periodic u-boundary point, and $z \in W^s(x) \cap W^u(y)$ be a special point. Since h is a conjugacy, h maps the laminations $W^s(K)$, $W^u(K)$ to the lamination $W^s(L)$, $W^u(L)$. This implies that h maps the boundary leaves of $W^s(K)$, $W^u(K)$ to the boundary leaves of $W^s(L)$, $W^u(L)$. Since h maps periodic points to periodic points, we see that $h(x)$ is a periodic s-boundary point of L, and $h(y)$ is a periodic u-boundary point of L. Finally, we have $]h(x), h(z)]^s \cap]h(y), h(z)]^u = h(]x, z]^s \cap]y, z]^u) = h(z)$. As a consequence, $h(z)$ is a special point of L. △

5. Construction of the set of geometrical types $\mathcal{T}(f, K, p)$.

Let f be a Smale diffeomoprhism of a compact surface, and K be a non-trivial saddle basic piece of f. The purpose of this section is to construct the set of geometrical types $\mathcal{T}(f, K, p)$ (for every p large enough). In subsection 5.1, we consider a special point z of K, and we construct a Markov partition $\mathcal{R}(z, p)$ of K. In subsection 5.2, we define the set of geometrical types $\mathcal{T}(f, K, p)$, and we prove that set is a complete invariant for topological conjugacy.

5.1. Construction of the Markov partition $\mathcal{R}(z, p)$.

In this subsection, we consider a Smale diffeomorphism f of a compact surface, a non-trivial saddle basic piece K of f, and a special point z of K (see definition). For every p large enough, we will construct a Markov partition $\mathcal{R}(z, p)$ of K. The construction, inspired by [3, exposé 10] and [2, chapter 4], is divided in several steps.

If $\mathcal{F}$ is a family of intervals, then we denote by $\cup\mathcal{F}$ the union of the elements of $\mathcal{F}$. A family $\mathcal{I}$ of intervals is said to be *positively invariant*, if $f(\cup\mathcal{I}) \subset (\cup\mathcal{I})$. Similarly, a family $\mathcal{J}$ of intervals is said to be *negatively invariant*, if $f^{-1}(\cup\mathcal{J}) \subset (\cup\mathcal{J})$.

Step 0. Construction of the family of unstable intervals $\mathcal{J}_0(z)$.

Definition of the family of unstable intervals $\mathcal{J}_0(z)$. Since z is a special point, z lies on the unstable manifold of a periodic u-boundary point y. We consider the unstable interval $J_0(z) :=]y, z]^u$, and the family of unstable intervals

$\mathcal{J}_0(z) := \{J_0(z),\ f^{-1}(J_0(z)), \dots,\ f^{-(2q-1)}(J_0(z))\}$ where q is the period of y

Remarks 2. *The period of each unstable separatrix of the point y is either q or $2q$. As a consequence, we have $f^{-2q}(J_0(z)) \subset int(J_0(z))$. In particular, this inclusion implies that:*
(i) *the family of unstable intervals $\mathcal{J}_0(z)$ is negatively invariant,*
(ii) *there are points of K in the interior of the interval $J_0(z)$.*

Step 1. Construction of the family of stable segments $\mathcal{I}_1(z)$.

Lemma 2. *Let x be a point in K, and W^s be one of the two stable separatrices of x. If the separatrix W^s is not free, then W^s intersects $\cup\mathcal{J}_0(z)$.*

Proof. On one hand, the lamination $W^s(K)$ intersects (transversaly) the interior of the unstable interval $J_0(z)$ (remark 2). On the other hand, the orbit of the separatrix W^s is dense in $W^s(K)$ (see property 5 of section 2). As a consequence, there exists an integer $k \geq 0$ such that $f^k(W^s)$ intersects $J_0(z)$. As a further consequence, W^s intersects $f^{-k}(J_0(z)) \subset (\cup\mathcal{J}_0(z))$. △

Construction of the segment $I_1(x, z)$. Let x be a periodic s-boundary point of K, and let W_1^s and W_2^s be the stable separatrices of x. For $i = 1, 2$, we consider the point x_i defined as follows:
— if W_i^s is not a free separatrix, then x_i is the unique point of $W_i^s \cap (\cup\mathcal{J}_0(z))$ such that the stable interval $]x, x_i[^s$ does not intersect $\cup\mathcal{J}_0(z)$;
— if W_i^s is a free separatrix, then $x_i = x$.
Then, we consider the stable segment $I_1(x, z) := [x_1, x_2]^s$ (see figure 5).

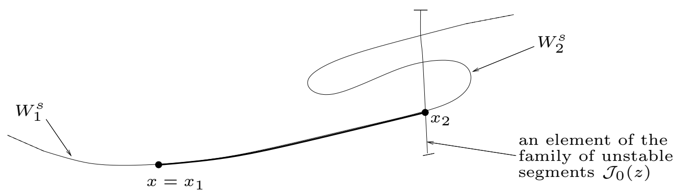

FIGURE 5. Construction of the segment $I_1(x, z)$ (the case where W_1^s is a free separatrix, but not W_2^s)

And then, we consider the finite family of stable segments

$$\mathcal{I}_1(z) := \{I_1(x, z) \mid x \text{ is a periodic s-boundary point of } K\}$$

Remarks 3. *By construction, the segment $I_1(x, z)$ is included in the stable manifold of the periodic s-boundary point x, and the periodic s-boundary point x lies in the segment $I_1(x, z)$. Moreover, $I_1(x, z)$ is non-trivial segment (*i.e. $I_1(x, z) \neq \{x\}$*), since at least one of the two unstable separatrices of the point x is not free (see property (vii) in section 2). Lastly, the both ends of the segment $I_1(x, z)$ are u-boundary points.*

Consequently, all the elements of the family of stable segments $\mathcal{I}_1(z)$ are included in the boundary leaves of $W^s(K)$, and all the periodic s-boundary points of K lie in $\cup\mathcal{I}_1(z)$. Moreover, $\mathcal{I}_1(z)$ is positively invariant (this follows from the negative invariance of $\mathcal{J}_0(z)$).

Step 1'. Construction of the family of unstable segments $\mathcal{J}_1(z)$. The proof of lemma 3 below is similar to the proof of lemma 2

Lemma 3. *Let y be a point in K, and W^u be one of the two unstable separatrices of y. If the separatrix W^u is not free, then W^u intersects $\cup\mathcal{I}_1(y)$*

Construction of the segment $J_1(y,z)$. Let y be a periodic u-boundary point of K, and let W_1^u, W_2^u the unstable separatrices of y. For $i = 1, 2$, we consider the point y_i defined as follows:
— if W_i^u is not free, then y_i is the unique point of $W_i^u \cap (\cup\mathcal{I}_1(z))$ such that $]y, y_i[^u \cap (\cup\mathcal{I}_1(z)) = \emptyset$;
— if W_i^u is free, then $y_i = y$.
Then, we consider the unstable segment $J_1(y,z) := [y_1, y_2]^u$.

Then, we consider the finite family of unstable segments

$$\mathcal{J}_1(z) := \{J_1(y,z) \mid y \text{ is a periodic u-boundary point of } K\}$$

Remarks 4. *By construction, the segment $J_1(y,z)$ is included in the unstable manifold of the periodic u-boundary point y, and the point y lies in the segment $J_1(y,z)$. Moreover, $J_1(y,z)$ is a non-trivial segment. Lastly, the both ends of the segment $J_1(y,z)$ are s-boundary points.*

Consequently, all the elements of the family of unstable segments $\mathcal{J}_1(z)$ are included in the boundary leaves of $W^u(K)$, and all the periodic u-boundary points of K lie in $\cup\mathcal{J}_1(z)$. Moreover, $\mathcal{J}_1(z)$ is negatively invariant.

Step 2. Construction of the family of externally isolated stable segments $\mathcal{I}_2(z)$.

Definition (externally isolated stable segment)**.** *The stable segment $[a,b]^s$ is said to be* externally isolated at b *if there exists a point b' such that $[a,b]^s \subset [a,b'[^s$ and $[a,b]^s \cap K = [a,b'[^s \cap K$. Similarly, the stable segment $[a,b]^s$ is said to be* externally isolated at a *if there exists a point a' such that $[a,b]^s \subset]a',b]^s$ and $[a,b]^s \cap K =]a',b]^s \cap K$. The stable segment $[a,b]^s$ is said to be* externally isolated *if it is externally isolated at a and b.*

For every periodic s-boundary point x, we will construct a stable segment $I_2(x,z)$, such that $I_2(x,z) \subset I_1(x,z)$ and such that $I_2(x,z)$ is externally isolated.

Construction of the segment $I_2(x,z)$. Let x be a periodic s-boundary point. Let us assume, for instance, that the stable segment $I_1(x,z) = [x_1,x_2]^s$ is externally isolated at x_1, but not externally isolated at x_2. Since x_2 is a u-boundary point (remarks 3), there exists a point $x_2' \in W^s(x)$ such that $]x_2', x_2[^s \cap K = \emptyset$. Since $x_1 \neq x_2$ (remarks 3), and since $[x_1,x_2]^s$ is not externally isolated at x_2, the point x_2' is necessarily in the segment $[x_1,x_2]^s$. Lastly, since K is compact, we can suppose that x_2' is in K. The segment $[x_1, {x_2'}^s]$ will be denoted by $I_2(x,z)$ (figure 6).

The segment $I_2(x,z)$ can also be characterized as follows:

Abstract characterization of the segment $I_2(x,z)$. Let x be a periodic s-boundary point of K. The segment $I_2(x,z)$ satisfies the three following properties: (i) $I_2(x,z) \subset I_1(x,z)$; (ii) the ends of $I_2(x,z)$ are in K ; (iii) $I_2(x,z)$ is externally isolated. Moreover, $I_2(x,z)$ is the biggest of all the segments which satisfies properties (i), (ii) and (iii).

Lemma 4. *The periodic s-boundary point x lies in the stable segment $I_2(x,z)$.*

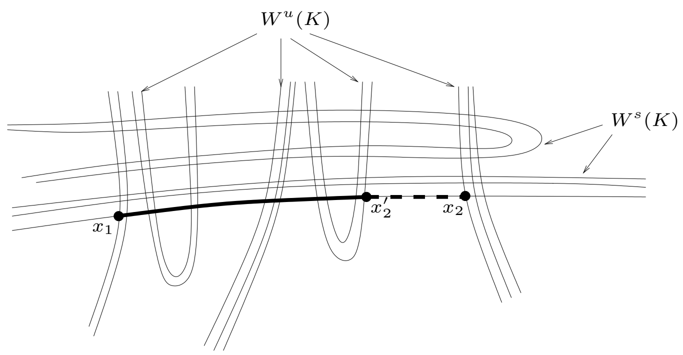

FIGURE 6. Construction of the segment $I_2(x,z)$

Proof. Let $I_1(x,z) = [x_1, x_2]^s$. Assume, for instance, that $I_1(x,z)$ is externally isolated at x_1, but is not externally isolated at x_2. Then, $I_2(x,z) = [x_1, x_2']^s$, where $]x_2', x_2[^s \cap K = \emptyset$. As a consequence, it suffices to prove that the points x and x_2 are not equal. This follows from the construction of the segment $I_1(x,z)$: if x is one of the ends of the segment $I_2(x,z)$, then one of the stable separatrices of x is free, and the stable segment $I_1(x,z)$ is externally isolated at x. △

Then, we consider the family of stable segments

$$\mathcal{I}_2(z) := \{I_2(x,z) \mid x \text{ is a periodic s-boundary point of } K\}$$

By construction, $\mathcal{I}_2(z)$ is a family of externally isolated segments, which are included in the boundary leaves of $W^s(K)$. By lemma 4, every periodic s-boundary point of K lies in $\cup\mathcal{I}_2(z)$. Moreover, the family $\mathcal{I}_2(z)$ is positively invariant (since $\mathcal{I}_1(z)$ is positively invariant).

Step 2'. Construction of the family of externally isolated unstable segments $\mathcal{J}_2(z)$. The definition of the notion of *externally isolated unstable segment* is similar to definition . The construction of the family of unstable segments $\mathcal{J}_2(z)$ is similar to the construction of the family of stable segments $\mathcal{I}_2(z)$.

Step 3. Construction of the saturated family of stable segments $\mathcal{I}_3(z)$.

Definition (unstable arch)**.** *An* unstable arch *is an unstable segment γ, such that the both ends of γ lie in K, and such that int$(\gamma) \cap K = \emptyset$.*

Definitions (saturated family of stable segments)**.** *Let $\mathcal{I}$ be a family of stable segments.*

A point $b \in K$ is said to be connected to $\mathcal{I}$ by an unstable arch *if b is one of the ends of an unstable arch $\gamma = [a,b]^u$, such that $a \in (\cup\mathcal{I})$.*

The family of stable segments $\mathcal{I}$ is said to be saturated *if it satisfies the following property: if b is connected to $\mathcal{I}$ by an unstable arch, then $b \in \cup\mathcal{I}$.*

Construction of the segment $I_3(x,z)$. For every periodic s-boundary point $x \in K$, we consider the stable segment $I_3(x,z)$ characterized by the following properties: (i) $I_2(x,z)$ is included in $I_3(x,z)$; (ii) if $b \in W^s(x)$ is connected to $\mathcal{I}_2(z)$ by an unstable arch, then $b \in I_3(x,z)$; (iii) $I_3(x,z)$ is the smallest of all the stable segments which verify (i) and (ii).

Then, we consider the family of stable segments

$$\mathcal{I}_3(z) := \{I_3(x,z) \mid x \text{ is a periodic s-boundary point of } K\}$$

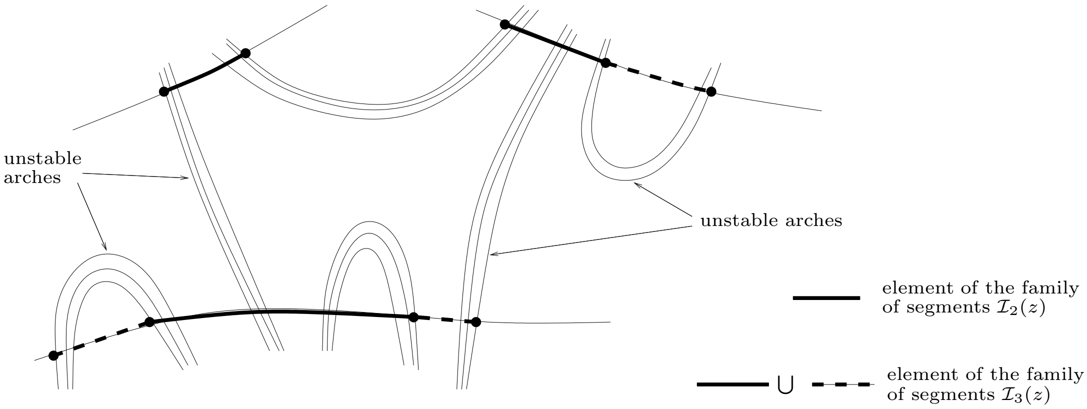

FIGURE 7. The family of stable segments $\mathcal{I}_3(z)$

Definition (adapted family of stable segments)**.** *Let $\mathcal{I}$ be a family of stable segments. Assume that every element of $\mathcal{I}$ is an externally isolated segment which is included in a boundary leaf of $W^s(K)$, and assume that every periodic s-boundary point of K lies in $\cup\mathcal{I}$, and assume that $\mathcal{I}$ positively invariant and saturated. Then, we say that $\mathcal{I}$ is an* adapted family of stable segments.

Lemma 5. *$\mathcal{I}_3(z)$ is an adapted family of stable segments.*

Proof. The only true difficulty is to prove that $\mathcal{I}_3(z)$ is saturated. This follows from the properties of the family $\mathcal{I}_2(z)$ and from [2, lemma 4.4.1, item (i)]. △

Step 3'. Construction of the saturated family of unstable segments $\mathcal{J}_3(z)$. .

If we exchange the stable and the unstable directions in definitions , and , we obtain the definition of a *stable arch*, and the definition of a *saturated family of unstable segments*, and the definition of an *adapted family of unstable segments.* Then, proceeding as in step 3, we can construct an adapted family of unstable segments $\mathcal{J}_3(z)$.

Step 4. Construction of the integer $p_{min}(z)$, of the family of stable segments $\mathcal{I}_4(z,p)$, and of the family of unstable segments $\mathcal{J}_4(z,p)$.

Definitions (rail, equivalence class of rails)**.** *Let $\mathcal{I}$ be a family of stable segments.*
• *A* rail leaning on *$\mathcal{I}$ is a non-trivial unstable segment γ, such that the both ends of γ lie in $\cup\mathcal{I}$, and such that γ is not an unstable arch (*i.e. *such that $int(\gamma)\cap K \neq \emptyset$).*
• *Two rails γ_1, γ_2 leaning on $\mathcal{I}$ are said to be* equivalent*, if γ_1 and γ_2 are the unstable sides of a rectangle R whose stable sides are included in $\cup\mathcal{I}$.*
• *If $\mathcal{C}$ is an equivalence class $\mathcal{C}$ of rails leaning on $\mathcal{I}$, then it is easy to see that the elements of $\mathcal{C}$ are the unstable cross bars of a rectangle $R_\mathcal{C}$ (*[2, proposition 4.2.1]*) ; the rectangle $R_\mathcal{C}$ is the* domain *of the equivalence class $\mathcal{C}$ (see figure 8).*

Definition (rail, equivalence class of rails)**.** *Let $\mathcal{J}$ be a family of unstable segments.*
• *An* rail leaning on *$\mathcal{J}$ is a non-trivial stable segment α, such that the both ends of α lie in $\cup\mathcal{J}$, and such that α is not a stable arch.*
• *Two rails α_1, α_2 leaning on $\mathcal{J}$ are said to be* equivalent*, if α_1 and α_2 are the stable sides of a rectangle R whose unstable sides are included in $\cup\mathcal{J}$.*

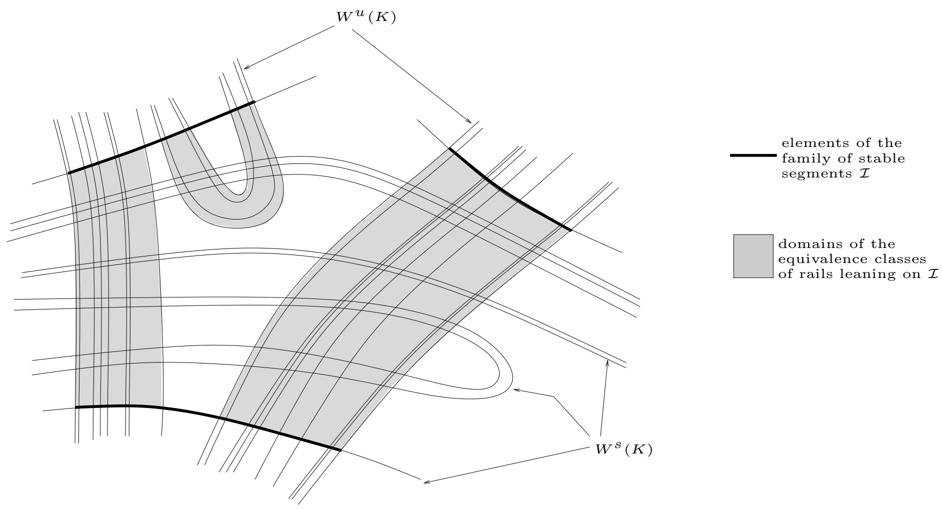

FIGURE 8. A family of stable segments $\mathcal{I}$, and the equivalence classes of rails leaning on $\mathcal{I}$

• *If $\mathcal{C}$ is an equivalence class of rails leaning on $\mathcal{J}$, then the elements of $\mathcal{C}$ are the stable cross bars of a rectangle $R_{\mathcal{C}}$; the rectangle $R_{\mathcal{C}}$ is the* domain *of the equivalence class $\mathcal{C}$.*

Let $\mathcal{I}$ be an adapted family of stable segments, and $\mathcal{J}$ be an adapted family of unstable segments. It is easy to prove that there exists an integer p, such that the unstable sides of the domains of the equivalence class of rails leaning on $\mathcal{I}$ are included in $f^p(\cup\mathcal{J})$, and such that the stable sides of the equivalence classes of rails leaning on $f^p(\mathcal{J})$ are included in $\cup\mathcal{I}$ ([2, corollary 4.3.9]). This suggests the following definition:

Definition of the integer $p_{min}(z)$. We denote by $p_{min}(z)$ the smallest of all the integers p which satisfy the following properties: (i) the unstable sides of the domains of the equivalence classes of unstable rails leaning on $\mathcal{I}_3(z)$ are included in $f^p(\cup\mathcal{J}_3(z))$; (ii) the stable sides of the domains of the equivalence classes of stable rails leaning on $f^p(\mathcal{J}_3(z))$ are included in $\cup\mathcal{I}_3(z)$.

Notation . *Let $\mathcal{I}$ be a finite family of stable segments and $\mathcal{J}$ be a finite family of unstable segments.*
— We denote by $\mathbb{D}_{\mathcal{J}}(\mathcal{I})$ the family of stable segments whose elements are the connected components of $\cup\mathcal{I}$ minus the interiors of all the stable arches whose both ends lie in $\mathcal{J}$ (see figure 9).
— We denote by $\mathbb{D}_{\mathcal{I}}(\mathcal{J})$ the family of unstable segments whose elements are the connected components of $\cup\mathcal{J}$ minus the interiors of all the unstable arches whose both ends lie in $\mathcal{I}$.

