Foliations of globally hyperbolic spacetimes locally modelled on AdS_3 by constant mean curvature surfaces

Thierry Barbot, François Béguin and Abdelghani Zeghib

Introduction

We recall that a *time function* on a spacetime M is a submersion $\tau : M \to \mathbb{R}$ such that τ is strictly increasing along every future-directed timelike curve. The fibers of a time function are always Cauchy surfaces. A *CMC time function* on a spacetime M is a time function $\tau : M \to \mathbb{R}$ such that, for every $\theta \in \mathbb{R}$, the set $\tau^{-1}(\theta)$ is a spacelike surface with constant mean curvature θ. The foliation defined by a CMC time function is sometimes called a *York slicing* (see, for exemple, [?]).

Using a kind of "maximum principle", it is not hard to show that a CMC time function with compact levels is always unique. More precisely, one has the following proposition:

Proposition 0.1. *Let M be a globally hyperbolic spacetime, with compact Cauchy hypersurfaces. Assume that M admits a CMC time function τ. Then, every CMC compact spacelike hypersurface in M is a fiber of τ. In particular, τ is unique.*

The aim of the paper is to prove the following theorem:

Theorem 0.2. *Let M be a maximal globally hyperbolic spacetime, locally modelled on AdS_3, with closed orientable Cauchy surfaces. Then, M admits a CMC time function.*

1 Unicity of CMC time functions with compact levels

The purpose of this section is to prove proposition 0.1. First of all, in order to avoid any ambiguity about signs convention, we want to recall the definition of the mean curvature of a spacelike surface in a lorentzian manifold.

Let Σ be a smooth spacelike hypersurface in a lorentzian manifold M, and p be a point of Σ. Let n be the future pointing unit normal of S at p. We recall that the *second fundamental form* of the surface S is the quadratic form Π_p on $T_p\Sigma$ defined by $\Pi_p(X,Y) = -g(\nabla_X n, Y)$, where g is the lorentzian metric and ∇ is the covariant derivative. The *mean curvature* of S at p is the trace of this quadratic form.

Remark 1.1. *Let us identify the tangent space of M at p with $\mathbb{R}^n$, in such a way that the tangent space of Σ at p is identified with $\mathbb{R}^{n-1} \times \{0\}$, and the vector n is identified with $(0,\dots,0,1)$. Let U be a neighbourhood of p in M. If U is small enough, the image of the surface $\Sigma \cap U$ under the inverse of the exponential map $\exp_p$ is the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f(0) = 0$ and $Df(0) = 0$. It is easy to verify that the second fundamental form of Σ at p is the opposite of the hessian of f at the origin. In particular, the mean curvature of Σ at p is the opposite of the trace of the hessian of f at the origin.*

The proof of proposition 0.1 relies on the following well-known lemma:

Lemma 1.2. *Let Σ and Σ' be two smooth spacelike hypersurfaces in a lorentzian manifold M. Assume that Σ and Σ' are tangent at some point p, and assume that Σ' is included in the future of Σ.*

Then, the mean curvature of Σ' at p is smaller or equal than those of Σ. Moreover, the mean curvatures of Σ and Σ' at p are equal only if Σ and Σ' have the same osculating quadric at p.

Proof. We identify the tangent space T_pM of M at p with $\mathbb{R}^n$, in such a way that the tangent hyperplane of Σ and Σ' is identified with $\mathbb{R}^{n-1} \times \{0\}$, and the future-pointing unit normal of Σ and Σ' is identified with the vector $(0,\dots,0,1)$. Let U be a neighbourhood of p in M. If U is small enough, the image of $\Sigma \cap U$ (resp. $\Sigma' \cap U$) under the inverse of the exponential map at p is the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ (resp. of a function $f' : \mathbb{R}^{n-1} \to \mathbb{R}$), such that $f(0) = 0$ and $Df(0) = 0$ (resp. $f'(0) = 0$ and $Df'(0) = 0$). Since Σ' is included in the future of Σ, we have $f' \geq f$. This implies that, for every $v \in \mathbb{R}^{n-1}$, we have

$D^2f'(0).(v,v) \geq D^2f(0)(v,v)$. According to remark 1.1, this implies that the mean curvature of Σ' at p is smaller or equal than those of Σ'.

The case of equality is a consequence of the following observation: given two functions $f, f' : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f(0) = f'(0) = 0$ and $Df(0) = Df'(0) = 0$, and such that $f' \geq f$, the the hessians of f and f' at p are equal if and only if they have the same trace. □

Proof of proposition 0.1. For every $s \in \tau(\mathbb{R})$, we denote by Σ_s the Cauchy surface $\tau^{-1}(s)$. Let Σ be a compact spacelike CMC surface in M. Let $s_1 := \inf\{s \in \mathbb{R} \mid \Sigma \cap \Sigma_s \neq \emptyset\}$ and $s_2 := \inf\{s \in \mathbb{R} \mid \Sigma \cap \Sigma_s \neq \emptyset\}$. The compacity of Σ implies that s_1 and s_2 are finite numbers, and that Σ does intersect the surfaces Σ_{s_1} and Σ_{s_2}. Moreover, by definition of s_1 and s_2, the surface Σ is included in the future the surface Σ_{s_1} and in the past of the surface Σ_{s_2}. Let p_1 be a point in $\Sigma \cap \Sigma_{s_1}$, and p_2 be a point in $\Sigma \cap \Sigma_{s_2}$. By lemma 1.2, the mean curvature of Σ at p_1 at most s_1, and the mean curvature of Σ at p_2 is at least s_2. Since Σ is a CMC surface, and since $s_1 \leq s_2$, this implies $s_1 = s_2$. Moreover, since Σ is in the future of Σ_{s_1} and in the past of Σ_{s_2}, this implies $\Sigma = \Sigma_{s_1} = \Sigma_{s_2}$. Hence, Σ is a fiber of the time function τ. □

2 The three dimensional anti-de Sitter space

In this section, we recall the construction of the different models of the three-dimensional anti-de Sitter space, and we study the geometrical properties (geodesics, causal structure, etc.) of this space.

2.1 The linear model of the anti-de Sitter space

We denote by (x_1, x_2, x_3, x_4) the standard coordinates on $\mathbb{R}^4$. We will also use the coordinates $(a, b, c, d) = (x_1 - x_3, -x_2 + x_4, x_2 + x_4, x_1 + x_3)$. We consider the quadratic form $Q = -x_1^2 - x_2^2 + x_3^2 + x_4^2 = -ad + bc$. We denote by B_Q the bilinear form associated with Q.

Let p be a point on the quadric of equation $(Q = -1)$ in $\mathbb{R}^4$. When we identify the tangent space of $\mathbb{R}^4$ at p with $\mathbb{R}^4$, the tangent space of the quadric $(Q = -1)$ at p is identified with the Q-orthogonal of p. Since Q is a non-degenerate quadratic form of signature $(-,-,+,+)$, and since $Q(p) = -1$, the restriction of Q to the Q-orthogonal of p is a non-degenerate quadratic form of signature $(-,+,+)$. This proves that the quadratic form Q induces a lorentzian metric of signature $(-,+,+)$ on the quadric $(Q = -1)$. In other words, the restriction of the pseudo-riemannian metric $-dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$ to the quadric $(Q = -1)$ is a lorentzian metric of signature $(-,+,+)$.

Definition 2.1. *The* linear model of the three-dimensional anti-de Sitter space*, denoted by AdS_3, is the quadric of equation $(Q = -1)$ in $\mathbb{R}^4$ endowed with the lorentzian metric induced by Q.*

One can easily verify that the anti-de Sitter space AdS_3 is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. More precisely, one can find a diffeomorphism $h : \mathbb{S}^1 \times \mathbb{R}^2 \to AdS_3$ such that the surface $h(\{\theta\} \times \mathbb{R}^2)$ is spacelike for every θ, and such that the circle $h(\mathbb{S}^1 \times \{x\})$ is timelike for every x. In particular, the anti-de Sitter space AdS_3 is time-orientable; from now on, we will assume that a time-orientation has been chosen.

The isometry group of the anti-de Sitter space AdS_3 is the group $O(2,2)$ of the linear transformations of $\mathbb{R}^4$ which preserve the quadratic form Q. The group $O(2,2)$ acts transitively on AdS_3 andthe stabilizer of any point is isomorphic to $O(2,1)$; hence, the anti-de Sitter space AdS_3 can be seen as the homogenous space $O(2,2)/O(2,1)$. We shall denote by $O_0(2,2)$ the connected component of the identity of $O(2,2)$; the elements of $O_0(2,2)$ preserve the three-dimensional orientation and the time-orientation of AdS_3.

Proposition 2.2. *The geodesics of AdS_3 are the connected components of the intersections of AdS_3 with the two-dimensional vector subspaces of $\mathbb{R}^4$.*

Proof. Let P be a two-dimensional vector subspace of $\mathbb{R}^4$. The geometry of $P \cap AdS_3$ depends on the signature of the restriction of Q to the plane P:

– If the restriction of Q to the plane P is a quadratic form of signature $(-,-)$, then there exists an element σ of $O(2,2)$ which maps P to the plane $(x_3 = 0, x_4 = 0)$. The intersection of AdS_3 with the plane $(x_3 = 0, x_4 = 0)$ is a closed timelike curve. This curve has to be a geodesic of AdS_3, since the symetry with respect to the plane $(x_3 = 0, x_4 = 0)$ is an isometry of AdS_3. Hence, the intersection of AdS_3 with the plane P is also a closed timelike geodesic of AdS_3.

– If the restriction of Q to the plane P is a quadratic form of signature $(-,+)$, then there exists an element of $O(2,2)$ which maps P to the plane $(x_1 = 0, x_3 = 0)$. Using the same arguments as in the first case, one can easily see that $P \cap AdS_3$ is the union of two disjoined non-closed spacelike geodesics of AdS_3.

– If the restriction of Q to the plane P is a degenerate quadratic form of signature $(0,-)$, then there

exists an element of $O(2,2)$ which maps P to the plane $(x_1 = x_3, x_4 = 0)$. Using the same arguments as in the first case, one can easily see that $P \cap AdS_3$ is a non-closed lightlike geodesic of AdS_3.
– Finally, if the restriction of Q to the plane P is a quadratic form of signature $(+,+)$, $(0,-)$ or $(0,0)$, then one can easily verify that the intersection $P \cap AdS_3$ is empty.

The discussion above implies that each connected component of the intersection of AdS_3 with a 2-dimensional vector subspace of $\mathbb{R}^4$ is a geodesic of AdS_3. The converse follow from the fact that a geodesic is uniquely determined by its tangent vector at some point. $\square$

Remark 2.3. *Let γ be a geodesic of AdS_3. According to proposition 2.2, there exists a 2-dimensional vector subspace P_γ of $\mathbb{R}^4$ such that γ is a connected component of $P_\gamma \cap AdS_3$. Moreover, reading again the proof of proposition 2.2, we notice that:*
– if γ is timelike, then the intersection of P_γ with the quadric $(Q = 0)$ is reduced to $(0,0,0,0)$;
– if γ is lightlike, then P_γ is tangent to the quadric $(Q = 0)$ along a line;
– if γ is spacelike, then P_γ intersects transversally the quadric $(Q = 0)$ along two lines.

Remark 2.4. *The proof of proposition 2.2 shows that all the timelike geodesics of AdS_3 are closed, so that a single point is not an achronal set in AdS_3. Moreover, one can prove that the past and the future in AdS_3 of any point $p \in AdS_3$ are both equal to the whole of AdS_3. So, the causal structure of AdS_3 is not very interesting. This is the reason why, instead of working in AdS_3 itself, we shall work in some "large" subsets of AdS_3 which do not contain any closed geodesics (see subsection 2.3).*

Using the same kind of arguments as in the proof of proposition 2.2, one can prove the following:

Proposition 2.5. *The two-dimensional totally geodesic space of AdS_3 are the connected components of the intersections of AdS_3 with the three-dimensional vector subspaces of $\mathbb{R}^4$.*

Remark 2.6. *In particular, given any point $p \in AdS_3$ and any vector plane P in T_pAdS_3, there exists a totally geodesic subspace of AdS_3 whose tangent space at p is the plane P.*

Let p be a point in AdS_3. We call *dual surface of the point p*, and we denote by p^*, the intersection of the hyperplane $p^\perp$ with AdS_3 (where $p^\perp = \{q \in \mathbb{R}^4 \mid B_Q(p,q) = 0\}$). According to proposition 2.5, each connected component of p^* is a two-dimensional totally geodesic subspace of AdS_3. Moreover, one can easily verify that p^* is made of two connected components, and that the restriction of Q to p^* is a quadratic form of signature $(+,+)$ (it is enough to consider the case where p is the point $(1,0,0,0)$ since $O(2,2)$ acts transitively on AdS_3). Hence, the surface p^* is the union of two disjoined spacelike totally geodesic subspaces of AdS_3.