Definition of the families of segments $\mathcal{I}_4(z,p)$ and $\mathcal{J}_4(z,p)$. For every $p \geq p_{min}(z)$, the family of stable segments $\mathcal{I}_4(z,p)$ and the family of unstable segments $\mathcal{J}_4(z,p)$ are defined by:

$$\mathcal{I}_4(z,p) := \mathbb{D}_{f^p(\mathcal{J}_3(z))}(\mathcal{I}_3(z)) \text{ and } \mathcal{J}_4(z,p) := \mathbb{D}_{\mathcal{I}_3(z)}(f^p(\mathcal{J}_3(z)))$$

Step 5. Definition of the Markov partition $\mathcal{R}(z,p)$. .

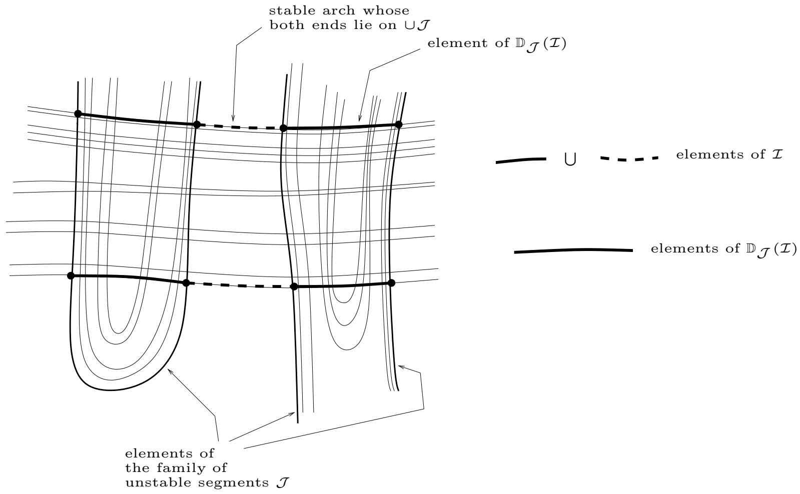

FIGURE 9. A family of stable segment $\mathcal{I}$, a family of unstable segments $\mathcal{J}$, and the family of stable segments $\mathbb{D}_{\mathcal{J}}(\mathcal{I})$

For every $p \geq p_{min}(z)$, the family of stable segments $\mathcal{I}_3(z)$ and the family of unstable segments $f^p(\mathcal{J}_3(z)$ satisfy the hypothesis of the following proposition:

Proposition 6. *Let $\mathcal{I}$ be an adapted family of stable segments, and $\mathcal{J}$ be an adapted family of unstable segments. Assume that the unstable sides of the domains of the equivalence classes of rails leaning on $\mathcal{I}$ are included in $\cup\mathcal{J}$, and assume that the stable sides of the domains of the equivalence classes of rails leaning on $\mathcal{J}$ are included in $\cup\mathcal{I}$. Then, there exists a Markov partition $\mathcal{R}$ of K, such that $\mathbb{D}_{\mathcal{J}}(\mathcal{I})$ is the family of the stable sides of the rectangles of $\mathcal{R}$, and $\mathbb{D}_{\mathcal{I}}(\mathcal{J})$ is the family of the unstable sides of the rectangles of $\mathcal{R}$.*

Sketch of the proof (see [2, theorem 4.3.3] **for a detailed proof).** Let $\mathcal{R}$ be the family of rectangles whose elements are the domains of the equivalence classes of rails leaning on $\mathbb{D}_{\mathcal{J}}(\mathcal{I})$. Let $\mathcal{R}'$ be the family of rectangles whose elements are the domains of the equivalence classes of stable rails leaning on $\mathbb{D}_{\mathcal{I}}(\mathcal{J})$. It is quite easy to see that $\mathcal{R}$ (resp. $\mathcal{R}'$) is made of a finite number of rectangles, and that these rectangles are disjoined. The hypothesis of proposition 6 implies that $\mathcal{R} = \mathcal{R}'$. It also imply that the elements of $\mathbb{D}_{\mathcal{J}}(\mathcal{I})$ are the stable sides of the rectangles of $\mathcal{R}$, and imply that the elements of $\mathbb{D}_{\mathcal{I}}(\mathcal{J})$ are the unstable sides of the rectangles of $\mathcal{R}' = \mathcal{R}$.

Let us denote by R be the union of the elements of $\mathcal{R}$. It is quite easy to prove that every point of K is on an rail leaning on $\mathbb{D}_{\mathcal{J}}(\mathcal{I})$; as a consequence, the basic piece K is included in R. Moreover, the positive invariance of the family $\mathcal{I}$ imply that we have $f(\partial^s R) \subset \partial^s R$, and the negative invariance of the family $\mathcal{J}$ imply that we have $f(\partial^u R) \supset \partial^u R$. As a consequence, $\mathcal{R}$ is a Markov partition of the compact set K. △

Definition of the Markov partition $\mathcal{R}(z,p)$. Let $p \geq p_{min}(z)$. Proposition 6 implies that there exists a Markov partition $\mathcal{R}(z,p)$ of K, such that the elements of $\mathcal{I}_4(z,p)$ are the stable sides of the rectangles of $\mathcal{R}(z,p)$, and such that the elements of $\mathcal{J}_4(z,p)$ are the unstable sides of the rectangles of $\mathcal{R}(z,p)$.

5.2. **Definition of the set of geometrical types $\mathcal{T}(f,K,p)$.** Using the construction of subsection 5.1, we are now going to define the set of geometrical types $\mathcal{T}(f,K,p)$. Let us begin by a remark:

Remark 4. *Let f_1, f_2 be Smale diffeomorphisms of compact surfaces, and K_1, K_2 be non-trivial saddle basic pieces of f_1, f_2. Assume that there exists an homeomorphism $h : \Delta(f_1, K_1) \to \Delta(f_2, K_2)$ which conjugates the restriction of f_1 and f_2. Let $\mathcal{R}_1$ be a Markov partition of the basic piece K_1. Then, $h(\mathcal{R}_1)$ is a Markov partition of the basic piece K_2 ; moreover, the Markov partitions $\mathcal{R}_1$ and $h(\mathcal{R}_1)$ have the same geometrical type.*
Particular case. *Let f be a Smale diffeomorphism of a compact surface, K be a non-trivial saddle basic piece of f, and $\mathcal{R}$ be a Markov partition of K. Then, $f(\mathcal{R})$ is a Markov partition of K ; moreover, the Markov partitions $\mathcal{R}$ and $f(\mathcal{R})$ have the same geometrical type.*

Lemma 6. *Let f_1, f_2 be Smale diffeomorphisms of compact surfaces, and K_1, K_2 be non-trivial saddle basic pieces of f_1, f_2. Assume that there exists an homeomorphism $h : \Delta(f_1, K_1) \to \Delta(f_2, K_2)$ which conjugates the restriction of f_1 and f_2. Let z_1 be a special point of K, and recall that $h(z_1)$ is a special point of K_2 (proposition 5). Then:*
— the integers $p_{min}(z_1)$ and $p_{min}(h(z_1))$ are equal,
— for every integer $p \geq p_{min}(z_1)$, we have $\mathcal{R}(h(z_1), p) = h(\mathcal{R}(z_1, p))$.

Proof. The lemma follows from the construction of the integer $p_{min}(z)$, from the costruction of the Markov partition $\mathcal{R}(z,p)$ in subsection 5.1, and from the fact that h maps $W^s(K_1)$ to $W^s(K_2)$, and maps $W^u(K_1)$ to $W^u(K_2)$. △

Corollary 1. *Let f be a Smale diffeomorphism of a compact surface, and K be a non-trivial saddle basic piece of f. For every special point $z \in K$, we have $p_{min}(f(z)) = p_{min}(z)$. Moreover, for every special point $z \in K$ and every integer $p \geq p_{min}(z)$, we have $\mathcal{R}(f(z), p) = f(\mathcal{R}(z, p))$.*

Proof. Take $f = f_1 = f_2 = h$ and $K = K_1 = K_2$ in lemma 6. △

Let K be a non-trivial saddle basic piece of a Smale diffeomorphism f.

Definition of the integer $p_{min}(f,K)$. Proposition 5 and corollary 1 imply that the set $\{p_{min}(z) \mid z \text{ is a special point of } K\}$ is a finite set. As a consequence, we may consider the integer

$$p_{min}(f,K) := \max\{p_{min}(z) \mid z \text{ is a special point of } K\}$$

Definition of the set of Markov partitions $\mathcal{R}(f,K,p)$. For every integer $p \geq p_{min}(f,K)$, we consider the set of Markov partitions

$$\mathcal{R}(f,K,p) := \{\mathcal{R}(z,p) \mid z \text{ is a special point of } K\}$$

Definition of the set of geometrical types $\mathcal{T}(f,K,p)$. For every integer $p \geq p_{min}(f,K)$, we consider the set $\mathcal{T}(f,K,p)$ made of the geometrical types of all the geometrizations of all the Markov partitions of the set $\mathcal{R}(f,K,p)$.

Remark 5. *For every integer $p \geq p_{min}(f,K)$, proposition 4, corollary 1, and remark 4 imply that the set $\mathcal{T}(f,K,p)$ is a finite set.*

The following proposition states the most important property of set $\mathcal{T}(f,K,p)$:

Proposition 7. *Let f_1, f_2 be some Smale difffeomorphisms of compact surfaces, and K_1, K_2 be some non-trivial saddle basic pieces of K_1 and K_2.*
(i) *Assume that there exists an homeomorphism $h : \Delta(f_1, K_1) \to \Delta(f_2, K_2)$ which conjugates the restrictions of f_1 and f_2. Then, we have $p_{min}(f_1, K_1) =$*

$p_{min}(f_2, K_2)$. Moreover, for every integer $p \geq p_{min}(f_1, K_1)$, we have $\mathcal{T}(f_1, K_1, p) = \mathcal{T}(f_2, K_2, p)$.
(ii) *Conversely, assume that there exists an integer $p \geq \max(p_{min}(f_1, K_1), p_{min}(f_2, K_2))$ such that $\mathcal{T}(f_1, K_1, p) = \mathcal{T}(f_2, K_2, p)$. Then, there exists an homeomorphism $h : \Delta(f_1, K_1) \to \Delta(f_2, K_2)$ which conjugates the restrictions of f_1 and f_2.*

Proof. Assume that the hypothesis of item (i) holds. Then, proposition 5 and lemma 6 imply that the integers $p_{min}(f_1, K_1)$ and $p_{min}(f_2, K_2)$ are equal. Moreover, proposition 5 and lemma 6 imply that, for every integer $p \geq p_{min}(f_1, K_1)$, the elements of the set of Markov partitions $\mathcal{R}(f_2, K_2, p)$ are the images under h of the elements of the set of Markov partitions $\mathcal{R}(f_1, K_1, p)$. Remark 4 completes the proof of item (i).

Item (ii) follows from theorem 1. △

6. Definition of the set of geometrical types $\mathcal{T}(T, p)$.

Preliminary: choice of a triple $(f_T, K_T, \mathcal{R}_T)$ for every realizable geometrical type T. For every realizable geometrical type T, we choose a Smale diffeomorphism f_T of a compact surface S_T, and a non-trivial saddle basic piece K_T of f_T, such that K_T admits a geometrized Markov partition $\mathcal{R}_T$ of geometrical type T. We denote by $R_{1,T}, \dots, R_{2,T}$ the rectangles of the Markov partition $\mathcal{R}_T$. We denote by R_T the union of these rectangles, and we denote by $\partial^s R_T$ (resp. $\partial^u R_T$) the union of the stable (resp. unstable) sides of these rectangles.

Notation . *For every realizable geometrical type T, we consider the integer $p_{min}(T) := p_{min}(f_T, K_T)$. Moreover, for every realizable geometrical type T and every integer $p \geq p_{min}(T)$, we consider the finite set of geometrical types $\mathcal{T}(T, p) := \mathcal{T}(f_T, K_T, p)$.*

Remark 6. *Let g_T be a Smale diffeomorphism of a compact surface and L_T be a non-trivial saddle basic piece of g_T, such that L_T admits a geometrized Markov partition of geometrical type T. Then, by item (i) of proposition 7, we have $p_{min}(T) = p_{min}(g_T, L_T)$. Moreover, proposition 7 implies that, for every integer $p \geq p_{min}(T)$, we have $\mathcal{T}(T, p) = \mathcal{T}(g_T, L_T, p)$.*

The most important property of the set of geometrical types $\mathcal{T}(T, p)$ is that this set is a complete invariant of strong equivalence. More precisely:

Proposition 8. *Let T_1 and T_2 be two realizable geometrical types.*
(i) *If the geometrical types T_1 and T_2 are strongly equivalent, then $p_{min}(T_1) = p_{min}(T_2)$, and $\mathcal{T}(T_1, p) = \mathcal{T}(T_2, p)$ for every $p \geq p_{min}(T_1)$,*
(ii) *Conversely, if there exists $p \geq \max(p_{min}(T_1), p_{min}(T_2))$ such that $\mathcal{T}(T_1, p) = \mathcal{T}(T_2, p)$, then the geometrical types T_1 and T_2 are strongly equivalent.*

Proof. Item (i) follows from the definition of strong equivalence, and from remark 6. Item (ii) follows from item (ii) of proposition 7 and remark 4. △

7. N-points and N-segments.

In this section, we consider a Smale diffeomorphism f of a compact surface, a non-trivial saddle basic piece K of f, and a geometrized Markov partition $\mathcal{R}$ of K. We denote by $R_1, \dots, R_n$ the rectangles of the Markov partition $\mathcal{R}$ (in particular, n is the number of rectangles of $\mathcal{R}$). We denote by R the union of the rectangles of $\mathcal{R}$. We denote by $\partial^s R$ (resp. $\partial^u R$) the union of the stable (resp. unstable) sides of the rectangles of $\mathcal{R}$.

Definition (N-point, N-segment)**.** *Let N be a non-negative integer.*
• *A* N-point of $(f,K,\mathcal{R})$ *is a point that lies in $f^{-N}(\partial^s R)\cap f^N(\partial^u R)$.*
• *A* stable N-segment of $(f,K,\mathcal{R})$ *is a segment which is included in $f^{-N}(\partial^s R)$, and whose both ends lie in $f^{-N}(\partial^s R)\cap f^N(\partial^u R)$.*
• *An* unstable N-segment of $(f,K,\mathcal{R})$ *is a segment which is included in $f^N(\partial^u R)$, and whose both ends lie in $f^{-N}(\partial^s R)\cap f^N(\partial^u R)$.*

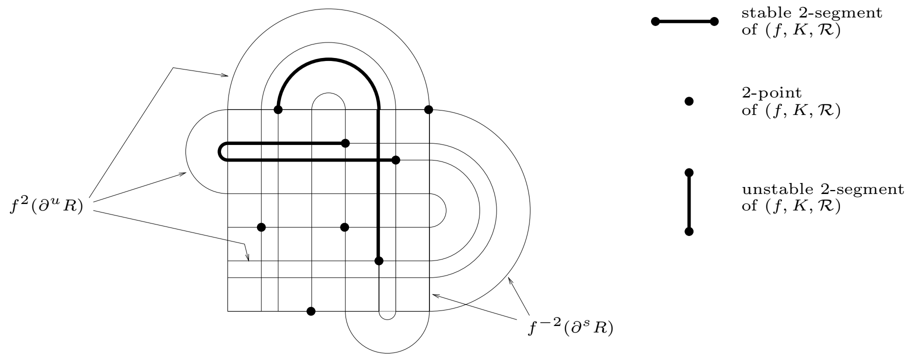

FIGURE 10. Some examples of 2-points and 2-segments of $(f,K,\mathcal{R})$, where f is the so-called "Smale's horseshoe" (see [8, chapter 4]), K is the unique non-trivial saddle basic piece of f, and $\mathcal{R}$ is a one-rectangle basic piece of K

Roughly speaking, the aim of the section is to find an integer N, such that all the segments involved in the construction of the set of Markov partitions $\mathcal{R}(f,K,p)$ are N-segments of $(f,K,\mathcal{R})$. This integer N will depend on the number n of rectangles of the Markov partition $\mathcal{R}$.

7.1. **Some technical results on N-points and N-segments of** $(f,K,\mathcal{R})$**.** In this subsection, we will prove some technical results on N-points and N-segments of $(f,K,\mathcal{R})$. These results will be used in subsections 7.3 and 7.4. Let us begin by some basic remarks:

Remarks 5. *Let N be a non-negative integer.*
(i) *Clearly, the ends of a N-segment of $(f,K,\mathcal{R})$ are N-points of $(f,K,\mathcal{R})$.*
(ii) *Every stable (resp. unstable) N-segment of $(f,K,\mathcal{R})$ is a stable (resp. unstable) segment. Every N-point of $(f,K,\mathcal{R})$ is a point of K.*
(iii) *There exist only finitely many N-points of $(f,K,\mathcal{R})$. As a consequence, there exist only finitely many N-segments of $(f,K,\mathcal{R})$.*
(iv) *The definition of a Markov partition implies that we have: $f^{-N}(\partial^s R)\subset f^{-(N+1)}(\partial^s R)$ and $f^N(\partial^u R)\subset f^{N+1}(\partial^u R)$. As a consequence, every N-point of $(f,K,\mathcal{R})$ is also a $(N+1)$-point of $(f,K,\mathcal{R})$, and every N-segment of $(f,K,\mathcal{R})$ is also a $(N+1)$-segment of $(f,K,\mathcal{R})$.*
(v) *The image under f or f^{-1} of a N-point of $(f,K,\mathcal{R})$ is a $(N+1)$-point of $(f,K,\mathcal{R})$. Similarly, the image under f or f^{-1} of a N-segment of $(f,K,\mathcal{R})$ is a $(N+1)$-segment of $(f,K,\mathcal{R})$.*

Lemma 7. *Let I be a stable N-segment of $(f,K,\mathcal{R})$, and δ^s be a stable cross bar of the rectangle $f^N(R_i)$ (for some $i\leq n$). Assume that $I\cap\delta^s\neq\emptyset$. Then, there are two possibilities: either $\delta^s\subset I$, or $I\cap\delta^s=\{x\}$ where x is an end of both I and δ^s.*

Proof. On one hand, the both ends of the segment I lie on $f^N(\partial^u R)$. On the other hand, the interior of the stable segment δ^s does not intersect $f^N(\partial^u R)$. As a consequence, none of the two ends of the segment I lies in the interior of the segment δ^s. This implies the lemma. △

Remark 7. *By definition of a Markov partition, the rectangles $f^N(R_1), \dots, f^N(R_n)$ are disjoined, and the set K is included in the union of these rectangles. This implies that every stable cross bar of the rectangle $f^N(R_i)$ is an externally isolated stable segment.*

Corollary 2. *Let I be a stable N-segment of $(f, K, \mathcal{R})$, and δ^s be a stable cross bar of the rectangle $f^N(R_i)$ (for some $i \leq n$). Assume that I is externally isolated, and that $I \cap \delta^s$ is non-empty. Then, δ^s is included in I.*

Proof. By lemma 7, there are two possibilities: either $\delta^s \subset I$, or $I \cap \delta^s = \{x\}$ where x is an end of both I and δ^s. To prove that the second possibility is absurd, we argue by contradiction: we suppose that $I \cap \delta^s = \{x\}$ where x is a end of both I and δ^s. Then, the point x lies in the interior of the stable segment $I \cup \delta^s$. Moreover, since the stable segments I and δ^s are both externally isolated (see remark 7), the point x is isolated in $K \cap (I \cup \delta^s)$. This contradicts property (vii) of section 2. △

Lemma 8. *If x is a periodic s-boundary point of K, then $x \in \partial^s R$.*

Proof. Let x be a periodic s-boundary point of K. Since $\mathcal{R}$ is a Markov partition of K, there exists an integer $i \leq n$ such that $x \in R_i$. By proposition 3, one of the two unstable separatrices of x is a free separatrix. In particular, one of the two unstable separatrices of x does not intersect $\partial^s R_i$ (since $\partial^s R_i \subset W^s(K)$). As a consequence, the point x cannot lie in $R_i \setminus \partial^s R_i$. This completes the proof. △

A stable side δ of a rectangle of the Markov partition $\mathcal{R}$ is said to be *periodic*, if there exists $p > 0$ such that $f^p(\delta) \subset \delta$. We denote by $\partial^s_{per} R$ the union of the periodic stable sides of $\mathcal{R}$.