Remark 2.7. *Every point of the surface p^* can be joined from p by a timelike geodesic segment.*

Proof. Let q be a point in p^*. We denote by P the 2-dimensional vector subspace spanned by p and q in $\mathbb{R}^4$. We have $Q(p) = Q(q) = -1$ and $B_Q(p,q) = 0$; this implies that the restriction of the quadratic form Q to the plane P is a quadratic form of signature $(-,-)$. Hence, according to the proof of proposition 2.2, the intersection of the plane P with AdS_3 is a timelike geodesic. This proves in particular that the points p and q are joined by a timelike geodesic segment. $\square$

2.2 The projective model of the anti-de Sitter space

We shall now define the "projective model of the anti-de Sitter space". An interesting feature of this model is that it allows to define attach a boundary to the anti-de Sitter space. This boundary will play a fundamental role in the proof of theorem 0.2.

We see the sphere $\mathbb{S}^3$ as the quotient of $\mathbb{R}^4 \setminus \{0\}$ by positive homotheties. We denote by π the natural projection of $\mathbb{R}^4 \setminus \{0\}$ on $\mathbb{S}^3$. We denote by $[x_1 : x_2 : x_3 : x_4]$ the "positively homogenous" coordinates on $\mathbb{S}^3$ induced by the coordinates (x_1, x_2, x_3, x_4) on $\mathbb{R}^4$: one has $[x_1 : x_2 : x_3 : x_4] = [y_1 : y_2 : y_3 : y_4]$ if and only if there exists $\lambda > 0$ such that $(x_1, x_2, x_3, x_4) = \lambda(y_1, y_2, y_3, y_4)$. Similarly, we denote by $[a : b : c : d]$ the positively homogenous coordinates on $\mathbb{S}^3$ induced by the coordinates (a, b, c, d) on $\mathbb{R}^4$. We endow $\mathbb{S}^3$ with its canonical riemannian metric.

Remark 2.8. *Given a point $p \in \mathbb{S}^3$, the quantity $Q(p)$ is defined up to multiplication by a positive number; this means that the sign of $Q(p)$ is well-defined. Similarly, given two points $p, q \in \mathbb{S}^3$, the sign of $B_Q(p,q)$ is well-defined.*

Definition 2.9. *The projection π maps diffeomorphically AdS_3 on its image $\pi(AdS_3) \subset \mathbb{S}^3$. The* projective model of the anti-de Sitter space*, that we denote by $\mathbb{A}\mathrm{d}\mathbb{S}_3$, is the image of AdS_3 under π, equipped with the image of the lorentzian metric of AdS_3. We denote by $\partial\mathbb{A}\mathrm{d}\mathbb{S}_3$ the boundary of $\mathbb{A}\mathrm{d}\mathbb{S}_3$ in $\mathbb{S}^3$.*

Observe that $\mathbb{A}d\mathbb{S}_3$ is made of the points of $\mathbb{S}^3$ which satisfy the inequation $(Q<0)$. Hence, $\partial\mathbb{A}d\mathbb{S}_3$ is the quadric of equation $(Q=0)$ in $\mathbb{S}^3$. This quadric admits two transversal rulings by families of great circles of $\mathbb{S}^3$. The first ruling, that we call *left ruling*, is the family of great circles $\{L_{(\lambda:\mu)}\}_{(\lambda:\mu)\in\mathbb{RP}^1}$ where $L_{(\lambda:\mu)} = \{[a:b:c:d]\in\partial\mathbb{A}d\mathbb{S}_3 \mid (a:c)=(b:d)=(\lambda:\mu) \text{ in } \mathbb{RP}^1\}$. The second ruling, that we call *right ruling*, is the family of great circles $\{R_{(\lambda:\mu)}\}_{(\lambda:\mu)\in\mathbb{RP}^1}$ where $R_{(\lambda:\mu)} = \{[a:b:c:d]\in\partial\mathbb{A}d\mathbb{S}_3 \mid (a:b)=(c:d)=(\lambda:\mu) \text{ in } \mathbb{RP}^1\}$. Through each point of $\partial\mathbb{A}d\mathbb{S}_3$ passes one circle of the left ruling and one circle of the right ruling. Any circle of the left ruling intersects any circle of the right ruling at two antipodal points.

The elements of $O_0(2,2)$ preserve the left and the right ruling of $\partial\mathbb{A}d\mathbb{S}_3$. Hence, for each element σ of $O_0(2,2)$, we can consider the action of σ on the left and the right rulings. This defines a morphism from $O_0(2,2)$ to $PSL(2,\mathbb{R})\times PSL(2,\mathbb{R})$. It is easy to see that this morphism is onto, and that the kernel of this morphism is a subgroup of order 2 of $O_0(2,2)$ (remember that two points of $\mathbb{A}d\mathbb{S}_3$ are on the same circle of the left and right rulings if and only if their are antipodal). As a consequence, we obtain an isomorphism from $O_0(2,2)$ to $SL(2,\mathbb{R})\times SL(2,\mathbb{R})/(-Id,-Id)$ such that the elements of $SL(2,\mathbb{R})\times\{Id\}/(-Id,-Id)$ preserve individually each circle of the right ruling, and the elements of $\{Id\}\times SL(2,\mathbb{R})/(-Id,-Id)$ preserve individually each circle of the left ruling.

Proposition 2.10. *The geodesics of $\mathbb{A}d\mathbb{S}_3$ are the connected components of the intersections of $\mathbb{A}d\mathbb{S}_3$ with the great circles of $\mathbb{S}^3$.*

Proof. By construction of $\mathbb{A}d\mathbb{S}_3$, the geodesics of $\mathbb{A}d\mathbb{S}_3$ are the images under π of the geodesics of AdS_3. By proposition 2.2, the geodesics of AdS_3 are the connected components of the intersections of AdS_3 with the two-dimensional vector subspaces of $\mathbb{R}^4$. The image under π of a two-dimensional vector subspace of $\mathbb{R}^4$ is a great circle of $\mathbb{S}^3$, *i.e.* a geodesic of $\mathbb{S}^3$. Putting everything together, we get proposition 2.10. □

Remark 2.11. *Let γ be a geodesic of $\mathbb{A}d\mathbb{S}_3$. By proposition 2.10, γ is a connected component of $\mathbb{A}d\mathbb{S}_3\cap\widehat{\gamma}$, where $\widehat{\gamma}$ is a geodesic of $\mathbb{S}^3$. Moreover, remark 2.3 and the proof of proposition 2.10 imply that:*
– if γ is a timelike geodesic, then the great circle $\widehat{\gamma}$ is included in $\mathbb{A}d\mathbb{S}_3$ and $\gamma=\widehat{\gamma}$,
– if γ is lightlike, then the great circle $\widehat{\gamma}$ is tangent to $\partial\mathbb{A}d\mathbb{S}_3$ at two antipodal points points $p,-p$, and γ is one of the two connected components of $\hat{\gamma}\setminus\{p,-p\}$,
– if γ is spacelike, then the great circle $\hat{\gamma}$ intersects $\partial\mathbb{A}d\mathbb{S}_3$ transversally at four points $\{p_1,-p_1,p_2,-p_2\}$, and γ is one of the four connected components of $\hat{\gamma}\setminus\{p_1,-p_1,p_2,-p_2\}$.