Remarks 6.
(i) *For every stable side δ of $\mathcal{R}$, there exists another stable side δ' of $\mathcal{R}$, such that $f(\delta) \subset \delta'$ (by definition of a Markov partition). Moreover, the set of the stable sides of $\mathcal{R}$ is a finite set of cardinal $2n$. As a consequence, the period of a stable side of $\mathcal{R}$ is at most $2n$. As another consequence, we have $f^{2n}(\partial^s R) \subset (\partial^s_{per} R)$.*
(ii) *There exists a periodic s-boundary point on each periodic stable side of $\mathcal{R}$.*
(iii) *By lemma 8, every periodic s-boundary point x of K lies on a stable side δ of a rectangle of $\mathcal{R}$. Moreover, it is clear that this stable side δ is necessarly periodic (with the same period as the point x). As a consequence, item (i) implies that the period of a periodic s-boundary point of K is at most $2n$.*
(iv) *Item (iii) implies that the period of a stable separatrix of a periodic s-boundary point of K is at most $4n$.*

Lemma 9. *Let I be a stable segment such that the both ends of I are N-points of $(f, K, \mathcal{R})$. Then, I is a $(N + 2n)$-segment of $(f, K, \mathcal{R})$.*

Proof. We have to prove that the stable segment I is included in $f^{-(N+2n)}(\partial^s R)$. By assumption, the both ends of the stable segment I lie in $f^{-N}(\partial^s R)$. Therefore, by item (i) of remark 6, the both ends of the stable segment I lie in $f^{-(N+2n)}(\partial^s_{per} R)$. Now, observe that two different connected components of $f^{-(N+2n)}(\partial^s_{per} R)$ cannot be included in the same leaf of $W^s(K)$ (since each connected component of $f^{-(N+2n)}(\partial^s_{per} R)$ contains a periodic s-boundary point). As a consequence, the two ends of the stable segment I must lie on the same connected component of $f^{-(N+2n)}(\partial^s_{per}R)$. As a further consequence, the segment I is included in $f^{-(N+2n)}(\partial^s_{per}\mathcal{R})$. △

Lemma 10. *Let $\mathcal{I}$ be a finite family of stable N-segments of $(f, K, \mathcal{R})$. If none of the elements of the family $\mathcal{I}$ is a trivial segment, and if all the periodic s-boundary points of K lie in $\cup\mathcal{I}$, then we have $(\cup\mathcal{I}) \supset f^N(\partial^s_{per}R) \supset f^{N+2n}(\partial^s R)$.*

Proof. Recall that the inclusion $f^N(\partial^s_{per}R) \supset f^{N+2n}(\partial^s R)$ was proved in remark 5. It remains to prove the inclusion $(\cup\mathcal{I}) \supset f^N(\partial^s_{per}R)$.
First step. Let x be a periodic s-boundary point of K. By assumption, there exists a stable segment $I \in \mathcal{I}$ such that $x \in I$. By lemma 8, the point x lies on a stable side δ of the rectangle $f^N(R_i)$ for some $i \leq n$. We will prove that $\delta \subset I$.

On the one hand, the segment δ is a stable cross bar of the rectangle $f^N(R_i)$ (since it is a stable side of this rectangle). On the other hand, $I \cap \delta^s$ is non-empty, since $x \in I \cap \delta^s$. Therefore, lemma 7 implies that, either $\delta^s \subset I$, or $I \cap \delta^s = \{x\}$. Let us prove that $I \cap \delta^s \neq \{x\}$:
— if $x \notin f^N(\partial^u R)$, then x is in the interior of I. Hence, $I \cap \delta^s \neq \{x\}$.
— if $x \in f^N(\partial^u R)$, then x is a periodic u-boundary point. As a consequence, one of the stable separatrices of x is a free separatrix. Let us denote by W^s this stable separatrix. Since the ends of I are in K, we have $I \cap W^s = \emptyset$. For the same reason, $\delta^s \cap W^s = \emptyset$. As a consequence, the stable segments I and δ lie on the same side of the point x on the leaf $W^s(x)$. As a further consequence, we have $\delta^s \subset I$ or $I \subset \delta^s$. In particular, $I \cap \delta^s \neq \{x\}$.
We have proved that $I \cap \delta^s \neq \{x\}$. Thus, by lemma 7, we have $\delta^s \subset I$.
Second step. Let δ^s be a connected component of $f^N(\partial^s_{per}R)$, that is a periodic stable side of the rectangle $f^N(R_i)$ for some i. According to item (ii) of remarks 6, there exists a periodic s-boundary point x of K which lies on δ^s. In the first step, we have proved that there exists an element I of the family $\mathcal{I}$, such that $\delta^s \subset I$. As a consequence, we have $f^N(\partial^s_{per}R) \subset (\cup\mathcal{I})$. △

Lemmas 7, 8, 9 and 10, corollary 2 and remarks 7 and 6 concern the *stable* N-segments of $(f, K, \mathcal{R})$. Of course, there exist some analogous statements concerning the *unstable* N-segments of $(f, K, \mathcal{R})$.

7.2. **N-embrionary separatrices of $(f, K, \mathcal{R})$.** Periodic s-boundary (resp. u-boundary) points of K play an important role in the construction of the set of Markov partitions $\mathcal{R}(f, K, p)$. Nevertheless, a periodic s-boundary (resp. u-boundary) point of K is not (in general) a N-point of $(f, K, \mathcal{R})$ for any integer N. These is the reason why we need to introduce some new objects, called N-embrionary separatrices of $(f, K, \mathcal{R})$. These objects will play an important role in the proofs of some of the results of subsections 7.3 and 7.4.

Definition (embrionary separatrix). *Let N be a non-negative integer.*
• *A* stable N-embrionary separatrix of $(f, K, \mathcal{R})$ *is an interval $I =]x, z]^s$ included in $f^{-N}(\partial^s R)$, such that x is a periodic s-boundary point of K and z is a N-point of $(f, K, \mathcal{R})$.*
• *An* unstable N-embrionary separatrix of $(f, K, \mathcal{R})$ *is an interval $J =]y, z]^u$ included in $f^N(\partial^u R)$,such that y is a periodic u-boundary point of K, and z is a N-point of $(f, K, \mathcal{R})$.*

Remark 8. *Every N-embrionary separatrix of $(f, K, \mathcal{R})$ is also a $(N + 1)$-embrionary separatrix of $(f, K, \mathcal{R})$. The image under f or f^{-1} of a N-embrionary separatrix of $(f, K, \mathcal{R})$ is a $(N + 1)$-embrionary separatrix of $(f, K, \mathcal{R})$.*

Lemma 11. *Let x be a periodic s-boundary point of K, and let W^s be a stable separatrix of x. Assume that W^s is not a free separatrix. Then, there exists a unique 0-embrionary separatrix $\widehat{W^s}$ of $(f, K, \mathcal{R})$ included in W^s.*

Proof. By lemma 8, the point x lies on a connected component δ^s of $\partial^s R$. Let $\widehat{W^s} = W^s \cap \delta^s$ (figure 11). Observe that $\widehat{W^s}$ is non-empty, since W^s is not a free separatrix. Clearly, $\widehat{W^s}$ is the unique 0-embrionary separatrix of $(f, K, \mathcal{R})$ which is included in W^s. △

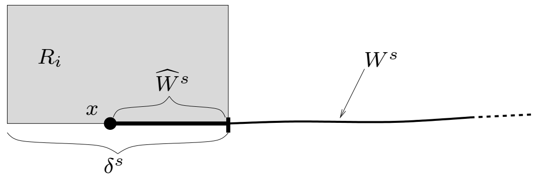

FIGURE 11. Illustration of the proof of lemma 11

Lemma 12. *Let x be a periodic s-boundary point of K, and W^s be a stable separatrix of x. We assume that W^s is not a free separatrix, and we denote by $\widehat{W^s}$ the unique 0-embrionary separatrix of $(f, K, \mathcal{R})$ such that $\widehat{W^s} \subset W^s$. If I is a N-embrionary separatrix of $(f, K, \mathcal{R})$ such that $I \subset f^N(W^s)$, then $I \supset f^N(\widehat{W^s})$.*

Proof. Let us first observe that $f^N(\widehat{W^s})$ and I are two stable intervals, which are both included in the stable separatrix $f^N(W^s)$. Moreover, the point x is an end of both the stable intervals $f^N(\widehat{W^s})$ and I. As a consequence, there are two possibilities: either $f^N(\widehat{W^s})$ is included in I, or I is included in the interior of $f^N(\widehat{W^s})$. We are left to prove that I cannot be included in the interior of $f^N(\widehat{W^s})$.

Since $\widehat{W^s}$ is a 0-embrionary separatrix, we have $\widehat{W^s} \subset \partial^s R$. In particular, $\partial^u R$ does not intersect the interior of $\widehat{W^s}$. Therefore, $f^N(\partial^u R)$ does not intersect the interior of $f^N(\widehat{W^s})$. Since the ends of I lie in $f^N(\partial^u R)$, this implies that I cannot be included in the interior of $f^N(\widehat{W^s})$. This completes the proof. △

Lemma 13. *Let x be a periodic s-boundary point, and z be a N-point of $(f, K, \mathcal{R})$ which lies on $W^s(x)$. Then, the stable interval $I =]x, z]^s$ is a $(N + 2n)$-embrionary separatrix of $(f, K, \mathcal{R})$.*

Proof. The arguments are the same as in the proof of lemma 9. △

Lemma 14. *Let W^s be a stable separatrix of a periodic s-boundary point of K. We assume that W^s is not a free separatrix. We denote by q^s be the period of W^s, and we denote by $\widehat{W^s}$ the unique 0-embrionary separatrix of $(f, K, \mathcal{R})$ such that $\widehat{W^s} \subset W^s$. Then, $f^{-q_s}(\widehat{W^s})$ contains a stable cross bar of a rectangle of the Markov partition $\mathcal{R}$.*

Proof. By definition of a 0-embrionary separatrix of $(f, K, \mathcal{R})$, we have $\widehat{W^s} =]x, a]^u$, where x is a periodic s-boundary point of K and $a \in (\partial^s R \cap \partial^u R)$ (*i.e.* a is a corner of a rectangle of the Markov partition $\mathcal{R}$).

Let $I = f^{-q_s}(\widehat{W^s}) \setminus \widehat{W^s} =]a, f^{-q_s}(a)]^s$. Since $\mathcal{R}$ is a Markov partition, we have $f^{-q_s}(\partial^u R) \subset \partial^u R$. In particular, the point $f^{-q^s}(a)$ lies in $\partial^u R$. As a consequence, I is a non-trivial stable segment whose both ends lie in $(\partial^u R)$. There are two possibilities: either the interior of I is disjoined from R, or the closure of I contains a stable cross bar of a rectangle of $\mathcal{R}$. The first possibility is absurd, since $]a, f^{-q_s}(a)[^s \cap K$ is clearly non-empty and $K \subset R$. △

Lemma 12, 13 and 14 concern *stable* N-embrionary separatrices of $(f, K, \mathcal{R})$. Of course, there exist analogous statements concerning *unstable* N-embrionary separatrices of $(f, K, \mathcal{R})$.

7.3. **The special points of K.** This subsection is devoted to the proof of the following result:

Proposition 9. *For every special point z of K, there exists an integer $k \in \mathbb{Z}$, such that $f^k(z)$ is a $(23n^2)$-point of $(f, K, \mathcal{R})$.*

Lemma 15. *Let δ^u be an unstable cross bar of the rectangle R_i. Assume that $f(R_i) \cap R_j$ is non-empty. Then0the unstable segment $f(\delta^u)$ contains an unstable cross bar of the rectangle R_j.*

Proof. Let V be a connected component of $f(R_i) \cap R_j$. By remark 3, V is a vertical subrectangle of R_j, and there exists an horizontal subrectangle H of R_i such that $V = f(H)$. Clearly, $\delta^u \cap H$ is a vertical cross bar of H. Thus, $f(\delta^u \cap H)$ is a vertical cross bar of $f(H) = V$. Now, observe that every vertical cross bar of V is a vertical cross bar of R_j. As a consequence, $f(\delta^u)$ contains a vertical cross bar of R_j. △

Corollary 3. *Let δ^u be an unstable cross bar of the rectangle R_i, and δ^s be a stable cross bar of the rectangle R_j. There exists an integer $k \leq n$, such that $f^k(\delta^u) \cap \delta^s$ is non-empty.*

Proof. Since K is transitive, there exists a finite sequence of integers $i = i_0, i_1, \dots, i_k = j$ such that, for every $l \leq k-1$, the intersection $f(R_{i_l}) \cap R_{i_{l+1}}$ is non-empty. We may assume that the integers $i_0, \dots, i_{k-1}$ are pairwise different ; as a consequence, we may assume that the integer k is less than n.

By lemma 15, there an unstable cross bar δ_1^u of the rectangle R_{i_1} such that $f(\delta^u) \supset \delta_1^u$. Using lemma 15 once again, we obtain an unstable cross bar δ_2^u of the rectangle R_{i_2} such that $f(\delta_1^u) \supset \delta_2^u$. By induction, we obtain an unstable cross bar δ_k^u of the rectangle $R_{i_k} = R_j$ such that $f^k(\delta^u) \supset \delta_k^u$. Now, we observe that the intersection $\delta_k^u \cap \delta^s$ is non-empty (since δ_k^u is an unstable cross bar of R_j and δ^s is a stable cross bar of R_j). Hence, $f^k(\delta^u) \cap \delta^s$ is non-empty. △

Corollary 4. *Let W^s be a stable separatrix of a periodic s-boundary point of (f, K), and W^u be an unstable separatrix of a periodic u-boundary point of (f, K). Assume that neither W^s, nor W^u is a free separatrix. Let $\widehat{W^s}$ be the unique 0-embrionary separatrix of $(f, K, \mathcal{R})$ such that $\widehat{W^s} \subset W^s$, and $\widehat{W^u}$ be a 0-embrionary separatrix of $(f, K, \mathcal{R})$ such that $\widehat{W^u} \subset W^u$. Then, there exists an integer $k \leq 9n$, such that $f^k(\widehat{W^u}) \cap \widehat{W^s}$ is non-empty.*

Proof. Firstly, lemma 14 implies that there exists an integer $k_1 \leq 4n$, an integer $i \leq n$, and an unstable cross bar δ^u of the rectangle R_i, such that $f^{k_1}(\widehat{W^u}) \supset \delta^u$. Secondly, a similar argument implies that there exists an integer $k_2 \leq 4n$, an integer $j \leq n$, and a stable cross bar δ^s of the rectangle R_j, such that $f^{-k_2}(\widehat{W^s}) \supset \delta^s$. Thirdly, corollary 3 implies that there exists an integer $k_3 \leq n$ such that $f^{k_3}(\delta^u) \cap \delta^s$ is non-empty. Hence, we have $f^k(\widehat{W^u}) \cap \widehat{W^s} \neq \emptyset$ where $k = k_1 + k_2 + k_3 \leq 9n$. △

Proof proposition 9. Let z be a special point of K. By definition of a special point, the point z lies in $W^s \cap W^u$, where W^s is a stable separatrix of a periodic s-boundary point x, and W^u is an unstable bseparatrix of a periodic u-boundary point y. Moreover, we have $]x,z]^s \cap]y,z]^u = \{z\}$. As usual, we denote by $\widehat{W^s}$ the unique 0-embrionary separatrix of $(f,K,\mathcal{R})$ such that $\widehat{W^s} \subset W^s$, and we denote by $\widehat{W^u}$ the unique embrionary separatrix of $(f,K,\mathcal{R})$ such that $\widehat{W^u} \subset W^u$. We denote by q_s and q_u the periods of the separatrices W^s and W^u. By item (iv) of remarks 6, the integers q_s, q_u are smaller than $4n$.

By corollary 4, there exists an integer $l \leq 9n$ such that $\widehat{W^s} \cap f^l(\widehat{W^u})$ is non-empty. We consider a point $z_0 \in \widehat{W^s} \cap f^l(\widehat{W^u})$. Then, the proof of proposition 4 implies that there exists an integer $k \in \mathbb{Z}$, such that $f^k(z) \in]x,z_0]^s \cap]y, f^{q_s q_u}(z_0)]^u$. By definition of the point z_0, we have

$$]x,z_0]^s \subset \widehat{W^s} \subset \partial^s R \text{ and }]y, f^{q_s q_u}(z_0)]^u \subset f^{l+q_s q_u}(\widehat{W^u}) \subset f^{l+q_s q_u}(\partial^u R)$$

Using these inclusions and the inequalities $l \leq 9n$ and $q_s, q_u \leq 4n$, we obtain

$$f^k(x) \in \partial^s R \cap f^{l+q_s q_u}(\partial^u R) \subset \partial^s R \cap f^{23n^2}(\partial^u R)$$

As a consequence, the point $f^k(z)$ is a $(23n^2)$-point of $(f,K,\mathcal{R})$. △

7.4. **Segments involved in the construction of the Markov partition $\mathcal{R}(z,p)$.** In this subsection, we consider a special point $z \in K$. Roughly speaking, the aim of the subsection is to prove the following informal statement: *Assume that the special point z is a N_0-point of $(f,K,\mathcal{R})$ for some non-negative integer N_0. Then, "every segment involved in the construction of the Markov partition $\mathcal{R}(z,p)$" is a $(N_0+46n+p)$-segment of $(f,K,\mathcal{R})$.*

Step 0. The family of unstable segments $\mathcal{J}_0(z)$.

Proposition 10. *Assume that the special point z is a N_0-point of $(f,K,\mathcal{R})$ for some non-negative integer N_0. Then, each element of the family of unstable intervals $\mathcal{J}_0(z)$ is a (N_0+6n)-embrionary separatrix of $(f,K,\mathcal{R})$.*

Proof. Recall that, by definition of a special point, the point z lies on the unstable manifold of a periodic u-boundary point y. Morover, recall that $\mathcal{J}_0(z) = \{J_0(z),\dots,f^{-2q}(J_0(z))\}$, where $J_0(z) =]y,z]^u$ and q is the period of y. Since z is N_0-point of $(f,K,\mathcal{R})$, lemma 13 implies that $J_0(z)$ is (N_0+2n)-embrionary separatrix of $(f,K,\mathcal{R})$. As a consequence, $f^{-i}(J_0(z))$ is a (N_0+2n+i)-embrionary separatrix of $(f,K,\mathcal{R})$ for every $i \in \mathbb{N}$ (see remark 5). Lastly, item (i) of remark 6 implies q is smaller than $2n$. This completes the proof. △

Step 1. The family of stable segments $\mathcal{I}_1(z)$. The family of unstable segments $\mathcal{J}_1(z)$.

Proposition 11. *Assume that the special point z is a N_0-point of $(f,K,\mathcal{R})$ for some non-negative integer N_0. Then, for every periodic s-boundary point $x \in K$, the stable segment $I_1(x,z)$ is a (N_0+21n)-segment of $(f,K,\mathcal{R})$.*

Before reading the proof of proposition 11, it is necessary to have in mind the construction of the segment $I_1(x,z)$ (see step 1 in subsection 5.1). The following lemma is the core of the proof of proposition 11.

Lemma 16. *Under the hypothesis of proposition 11, the ends of the segment $I_1(x,z)$ are (N_0+19n)-points of $(f,K,\mathcal{R})$.*

Proof. Recall that $J_0(z) =]y,z]^s$, where y is a periodic u-boundary point. We denote by W^u the unstable separatrix of y such that $J_0(z) \subset W^u$. We denote by q_u the period of the separatrix W^u, and we denote by $\widehat{W}^u$ the unique 0-embrionary separatrix of $(f,K,\mathcal{R})$ such that $\widehat{W}^u \subset W^u$.

We denote by W_1^s and W_2^s the stable separatrices of x, and we denote by x_1 and x_2 the ends of the unstable segment $I_1(x,z)$ such that $x_1 \in W_1^s \cup \{x\}$ and $x_2 \in W_2^s \cup \{x\}$. We denote by $\widehat{W_1^s}$ and $\widehat{W_2^s}$ the unique 0-embrionary separatrices of $(f,K,\mathcal{R})$ such that $\widehat{W_1^s} \subset W_1^s$ and $\widehat{W_2^s} \subset W_2^s$.

Let $i \in \{1,2\}$. We have to prove that the point x_i is a $(19n^2)$-point of $(f,K,\mathcal{R})$. Recall that the definition of x_i is divided in two cases (see subsection 5.1):

First case: ***W_i^s is a free separatrix.*** Then $x_i = x$. By assumption, x is a periodic s-boundary point. Thus, x lies in $\partial^s R$ (lemma 8). Moreover, x is a periodic u-boundary point (since W_i^s is a free separatrix). Thus, x lies in $\partial^u R$ (lemma 8). Hence, the point $x_i = x$ lies in $\partial^s R \cap \partial^u R$, *i.e.* is a 0-point of $(f,K,\mathcal{R})$.

Second case: ***W_i^s is not a free separatrix.*** Then, x_i is the unique point of $W_i^s \cap (\cup \mathcal{J}_0(z))$, such that $]x,x_i[^s$ does not intersect $\cup\mathcal{J}_0(z)$. By corollary 4, there exists $k \leq 9n$, such that $f^{-k}(\widehat{W_i^s})$ intersects $\widehat{W^u}$. Moreover, proposition 10 and lemma 12 imply that $f^{N_0+6n}(\widehat{W}^u)$ is included in $\cup\mathcal{J}_0(z)$. Thus, $f^{-(k+N_0+6n)}(\widehat{W_i^s})$ intersects $\cup\mathcal{J}_0(z)$. Since $\mathcal{J}_0(z)$ is positively invariant, we obtain the following:
Fact 1: $f^{-(k'+N_0+6n)}(\widehat{W_i^s})$ intersects $\cup\mathcal{J}_0(z)$ for every integer $k' \geq k$.
Besides, item (iv) of remark 6 implies the following:
Fact 2: there exists k', such that $k \leq k' \leq k+4n$, and such that $f^{-(k'+N_0+6n)}(\widehat{W_i^s}) \subset W_i^s$.
The two properties stated above and the definition of x_i imply that x_i must lie in $f^{-(k'+N_0+6n)}(\widehat{W_i^s}) \cap (\cup\mathcal{J}_0(z))$. On the other hand, $\widehat{W_i^s}$ is included in $\partial^s R$, and $\cup\mathcal{J}_0(z)$ is included in $f^{N_0+6n}(\partial^u R)$ (because $\widehat{W_i^s}$ is a 0-embrionary separatrix and $\mathcal{J}_0(z)$ is a family of (N_0+6n)-embrionary separatrices). As a consequence, the point x_i lies in $f^{-(9n+4n+N_0+6n)}(\partial^s R) \cap f^{N_0+6n}(\partial^u R)$. In particular, x_i is a (N_0+19n)-point of $(f,K,\mathcal{R})$.