Remark 2.12. *Let q be a point of $\partial\mathbb{A}d\mathbb{S}_3$, and p be a point in $\mathbb{A}d\mathbb{S}_3$. The great 2-sphere S_q of $\mathbb{S}^3$ which is tangent to the quadric $\partial\mathbb{A}d\mathbb{S}_3$ at q is $S_q=\{r\in\mathbb{S}^3 \mid B_Q(q,r)=0\}$. Consequently, remark 2.11 implies that there exists a lightlike geodesic γ passing through p and such that the ends of γ in $\partial\mathbb{A}d\mathbb{S}_3$ are the points q and $-q$ if and only if $B_Q(q,r)=0$.*

Using proposition 2.5 and the same arguments as in the proof of proposition 2.10, we obtain:

Proposition 2.13. *The two-dimensional totally geodesic subspaces of $\mathbb{A}d\mathbb{S}_3$ are the connected components of the intersections of $\mathbb{A}d\mathbb{S}_3$ with the great 2-spheres of the sphere $\mathbb{S}^3$.*

For every $p\in\mathbb{A}d\mathbb{S}_3$, we call *dual surface of p*, and denote by p^*, the image under π of the dual surface of the $\widehat{p}$ (defined in subsection 2.1), where $\widehat{p}$ is the unique point of AdS_3 such that $\pi(\widehat{p})=p$. Of course, p^* is also the intersection of $p^\perp$ with $\mathbb{A}d\mathbb{S}_3$, where $p^\perp=\{q\in\mathbb{S}^3 \mid B_Q(p,q)=0\}$. We denote by $\overline{p^*}$ the closure on p^* in $\mathbb{A}d\mathbb{S}_3\cup\partial AdS_3$. Of course, $\overline{p^*}$ is also the intersection of $p^\perp$ with $\mathbb{A}d\mathbb{S}_3\cup\partial\mathbb{A}d\mathbb{S}_3$.

2.3 The locally affine structure of the anti-de Sitter space

By an *open hemisphere* of $\mathbb{S}^3$, we mean anyone of the two connected components of $\mathbb{S}^3$ minus any great 2-sphere. Given an open hemisphere U, we say that a diffeomorphism $\varphi:U\to\mathbb{R}^3$ is an *affine chart* is φ maps the great circles of $\mathbb{S}^3$ (intersected with U) to the affine lines of $\mathbb{R}^3$. It is well-known that, for every open hemisphere U of $\mathbb{S}^3$, there exists an affine chart $\varphi:U\to\mathbb{R}^3$. This defines a locally affine structure on $\mathbb{S}^3$, which induces a locally affine structure on $\mathbb{A}d\mathbb{S}_3$. The purpose of this subsection is to define some particular affine charts of $\mathbb{A}d\mathbb{S}_3$.

For every point $p\in\mathbb{A}d\mathbb{S}_3$, we consider the sets

$$U_p:=\{q\in\mathbb{S}^3 \mid B_Q(p,q)>0\}\ ,\ \mathcal{A}_p:=\{q\in\mathbb{A}d\mathbb{S}_3 \mid B_Q(p,q)>0\} \text{ and } \partial\mathcal{A}_p:=\{q\in\partial\mathbb{A}d\mathbb{S}_3 \mid B_Q(p,q)>0\}$$

Note that U_p is a open hemisphere of $\mathbb{S}^3$, and note that $\mathcal{A}_p=\mathbb{A}d\mathbb{S}_3\cap U_p$ and $\partial\mathcal{A}_p=\mathbb{A}d\mathbb{S}_3\cap U_p$. Also note that $\partial\mathcal{A}_p$ is *not* the boundary of $\mathcal{A}_p$ in $\mathbb{S}^3$: it is the boundary of $\mathcal{A}_p$ in U_p. It is also interesting to observe that the hemisphere U_p is the connected component of $\mathbb{S}^3 \setminus p^\perp$ containing p. Similarly, the domain $\mathcal{A}_p$ is the connected component of $\mathbb{A}d\mathbb{S}_3 \setminus p^*$ containing p, and $\mathcal{A}_p \cup \partial\mathcal{A}_p$ is the connected component of $(\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3) \setminus \overline{p^*}$.

Let p_0 be the point of coordinates $[1:0:0:0]$ in $\mathbb{S}^3$. We observe that

$$U_{p_0} = \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{S}^3 \mid x_1 > 0\}$$

and we consider the diffeomorphism

$$\begin{array}{llll} \Phi_{p_0}: & U_{p_0} & \longrightarrow & \mathbb{R}^3 \\ & [x_1 : x_2 : x_3 : x_4] & \longmapsto & (x,y,z) = \left(\frac{x_3}{x_1}, \frac{x_4}{x_1}, \frac{x_2}{x_1}\right) \end{array}$$

Now, given any point $p \in \mathbb{A}d\mathbb{S}_3$, we can find an element σ_p of $O_0(2,2)$, such that $\sigma_p(p) = p_0$. Then, we consider the diffeomorphism $\Phi_p : U_p \to \mathbb{R}^3$ defined by $\Phi_p = \Phi_{p_0} \circ \sigma_p$.

For every $p \in \mathbb{A}d\mathbb{S}_3$, the diffeomorphism Φ_p maps the domain $\mathcal{A}_p$ on the region of $\mathbb{R}^3$ defined by the inequation $(x^2+y^2-z^2 < 1)$, and maps $\partial\mathcal{A}_p$ on the one-sheeted hyperboloid $(x^2+y^2-z^2 = 1)$. Moreover, it is well-known that Φ_{p_0} is an affine chart, *i.e.* that it maps the great circles of $\mathbb{S}^3$ to the affine lines of $\mathbb{R}^3$. Combining this with proposition 2.10, we obtain that, for every $p \in \mathbb{A}d\mathbb{S}_3$, the diffeomorphism Φ_p maps the geodesics of $\mathbb{A}d\mathbb{S}_3$ to the intersections of the affine lines of $\mathbb{R}^3$ with the set $(x^2+y^2-z^2 < 1)$. Similarly, Φ_p maps the totally geodesic subspaces of $\mathbb{A}d\mathbb{S}_3$ to the intersections of the affine planes of $\mathbb{R}^3$ with the set $(x^2+y^2-z^2 < 1)$.