In both cases, we have proved that the point x_i is a (N_0+19n)-point of $(f,K,\mathcal{R})$. This completes the proof of lemma 16 △

Proof of proposition 11. The proposition follows from lemmas 16 and 9. △

Proposition 12. *Assume that the special point z is a N_0-point of $(f,K,\mathcal{R})$ for some non-negative integer N_0. Then, for ever periodic u-boundary point y, the unstable segment $J_1(y,z)$ is a (N_0+42n)-segment of $(f,K,\mathcal{R})$.*

Proof. Similar to the proof of proposition 11. △

Step 2. The family of stable segments $\mathcal{I}_2(z)$. The family of unstable segments $\mathcal{J}_2(z)$.

Proposition 13. *Let x be a periodic s-boundary point of K. Assume that the special point z is N_0-point of $(f,K,\mathcal{R})$ for some non-negative integer N_0. Then, the stable segment $I_2(x,z)$ is also a (N_0+21n)-segment of $(f,K,\mathcal{R})$.*

Lemma 17. *Let $\alpha \subset W^s(K)$ be a stable arch.*
(i) *For every integer $k \in \mathbb{Z}$, either $\alpha \subset f^k(R) \setminus f^k(\partial^u R)$, or int$(\alpha) \cap f^k(R) = \emptyset$.*
(ii) *Moreover, if int$(\alpha) \cap f^k(R) = \emptyset$, then the both ends of α lie on $f^k(\partial^u R)$.*

Proof of item (i) of lemma 17. We argue by contradiction: we suppose that α is not included in $f^k(R) \setminus f^k(\partial^u R)$, and that $\text{int}(\alpha) \cap f^k(R)$ is non-empty. In particular, α intersects $f^k(\partial^u R)$.
— If $\text{int}(\alpha) \cap f^k(\partial^u R) \neq \emptyset$, then $\text{int}(\alpha) \cap K \neq \emptyset$ (since $\alpha \cap f^k(\partial^u R) \subset W^s(K) \cap W^u(K)$ and $K = W^s(K) \cap W^u(K)$). This is in contradiction with the fact that α is a stable arch.

— If $\mathrm{int}(\alpha) \cap f^k(\partial^u R) = \emptyset$, then α is included in the rectangle $f^k(R_i)$ for some $i \leq n$, and one of the ends of α lies on $f^k(\partial^u R_i)$. Then, property (vii) of section 2 implies that $\mathrm{int}(\alpha) \cap K$ is non-empty. Once again, this contradicts the fact that α is a stable arch.
In each case, we have obtained a contradiction ; this completes the proof. △

Corollary 5. *Let $\alpha \subset W^s(K)$ be a stable arch.*
(i) *If $\alpha \not\subset f^{-(N_1+2n)}(R)$, then the both ends of α lie in $f^{-(N_1+2n)}(\partial^u R)$.*
(ii) *If $int(\alpha) \cap f^{N_1}(R) \neq \emptyset$, then none of the ends of α lie in $f^{N_1}(\partial^u R)$.*

Proof. To prove item (i), apply lemma 17 with $k = -(N_1+2n)$. To prove item (ii), apply lemma 17 with $k = N_1$. △

Proof of item (ii) of lemma 17. Let us assume that $\mathrm{int}(\alpha) \cap f^k(R)$ is non-empty.

Since α is a stable arch, the both ends of α lie in K ; in particular, the both ends of α lie in $f^k(R)$. On the other hand, $\mathrm{int}(\alpha) \cap f^k(R)$ is empty (by assumption). As a consequence, the both ends of α lie in $f^k(\cup\partial\mathcal{R}) = f^k(\partial^s R) \cup f^k(\partial^u R)$. Moreover, none of the ends of α can lie in $f^k((\partial^s R) \setminus (\partial^u R))$ (since α is a stable segment and $\mathrm{int}(\alpha) \cap f^k(R) = \emptyset$). As a consequence, the both ends of α lie in $f^k(\partial^u R)$. △

Proof of proposition 13. Let $N_1 = N_0 + 21n$. By proposition 11, the segment $I_1(x,z)$ is a N_1-segment of $(f,K,\mathcal{R})$. Let a and b be the ends of the stable segment $I_1(x,z)$. Since $I_1(x,z)$ is a N_1-segment of $(f,K,\mathcal{R})$, the segment $I_1(x,z)$ is included in $f^{-N_1}(\partial^s R)$, and the points a,b lie in $f^{N_1}(\partial^u \mathcal{R})$.

As a example, we treat the case where the segment $I_1(x,z) = [a,b]^s$ is externally isolated near a, but is not externally isolated near b. Then, $I_2(x,z) = [a,b']^s$, where b' is the unique point such that $b' \in]a,b[^s \cap K$ and $]b',b[^s \cap K = \emptyset$ (see the construction of the segment $I_2(x,z)$ in subsection 5.1). In particular, $[b',b]^s$ is a stable arch. Since $b \in f^{N_1}(\partial^u R)$, lemma 17 implies that $b' \in f^{N_1}(\partial^u R)$. As a consequence, $I_2(x,z) = [a,b']^s$ is a N_1-segment of $(f,K,\mathcal{R})$. △

The proof of the following result is similar to the proof of proposition 13:

Proposition 14. *Let y be a periodic u-boundary point of K. Assume that the special point z is a N_0-point of $(f,K,\mathcal{R})$ for some non-negative integer N_0. Then, the unstable segment $J_2(y,z)$ is also a (N_0+42n)-segment of $(f,K,\mathcal{R})$.*

Step 3. The family of stable segments $\mathcal{I}_3(z)$. The family of unstable segments $\mathcal{J}_3(z)$.

Definition (unstable N-arch)**.** *Let $\gamma \subset W^u(K)$ be an unstable arch. If γ is a N-segment of $(f,K,\mathcal{R})$ for some integer N, then we say that γ is an* unstable N-arch *of $(f,K,\mathcal{R})$.*

We will prove the following proposition:

Proposition 15. *Assume that the special point z is a N_0-point of $(f,K,\mathcal{R})$ for some non-negative integer N_0, and let $N_1 = N_0 + 21n$. Then, for every periodic s-boundary point $x \in K$:*
(i) *in the definition of the segment $I_3(x,z)$ (see step 3 in subsection 5.1), we can replace "unstable arches" by "unstable (N_1+2n)-arches",*
(ii) *the stable segment $I_3(x,z)$ is a (N_1+4n)-segment of $(f,K,\mathcal{R})$.*

Lemma 18 and corollary 6 below are the key of the proof of proposition 15. The proofs of this lemma and this corollary are completely similar to the proofs of lemma 17 and corollary 5.

Lemma 18. *Let $\gamma \subset W^u(K)$ be a unstable arch.*
(i) *For every integer $k \in \mathbb{Z}$, either $\gamma \subset f^k(R) \setminus f^k(\partial^s R)$, or $int(\gamma) \cap f^k(R) = \emptyset$.*
(ii) *Moreover, if $int(\gamma) \cap f^k(R) = \emptyset$, then the both ends of γ lie in $f^k(\partial^s R)$.*

Corollary 6. *Let $\gamma \subset W^u(K)$ be an unstable arch.*
(i) *If $\gamma \not\subset f^{N_1+2n}(R)$, then the both ends of γ lie in $f^{N_1+2n}(\partial^s R)$.*
(ii) *If $int(\gamma) \cap f^{-N_1}(R) \neq \emptyset$, then none of the ends of γ lie in $f^{-N_1}(\partial^s R)$.*

Let N be a non-negative integer. An *unstable N-ribbon* of $(f, K, \mathcal{R})$ is the closure of a connected component of $f^N(R) \setminus f^{-N}(R)$.

Remark 9.
(i) *The definition of a Markov partition implies that every unstable N-ribbon of $(f, K, \mathcal{R})$ is a horizontal subrectangle of $f^N(R_i)$ for some integer $i \leq n$. In particular, every stable side of a N-ribbon of $(f, K, \mathcal{R})$ is a stable cross bar of $f^N(R)$.*
(ii) *If γ is an unstable side of an unstable N-ribbon, then γ is included in $f^N(\partial^u R)$, the both ends of γ lie in $f^{-N}(\partial^s R)$, and the interior of γ is disjoined from $f^{-N}(R) \supset K$. Consequently, every unstable side of an unstable N-ribbon of $(f, K, \mathcal{R})$ is an unstable N-arch of $(f, K, \mathcal{R})$.*
(iii) *Let γ be an unstable segment. Then, γ is an unstable cross bar of a N-ribbon of $(f, K, \mathcal{R})$ if and only if γ is included in $f^N(R)$, the interior of γ is disjoined from $f^{-N}(R)$, and the both ends of γ lie in $f^{-N}(\partial^s R)$.*

Proof of item (i) of proposition 15. By proposition 13, every element of the family of stable segments $\mathcal{I}_2(z)$ is a N_1-segment of $(f, K, \mathcal{R})$. As a consequence, we have $(\cup\mathcal{I}_2(z)) \subset f^{-N_1}(\partial^s R)$ (by definition of N_1-segments), and $(\cup\mathcal{I}_2(z)) \supset f^{N_1+2n}(\partial^s R)$ (by lemma 10).
First observation. Let γ be an unstable arch, such that one (and only one) of the ends of γ lies in $\cup\mathcal{I}_2(z)$. Using the inclusions proved above, corollary 5, and item (iii) of remark 9, we see that γ is an unstable cross bar of an unstable $(N_1 + 2n)$-ribbon of $(f, K, \mathcal{R})$.
Second observation. Let α be a stable side of an unstable $(N_1 + 2n)$-ribbon of $(f, K, \mathcal{R})$. The first item of remark 9 and corollary 2 imply that the following dichotomy holds: either α does not intersect $(\cup\mathcal{I}_2(z))$, or α is included in $(\cup\mathcal{I}_2(z))$.

The two observations above imply that, in the definition of the segment $I_3(x, z)$, we can replace "unstable arches" by "unstable sides of unstable $(N_1 + 2n)$-ribbons of $(f, K, \mathcal{R})$". Since every unstable sides of unstable $(N_1 + 2n)$-ribbons of $(f, K, \mathcal{R})$ is an unstable $(N_1 + 2n)$-arch of $(f, K, \mathcal{R})$ (remark 9), this completes the proof. △

Proof of item (ii) of proposition 15. The definition of the segment $I_3(x, z)$ and item (i) of the proposition imply that each end of the segment $I_3(x, z)$ is either an end of the segment $I_2(x, z)$, or an end of an unstable $(N_1 + 2n)$-arch of $(f, K, \mathcal{R})$. In particular, each end of the segment $I_3(x, z)$ is a $(N_1 + 2n)$-point of $(f, K, \mathcal{R})$. Then, lemma 9 implies that $I_3(x, z)$ is a $(N_1 + 4n)$-segment of $(f, K, \mathcal{R})$. △

The proof of the following proposition is similar to the proof of proposition 15:

Proposition 16. *Assume that the special point z is a N_0-point of $(f, K, \mathcal{R})$ for some non-negative integer N_0, and consider the integer $N_1 = N_0 + 42n$. Then, for every periodic u-boundary point $y \in K$:*
(i) *in the definition of the segment $J_3(y, z)$ (see step 3 in subsection 5.1), we can replace "stable arches" by "stable $(N_1 + 2n)$-arches",*
(ii) *the unstable segment $J_3(y, z)$ is a $(N_1 + 4n)$-segment of $(f, K, \mathcal{R})$.*

Step 4. The integer $p_{min}(f,K)$. The family of stable segments $\mathcal{I}_4(z,p)$. The family of unstable segments $\mathcal{J}_4(z,p)$.

Proposition 17. *Assume that the special point z is a N_0-point of $(f,K,\mathcal{R})$ for some non-negative integer N_0, and consider the integer $N_2 = N_0 + 46n$. Then $p_{min}(z) \leq 2(N_2 + 2n)$.*

Proof. Let us recall that $p_{min}(z)$ is the smallest of all the integers p that satisfy the two following properties:
(i) the unstable sides of the domain of every equivalence class of rails leaning on $\cup\mathcal{I}_3(z)$ is included in $f^p(\cup\mathcal{J}_3(z))$,
(ii) the stable sides of the domain of every equivalence class of rails leaning on $f^p(\cup\mathcal{J}_3(z))$ is included in $\cup\mathcal{I}_3(z)$.
We will prove that properties (i) and (ii) are satisfied provided that the integer p is greater or equal than $2(N_2 + 2n)$.

First step. Let γ be a rail leaning on $\mathcal{I}_3(z)$. We will prove that there exists an integer $i \leq n$, and an horizontal subrectangle H_γ of the rectangle $f^{(N_2+2n)}(R_i)$, such that γ is an unstable cross bar of H_γ.

On the one hand, since γ is a rail leaning on $\mathcal{I}_3(z)$, we have $\mathrm{int}(\gamma)\cap(\cup\mathcal{I}_3(z)) = \emptyset$. On the other hand, since every element of the family of stable segments $\mathcal{I}_3(z)$ is a N_2-segment of $(f,K,\mathcal{R})$ (proposition 15), we have $(\cup\mathcal{I}_3(z)) \supset f^{N_2+2n}(\partial^s R)$ (lemma 10). Putting these two facts together, we obtain $\mathrm{int}(\gamma)\cap f^{N_2+2n}(\partial^s R) = \emptyset$. Moreover, since γ is not an unstable arch, we have $\mathrm{int}(\gamma)\cap f^{N_2+2n}(R) \neq \emptyset$. As a consequence, we have $\gamma \subset f^{N_2+2n}(R_i)$ for some integer $i \leq n$.

We have proved that γ is an unstable segment included in the rectangle $f^{N_2+n}(R_i)$ for some i. Hence, there exists a (unique) horizontal subrectangle H_γ of the rectangle $f^{N_2+2n}(R_i)$ such that γ is an unstable cross bar of H_γ.

Second step. We consider the rail γ and the horizontal subrectangle H_γ introduced in the first step. For every unstable cross bar γ' of H_γ, we will prove that γ' is a rail leaning on $\mathcal{I}_3(z)$, and that the rails γ and γ' are equivalent.

Let us first remark that each of the two sides sides of H_γ has a non-empty intersection with $\cup\mathcal{I}_3(z)$ (since the ends of γ lie on $\cup\mathcal{I}_3(z)$ and γ is an unstable cross bar of H_γ). Therefore, by corollary 2, we have

$$\cup\mathcal{I}_3(z) \supset \partial^s H_\gamma \tag{1}$$

Now, we prove that $\cup\mathcal{I}_3(z)\cap(H_\gamma \setminus \partial^s H_\gamma) = \emptyset$. For that purpose, let $x \in (\cup\mathcal{I}_3(z))\cap H_\gamma$, and let δ^s be the stable cross bar of H_γ, such that $x \in \delta^s$. We have:
— $\delta^s \subset (\cup\mathcal{I}_3(z))$ (by corollary 2),
— $\gamma\cap\delta^s \neq \emptyset$ (since γ and δ^s are respectively stable and unstable cross bars of H_γ),
— $\mathrm{int}(\gamma)\cap(\cup\mathcal{I}_3(z)) = \emptyset$ (since γ is a rail leaning on $\mathcal{I}_3(z)$).
The three properties above imply that the stable cross bar δ^s is one of the two stable sides of the subrectangle H_γ. In particular, $x \in \partial^s H_\gamma$. Since x is any point in $(\cup\mathcal{I}_3(z))\cap H_\gamma$, we have proved that

$$(\cup\mathcal{I}_3(z))\cap(H_\gamma \setminus \partial^s H_\gamma) = \emptyset \tag{2}$$

Properties (1) and (2) imply that we have $(\cup\mathcal{I}_3(z))\cap H_\gamma = \partial^s H_\gamma$. As a consequence, every unstable cross bar γ' of H_γ is a rail leaning on $\mathcal{I}_3(z)$. Moreover, the rails γ and γ' are equivalent.

Third step. By proposition 16, every element of the family of unstable segments $\mathcal{J}_3(z)$ is a N_2-segment of $(f,K,\mathcal{R})$. Then, by corollary 10, we have $f^p(\cup\mathcal{J}_3(z)) \supset f^{p-N_2}(\partial^u R)$ for every integer $p \geq 0$ On the other hand, the first and second step of the proof imply that the unstable sides of the domain of every equivalence class of rails leaning on $\mathcal{I}_3(z)$ is included in $f^{N_2+2n}(\partial^u R)$. As a consequence, for every $p \geq 2(N_2+n)$, the unstable sides of the domain of every equivalence class of rails leaning on $\mathcal{I}_3(z)$ is included in $f^p(\cup\mathcal{J}_3(z))$.

End of the proof. We have proved that property (i) is satisfied by every integer $p \geq 2(N_2+n)$. By similar arguments, property (ii) is also satisfied by every integer $p \geq 2(N_2+n)$. Hence, we have $p_{min}(z) \leq 2(N_2+n)$. △

Proof of proposition 1. Proposition 1 follows from proposition 9 and 17. △

Proposition 18. *Assume the special point z is a N_0-point of $(f, K, \mathcal{R})$ for some non-negative integer N_0, and consider the integer $N_2 = N_0 + 46n$. Then, for every $p \geq p_{min}(z)$, every element of the families of segments $\mathcal{I}_4(z,p), \mathcal{J}_4(z,p)$ is a (N_2+p)-segments of $(f, K, \mathcal{R})$.*

Proof. By propositions 15 and 16, every element of the families of segments $\mathcal{I}_3(z)$, $\mathcal{J}_3(z)$ is a N_2-segment of $(f, K, \mathcal{R})$. Therefore, we have $\cup\mathcal{I}_3(z) \subset f^{-N_2}(\partial^s R)$ and $\cup\mathcal{J}_3(z) \subset f^{N_2}(\partial^u R)$.

Now, let $p \geq p_{min}(z)$, and I be an element of the family of stable segments $\mathcal{I}_4(z,p)$. The definition of the family of stable segments $\mathcal{I}_4(z,p)$ (see step 4 in subsection 5.1) implies that the segment I is included in $\cup\mathcal{I}_3(z)$, and implies that the ends of I lie in $f^p(\cup\mathcal{J}_3(z))$. As a consequence, the segment I is included in $f^{-N_2}(\partial^s R) \subset f^{-(N_2+p)}(\partial^s R)$, and the ends of I lie in $f^{N_2+p}(\partial^u R)$. In other words, the stable segment I is a (N_2+p)-segment of $(f, K, \mathcal{R})$.

Similar arguments imply that every element of the family of unstable segments $\mathcal{J}_4(z,p)$ is a (N_2+p)-segment of $(f, K, \mathcal{R})$. △

8. **Elementary combinatorial objects.** Proposition 2 claims that there exists an algorithm taking a geometrical type T and an integer $p \geq p_{min}(T)$ as input, and giving back the set of geometrical type $\mathcal{T}(T,p)$. The purpose of the present section is to define the "elementary combinatorial objects" that this algorithm manipulates.

More precisely, for every realizable geometrical type T, we will define the *N-s-code* of a N-point x of $(f_T, K_T, \mathcal{R}_T)$: this is quadruple of integers and binary symbols, which encodes the position of the point x. We will also define the *N-u-code* of a N-point of $(f_T, K_T, \mathcal{R}_T)$, and the N-code of a N-segment of $(f_T, K_T, \mathcal{R}_T)$.

Before that, we need to define the *powers of a geometrical type*, we need to introduce the partial orders $\prec^s_N$ and $\prec^u_N$, and we need to define the *left, right, bottom and top sides* of a rectangle.

8.1. **Powers of a geometrical type.** Let T be a realizable geometrical type, and p be a positive integer. We consider the diffeomorphism f_T, the basic piece K_T, and the geometrized Markov partition $\mathcal{R}_T$, defined in the preliminary of section 6. Recall that the rectangles of the geometrized Markov partition $\mathcal{R}_T$ are denoted by $R_{1,T}, \dots, R_{n_T,T}$.

It is easy to verify that the diffeomorphism f_T^p is a Smale diffeomorphism, and that the compact set K_T is a non-trivial saddle basic piece of the Smale diffeomorphism f_T^p. Moreover, it is easy to verify that $\mathcal{R}_T$ is still a geometrized Markov partition of K_T, when K_T is considered as a basic piece of the diffeomorphism f_T.