Remark 2.14. *Let γ_0 be a geodesic of $\mathbb{A}d\mathbb{S}_3$. Let γ_p be the image under Φ_p of $\gamma \cap \mathcal{A}_p$. According to the above remark, γ_p is included in an affine line $\widehat{\gamma}_p$ of $\mathbb{R}^3$. Moreover, using remark 2.11, we see that:*
– if γ is timelike, then the line $\widehat{\gamma}_p$ does not intersect the hyperboloid $(x^2+y^2-z^2 = 1)$ and $\gamma_p = \widehat{\gamma}_p$,
- if γ is lightlike, then the affine line $\widehat{\gamma}_p$ is tangent to the hyperboloid $(x^2+y^2-z^2 = 1)$ at one point q and γ_p is one of the two connected components of $\widehat{\gamma}_p \setminus q$,
- if γ is spacelike, then the line γ intersects transversally the hyperboloid $(x^2+y^2tz^2 = 1)$ at two points q_1, q_2 and γ is the bounded connected component of $\widehat{\gamma} \setminus \{q_1, q_2\}$.

The image under Φ_p of any geodesic of $\mathbb{A}d\mathbb{S}_3$ is included in an affine line of $\mathbb{R}^3$. This implies in particular that there is no closed geodesic of $\mathbb{A}d\mathbb{S}_3$ included in $\mathcal{A}_p$. Moreover, it is not hard to prove that there is no closed timelike curve in $\mathcal{A}_p$. So, the causal structure of $\mathcal{A}_p$ is more interesting than those of $\mathbb{A}d\mathbb{S}_3$ (see remark 2.4).

2.4 Convex subsets of $\mathbb{A}d\mathbb{S}_3$

Using the locally affine structures on $\mathbb{A}d\mathbb{S}_3$, we will define a notion of convexity for subsets of $\mathbb{A}d\mathbb{S}_3$.

First, we define a *convex subset of* $\mathbb{S}^3$ to be a set C included in some open hemisphere U of $\mathbb{S}^3$, such that there exists some affine chart[1] $\varphi : U \to \mathbb{R}^3$ such that the set $\varphi(C)$ is convex subset of $\mathbb{R}^3$.

Observe that, if C is a convex subset of $\mathbb{S}^3$, then, for *every* open hemisphere V of $\mathbb{S}^3$ containing C, and *every* affine chart $\psi : V \to \mathbb{R}^3$, the set $\varphi(C)$ is a convex subset of $\mathbb{R}^3$. Also observe that, a set C included in some open hemisphere of $\mathbb{S}^3$ is a convex subset of $\mathbb{S}^3$ if and only if the positive cone $\pi^{-1}(C)$ is a convex subset of $\mathbb{R}^4$ (recall that π is the natural projection of $\mathbb{R}^4 \setminus \{0\}$ on $\mathbb{S}^3$).

Now, given a subset E of $\mathbb{S}^3$ such that C is included in some open hemisphere of $\mathbb{S}^3$, we define the *convex hull* $\mathrm{Conv}(C)$ of the set C to be the intersection of all the convex subsets of $\mathbb{S}^3$ containing C. Observe that this is well-defined since an open hemisphere is a convex subset of $\mathbb{S}^3$. Also observe that, if U is an open hemisphere containing C and $\Phi : U \to \mathbb{R}^3$ is an affine chart, the set $\mathrm{Conv}(C)$ is the image under Φ^{-1} of the convex hull in $\mathbb{R}^3$ of the set $\Phi(C)$. Lastly, observe that $\mathrm{Conv}(C)$ is also the image under π of the convex hull in $\mathbb{R}^4$ of the positive cone $\pi^{-1}(C)$.

Now, recall that $\mathbb{A}d\mathbb{S}_3$ is included in the sphere $\mathbb{S}^3$, and let C be a subset of $\mathbb{A}d\mathbb{S}_3$. We say that C is a *convex subset of* $\mathbb{A}d\mathbb{S}_3$ if it is convex as a subset of $\mathbb{S}^3$. We say that C is a *relatively subset C of* $\mathbb{A}d\mathbb{S}_3$ if C is the intersection of $\mathbb{A}d\mathbb{S}_3$ with a convex subset of $\mathbb{S}^3$. Equivalently, C is a convex subset of $\mathbb{A}d\mathbb{S}_3$ if $C = \mathrm{Conv}(C)$, and C is a relatively convex subset of $\mathbb{A}d\mathbb{S}_3$ if $C = \mathrm{Conv}(C) \cap \mathbb{A}d\mathbb{S}_3$.

[1] recall that this means that $\varphi : U \to \mathbb{R}^3$ is a diffeomorphism which maps the great circles of $\mathbb{S}^3$ (intersected with U) to the affine lines of $\mathbb{R}^3$.

2.5 The $SL(2,\mathbb{R})$-model of the anti-de Sitter space

Recall that, if we use the coordinates a, b, c, d on $\mathbb{R}^4$, the linear model of the anti-de Sitter space is nothing but the quadric

$$\{(a,b,c,d) \in \mathbb{R}^4 \mid -ad + bc = -1\}$$

endowed with the lorentzian metric induced by the quadratic from $Q(a,b,c,d) = -ad + bc$. Hence, the linear model of the anti-de Sitter space can be identified with

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) \mid ad - bc = 1 \right\}$$

endowed with the lorentzian metric induced by the quadratic form $-\det$ on $M(2,\mathbb{R})$.

Observe that the quadratic form $-\det$ on $M(2,\mathbb{R})$ is invariant under left and right multiplication by elements of $SL(2,\mathbb{R})$ (actually, the lorentzian metric induced by $-det$ on $SL(2,\mathbb{R})$ is a multiple of the Killing form of the Lie group $SL(2,\mathbb{R})$). Using this it is easy to see that the isometry group of $(SL(2,\mathbb{R}), -det)$ is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ acting on $SL(2,\mathbb{R})$ by left and right multiplication, *i.e.* acting by $(g_L, g_R) \cdot g = g_L\, g\, g_R^{-1}$.