Definition (powers of a geometrical type)**.** *The p^{th}* power *of the geometrical type T is the geometrical type of the geometrized Markov partition $\mathcal{R}_T$, where $\mathcal{R}_T$ is considered as a Markov partition of the basic piece K_T, and K_T is considered as a basic piece of the diffeomorphism f_T^p.*

Notations . *The p^{th} power of the geometrical type T will be denoted by*

$$T^{(p)} = \left(n_T,\, h_T^{(p)},\, v_T^{(p)},\, \Phi_T^{(p)},\, \varepsilon_T^{(p)}\right)$$

wit $h_T^{(p)} = \left(h_{1,T}^{(p)}, \dots, h_{n_T,T}^{(p)}\right)$ *and* $v_T^{(p)} = \left(v_{1,T}^{(p)}, \dots, v_{n_T,T}^{(p)}\right)$
For every $i \leq n_T$, the connected components of $(R_{1,T} \cup \cdots \cup R_{n_T,T}) \cap f_T^p(R_{i,T})$ are the images under f_T^p of some horizontal subrectangles of the rectangle $R_{i,T}$. These horizontal subrectangles will be denoted by $H_{i,T}^{1,(p)}$, $H_{i,T}^{2,(p)}, \dots$, the order being induced by the orientation of the unstable cross bars of the rectangle $R_{i,T}$.
For every $k \leq n_T$, the connected components of $R_{k,T} \cap f(R_{1,T} \cup \cdots \cup R_{n_T,T})$ are some vertical subrectangles of the rectangle $R_{k,T}$. These vertical subrectangles will be denoted by $V_{k,T}^{1,(p)}$, $V_{k,T}^{2,(p)}, \dots$, the order being induced by the orientation of the stable cross bars of $R_{k,T}$.

Let us recall what these notations mean:
— $h_{i,T}^{(p)}$ is the number of connected components of $(R_{1,T} \cup \cdots \cup R_{n_T,T}) \cap f_T^p(R_{i,T})$. These components are the image of the horizontal subrectangles $H_{i,T}^{1,(p)}$, $H_{i,T}^{2,(p)}, \dots$.
— $v_{k,T}^{(p)}$ is the number of connected components of $R_{k,T} \cap f(R_{1,T} \cup \cdots \cup R_{n_T,T})$. These components are the vertical subrectangles $V_{k,T}^{1,(p)}$, $V_{k,T}^{2,(p)}, \dots$.
— $\Phi_T^{(p)}(i,j) = (k,l)$ if the diffeomorphism f_T^p maps the horizontal subrectangle $H_{i,T}^{j,(p)}$ to the vertical subrectangle $V_{k,T}^{l,(p)}$,
— $\varepsilon_T^{(p)}(i,j) = +$ if and only if the diffeomorphism f_T^p maps the orientation of the unstable cross bars of $H_{i,T}^{j,(p)}$ to the orientation of the unstable cross bars of $V_{k,T}^{l,(p)}$.

Remark 10. *Theorem 1 implies that the geometrical type $T^{(p)}$ does not depend on the choice of the triple $(f_T, K_T, \mathcal{R}_T)$.*

8.2. **The partial orders $\prec^s_N$ and $\prec^u_N$.** Let T be a realizable geometrical type, and N be a non-negative integer. We consider the diffeomorphism f_T, the basic piece K_T, and the geometrized Markov partition $\mathcal{R}_T$, defined in the preliminary of section 6. Recall that we denote by $\partial^s R_T$ (resp. $\partial^u R_T$) the union of the stable (resp. unstable) sides of the rectangles of the Markov partition $\mathcal{R}_T$.

On the one hand, recall that the stable cross bars of the rectangles of $\mathcal{R}_T$ are oriented (since $\mathcal{R}_T$ is a *geometrized* Markov partition). In particular, the stable sides of the rectangles of $\mathcal{R}_T$ are oriented. On the other hand, recall that the N-points of $(f_T, K_T, \mathcal{R}_T)$ lie on the connected components of $f_T^{-N}(\partial^s R_T)$. These two observations allow us to define the partial order $\prec^s_N$:

Definition (the partial order $\prec^s_N$). *The* N-orientation *of the connected components of $f^{-N}(\partial^s R_T)$ is the image under f_T^{-N} of the orientation of the stable sides of the rectangles of $\mathcal{R}_T$. This orientation induces a partial order on the set of the N-points of $(f_T, K_T, \mathcal{R}_T)$; we denote this partial order by $\prec^s_N$.*

Similarly, we define the partial order $\prec^u_N$:

Definition (the partial order $\prec^u_N$). *The* N-orientation *of the connected components of $f^N(\partial^u R)$ is the image under f_T^N of the orientation of the unstable sides of the rectangles of $\mathcal{R}_T$. This orientation induces a partial order on the set of the N-points of $(f_T, K_T, \mathcal{R}_T)$; we denote this partial order by $\prec^u_N$.*

Remark 11. *Two N-points of $(f_T, K_T, \mathcal{R}_T)$ are comparable for the order $\prec^s_N$ if and only if they lie on the same connected component of $f_T^{-N}(\partial^s R_T)$. Two N-points* *of $(f_T, K_T, \mathcal{R}_T)$ are comparable for the order $\prec^u_N$ if and only if they lie on the same connected component of $f_T^N(\partial^u R_T)$.*

8.3. **The N-s-code and the N-u-code of a N-point. The N-code of a N-segment.** Let T be a realizable geometrical type, and N be a non-negative integer. We consider the diffeomorphism f_T, the basic piece K_T, and the geometrized Markov partition $\mathcal{R}_T$, defined in the preliminary of section 6. We use the notations defined in subsection 8.1.

Definition (left, right, bottom and top side of a rectangle)**.** *Let Q be a rectangle such that the stable (resp. unstable) cross bars of the rectangle Q are oriented.*
The unstable sides of the rectangle Q will be called respectively left side of the rectangle Q *and* right side of the rectangle Q*, in such a way that the orientation of the stable cross bars of Q goes from the left side of Q towards the right side of Q. The left side and right side of the rectangle Q will be denoted by $\partial^{\text{left}}Q$ and $\partial^{\text{right}}Q$.*
The stable sides of the rectangle Q will be called respectively bottom side of the rectangle Q *and* top side of the rectangle Q*, in such a way that the orientation of the unstable cross bars of Q goes from the bottom side of Q towards the top side of Q. The bottom and the top side of the rectangle Q will be denoted respectively by $\partial^{\text{bottom}}Q$ and $\partial^{\text{top}}Q$.*

Remark 12. *For every i, j, k, l, p, our definition allow us to speak of the left, right, bottom and top sides of the subrectangles $V_{i,T}^{j,(p)}$ and $H_{k,T}^{l,(p)}$.*

Let x be a N-point of $(f_T, K_T, \mathcal{R}_T)$. We recall this means that the point x lies in $f_T^{-N}(\partial^s R_T) \cap f_T^N(\partial^u R_T)$.
• Let $y = f_T^N(x)$. On the one hand, the point x lies in $f_T^{-N}(\partial^s R_T)$. As a consequence, there exists an integer $i \leq n_T$, and there exists $\eta =$ *bottom* or *top*, such that the point y lies in $\partial^\eta R_{i,T}$. On the other hand, the point x lies in $f_T^N(\partial^u R_T)$ (and thus, the point y lies in $f_T^{2N}(\partial^u R_T)$). As a consequence, there exists an integer $j \leq v_{i,T}^{(2N)}$, and there exists $\xi =$ *left* or *right*, such that the point y lies in $\partial^\xi V_{i,T}^{j,(2N)}$. Observe that y is the unique point in $\partial^\eta R_{i,T} \cap \partial^\xi V_{i,T}^{j,(2N)} = \partial^\eta V_{i,T}^{j,(2N)} \cap \partial^\xi V_{i,T}^{j,(2N)}$ (more precisely, y is the (η, ξ)-corner of the subrectangle $V_{i,T}^{j,(2N)}$).
• Let $z = f_T^{-N}(x)$. On one hand, the point x lies in $f_T^N(\cup\partial^u \mathcal{R}_T)$. As a consequence, there exists an integer $k \leq n_T$, and there exists $\zeta =$ *left* or *right*, such that the point z lies in $\partial^\zeta R_{k,T}$. On the other hand, the point x lies in $f_T^{-N}(\cup\partial^s \mathcal{R}_T)$ (and thus, the point z lies in $f_T^{-2N}(\cup\partial^s \mathcal{R}_T)$). As a consequence, there exists $l \leq v_{k,T}^{(2N)}$, and there exists $\mu =$ *bottom* or *top*, such that the point z lies in $\partial^\mu H_{k,T}^{l,(2N)}$. Observe that z is the unique point in $\partial^\zeta R_{k,T} \cap \partial^\mu H_{k,T}^{l,(2N)} = \partial^\mu H_{k,T}^{l,(2N)} \cap \partial^\zeta H_{k,T}^{l,(2N)}$ (more precisely, z is the (μ, ζ)-corner of the subrectangle $H_{k,T}^{l,(2N)}$).

Definition (N-s-code and N-u-code of a N-point)**.** *Let x be a N-point of $(f_T, K_T, \mathcal{R}_T)$.*
• *Let $y = f_T^N(x)$. The* N-s-code of x *is the 4-uple (i, η, j, ξ), where i is an integer smaller than n_T, where $\eta =$* bottom *or $\eta =$* top*, where j is an integer smaller than $v_{i,T}^{(2N)}$, where $\xi =$* left *or $\xi =$* right*, and where y is the unique point in $\partial^\eta R_{i,T} \cap \partial^\xi V_{i,T}^{j,(2N)} = \partial^\eta V_{i,T}^{j,(2N)} \cap \partial^\xi V_{i,T}^{j,(2N)}$.*
• *Let $z = f_T^{-N}(x)$. The* N-u-code of x *is the 4-uple (k, ζ, l, μ), where k is an integer smaller than n_T, where $\zeta =$* left *or $\zeta =$* right*, where l is an integer smaller*

than $h_{k,T}^{(2N)}$*, where* $\mu = \text{bottom}$ *or* $\mu = \text{top}$*, and where* z *is the unique point in* $\partial^\zeta R_{k,T} \cap \partial^\mu H_{k,T}^{l,(2N)} = \partial^\mu H_{k,T}^{l,(2N)} \cap \partial^\zeta H_{k,T}^{l,(2N)}$.

Lemma 19. *Let* x *and* y *be two* N*-points of* $(f_T, K_T, \mathcal{R}_T)$.
• *Let* $(i_1, \eta_1, j_1, \xi_1)$ *and* $(i_2, \eta_2, j_2, \xi_2)$ *be the* N*-s-codes of the points* x *and* y*. The points* x *and* y *are comparable with respect to the order* $\prec_{N,s}$ *if and only if* $i_1 = i_2$ *and* $\eta_1 = \eta_2$*. Moreover, if* x *and* y *are comparable, then* $x \prec_{N,s} y$ *if and only if* $(j_2 > j_1)$ *or* $(j_1 = j_2,\ \xi_1 = \text{left}$ *and* $\xi_2 = \text{right})$.
• *Let* $(k_1, \zeta_1, l_1, \mu_1)$ *and* $(k_2, \zeta_2, l_2, \mu_2)$ *be the* N*-u-codes of the points* x *and* y*. The points* x *and* y *are comparable with respect to the order* $\prec_{N,u}$ *if and only if* $k_1 = k_2$ *and* $\zeta_1 = \zeta_2$*. Moreover, if* x *and* y *are comparable, then* $x \prec_{N,u} y$ *if and only if* $(l_2 > l_1)$ *or* $(l_1 = l_2,\ \mu_1 = \text{bottom}$ *and* $\mu_2 = \text{top})$.

Proof. This directly follows from the definition of the N-s-code (resp. the N-u-code) of a N-point, and from the definition of the orders $\prec_N^s$ and $\prec_N^u$. △

Definition (N-code of a N-segment)**.** *Let* I *be a stable* N*-segment of* $(f_T, K_T, \mathcal{R}_T)$*. Let* x *and* y *be the ends of* I *such that* $x \prec_N^s y$*. Let* (i, η, j_1, ξ_1) *and* (i, η, j_2, ξ_2) *be the* N*-s-codes of* x *and* y*. Then, the* N-code of the segment I *is the 6-uple* $(i, \eta, j_1, \xi_1, j_2, \xi_2)$.
Let J *be an unstable* N*-segment of* $(f_T, K_T, \mathcal{R}_T)$*. Let* x *and* y *be the ends of* J *such that* $x \prec_N^u y$*. Let* (k, ζ, l_1, μ_1) *and* (k, ζ, l_2, μ_2) *be the* N*-u-codes of* x *and* y*. Then, the* N-code of the segment J *is the 6-uple* $(k, \zeta, l_1, \mu_1, l_2, \mu_2)$.

9. **Elementary operations.** In the previous section, we have define some "elementary combinatorial objects" (the N-s-code and the N-u-code of a N-point, and the N-code of a N-segment). The purpose of the present section is to define a few "elementary combinatorial operations" on these elementary objects. In subsection 9.1, we define the elementary operations. Then, in subsection 9.2, we prove that these elementary operations are algorithmic.

9.1. **Definition of the elementary operations.** Recall that, for every realizable geometrical type T, we have choosen a Smale diffeomorphism f_T and a non-trivial saddle basic piece K_T of f_T, such that K_T admits a geometrized Markov partition $\mathcal{R}_T$ of geometrical type T. Here is a list of elementary operations:

`Power` — Takes a realizable geometrical type T and an integer p as input. Gives back the p^{th} power $T^{(p)}$ of the geometrical type T (see subsection 8.1).

`UnstableCode` — Takes a realizable geometrical type T, an integer N, and the N-s-code of a N-point x of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-u-code of x.

`StableCode` — Takes a realizable geometrical type T, an integer N, and the N-u-code of a N-point x of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-s-code of x.

`Ends` — Takes a realizable geometrical type T, an integer N, and the N-code of a stable (resp. unstable) N-segment I of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-s-codes (resp. the N-u-codes) of the two ends of I.

`StableSegment` — Takes a realizable geometrical type T, an integer N, and the N-s-codes of two N-points x, y of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-code of the N-segment $[x, y]^s$ (provided that x and y lie on the same connected component of $f_T^{-N}(\partial^s R_T)$).

`UnstableSegment` — Takes a realizable geometrical type T, an integer N, and the N-u-codes of two N-points x, y of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-code of the N-segment $[x,y]^u$ (provided that x and y lie on the same connected component of $f_T^N(\partial^u R_T)$).

`StableSides` (resp. `UnstableSides`) — Takes a realizable geometrical type T and an integer N as input. Gives back the list of the N-codes of all the stable (resp. unstable) sides of the rectangles Markov partition $\mathcal{R}_T$.

`Image` — Takes a realizable geometrical type T, an integer N and the N-code of a (stable or unstable) N-segment I of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-code of the segment $f_T(I)$ (provided that $f_T(I)$ is a N-segment of $(f_T, K_T, \mathcal{R}_T)$).

`InverseImage` — Takes a realizable geometrical type T, an integer N and the N-code of a (stable or unstable) N-segment I of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-code of the stable segment $f_T^{-1}(I)$ (provided that $f_T^{-1}(I)$ is a N-segment of $(f_T, K_T, \mathcal{R}_T)$).

`PeriodicStableSides` (resp. `PeriodicUnstableSides`) — Takes a realizable geometrical type T, an integer $N \geq 2n_T$ as input. Gives back the list of the N-codes of the periodic stable (resp. unstable) sides of the rectangles of the Markov partition $\mathcal{R}_T$.

`StablePredecessor` — Takes a realizable geometrical type T, an integer N and the N-s-code of a N-point x of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-s-code of the predecessor of x with respect to the order $\prec_{s,N}$ (provided that this predecessor does exist).

The elementary operations `StableSuccessor` , `UnstablePredecessor` and `UnstableSuccessor` are defined similarly.

`StableOrder` — Takes a realizable geometrical type T, an integer N and the N-s-codes of two N-points x, y of $(f_T, K_T, \mathcal{R}_T)$ as input. Returns `yes` if $x \prec_{N,s} y$, and `no` otherwise.

The elementary operation `UnstableOrder` is defined similarly.

`PointsOnStableSegment` — Takes a realizable geometrical type T, an integer N and the N-code of a stable N-segment I of $(f_T, K_T, \mathcal{R}_T)$ as input. Gives back the N-s-codes of all the N-points of $(f_T, K_T, \mathcal{R}_T)$ that lie on the segment I.

The elementary operation `PointsOnUnstableSegment` is defined similarly.

`Intersection` — Takes a realizable geometrical type T, an integer N, the N-code of stable N-segment I and the N-code of an unstable N-segment J of T as input. Gives back the list of the N-s-codes of the N-points that lie in $I \cap J$.

`StableArch` (resp. `UnstableArch`) — Takes a realizable geometrical type T, an integer N and the N-code of a stable (resp. unstable) N-segment I of $(f_T, K_T, \mathcal{R}_T)$ as input. Returns `yes` if I is a stable (resp. unstable) arch, and `no` otherwise.

9.2. **The elementary operations are algorithmic.**

Proposition 19. *The operation* `Power` *is algorithmic.*

Proof. See appendix A. △

Proposition 20. *The operations* `StableCode` *,* `UnstableCode` *are algorithmic.*

To prove proposition 20, we need to introduce some notations:

Notation . *If $\varepsilon \in \{+,-\}$ and $\eta \in \{\text{bottom}, \text{top}\}$, then we consider the binary symbol $\varepsilon \star \eta$ defined in the first array below. Similarly, if $\varepsilon \in \{+,-\}$ and $\xi \in \{\text{left}, \text{right}\}$, then we consider the binary symbol $\varepsilon \star \xi$ defined in the second array below.*

	$\eta=$ bottom	$\eta=$ top
$\varepsilon=+$	$\varepsilon\star\eta=$ bottom	$\varepsilon\star\eta=$ top
$\varepsilon=-$	$\varepsilon\star\eta=$ top	$\varepsilon\star\eta=$ bottom

	$\xi=$ left	$\xi=$ right
$\varepsilon=+$	$\varepsilon\star\xi=$ left	$\varepsilon\star\xi=$ right
$\varepsilon=-$	$\varepsilon\star\xi=$ right	$\varepsilon\star\xi=$ left

The following lemma is the key of the proof of proposition 20:

Lemma 20. *Let T be a realizable geometrical type, N be a non-negative integer, and x be a N-point of $(f_T, K_T, \mathcal{R}_T)$. Let (i,η,j,ξ) and (k,ζ,l,μ) be respectively the N-s-code and the N-u-code of x. Then, the following equalities hold:*

$$(i,j)=\Phi_T^{(2N)}(k,l)\quad \eta=\varepsilon_T^{(2N)}(k,l)\star\mu\quad \xi=\varepsilon_T^{(2N)}(k,l)\star\zeta$$

Proof. The definition of the N-u-code of the point x implies that the point $f_T^{-N}(x)$ lies in $\partial^\zeta H_{k,T}^{l,(2N)}\cap\partial^\mu H_{k,T}^{l,(2N)}$. The definition of the N-s-code of the point x implies that the point $f_T^N(x)$ lies in $\partial^\eta V_{i,T}^{j,(2N)}\cap\partial^\xi V_{i,T}^{j,(2N)}$. The definitions of the bijection $\Phi_T^{(2N)}$ and of the function $\varepsilon_T^{(2N)}$ implies that:
(i)The diffeomorphism f_T^{2N} maps the horizontal subrectangle $H_{k,T}^{l,(2N)}$ to the vertical subrectangle $V_{i,T}^{j,(2N)}$ where $(i,j)=\Phi_T^{(2N)}(k,l)$.
(ii)The diffeomorphism f_T^{2N} maps $\partial^\mu H_{k,T}^{l,(2N)}$ to $\partial^\eta V_{i,T}^{j,(2N)}$, where $\eta=\varepsilon_T^{(2N)}(k,l)\star\mu$.
(iii)The diffeomorphism f_T^{2N} maps $\partial^\zeta H_{k,T}^{l,(2N)}$ to $\partial^\xi V_{i,T}^{j,(2N)}$, where $\xi=\varepsilon^{(2N)}(k,l)\star\zeta$.
This proves the lemma. △

Proof of proposition 20. This follows from lemma 20 and proposition 19. △

Proposition 21. *The operations* `Ends` , `StableSegment` , `UnstableSegment` *are algorithmic.*

Proof. This trivially follows from the definition of the N-code of a N-segment. △

Proposition 22. *The operations* `StableSides` , `UnstableSides` *are algorithmic.*

To prove proposition 22, we need to introduce some notations:

Notation . *Let p and q be two positive integers such that $p\le q$.*
For every $i\le n_T$, and every $j\le h_{i,T}^{(p)}$, let $\theta_{i,T}^{j,(p),(q)}$ be the integer defined as follows:

$$\theta_{i,T}^{j,(p),(q)}=\sharp\left\{m \mid \text{ the subrectangle } H_{i,T}^{m,(q)} \text{ is included in the subrectangle } H_{i,T}^{j,(p)}\right\}$$

For every $k\le n_T$, and every $l\le v_{k,T}^{l,(p)}$, let $\nu_{k,T}^{l,(p),(q)}$ be the integer defined as follows:

$$\nu_{k,T}^{l,(p),(q)}=\sharp\left\{m \mid \text{ the subrectangle } V_{k,T}^{m,(q)} \text{ is included in the subrectangle } V_{k,T}^{l,(q)}\right\}$$

Lemma 21. *The following equalities hold:*
- $\theta_{i,T}^{j,(p),(q)}=h_{k,T}^{(q-p)}$, *where k is defined by $\Phi_T^{(p)}(i,j)=(k,l)$.*
- $\nu_{k,T}^{l,(p),(q)}=v_{i,T}^{(q-p)}$, *where i is defined by $(\Phi_T^{(p)})^{-1}(k,l)=(i,j)$.*

Proof. By definition, $\theta_{i,T}^{j,(p),(q)}$ is the number of value of the integer m, such that the horizontal subrectangle $H_{i,T}^{m,(q)}$ is included in the horizontal subrectangle $H_{i,T}^{j,(p)}$. Now, let us recall that, for every m, the image under $f^{(q)}$ of the horizontal subrectangle $H_{i,T}^{m,(q)}$ is a connected component of $f_T^q(H_{i,T}^{j,(p)})\cap R_T$. As a consequence, $\theta_{i,T}^{j,(p),(q)}$ is equal to the number of connected components of $f_T^q(H_{i,T}^{j,(p)})\cap R_T$. On the other hand, we have:

$$f_T^q(H_{i,T}^{j,(p)})=f_T^{q-p}(f_T^p(H_{i,T}^{j,(p)}))=f_T^{q-p}(V_{k,T}^{l,(p)}) \text{ where } (k,l)=\Phi_T^{(p)}(i,j)$$

This implies that $\theta_{i,T}^{j,(p),(q)}$ is equal to the number of connected components of $f_T^{q-p}(V_{k,T}^{l,(p)}) \cap R_T$. This number is equal to the number of connected component of $f_T^{q-p}(R_{T,k}) \cap R_T$. By definition, the number $h_{T,k}^{(q-p)}$ is the number of connected component of $f_T^{q-p}(R_{T,k}) \cap R_T$. This completes the proof of the first item ; the proof of second item is similar. △

Corollary 7. *The operations* $(T, i, j, p, q) \mapsto \theta_{T,i}^{j,(p),(q)}$ *and* $(T, k, l, p, q) \mapsto \nu_{T,k}^{l,(p),(q)}$ *are algorithmic.*

Proof. This follows immediately from proposition 19 and 21. △

Proof of proposition 22.
First step: the N-s-code of the bottom-left corner of the rectangle $R_{k,T}$. Let T be a geometrical type, and N be a non-negative integer. Given an integer $k \leq n_T$, we denote by x the bottom-left corner of the rectangle $R_{k,T}$. We want to find the N-s-code of the point x.

On one hand, the point x lies on the bottom side of the horizontal subrectangle $H_{k,T}^{1,(N)}$. Thus, the point $f_T^N(x)$ lies on $\partial^\eta V_{T,i}^{j',(N)} \subset \partial^\eta R_{T,i}$ where

$$\begin{cases} (i, j') = \Phi_T^{(N)}(k, 1) & \\ \eta = \textit{bottom} & \text{if } \varepsilon_T^{(N)}(k, 1) = + \\ \eta = \textit{top} & \text{if } \varepsilon_T^{(N)}(k, 1) = - \end{cases} \tag{3}$$

On the other hand, the point x lies on the left side of the horizontal subrectangle $H_{T,k}^{1,(N)}$. As a consequence, the point $f_T^N(x)$ lies on $\partial^\xi V_{T,i}^{j',(N)}$ where

$$\begin{cases} \xi = \textit{left} & \text{if } \varepsilon_T^{(N)}(k, 1) = + \\ \xi = \textit{right} & \text{if } \varepsilon_T^{(N)}(k, 1) = - \end{cases} \tag{4}$$

As a further consequence, the point $f_T^N(x)$ lies on $\partial^\xi V_{T,i}^{j,(2N)}$ where

$$\begin{cases} j = \sum_{\alpha=1}^{\alpha=j'-1} \nu_{T,i}^{\alpha,(N),(2N)} + 1 & \text{if } \xi = l \\ j = \sum_{\alpha=1}^{\alpha=j'} \nu_{T,i}^{\alpha,(N),(2N)} & \text{if } \xi = r \end{cases} \tag{5}$$

By definition, the N-s-code of the point x is (i, η, j, ξ).

Second step: end of the proof. Let us consider the operation which takes a geometrical type T, an integer N and an integer $k \leq n_T$ as input, and gives back the N-s-code (i, η, j, ξ) of the bottom-left corner of the rectangle $R_{k,T}$. Formulas (3), (4), (5) above imply that this operation is algorithmic. Of course, the similar operation which deal with the right-bottom, the left-top and the right-top corners are algorithmic as well. Using proposition 21, this implies that the operations `StableSides` and `UnstableSides` are algorithmic. △

Proposition 23. *The operations* `Image` *and* `InverseImage` *are algorithmic.*

Proof.
First step: N-s-code of the image of a N-point. Let T be a geometrical type. Let x be a N-point of $(f_T, K_T, \mathcal{R}_T)$ such that $f_T(x)$ is also a N-point of $(f_T, K_T, \mathcal{R}_T)$. Let (i, η, j, ξ) be the N-s-code of the point x. We want to find the N-s-code of the point $f_T(x)$.

On one hand, by definition of the N-s-code of the point x, the point $f_T^N(x)$ lies on $\partial^\eta V_{i,T}^{j,(2n)} \subset \partial^\eta R_{i,T}$. Thus, if we consider the integer m defined by:

$$\begin{cases} m = 1 & \text{if } \eta = \textit{bottom} \\ m = h_{i,T}^{(1)} & \text{if } \eta = \textit{top} \end{cases} \tag{6}$$

then the point $f_T^N(x)$ lies on $\partial^\eta H_{i,T}^{m,(1)}$. And thus, the point $f_T^N(f_T(x))$ lies on $\partial^\mu V_{k,T}^{r,(1)} \subset \partial^\mu R_{k,T}$ where:

$$\begin{cases} (k,r) = \Phi_T^{(1)}(i,m) \\ \mu = \varepsilon_T^{(1)}(i,m) \star \eta \end{cases} \tag{7}$$

On the other hand, by definition of j and ξ, the point $f_T^N(x)$ lies on $\partial^\xi V_{i,T}^{j,(2N)}$. Since the point $f_T(x)$ is a N-point of $(f_T, K_T, \mathcal{R}_T)$ (by assumption), the point $f_T^N(x)$ lies on $f_T^{2N-1}(\partial^u R_T)$. As a consequence, the point $f_T^N(x)$ lies on $\partial^\xi V_{T,i}^{j',(2N-1)}$, where the integer j' is defined by:

$$\begin{cases} \text{If } \xi = \textit{left} & \text{then } j = \sum_{\alpha=1}^{\alpha=j'-1} \nu_{T,i}^{\alpha,(2N-1),(2N)} + 1 \\ \text{If } \xi = \textit{right} & \text{then } j = \sum_{\alpha=1}^{\alpha=j'} \nu_{T,i}^{\alpha,(2N-1),(2N)} \end{cases} \tag{8}$$

The connected components of $H_{i,T}^{m,(1)} \cap f_T^{2N-1}(R_T)$ are vertical subrectangles of the horizontal subrectangle $H_{i,T}^{m,(1)}$. The point $f_T^N(x)$ is in the $(j')^{th}$ of these vertical subrectangles (the order of the subrectangles being induced by the orientation of the horizontal cross bars of $H_{i,T}^{m,(1)}$). The diffeomorphism f_T maps the horizontal subrectangle $H_{i,T}^{m,(1)}$ to the vertical subrectangle $V_{k,T}^{r,(1)}$. Moreover, the diffeomorphism f_T maps the orientation of the horizontal cross bars of $H_{i,T}^{m,(1)}$ to the orientation of the horizontal cross bars of $V_{k,T}^{r,(1)}$ if and only if $\varepsilon_T^{(1)}(i,m) = +$. As a consequence, the point $f_T^N(f_T(x)) = f_T(f_T^N(x))$ lies in the vertical subrectangle $V_{k,T}^{l,(2N)}$ where l is defined by:

$$\begin{cases} l = \sum_{\alpha=1}^{\alpha=r-1} \nu_{T,k}^{\alpha,(1),(2N)} + j' & \text{if } \varepsilon_T^{(1)}(i,m) = + \\ l = \sum_{\alpha=1}^{\alpha=r} \nu_{T,k}^{\alpha,(1),(2N)} + 1 - j' & \text{if } \varepsilon_T^{(1)}(i,m) = - \end{cases} \tag{9}$$

Moreover, since the point $f_T^N(x)$ lies on $\partial^\xi V_{T,i}^{j',(2N-1)}$, the point $f_T^N(f_T(x))$ lies on $\partial^\zeta V_{T,k}^{l,(2N)}$ where ζ is defined by:

$$\zeta = \varepsilon_T^{(1)}(i,m) \star \xi \tag{10}$$

By definition, the N-s-code of the N-point $f_T(x)$ is (k, μ, l, ζ) where k, μ, l and ζ are defined by formulas (6)...(10).

Second step: end of the proof. Let us consider the operation which takes a geometrical type T, an non-negative integer N, and the N-s-code (i, η, j, ξ) of a N-point x of $(f_T, K_T, \mathcal{R}_T)$ as input, and gives back the N-s-code (k, μ, l, ζ) of the point $f_T(x)$. Formulas (6)...(10) obtained above and proposition 19 imply that this operation is algorithmic. Using proposition 21, this immediately implies that the operation `Image` is algorithmic. △

Proposition 24. *The operations* `StableOrder` *,* `UnstableOrder` *,* `StablePredecessor` *,* `StableSuccessor` *,* `UnstablePredecessor` *and* `UnstableSuccessor` *are algorithmic.*

Proof. This directly follows from lemma 19. △

Proposition 25. *The operations* `PointsOnStableSegment` *and* `PointsOnUnstableSegment` *are algorithmic.*

Proof. Given a realizable geometrical type T, a non-negative integer N, and the N-code of a stable N-segment I of $(f_T, K_T, \mathcal{R}_T)$,

- Let x and y be the ends of the segment I, such that $x \prec^s_N y$. The N-s-codes of the points x and y are given by the operation `Ends` .
- Observe that a N-point z of $(f_T, K_T, \mathcal{R}_T)$ lies on the segment I if and only if $x \prec^s_N z \prec^s_N y$. Thus, we can use the operation `StableSuccessor` to enumerate the N-s-codes of the N-points which lie on the segment I. △

Proposition 26. *The operations* `PeriodicStableSides` *and* `PeriodicUnstableSides` *are algorithmic.*

Proof. This follows from propositions 22 and 23, and item (i) of remark 6. △

Proposition 27. *The operation* `Intersection` *is algorithmic.*

Proof. Given a realizable geometrical type T, a poistive integer N, the N-code of a stable N-segment I and the N-code of an unstable N-segment J of $(f_T, K_T, \mathcal{R}_T)$,

- The operation `PointsOnStableSegment` gives the list $\mathcal{L}_I$ of the N-s-codes of the N-points that lie on the stable segment I,
- The operation `PointsOnUnstableSegment` and `StableCode` give the list $\mathcal{L}_J$ of the N-s-codes of the N-points that lie on the unstable segment J,
- Considering the intersection of the lists $\mathcal{L}_I$ and $\mathcal{L}_J$, we obtain the list of the N-s-codes of the N-points that lie in $I \cap J$. △

Proposition 28. *The operations* `StableArch` , `UnstableArch` *are algorithmic.*

To prove proposition 28, we need to introduce the notion of *minimal N-segment*:

Definition (minimal N-segment)**.** *Let T be a realizable geometrical type. A N-segment I of $(f_T, K_T, \mathcal{R}_T)$ is said to be* minimal*, if I is non-trivial and there does not exist any N-point of $(f_T, K_T, \mathcal{R}_T)$ in the interior of I.*

Lemma 22. *Let I be a stable N-segment of $(f_T, K_T, \mathcal{R}_T)$. Then, I is a stable arch of (f_T, K_T) if and only if I is minimal and I is not included in $f_T^N(R_T)$.*

Proof. If I is a stable arch, then there does not exist any point of K_T in the interior of I. In particular, if I is a stable arch, then I is a minimal.

From now on, we assume that I is minimal. Since I is a minimal N-segment of $(f_T, K_T, \mathcal{R}_T)$, we have $\text{int}(I) \cap f_T^N(\partial^u R_T) = \emptyset$. As a consequence, one of the two following possibilities holds: either $\text{int}(I) \cap f_T^N(R_T) = \emptyset$, or $I \subset f_T^N(R_T)$

Let us first consider the case where $\text{int}(I) \cap f_T^N(R_T) = \emptyset$. Then, there is no point of K_T in the interior of I (since $K_T \subset f_T^N(R_T)$), *i.e.* I is a stable arch of (f_T, K_T).

Now, let us consider the case where $I \subset f_T^N(R_T)$. Recall that I is a non-trivial stable segment of (f_T, K_T), such that the both ends of I lie on $f_T^N(\partial^u R_T)$. Therefore, there exists an integer $i \leq n$ such that I is a stable cross bar of the rectangle $f_T^N(R_{T,i})$. As a consequence, the lamination $W^u(K_T)$ does intersect the interior of I. As a further consequence, the segment I cannot be a stable arch. △

Proof of proposition 28.
First step. Let T be a geometrical type, N be a non-negative integer, and I be a stable N-segment of $(f_T, K_T, \mathcal{R}_T)$. Let $(i, \eta, j_1, \xi_1, j_2\xi_2)$ be the N-code of I. We will prove that I is a stable arch if and only if $j_2 = j_1 + 1$, $\xi_1 =$ *right* and $\xi_2 =$ *left.*

Let us denote by x and y the ends of the segment I (where $x \prec_N^s y$). The N-s-codes of x and y are respectively (i, η, j_1, ξ_1) and (i, η, j_2, ξ_2). Now, let us observe that the N-segment I is minimal if and only if y is the stable successor of x. Therefore, by lemma 19, the N-segment I is minimal if and only if one of the two following possibilities hold:
(i) $j_2 = j_1$ $\eta_1 = \textit{left}$ $\eta_2 = \textit{right}$
(ii) $j_2 = j_1 + 1$ $\eta_1 = \text{right}$ $\eta_2 = \text{left}$
• If (i) holds, then $f_T^N(I) = \partial^\eta V_{i,T}^{j,(2N)}$. In particular, $f_T^N(I) \subset f_T^{2N}(R_T)$, *i.e.* $I \subset f_T^N(R_T)$. Then, lemma 22 implies that I is not a stable arch.
• If (ii) holds, then $f_T^N(I)$ is the subsegment of the stable side $\partial^\eta R_{i,T}$ located between the vertical subrectangles $V_{i,T}^{j,(2N)}$ and $V_{i,T}^{j+1,(2N)}$. In particular, the interior of the stable segment $f_T^N(I)$ is disjoined from $f_T^{2N}(R_T)$, *i.e.* the interior of the segment I is disjoined from $f_T^N(R_T)$. Then, lemma 22 imply that I is a stable arch.

Second step: end of the proof. We have proved that the operation `StableArch` takes a geometrical type T, a non-negative integer N and a N-code $(i, \eta, j_1, \xi_1, j_2, \xi_2)$ as input, and returns `yes` if $j_2 = j_1 + 1$, $\xi_1 = \textit{right}$ and $\xi_2 = \textit{left}$, and returns `no` otherwise. In particular, the operation `StableArch` is algorithmic. Similar arguments imply that the operation `UnstableArch` is algorithmic. △

10. **Description of the algorithm.** In this last section, we will describe step by step an algorithm which takes a realizable geometrical type T, and an integer $p \geq p_{min}(T)$ as input, and gives back the finite set of geometrical type $\mathcal{T}(T, p)$. This will complete the proof of proposition 2 and theorem 2.

For every geometrical type T and every integer $p \geq p_{min}(T)$, we consider the integers $N_0(T) := 23n_T^2$ and $N(T, p) := N_0(T) + 46n_T + p$.

10.1. **Construction of the $N(T,p)$-codes of some special points of (f_T, K_T).** For every realizable geometrical type T, we consider the set

$$\text{Sp}(T) := \{z \mid z \text{ is a special point of } K_T \text{ and a } N_0(T)\text{-point of } (f_T, K_T, \mathcal{R}_T)\}$$

Recall that, for every special point z of (f_T, K_T), there exists an integer $k \in \mathbb{Z}$ such that $f_T^k(z) \in \text{Sp}(T)$ (proposition 9). The aim of this subsection is to prove the following result:

Proposition 29. *There exists an algorithm which takes a realizable geometrical type T and an integer $p \geq p_0(T)$ as input, and gives back the list of the $N(T,p)$-codes of the elements of Sp(T).*

Let T be a realizable geometrical type, and z be a $N_0(T)$-point of $(f_T, K_T, \mathcal{R}_T)$. By item (i) and (ii) of remark 6, the point z lies on the stable manifold of a periodic s-boundary point x ; we consider the stable interval $I_0(z) :=]x, z]^s$. Similarly, z lies on the unstable manifold of a periodic u-boundary y ; we consider the unstable interval $J_0(z) :=]y, z]^u$. By definition, z is special point (*i.e.* $z \in \text{Sp}(T)$) if and only if $I_0(z) \cap J_0(z) = \{z\}$.

The following technical difficulty is arising: in general, the points x and y are not N-points of $(f_T, K_T, \mathcal{R}_T)$ for any integer N ; as a consequence, there is no straightforward algorithm to decide whether the equality $I_0(z) \cap J_0(z) = \{z\}$ does or does not hold. To get round this technical difficulty, we will introduce a stable segment $\widetilde{I_0}(z)$ and an unstable segment $\widetilde{J_0}(z)$:
• We denote by $\widetilde{I_0}(z)$ the union of all the stable $N(T,p)$-segments of $(f_T, K_T, \mathcal{R}_T)$ which are included in the stable interval $I_0(z)$. Observe that $\widetilde{I_0}(z)$ is a stable

$N(T,p)$-segment of $(f_T, K_T, \mathcal{R}_T)$ (since, by lemma 9, the stable interval $I_0(z)$ is included in a connected component of $f_T^{N(T,p)}(\partial^s R_T)$). We say that $\widetilde{I_0}(z)$ is the *stable $N(T,p)$-segment associated with z*.

• We denote by $\widetilde{J_0}(z)$ the union of all the unstable $N(T,p)$-segments of $(f_T, K_T, \mathcal{R}_T)$ that are included in the unstable interval $J_0(z)$. Then, $\widetilde{J_0}(z)$ is an unstable $N(T,p)$-segment of $(f_T, K_T, \mathcal{R}_T)$. We say that $\widetilde{J_0}(z)$ is the *unstable $N(T,p)$-segment associated with z*.

Remarks 7. *Here are the main properties of the segments $\widetilde{I_0}(z)$ and $\widetilde{J_0}(z)$:*
(i) *If w is a $N(T,p)$-point of $(f_T, K_T, \mathcal{R}_T)$ such that $w \in I_0(z)$ (resp. such that $w \in J_0(z)$), then $w \in \widetilde{I_0}(z)$ (resp. $w \in \widetilde{J_0}(z)$) (this follows directly from the definition of the segment $\widetilde{I_0}(z)$).*
(ii) *Recall that the point z is a special point of K_T, if and only if $I_0(z) \cap J_0(z) = \{z\}$. Then, using item (i) and (ibis), we obtain the following fact: the point z is a special point of K_T, if and only if $\widetilde{I_0}(z) \cap \widetilde{J_0}(z) = \{z\}$.*

Lemma 23. *There exists an algorithm wich takes a realizable geometrical type T, an integer $p \geq p_{min}(T)$ and the $N(T,p)$-s-code of a $N_0(T)$-point z of $(f_T, K_T, \mathcal{R}_T)$ as input, and gives back the $N(T,p)$-code of the stable $N(T,p)$-segment $\widetilde{J_0}(z)$ associated with z.*

Proof. Let us consider a realizable geometrical type T, an integer $p \geq p_0(T)$ and a $N_0(T)$-point z of $(f_T, K_T, \mathcal{R}_T)$. We denote by x the unique periodic s-boundary point such that $z \in W^s(x)$, and we denote by y the unique periodic u-boundary point such that $z \in W^u(y)$. We consider the stable interval $I_0(z) :=]x, z]^s$ and the unstable interval $J_0(z) :=]y, z]^s$. We denote by $\widetilde{I_0}(z)$ and $\widetilde{J_0}(z)$ the stable and the unstable $N(T,p)$-segments associated with z.

Observation 1. The point z is a $N(T,p)$-point of $(f_T, K_T, \mathcal{R}_T)$; as a consequence, z is one of the two ends of the segment $\widetilde{I_0}(z)$ (item (i) of remark 7). As a further consequence, we have $\widetilde{I_0}(z) =]\widetilde{x}, z]^s$ where $\widetilde{x}$ is the unique $N(T,p)$-point of $(f_T, K_T, \mathcal{R}_T)$, such that $\widetilde{z} \in]x, z]^s$, and such that there does not exist any $N(T,p)$-point of $(f_T, K_T, \mathcal{R}_T)$ in $]x, \widetilde{x}[^s$.