2.6 Causal structure of the anti-de Sitter space

Let dt^2 be the standard riemannian metric on the circle $\mathbb{S}^1$; let ds^2 be the standard riemannian metric on the 2-dimensional sphere $\mathbb{S}^2$; let $\mathbb{D}^2$ be the open upper-hemisphere of $\mathbb{S}^2$, and $\overline{\mathbb{D}^2}$ be the closure of $\mathbb{D}^2$. We will prove that $\mathbb{A}d\mathbb{S}_3$ has the same causal structure as $(\mathbb{S}^1 \times \mathbb{D}^2\ ,\ -dt^2 + ds^2)$. More precisely:

Proposition 2.15. *There exists a diffeomorphism $\Psi : \mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1 \times \mathbb{D}^2$ such that the pull back by Ψ of the lorentzian metric $-dt^2 + ds^2$ defines the same causal structure as the original metric of $\mathbb{A}d\mathbb{S}_3$. Moreover, the diffeomorphism Ψ can be extended to a diffeomorphism $\overline{\Psi} : \mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1 \times \overline{\mathbb{D}^2}$.*

To prove proposition 2.15, we will embed $\mathbb{A}d\mathbb{S}_3$ in the so-called *three-dimensional Einstein space.* Let $(x_1, x_2, x_3, x_4, x_5)$ be the standard coordinates on $\mathbb{R}^5$; let $\widetilde{Q}$ be the quadratic form on $\mathbb{R}^5$ defined by $\widetilde{Q}(x_1, x_2, x_3, x_4, x_5) = -x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2$; let $\mathbb{S}^4$ be the quotient of $\mathbb{R}^5 \setminus \{0\}$ by positive homotheties, and $\widetilde{\pi}$ be the natural projection of $\mathbb{R}^5 \setminus \{0\}$ on $\mathbb{S}^4$. The *three-dimensional Einstien space*, denoted by Ein_3, is the image under $\widetilde{\pi}$ of the quadric $(\widetilde{Q} = 0)$. There is a natural conformal class of lorentzian metric on Ein_3, defined as follows:

— Given an open subset U of Ein_3, and a local section $\sigma : U \to \mathbb{R}^5 \setminus \{0\}$ of the projection $\widetilde{\pi}$, we define a lorentzian metric g_σ on U as follows. For every point $p \in U$ and every vector $v \in T_p Ein_3$, we choose a vector $\hat{v} \in T_{\sigma(p)}\mathbb{R}^5$ such that $d\widetilde{\pi}(\sigma(p)).\hat{v} = v$. The quantity $\widetilde{Q}(\hat{v})$ does not depend on the choice of the vector $\hat{v}$: indeed, the vector $\hat{v}$ is tangent to the quadric $(\widetilde{Q} = 0)$, the vector $\hat{v}$ is defined up to the addition of an element of $\widetilde{\pi}^{-1}(p)$, and the half-line $\widetilde{\pi}^{-1}(p)$ is included the $\widetilde{Q}$-orthogonal of the tangent space of the quadric $(\widetilde{Q} = 0)$ at σ_p. We set $g_\sigma(v) := \widetilde{Q}(\hat{v})$.

— The conformal class of the metric g_σ does not depend on the section σ. Indeed, if σ and σ' are two sections of the projection $\widetilde{\pi}$ defined on U, then we have $g_{\sigma'} = \lambda^2.g_\sigma$, where $\lambda : U \to \mathbb{R}$ is the function such that $\sigma' = \lambda.\sigma$.

Proof of proposition 2.15. Let $A = \{[x_1 : x_2 : x_3 : x_4 : x_5] \in Ein_3 \mid x_5 > 0\}$, and let ∂A be the boundary of A. We will consider two particular sections of the projection $\widetilde{\pi}$. On the one hand, we consider the section σ, defined on A, whose image is included in the affine hyperplane $(x_5 = 1)$. The anti-de Sitter space $\mathbb{A}d\mathbb{S}_3$ is isometric to the set A equipped with the lorentzian metric g_σ: the most natural isometry is the diffeomorphism Φ defined by $\Phi([x_1 : x_2 : x_3 : x_4]) = [x_1 : x_2 : x_3 : x_4 : 1]$. On the other hand, we consider the section σ', defined on the whole of Ein_3, whose image is included in the euclidian sphere $(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 2)$. The set A equipped with the lorentzian metric $g_{\sigma'}$ is isometric to the set

$$\left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 = 1\ ,\ x_3^2 + x_4^2 + x_5^2 = 1\ ,\ x_5 > 0\right\} \simeq \mathbb{S}^1 \times \mathbb{D}^2$$

equipped with the lorentzian metric $-(dx_1^2 + dx_2^2) + (dx_3^2 + dx_4^2 + dx_5^2) \simeq -dt^2 + ds^2$: the most natural isometry is the diffeomorphism $\Phi' = \sigma'_{|A}$. We consider the diffeomorphism $\Psi := \Phi' \circ \Phi : \mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1 \times \mathbb{D}^2$. Since the metric g_σ and $g_{\sigma'}$ are conformally equivalent, the pull back by Ψ of the metric $-dt^2 + ds^2$ is conformally equivalent to the original metric of $\mathbb{A}d\mathbb{S}_3$.

The diffeomorphism Φ can be extended to a diffeomorphism $\overline{\Phi} : \mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3 \to A \cup \partial A$: for every $[x_1 : x_2 : x_3 : x_4]$ in $\partial\mathbb{A}d\mathbb{S}_3$, we have $\overline{\Phi}([x_1 : x_2 : x_3 : x_4]) = [x_1 : x_2 : x_3 : x_4 : 0]$. The diffeomorphism Φ' can be extended to a diffeomorphism $\overline{\Phi'} : A \cup \partial A \to \mathbb{S}^1 \times \overline{\mathbb{D}^2}$: we have $\overline{\Phi'} = \sigma'_{A \cup \partial A}$. Hence, the diffeomorphism Ψ can be extended to a diffeomorphism $\overline{\Psi} = \overline{\Phi} \circ \overline{\Phi'} : \mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1 \times \overline{\mathbb{D}^2}$. $\square$

Causal structure on $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. Let $\overline{g}$ be the lorentzian metric on $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$, obtained by pulling back the lorentzian metric $-dt^2+ds^2$ on $\mathbb{S}^1 \times \overline{\mathbb{D}^2}$ by the diffeomorphism $\overline{\Psi}$. The lorentzian metric $\overline{g}$ defines the same causal structure on $\mathbb{A}d\mathbb{S}_3$ as the original metric of $\mathbb{A}d\mathbb{S}_3$. From now on, we endow $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ with the causal structure defined by the metric $\overline{g}$. This causal structure allows us to speak of timelike, lightlike and spacelike objects in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. In particular, we can consider the causal structure induced on the quadric $\partial\mathbb{A}d\mathbb{S}_3$. Given a point $q \in \partial\mathbb{A}d\mathbb{S}_3$, it is easy to verify that the lightcone of q for this conformally lorentzian structure is made of the circle of the left ruling and of the circle of the right ruling passing through q.