Observation 2. According to lemma 8, the periodic s-boundary point x lies in a connected component δ^s of $f_T^{N(T,p)}(\partial^s_{per} R_T)$. Observe that δ^s is the unique connected component of $f_T^{N(T,p)}(\partial^s_{per} R_T)$, such that the ends of δ^s are comparable to z for the order $\prec^s_{N(T,p)}$. Moreover, by lemma 7, there does not exist any $N(T,p)$-point of (f_T, K_T, f_T) in the interior of the segment δ^s. As a consequence, the point $\widetilde{x}$ is one of the ends of the segment δ^s.

Description of the algorithm. Given a realizable geometrical type T, and the $N(T,p)$-s-code of a point $z \in \mathrm{Sp}(T)$,

- Using the operations `PeriodicStableSides` and `Image` , we obtain the $N(T,p)$-codes of the connected components of $f_T^{N(T,p)}(\partial^s_{per} R_T)$.
- Let $\mathcal{E}(T)$ be the set made of the ends of the connected components of $f_T^{N(T,p)}(\partial^s_{per} R_T)$. since we know the $N(T,p)$-codes of the connected components of $f_T^{N(T,p)}(\partial^s_{per} R_T)$, we can use the operation `Ends` to obtain the $N(T,p)$-s-codes of the elements of $\mathcal{E}(T)$.
- We denote by a and b the two elements of $\mathcal{E}(T)$ that are comparable to z for the order $\prec^s_{N(T,p)}$. The $N(T,p)$-s-codes of a and b can be found (among the

$N(T,p)$-s-codes of the elements of $\mathcal{E}(T)$) thanks to the operation `StableOrder` .

- We can use the operation `StableOrder` to decide whether $a \in [b,z]^s$, or $b \in [a,z]^s$. If $a \in [b,z]^s$, then we consider the point $\widetilde{x} := a$. If $b \in [a,z]^s$, then we consider the point $\widetilde{x} := b$. In both case, we know the $N(T,p)$-s-code of $\widetilde{x}$.
- The segment $[\widetilde{x},z]^s$ is the stable $N(T,p)$-segment associated with z (this follows from *observation 1* and *observation 2* above). Since we know the N-s-codes of the points $\widetilde{x}$ and z, we can use the operation `StableSegment` to get the $N(T,p)$-code of this segment. △

Proof of proposition 29. Recall that a point z is in the set $\mathrm{Sp}(T)$, if and only if z is a $N_0(T)$-point of $(f_T, K_T, \mathcal{R}_T)$, and z is a special point of K_T. Recall that a $N_0(T)$-point of $(f_T, K_T, \mathcal{R}_T)$ is a point that lies in $f_T^{-N_0(T)}(\partial^s R_T) \cap f_T^{N_0(T)}(\partial^u R_T)$. Finally, recall that a $N_0(T)$-point z of $(f_T, K_T, \mathcal{R}_T)$ is a special point of K_T, if and only if $\widetilde{I_0}(z) \cap \widetilde{J_0}(z) = \{z\}$ (see remark 7). This leads to the following algorithm:

Description of the algorithm. Given a geometrical type T and $p \geq p_0(T)$,

- The operations `StableSides` give the $N(T,p)$-codes of the stable sides of the rectangles of the Markov partition $\mathcal{R}_T$, *i.e.* the connected components of $\partial^s R_T$. Then, using the operation `InverseImage` , we obtain the $N(T,p)$-codes of the connected components of $f_T^{-N_0(T)}(\partial^s R_T)$.
- Similarly, using the operation `UnstableSides` and `Image` , we obtain the $N(T,p)$-codes of the connected components of $f_T^{N_0(T)}(\partial^u R_T)$.
- Then, using the operation `Intersection` , we obtain the $N(T,p)$-s-codes of all the points that lie in $f_T^{-N_0(T)}(\partial^s R_T) \cap f_T^{N_0(T)}(\partial^u R_T)$, that is the $N(T,p)$-s-codes of all the $N_0(T)$-points of $(f_T, K_T, \mathcal{R}_T)$.
- Given the $N(T,p)$-s-code of a $N_0(T)$-point z of $(f_T, K_T, \mathcal{R}_T)$,
  - By lemma 23, there exists an algorithm which gives back the $N(T,p)$-code of the stable $N(T,p)$-segment $\widetilde{I_0}(z)$ associated with z,
  - Similarly, there exists an algorithm which gives back the $N(T,p)$-code of the unstable $N(T,p)$-segment $\widetilde{J_0}(z)$ associated with z,
  - Using the operation `Intersection` , we obtain the list of the $N(T,p)$-s-codes of the points that lie in $\widetilde{I_0}(z) \cap \widetilde{J_0}(z)$; in particular, we can decide whether the equality $\widetilde{I_0}(z) \cap \widetilde{J_0}(z) = \{z\}$ does or does not hold. The point z is an element of the set $\mathrm{Sp}(T)$ if and only if this equality does hold. △

10.2. Construction of the $N(T,p)$-codes of the sides of the rectangles of the Markov partition $\mathcal{R}(z,p)$.

Step 1. The $N(T,p)$-codes of the elements of the families of segments $\mathcal{I}_1(z)$ and $\mathcal{J}_1(z)$. For realizable geometrical type T and every point $z \in \mathrm{Sp}(T)$, proposition 11 implies that each element of the family of stable segments $\mathcal{I}_1(z)$ is a $N(T,p)$-segment of $(f_T, K_T, \mathcal{R}_T)$. We will prove the following result:

Proposition 30. *There exists an algorithm which takes a geometrical type T, an integer $p \geq p_0(T)$, and the $N(T,p)$-code of a point point $z \in Sp(T)$ as input, and gives back the $N(T,p)$-codes of the elements of the family of stable segments $\mathcal{I}_1(z)$.*

Proof of proposition 30.
Notations. Let T be a realizable geometrical type, $p \geq p_0(T)$ be an integer, and z be a point in $\mathrm{Sp}(T)$. We will use the following notations:

— Let y be the periodic u-boundary point of K_T, such that $z \in W^u(y)$. Recall that $\mathcal{J}_0(z)$ is the family of unstable intervals $\{J_0(z),\ f_T(J_0(z)), \dots,\ f_T^{2q-1}(J_0(z))\}$, where $J_0(z) =]y, z]^u$ and q is the period of y.
— For every $i \in \mathbb{N}$, we denote by $\widetilde{J}_0^i(z)$ the unstable $N(T,p)$-segment associated with the point $f_T^i(z)$, as defined in subsection 10.1. By definition, $\widetilde{J}_0^i(z)$ is equal to the unio of all the $N(T,p)$-segments of $(f_T, K_T, \mathcal{R}_T)$ which are included in the unstable interval $f_T^i(J_0(z))$. Then, we consider the family of unstable $N(T,p)$-segments $\widetilde{\mathcal{J}}_0(z) := \{\widetilde{J}_0(z), \dots, \widetilde{J}_0^{2q-1}(z))\}$.
— We denote by $\mathcal{E}_0(z)$ the set of all the $N(T,p)$-points of $(f_T, K_T, \mathcal{R}_T)$ that lie on $\bigcup \mathcal{J}_0(z)$. Observe that $\mathcal{E}_0(z)$ is also the set of all the $N(T,p)$-points of $(f_T, K_T, \mathcal{R}_T)$ that lie on $\bigcup \widetilde{\mathcal{J}}_0(z)$ (see item (i) of remark 7).
— Let I be a stable $N(T,p)$-segment of $(f_T, K_T, \mathcal{R}_T)$. Let a and b be the ends of I, such that $a \prec^s_{N(T,p)} b$. We consider the sets $\mathcal{E}_0^-(z,I)$ and $\mathcal{E}_0^+(z,I)$ defined as follows:

$$\mathcal{E}_0^-(z,I) := \{x \in \mathcal{E}_0(z) \mid x \prec^s_{N(T,p)} a\} \text{ and } \mathcal{E}_0^+(z,I) = \{x \in \mathcal{E}_0(z) \mid b \prec^s_{N(T,p)} x\}$$

Preliminary observations. Let T be a realizable geometrical type, $p \geq p_0(T)$ be an integer, and z be a point in $\mathrm{Sp}(T)$. The following observations are the key of the proof of proposition 30:
(i) By item (iii) remark 6, we have $q \leq 2n_T$. As a consequence, we have

$$\bigcup \mathcal{J}_0(z) = \mathcal{J}_0(z) \cup \cdots \cup f_T^{4n_T-1}(J_0(z)) \text{ and } \bigcup \mathcal{J}_0(z) = \widetilde{J}_0(z) \cup \cdots \cup \widetilde{J}_0^{4n_T-1}(z)$$

(ii) Proposition 11 implies that every element of the family $\mathcal{I}_1(z)$ is a stable $N(T,p)$-segment of $(f_T, K_T, \mathcal{R}_T)$. Then, lemma 10 implies that $f_T^{N(T,p)}(\partial^s_{per} R_T) \subset \bigcup \mathcal{I}_1(z)$.
(iii) Let I be a connected coomponent of $f_T^{N(T,p)}(\partial^s_{per} R_T)$. Let a and b be the ends of the segment I such that $a \prec^s_{N(T,p)} b$. By item (ii) above, there exists a periodic s-boundary point $x \in K_T$ such that $I \subset I_1(x,z)$. Let $\widetilde{a}$ and $\widetilde{b}$ be the ends of the stable segment $I_1(x,z)$, such that $\widetilde{a} \prec^s_{N(T,p)} \widetilde{b}$. Since I is included in $I_1(y,z)$, we have $\widetilde{a} \prec^s_{N(T,p)} a$ and $b \prec^s_{N(T,p)} \widetilde{b}$. Using the definition of the segment $I_1(x,z)$ (see subsection 5.1), we see that the points $\widetilde{a}$ and $\widetilde{b}$ can be defined as follows:
— If $\mathcal{E}_0^-(z,I)$ is empty, then $\widetilde{a} = a$.
— If $\mathcal{E}_0^-(z,I)$ is non-empty, then $\widetilde{a}$ is the unique element of $\mathcal{E}_0^-(I)$ such that $]\widetilde{a}, a[^u \cap \bigcup \mathcal{J}_0(z) = \emptyset$, *that is* $\widetilde{a}$ is the element of $\mathcal{E}_0^-(z,I)$ which is maximal with respect to the order $\prec^s_{N(T,p)}$.
— If $\mathcal{E}_0^+(z,I)$ is empty, then $\widetilde{b} = b$.
— If $\mathcal{E}_0^+(z,I)$ is non-empty, then $\widetilde{b}$ is the element of $\mathcal{E}_0^-(z,I)$ which is minimal with respect to the order $\prec^s_{N(T,p)}$.

Description of the algorithm. Given a realizable geometrical type T, an integer $p \geq p_0(T)$ and the $N(T,p)$-s-code of a $N_0(T)$-point z of $(f_T, K_T, \mathcal{R}_T)$,

- Using the operation `Image` , we obtain the $N(T,p)$-s-codes of the points $f_T(z), \dots, f_T^{4n_T-1}(z)$.
- Then, using the operation `UnstableCode` , we obtain the $N(T,p)$-u-codes of the points $z, f_T(z), \dots, f_T^{4n_T-1}(z)$.
- By lemma 23, there exists an algorithm which gives back the $N(T,p)$-codes of the segments $\widetilde{J}_0(z), \dots, \widetilde{J}_0^{4n_T-1}(z)$.

- Using the operation `PointsOnUnstableSegment` , we obtain the list of the $N(T,p)$-u-codes of the elements of $\mathcal{E}_0(z)$. Using the operation `StableCode` , we obtain the list of the $N(T,p)$-s-codes of the elements of $\mathcal{E}_0(z)$.
- Using the operation `PeriodicStableSides` and `Image` , we obtain the $N(T,p)$-codes of the connected components of $f_T^{N(T,p)}(\partial^s_{per}R_T)$.
- Given the $N(T,p)$-code of a connected component I of $f_T^{-N(T,p)}(\partial^s_{per}R_T)$,
  - Let a and b be the ends of the unstable segment I, such that $a \prec^s_{N(T,p)} b$. Since we know the $N(T,p)$-code of the segment I, the $N(T,p)$-s-codes of the points a and b can be obtained using the operation `Ends` .
  - Given the $N(T,p)$-s-code of any element x of $\mathcal{E}_0(z)$, the operation `StableOrder` decides whether x is in $\mathcal{E}_0^-(z,I)$ (resp. $\mathcal{E}_0^+(z,I)$) or not. Therefore, we know the list of the $N(T,p)$-s-codes of the elements of the set $\mathcal{E}_0^-(z,I)$ (resp. $\mathcal{E}_0^+(z,I)$).
  - We consider the $N(T,p)$-points $\widetilde{a}$ and $\widetilde{b}$ defined as follows:
    - If the set $\mathcal{E}_0^-(z,I)$ is empty, then $\widetilde{a} := a$.
    - Otherwise, $\widetilde{a}$ is the biggest element of the set $\mathcal{E}_0^-(z,I)$ for the order $\prec^s_{N(T,p)}$. Observe that we can find the $N(T,p)$-s-code of the point $\widetilde{a}$ using the operation `StableOrder` .
    - If the set $\mathcal{E}_0^+(z,I)$ is empty, then $\widetilde{b} := b$.
    - Otherwise, $\widetilde{b}$ is the smallest element of the set $\mathcal{E}_0^+(z,I)$ for the order $\prec^s_{N(T,p)}$. Observe that we can find the $N(T,p)$-s-code of the point $\widetilde{b}$ using the operation `StableOrder` .
  - According to item (iv) of the "preliminary observations", the segment $[\widetilde{a},\widetilde{b}]^s$ is an element of the family $\mathcal{I}_1(z)$. Since we know the $N(T,p)$-codes of the points $\widetilde{a}$ and $\widetilde{b}$, we can use the operation `UnstableSegment` to obtain the $N(T,p)$-code of this segment. △

Proposition 31. *There exists an algorithm which takes a realizable geometrical type T, an integer $p \geq p_{min}(T)$, and the $N(T,p)$-code of a point $z \in Sp(T)$ as imput, and gives back the $N(T,p)$-codes of the elements of the family of unstable $N(T,p)$-segments $\mathcal{J}_1(z)$.*

Proof. The proof is similar to those of proposition 30 and is left to the reader. △

Step 2. The $N(T,p)$-codes of the elements of the families of segments $\mathcal{I}_2(z)$, $\mathcal{J}_2(z)$.

Proposition 32. *There exists an algorithm which takes a geometrical type T, and the $N(T,p)$-code of a point $z \in Sp(T)$ as input, and gives back the $N(T,p)$-codes of the elements of the families of segments $\mathcal{I}_2(z)$, $\mathcal{J}_2(z)$.*

Proof. We will describe an algorithm that gives back the $N(T,p)$-codes of the elements of the family of stable segments $\mathcal{I}_2(z)$.

Preliminary observations. Let $I_1(x,z)$ be an element of the family of stable segments $\mathcal{I}_1(z)$. Let a and b be the ends of $I_1(x,z)$, such that $a \prec^s_{N(T,p)} b$.

Let us first assume that the segment $I_1(x,z) = [a,b]^s$ is externally near a. Recall that b is $N(T,p)$-points of $(f_T, K_T, \mathcal{R}_T)$ (see proposition 11). We denote by $\tilde{b}$ the $N(T,p)$-stable predecessor of the $N(T,p)$-point b for the order $\prec^s_{N(T,p)}$.

If the stable segment $I_1(x,z) = [a,b]^s$ is not externally near b, then there exists a (unique) point $b' \in [a,b[^s$ such that $[b',b]^s$ is a stable arch, and we have $I_2(x,z)^s = [a,b'[^s$ (see the construction of the segment $I_2(x,z)$ in subsection 5.1). Corollary 5 implies that b' is a $N(T,p)$-point of $(f_T, K_T, \mathcal{R}_T)$. Then, since $[b',b]^s$ is a stable arch, we necessarily have $b' = \tilde{b}$. By contraposition, we obtain: if $[\tilde{b}, b]$ is not a stable arch, then the segment $I_1(x,z) = [a,b]^s$ is externally near b and we have $I_2(x,z) = I_1(x,z)$.

Conversely, if $[\tilde{b}, b]^s$ is stable arch, then the stable segment $I_1(x,z) = [a,b]^s$ is not externally isolated near b (property 7 of section 2), and we have $I_2(x,z) = [a,\tilde{b}]^s$.

If the segment $I_1(x,z)$ is not externally isolated near a, the discussion is the same, exept that we also have to consider the $N(T,p)$-stable successor $\tilde{a}$ of the $N(T,p)$-point a for the order $\prec^s_{N(T,p)}$. This leads to the following algorithm:

Description of the algorithm. Given a realizable geometrical type T and an integer $p \geq p_0(T)$,

- By proposition 30, there exists an algorithm which gives the $N(T,p)$-codes of the elements of the family of stable segments $\mathcal{I}_1(z)$.
- Given the $N(T,p)$-code of an element $I_1(x,z)$ of the family $\mathcal{I}_1(z)$,
  - Using the operation `Ends` , we obtain the $N(T,p)$-codes of the ends a, b of $I_1(x,z)$,
  - Using the operation `StableSuccessor` , we obtain the $N(T,p)$-code of the successor $\tilde{a}$ of the point a for the order $\prec^s_{N(T,p)}$. Similarly, using the operation `StablePredecessor` , we obtain the $N(T,p)$-code of the predecessor $\tilde{b}$ of the point b for the order $\prec^s_{N(T,p)}$.
  - Then, using the operation `StableSegment` , we obtain the $N(T,p)$-codes of the stable segments $[a,\tilde{a}]^s$ and $[\tilde{b},b]^s$.
  - The operation `StableArch` decides whether $[a,\tilde{a}]^s$ (resp. $[\tilde{b},b]^s$) is a stable arch or not.
  - If neither $[a,\tilde{a}]^s$ nor $[b,\tilde{b}]^s$ is a stable arch, then $I_2(x,z) = I_1(x,z)$ (in particular, we already know the $N(T,p)$-code of the segment $I_2(x,z)$).
  - If $[a,\tilde{a}]^s$ is a stable arch and $[\tilde{b},b]^s$ is not a stable arch, then $I_2(x,z) = [\tilde{a},b]^s$. Since we know the $N(T,p)$-codes of $\tilde{a}$ and b, we can use the operation `StableSegment` to get the $N(T,p)$-code of the segment $I_2(x,z)$.
  - If $[\tilde{b},b]^s$ is a stable arch and $[a,\tilde{a}]^s$ is not a stable arch, then $I_2(x,z) = [a,\tilde{b}]^s$. We can use the operation `StableSegment` to get the $N(T,p)$-code of the segment $I_2(x,z)$.
  - If $[a,\tilde{a}]^s$ and $[\tilde{b},b]^s$ are stable arches, then $I_2(x,z) = [\tilde{a},\tilde{b}]^s$. The operation `StableSegment` gives the $N(T,p)$-code of the segment $I_2(x,z)$. △

Step 3. The $N(T,p)$-codes of the elements of the families of segments $\mathcal{I}_3(z)$, $\mathcal{J}_3(z)$.

Proposition 33. *There exists an algorithm which takes a realizable geometrical type T, an integer $p \geq p_0(T)$, and the $N(T,p)$-code of a point $z \in Sp(T)$ as input, and gives back the $N(T,p)$-codes of the elements of $\mathcal{I}_3(z)$, $\mathcal{J}_3(z)$.*

Proof. We will describe an algorithm which gives back the $N(T,p)$-codes of the elements of the family of stable segments $\mathcal{I}_3(z)$. Let us begin by a two observations:
(i) In the definition of the family of stable segments $\mathcal{I}_3(z)$, we can replace "unstable arches" by "unstable $N(T,p)$-arches of $(f_T, K_T, \mathcal{R}_T)$" (see proposition 15).
(ii) If $\gamma = [a,b]^u$ is an unstable $N(T,p)$-arch of $(f_T, K_T, \mathcal{R}_T)$, then b is either the successor or the predecessor of a for the order $\prec^s_{N(T,p)}$. Hence, if b is connected to $\mathcal{I}_2(z)$ by an unstable $N(T,p)$-arch, then there exists a $N(T,p)$-point $a \in (\cup\mathcal{I}_2(z))$ such that b is either the successor or the predecessor of a for the order $\prec^s_{N(T,p)}$.
The two observations above prove that the $N(T,p)$-codes of the elements of the family $\mathcal{I}_3(z)$ can be obtained by the algorithm described below.

Description of the algorithm. Given a realizable geometrical type T, an integer $p \geq p_0(T)$, and the $N(T,p)$-code of a point $z \in \mathrm{Sp}(T)$,

- According to proposition 32, there exists an algorithm which gives back the $N(T,p)$-codes of the elements of the family of stable segments $\mathcal{I}_2(z)$. We denote by $\mathcal{L}(z)$ the set of all the $N(T,p)$-points which are linked with $\mathcal{I}_2(z)$ by an unstable $N(T,p)$-arch.

First step. Construction of the list of the $N(T,p)$-s-codes of the elements of $\mathcal{L}(z)$.

- We start with the empty list.
- The operation `PointsOnStableSegment` provides the list of the $N(T,p)$-codes of all the $N(T,p)$-points which lie on $\bigcup \mathcal{I}_2(z)$.
- For every $N(T,p)$-point $x \in \bigcup \mathcal{I}_2(z)$,
  - Using the operation `UnstableSuccessor` and `UnstablePredecessor` , we obtain the $N(T,p)$-codes of the predecessor x_1 and the successor x_2 of the point x for the order $\prec^u_{N(T,p)}$.
  - Using the operation `UnstableSegment` , we obtain the $N(T,p)$-codes of the unstable $N(T,p)$-segments $[x_1,x]^u$ and $[x,x_2]^u$. The elemantary operation `UnstableArch` decides whether the segment $[x_1,x]^u$ (resp. $[x,x_2]^u$) is an unstable arch or not.
  - If $[x_1,x]^u$ is a u-arch, then we add the $N(T,p)$-code of x_1 to the list $\mathcal{L}(z)$. If $[x,x_2]^u$ is a u-arch, then we add the $N(T,p)$-code of x_2 to the list $\mathcal{L}(z)$.

Second step. Construction of the $N(T,p)$-codes of the segment $I_3(x,z)$.

- For every element $I_2(x,z)$ of the family $\mathcal{I}_2(z)$,
  - Let $\mathcal{L}(x,z)$ be the set made of the elements of $\mathcal{L}(z)$ which are comparable to the ends of the segment $I_2(x,z)$ for the order $\prec^s_{N(T,p)}$. Using the operation `StableOrder` , we can find the $N(T,p)$-s-codes of the elements of $\mathcal{L}(x,z)$ (among the $N(T,p)$-s-codes of the elements of $\mathcal{L}(z)$)
  - By definition, the segment $I_3(x,z)$ is the smallest of all the stable $N(T,p)$-segments which contains the both ends of $I_2(x,z)$, and all the points of $\mathcal{L}(x,z)$. As a consequence, the $N(T,p)$-codes of the ends of the segment $I_2(x,z)$ can be found thanks to the operation `StableOrder` . Then, it remains to use the operation `StableSegment` to obtain the $N(T,p)$-code of the segment $I_3(x,z)$. $\triangle$

Step 4. The $N(T,p)$-codes of the families of segments $\mathcal{I}_4(z,p)$, $\mathcal{J}_4(z,p)$.

Proposition 34. *There exists an algorithm which takes a realizable geometrical type T, an integer $p \geq p_{min}(T)$, and the $N(T,p)$-code of a point $z \in Sp(T)$ as input, and gives back the $N(T,p)$-codes of the elements of $\mathcal{I}_4(z,p)$, $\mathcal{J}_4(z,p)$.*

Proof. Let us consider a realizable geometrical type T, an integer $p \geq p_{min}(T)$, and a point $z \in \mathrm{Sp}(T)$. By definition, the elements of the family $\mathcal{I}_4(z,p)$ are the connected components of $\cup \mathcal{I}_3(z)$ minus the interiors of all the stable arches whose both ends lie in $f^p(\mathcal{J}_3(z))$ As a consequence, the following algorithm provides the $N(T,p)$-codes of the elements of the family of stable segments $\mathcal{I}_4(z,p)$:
Description of the algorithm. Given a realizable geometrical type T, an integer $p \geq p_0(T)$ and the $N(T,p)$-code of a point $z \in \mathrm{Sp}(T)$,

- We will describe an algorithmic construction of the list of the $N(T,p)$-codes of the elements of the family $\mathcal{I}_4(z,p)$. We start with the empty list.
- According to proposition 32, there exists an algorithm which gives back the $N(T,p)$-codes of the elements of the families of segments $\mathcal{I}_3(z)$ and $\mathcal{J}_3(z)$.

- Then, using the operation `Image` , we obtain the $N(T,p)$-codes of the elements of the family of unstable segments $f_T^p(\mathcal{J}_3(z))$.
- Given the $N(T,p)$-code of an element I of the family $\mathcal{I}_3(z)$,
  - Let $x_1,\dots,x_r$ be the elements of $I\cap f^p(\cup\mathcal{J}_3(z))$. The $N(T,p)$-codes of the points $x_1,\dots,x_r$ can be obtained using the operation `Intersection` . Using the operation `StableOrder` , we can assume that the points $x_1,\dots,x_r$ are arranged in ascending order for the order $\prec^s_{N(T,p)}$
  - For $i=1$ to $r-1$,
    - Using the operation `StableSegment` , we get the $N(T,p)$-code of the segment $[x_i,x_{i+1}]^s$
    - The operation `StableArch` decides whether the segment $[x_i,x_{i+1}]^s$ is a stable arch or not. If it is not a stable arch, then we add the $N(T,p)$-code of this segment to the list of the $N(T,p)$-codes of the elements of the family $\mathcal{I}_4(z,p)$. △

10.3. **Algorithmic construction of the geometrical types of the Markov partition $\mathcal{R}(z,p)$.**

Proposition 35. *There exists an algorithm which takes a realizable geometrical type T, an integer $p\geq p_{min}(T)$, and the $N(T,p)$-s-code of a point $z\in Sp(T)$ and gives back the list of the geometrical types of all the geometrizations of the Markov partition $\mathcal{R}(z,p)$.*

Sketch of the proof According to proposition 34, there exists an algorithm, which takes a realizable geometrical type T, an integer $p\geq p_{min}(T)$, and the $N(T,p)$-s-code of a point $z\in \mathrm{Sp}(T)$, and gives back the $N(T,p)$-codes of the elements of the families of segments $\mathcal{I}_4(z,p)$, $\mathcal{J}_4(z,p)$. The elements of the families $\mathcal{I}_4(z,p)$, $\mathcal{J}_4(z,p)$ are known to be the sides of the rectangles of the Markov partition $\mathcal{R}(z,p)$.

Using the elementary operations `Image` , `Intersection` , `StableOrder` and `UnstableOrder` , it is very easy to write an algorithmic procedure, which takes the $N(T,p)$-codes of the sides of the rectangles of the Markov partition $\mathcal{R}(z,p)$ as input and gives back the geometrical types of all the geometrizations of the Markov partition $\mathcal{R}(z,p)$. This completes the proof. △

10.4. **Proofs of proposition 2 and theorem 2.**

Proof of proposition 2. Let T be a realizable geometrical type, and p be an integer greater or equal than $p_{min}(T)$. Recall that the set of geometrical types $\mathcal{T}(T,p)$ is defined as follows:

$$\mathcal{T}(T,p)=\{T(z,p)\mid z \text{ is a special point of } K_T\}$$

On the one hand, for every special point z, there exists an integer $k\in\mathbb{Z}$ such that the point $f_T^k(z)$ is a $N_0(T)$-point of $(f_T,K_T,\mathcal{R}_T)$, *i.e.* such that $f_T^k(z)\in\mathrm{Sp}(T)$ (proposition 9). On the other hand, for every special point z of K_T, and every integer $k\in\mathbb{Z}$, we have $T(z,p)=T(f_T^k(z),p)$. As a consequence, we have

$$\mathcal{T}(T,p)=\{T(z,p)\mid z\in\mathrm{Sp}(T)\}$$

Propositions 29 and 35 imply there exists an algorithm which takes a realizable geometrical type T and an integer $p\geq p_{min}(T)$ as input, and gives back the set of geometrical types $\mathcal{T}(T,p)=\{T(z,p)\mid z\in\mathrm{Sp}(T)\}$. This completes the proof. △

Proof of theorem 2. Let T_1 and T_2 be two realizable geometrical types. We consider the integer $p=50\max(2(N_0(T_1)+48n_{T_1}),2(N_0(T_2)+48n_{T_2}))$. By proposition 1, p is greater or equal than $\max(p_{min}(T_1),p_{min}(T_2))$. By proposition 8, the geometrical types T_1 and T_2 are equivalent if and only if the finite sets of geometrical types $\mathcal{T}(T_1,p)$ and $\mathcal{T}(T_2,p)$ are equal. Besides, proposition 2 implies that there exists an algorithm which takes T_1 and T_2 as input, and gives back the sets of geometrical types $\mathcal{T}(T_1,p)$ and $\mathcal{T}(T_2,p)$. This completes the proof. $\triangle$

Appendix A. Powers of a geometrical type. The aim of this appendix is to prove proposition 19. For that purpose, we consider a realizable geometrical type T, and some positive integers p and q. We will establish some formulas that relates the geometrical type $T^{(p+q)} = (n_T, h_T^{(p+q)}, v_T^{(p+q)}, \Phi_T^{(p+q)}, \varepsilon_T^{(p+q)})$ to the geometrical types $T^{(p)} = (n_T, h_T^{(p)}, v_T^{(p)}, \Phi_T^{(p)}, \varepsilon_T^{(p)})$ and $T^{(q)} = (n_T, h_T^{(q)}, v_T^{(q)}, \Phi_T^{(q)}, \varepsilon_T^{(q)})$.

A.1 The integers $\theta_{i,m,T}^{(p),(p+q)}$ and $\nu_{k,m,T}^{(q),(p+q)}$. The two following formulas are direct consequences of lemma 21:

— for every $i \leq n_T$, and every $m \leq h_{i,T}^{(p)}$, we have

$$\theta_{i,m,T}^{(p),(p+q)} = h_{k_m,T}^{(q)} \text{ where } k_m \text{ is given by the equality } \Phi_T^{(p)}(i,m) = (k_m, l_m) \quad (11)$$

— for every $k \leq n_T$, and every $m \leq v_{i,T}^{(q)}$, we have

$$\nu_{k,m,T}^{(q),(p+q)} = v_{i_m,T}^{(p)} \text{ where } i_m \text{ satisfies the equality } (\Phi_T^{(q)})^{-1}(k,m) = (i_m, j_m) \quad (12)$$

A.2 The integers $h_{i,T}^{(p+q)}$ and $v_{k,T}^{(p+q)}$. For every $i \leq n_T$ and every $j \leq h_{i,T}^{(p+q)}$, there exists m such that $H_{i,T}^{j,(p+q)} \subset H_{i,T}^{m,(p)}$. Thus, for every $i \leq n_T$, we have

$$h_{i,T}^{(p+q)} = \sum_{m=1}^{m=h_{i,T}^{(p)}} \theta_{i,m,T}^{(p),(p+q)} \quad (13)$$

Similarly, for every $k \leq n_T$, we have

$$v_{k,T}^{(p+q)} = \sum_{m=1}^{m=v_{i,T}^{(q)}} \nu_{k,m,T}^{(q),(p+q)} \quad (14)$$

A.3 The function $\Phi_T^{(p+q)}$. Let i and j be positive integers such that $i \leq n_T$ and $j \leq h_{i,T}^{(p+q)}$. We want to find the integers k and l, such that $\Phi_T^{(p+q)}(i,j) = (k,l)$. In other words, we want to find the integers k and l, such that the vertical subrectangle $V_{k,T}^{l,(p+q)}$ is the image of the horizontal subrectangle $H_{i,T}^{j,(p+q)}$ under the diffeomorphism f_T^{p+q}.

First step: the image of the horizontal subrectangle $H_{i,T}^{j,(p+q)}$ under f_T^p

The horizontal subrectangke $H_{i,T}^{j,(p+q)}$ is a connected component of $f_T^{-(p+q)}(R_T) \cap R_T = f_T^{-(p+q)}(R_T) \cap f_T^{-p}(R_T) \cap R_T$. Thus, the subrectangle $f_T^p(H_{i,T}^{j,(p+q)})$ is a connected component of $f_T^{-q}(R_T) \cap R_T \cap f_T^q(R_T) = (f_T^p(R_T) \cap R_T) \cap (f_T^{-q}(R_T) \cap R_T)$. Consequently, there exist an integer $\alpha \leq n_T$, an integer $\beta \leq v_{\alpha,T}^{(p)}$, and an integer $\gamma \leq h_{\alpha,T}^{(q)}$ such that $f_T^p(H_{i,T}^{j,(p+q)}) = V_{\alpha,T}^{\beta,(p)} \cap H_{\alpha,T}^{\gamma,(q)}$. The aim of this first step is to find the integers α, β, γ.

Let j_1 be the integer such that $H_{i,T}^{j,(p+q)} \subset H_{i,T}^{j_1,(p)}$. From an algorithmical point of vue, j_1 is the unique integer such that

$$\sum_{m=1}^{m=j_1-1} \theta_{i,m,T}^{(p),(p+q)} < j < \sum_{m=1}^{m=j_1} \theta_{i,m,T}^{(p),(p+q)} \quad (15)$$

By definition of the function $\Phi^{(p)}$, we have $f_T^p(H_{i,T}^{j_1,(p)}) = V_{\alpha,T}^{\beta,(p)}$ where

$$(\alpha,\beta) = \Phi_T^{(p)}(i,j_1) \tag{16}$$

Moreover, if we consider the integer j_2 defined by

$$j_2 := j - \sum_{m=1}^{m=j_1-1} \theta_{i,m,T}^{(p+q),(p)} \tag{17}$$

then we have $f_T^p(H_{i,T}^{j,(p+q)}) = V_{\alpha,T}^{\beta,(p)} \cap H_{\alpha,T}^{\gamma,(q)}$ where γ is defined by

$$\begin{cases} \gamma = j_2 & \text{if } \varepsilon_T^{(p)}(i,j_1) = + \\ \gamma = \theta_{i,j_1,T}^{(p),(p+q)} + 1 - j_2 & \text{if } \varepsilon_T^{(p)}(i,j_1) = - \end{cases} \tag{18}$$

Second step: the image of the subrectangle $H_{\alpha,T}^{\gamma,(q)} \cap V_{\alpha,T}^{\beta,(p)}$ under f_T^q
We have $f_T^{(p+q)}(H_{i,T}^{j,(p+q)}) = f_T^q(f_T^p(H_{i,T}^{j,(p+q)})) = f_T^q(H_{\alpha,T}^{\gamma,(q)} \cap V_{\alpha,T}^{\beta,(p)})$. As a consequence, we are left to compute the integers k and l such that $f_T^q(H_{\alpha,T}^{\gamma,(q)} \cap V_{\alpha,T}^{\beta,(p)}) = V_{k,T}^{l,(p+q)}$. For that purpose, let us first observe that the diffeomorphism f_T^q maps the horizontal subrectangle $H_{\alpha,T}^{\gamma,(q)}$ to the vertical subrectangle $V_{k,T}^{l_1,(q)}$, where the integers k and l_1 are given by

$$\Phi_T^{(q)}(\alpha,\gamma) = (k,l_1) \tag{19}$$

Now, if we consider the integer l_2 defined by

$$\begin{cases} l_2 := \beta & \text{if } \varepsilon_T^{(q)}(k,l_1) = + \\ l_2 := \nu_{k,l_1,T}^{(q),(p+q)} + 1 - \beta & \text{if } \varepsilon_T^{(q)}(k,l_1) = - \end{cases} \tag{20}$$

then the integer l is the l_2^{th} integer such that the vertical subrectangle $V_{k,T}^{l,(p+q)}$ is included in the vertical subrectangle $V_{k,T}^{l_1,(q)}$. In particular, the integer l is given by the following formula:

$$l = \sum_{m=1}^{m=l_1-1} \nu_{k,m,T}^{(q),(p+q)} + l_2 \tag{21}$$

A.4 The function $\varepsilon_T^{(p+q)}$. Let i and j be two integers such that $1 \leq i \leq n$ and $1 \leq j \leq h_{i,T}^{(p+q)}$. We want to compute $\varepsilon_T^{(p+q)}(i,j)$. For that purpose, we consider the integers j_1, α and β, such that the horizontal subrectangle $H_{i,T}^{j,(p+q)}$ is included in the horizontal subrectangle $H_{i,T}^{j_1,(p)}$, and such that the subrectangle $f_T^p(H_{i,T}^{j,(p+q)})$ is included in the horizontal subrectangle $H_{\alpha,T}^{\beta,(q)}$. Explicit values of the integers j_1, α and β are given by formulas (13) and (14). Moreover, we clearly have:

$$\varepsilon_T^{(p+q)}(i,j) = \varepsilon_T^{(p)}(i,j_1).\varepsilon_T^{(q)}(\alpha,\beta) \tag{22}$$

A.5 Proof of proposition 19. If we are given the geometrical type $T^{(p)} = (n_T, h_T^{(p)}, v_T^{(p)}, \Phi_T^{(p)}, \varepsilon_T^{(p)})$ and the geomtrical type $T^{(q)} = (n_T, h_T^{(q)}, v_T^{(q)}, \Phi_T^{(q)}, \varepsilon_T^{(q)})$, then formulas (11)...(22) provide an algorithmic way to compute the geometrical type $T^{(p+q)} = (n_T, h_T^{(p+q)}, v_T^{(p+q)}, \Phi_T^{(p+q)}, \varepsilon_T^{(p+q)})$. This clearly implies proposition 19.

Appendix B. Construction of the set of Markov partitions $\mathcal{R}(f, K, p)$ in the case of Smale's horseshoe. Let f be the so-called *Smale's horseshoe diffeomorphism* (see, for example, [8, chapter 4] for the construction of f). Recall that f is a Smale diffeomorphism of the sphere $\mathbb{S}^2$, with three basic pieces : a source, a sink, and a non-trivial saddle basic piece that we denote by K.

Using the construction of f, it is easy to verify that there exists a unique periodic s-boundary point x in K. Moreover, x is also the unique periodic u-boundary point of K. The stable manifold and the unstable manifold of the point x are represented on figure 12.

One can see on figure 12 that there are exactly two orbits of special points in K (the orbit of the point z_1 and the orbit of the point z_2). The construction of the Markov partition $\mathcal{R}(z_2, 2)$ (following the process of construction described in section 5.1) is represented on figures 12, 13 and 14. We leave the construction of the Markov partition $\mathcal{R}(z_1, 2)$ as an exercice.

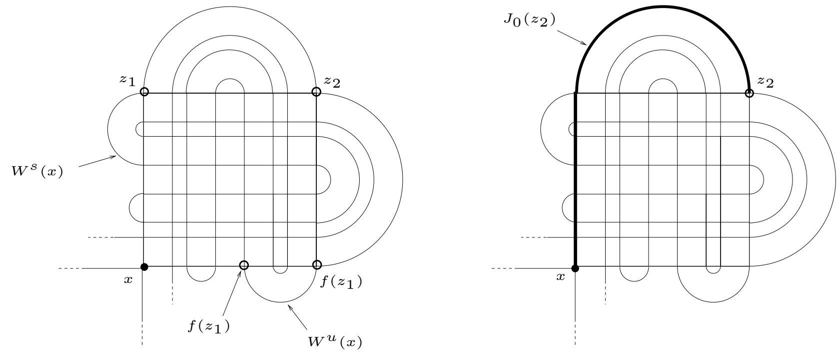

FIGURE 12. On the left: the periodic s and u-boundary point x, the stable and the unstable manifold of x, and the special points z_1, z_2. On the right: the segment $J_0(z_1)$.

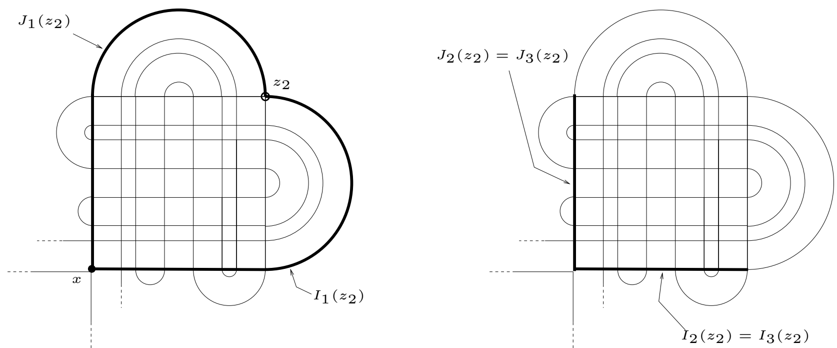

FIGURE 13. On the left: the segments $I_1(x_1, z_1)$ and $J_1(x_1, z_1)$. On the right: the segments $I_2(x, z_1) = I_3(x, z_1)$ and $J_2(x, z_1) = J_3(x, z_1)$.

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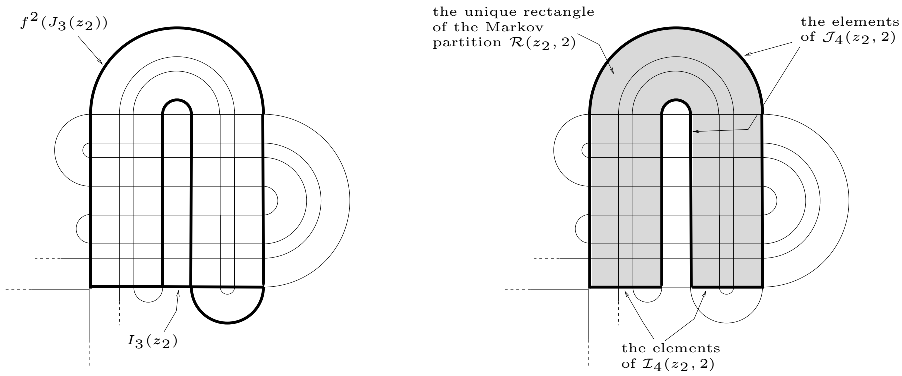

FIGURE 14. On the left: the segments $I_3(x, z_1)$ and $f^2(J_3(x, z_1))$. On the right: the elements of the family of segments $\mathcal{I}_4(z_1, 2)$ and $\mathcal{J}_4(z_1, 2)$, which are the sides of the unique rectangle of the Markov partition $\mathcal{R}(z_2, 2)$.

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