Remark 2.16. *Let p_0 be the point of coordinates $[1:0:0:0]$ in $\mathbb{S}^3$. Recall that $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$ is the subset of $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ defined by the inequation $(x_1 > 0)$. Hence, the diffeomorphism $\overline{\Psi}$ defined above maps $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$ on $\{(x_1,x_2,x_3,x_4,x_5) \mid x_1^2+x_2^2=1 \ , \ x_1>0 \ , \ x_3^2+x_4^2+x_5^2=1 \ , \ x_5 \geq 0\} \simeq (-\pi/2,\pi/2)\times\overline{\mathbb{D}^2}$.*

Corollary 2.17. *For every $p \in \mathbb{A}d\mathbb{S}_3$, the domain $\mathcal{A}_p \cup \partial\mathcal{A}_p$ has the same causal structure as the lorentzian space $\left((-\pi/2,\pi/2)\times\overline{\mathbb{D}^2} \ , \ -dt^2+ds^2\right)$.*

Proof. Since $O(2,2)$ acts transitively on $\mathbb{A}d\mathbb{S}_3$, it is enough to consider the case where p is the point of coordinates $[1:0:0:0]$. This case follows from proposition 2.15 and remark 2.16. □

The two following propositions will play some fundamental roles in the proof of theorem 0.2:

Proposition 2.18. *Let p be a point in $\mathbb{A}d\mathbb{S}_3$, and q be a point in $\partial\mathcal{A}_p$. A point $r \in \mathcal{A}_p \cup \partial\mathcal{A}_p$ can be joined from q by a timelike (resp. causal) curve if and only if $B_Q(q,r)$ is positive (resp. non-negative).*

Proof. Since $O_0(2,2)$ acts transitively on $\mathbb{A}d\mathbb{S}_3$, we can assume that $p=[1:0:0:0]$. There exists a timelike curve joining q to r in $\mathcal{A}_p \cup \partial\mathcal{A}_p$ if and only if there exists a timelike curve joining $\Psi_p(q)$ to $\Psi_p(r)$ in $\left((-\pi/2,\pi/2)\times\overline{\mathbb{D}^2} \ , \ -dt^2+ds^2\right)$. We see $\left((-\pi/2,\pi/2)\times\overline{\mathbb{D}^2} \ , \ -dt^2+ds^2\right)$ as the set

$$\left\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5 \mid x_1^2+x_2^2=1 \ , \ x_1>0 \ , \ x_3^2+x_4^2+x_5^2=1 \ , \ x_5>0\right\}$$

equipped with the metric $-(dx_1^2+dx_2^2)+(dx_3^2+dx_4^2+dx_5^2)$. First, we observe that $B_Q(q,r)$ and $B_{\widetilde{Q}}(\Psi_p(q),\Psi_p(r))$ have the same sign (this follows from the definition of the diffeomorphism Ψ_p, see the proof of proposition 2.15). Second, we observe that the points $\Psi_p(q)$ and $\Psi_p(r)$ are joined by a timelike (resp. causal) curve in $\left((-\pi/2,\pi/2)\times\overline{\mathbb{D}^2} \ , \ -dt^2+ds^2\right)$ if and only if $\widetilde{Q}(\Psi_p(q)-\Psi_p(r))$ is negative (resp. non-positive). Third, we observe that the quantity $\widetilde{Q}\left(\Psi_p(q)-\Psi_p(r)\right)$ and $B_{\widetilde{Q}}\left(\Psi_p(q),\Psi_p(r)\right)$ have opposite signs (since $\widetilde{Q}\left(\Psi_p(q)\right) = \widetilde{Q}\left(\Psi_p(q)\right) = 0$). Putting everything together, we obtain the proposition. □

Remark 2.19. *Let p be a point in $\mathbb{A}d\mathbb{S}_3$. Let P be a spacelike totally geodesic subspace of $\mathcal{A}_p$ (by such we mean the intersection of $\mathcal{A}_p$ with a spacelike totally geodesic subspace of $\mathbb{A}d\mathbb{S}_3$). Then, P divides $\mathcal{A}_p$ into two closed regions: the past of P in $\mathcal{A}_p$ and the future of P in $\mathcal{A}_p$.*

Proof. We identify $\mathcal{A}_p$ and P with their images under the embedding Φ_p. Then, P is the intersection of $\mathcal{A}_p$ (*i.e.* of the set $(-x^2+y^2+z^2<1)$) with an affine plane $\widehat{P}$ of $\mathbb{R}^3$. We consider the two regions of $\mathcal{A}_p$ defined as the intersections of $\mathcal{A}_p$ with the closures two connected components of $\mathbb{R}^3\setminus\widehat{P}$. Since P is spacelike and connected, the past (resp. the future) of P in $\mathcal{A}_p$ is necessarily included in one of these two regions. Finally, using remark 2.14, it is elementary exercice to verify that, for point $q\in\mathcal{A}_p$, there exists a timelike geodesic joining q to a point of P. Hence, the union of the past and the future of P must be equal to $\mathcal{A}_p$. The proposition follows. □

3 Proof of theorem 0.2 in the case $g \geq 2$

The purpose of this section is to prove theorem 0.2 in the case where the genus of the Cauchy surfaces of the globally hyperbolic spacetime under consideration is at least 2.

All along the section, we consider a maximal globally hyperbolic spacetime M, locally modelled on AdS_3, with closed orientable Cauchy surfaces of genus $g\geq 2$. We denote by $\widetilde{M}$ the universal covering of M. We choose a Cauchy surface Σ_0 in M, and the lift $\widetilde{\Sigma}_0$ of Σ_0 in $\widetilde{M}$. Since M is locally modelled on AdS_3, we have a locally isometric developping map $\mathcal{D}:\widetilde{\rightarrow} AdS_3$. We denote $S_0=\mathcal{D}(\widetilde{\Sigma}_0)$. The developping map induces a representation ρ of the fundamental group $\pi_1(M)=\pi_1(\Sigma_0)$ in $O_0(2,2)$. We denote by

$\Gamma = \rho(\Pi_1(M))$. Identifying $O_0(2,2)$ with $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/(-Id,-Id)$ (see subsection 2.2), we can see ρ as a representation of $O_0(2,2)$ in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Then, we will denote by ρ_L and ρ_R the representations of $\pi_1(M)$ in $SL(2,\mathbb{R})$ such that $\rho = \rho_L \times \rho_R$.

We have to prove that M admits a CMC time function. In subsection 3.1, we will explain why these reduces to the proof of the existence of a pair of barriers in M. In subsection 3.2, we will study the surface S_0 and its boundary ∂S_0 in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. In subsection 3.3, we study the Cauchy development $D(S_0)$ of the surface S_0. In particular, we prove that M is isometric to the quotient $\Gamma\backslash D(S_0)$. In subsection 3.4, we study the intersection C_0 of $\mathbb{A}d\mathbb{S}_3$ with the convex hull of the curve ∂S_0. In particular, we prove that the C_0 is included in the Cauchy development $D(S_0)$, so that we may consider the projection $\Gamma\backslash C_0$ of C_0 in $\Gamma\backslash D(S_0) \simeq M$. In subsection 3.5, we define some notion of convexity and concavity for spacelike surfaces in $\mathbb{A}d\mathbb{S}_3$, and we prove that the boundary of C_0 in $\mathbb{A}d\mathbb{S}_3$ is the union of two disjoined spacelike topological surfaces S_0^- and S_0^+, repectively convex and concave. The projections $\Sigma_0^- = \Gamma\backslash S_0^-$ and $\Sigma_0^+ = \Gamma\backslash S_0^+$ of these surfaces in $\Gamma\backslash D(S_0) \simeq M$ is "almost a pair of barriers". There are still two small problems: in general, the surfaces Σ_0^- and Σ_0^+ have flat regions (whereas, for barriers, we need surfaces with positive and negative mean curvature), and in general, these are only topological surfaces (whereas, for barriers, we need surfaces of class C^2). The purpose of subsections 3.6 and 3.7 is to approximate the surfaces Σ_0^- and Σ_0^+ by a true pair of barriers.

3.1 Reduction of theorem 0.2 to the existence of a pair of barriers

V. Moncrief has proved that the solutions of the vacuum Einstein equation in dimension $2+1$ with compact Cauchy surface can be described as the orbits of a non-autonomous hamiltonian flow on a finite-dimensional space (namely the Teichmüller space of the Cauchy surface). Using this hamiltonian flow, L. Andersson, Moncrief and A. Tromba have obtained the following theorem ([2, corollary 7]):

Theorem 3.1 (Andersson, Moncrief, Tromba). *Let N be a 3-dimensional maximal globally hyperbolic spacetime, with constant curvature, and with closed Cauchy surfaces of genus $g \geq 2$. Then, N admits a CMC time function.*

Thanks to theorem 3.1, the proof of theorem 0.2 is reduced to the proof of the existence of a CMC Cauchy surface. The existence of CMC surfaces, in particular the existence of surfaces with zero mean curvature, has been studied in many context. The problem usually splits into two disjoint steps : a geometrical step which consists in constructing some surfaces with (non-constant) negative and positive mean curvature called *barriers*, and an analitical step which consists in solving the appropriate PDE to prove the existence of a surface with zero mean curvature assuming the existence of barriers. In our context, the second step was performed by C. Gerhardt (see [4, theorem 6.1][2]):

Definition 3.2. *We define a* pair of barriers *in a three-dimensional globally hyperbolic lorentzian manifold N is a pair of disjoint Cauchy surfaces Σ^- and Σ^+ in N, such that Σ^+ is in the future of Σ^-, the supremum of the mean curvature of Σ^- is negative, and the infimum of the mean curvature of Σ^+ is positive.*

Theorem 3.3 (Gerhardt). *Let N be a three-dimensional globally hyperbolic lorentzian manifold, with compact Cauchy surfaces. Assume that there exists a pair of barriers in N. Then, there exists a Cauchy surface with zero mean curvature in N.*

Using proposition 0.1, and the results of Andersson-Moncrief-Tromba and Gerhardt stated above, the proof of our main theorem reduces to the proof of the existence of a pair of barriers in our spacetime M.

3.2 The spacelike surface S_0

The purpose of this subsection is to collect as many information as possible on the surface S_0. In particular, we will prove that S_0 is an open disc properly embedded in $\mathbb{A}d\mathbb{S}_3$, that the closure $\overline{S_0}$ of S_0 in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ is a closed topological disc, and that $\overline{S_0}$ is an achronal set. The results of this subsection are not original: most of them can be found in the IHES preprint of G. Mess ([7]). Yet, we will provide a proof of each result to keep our paper as self-contained as possible[3].

The lorentzian metric of M induces a riemannian metric on the Cauchy surface Σ_0, which can be lifted to get a riemannian metric on $\widetilde{\Sigma}_0$. Since Σ_0 is compact, the riemannian metrics on Σ_0 and $\widetilde{\Sigma}_0$

[2]The result proved by Gerhardt in [4] is more general than the statement that we give here.

[3]By the way, using the conformal equivalence of $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ with $(\overline{\mathbb{D}}^2 \times \mathbb{S}^1, -dt^2 + ds^2)$, we were able to simplify some of the proofs of Mess's preprint.

are complete. The developping map $\mathcal{D}$ induces a locally isometric immersion of the surface $\widetilde{\Sigma}_0$ in AdS_3. Actually, it appears that this immersion is automatically a proper embedding:

Proposition 3.4. *The surface S_0 is an open disc properly embedded in $\mathbb{A}d\mathbb{S}_3$. Moreover, every timelike geodesic of $\mathbb{A}d\mathbb{S}_3$ intersects the surface S_0 at exactly one point.*

Proof. We consider the projection $\zeta : AdS_3 \to \mathbb{R}^2$, defined by $\zeta(x_1, x_2, x_3, x_4) = (x_3, x_4)$. Observe that the fibers of the projection ζ are the orbits of a timelike killing vector field of $\mathbb{A}d\mathbb{S}_3$. We endow $\mathbb{R}^2$ with the riemanian metric g_ζ defined as follows. Given a point $q \in \mathbb{R}^2$ and a vector $v \in T_q\mathbb{R}^2$, we choose a point $\hat{q} \in \zeta^{-1}(q)$, and we consider the unique vector $\hat{v} \in T_{\hat{q}}AdS_3$ such that $d\zeta_{\hat{q}}.\hat{v} = v$ and such that $\hat{v}$ is orthogonal to the fibers $\zeta^{-1}(q)$. We define $g_\zeta(v)$ to be to the norm of the vector $\hat{v}$ for the lorentzian metric of AdS_3. This definition does not depend on the choice of the point $\hat{q}$, since the fibers of ζ are the orbits of a killing vector field. It is easy to verify that $\mathbb{R}^2$ endowed with the metric g_ζ is isometric to the hyperbolic plane.

Claim 1. Given any point $q \in AdS_3$ and any spacelike vector v in T_qAdS_3, the norm of the vector $d\zeta_q(v)$ for the metric g_ζ is bigger than the norm of v in AdS_3.

Indeed, write $v = u + w$ where u is tangent to the fiber of the projection ζ (in particular, u is timelike) and w is orthogonal to this fiber. On the one hand, by definition of g_ζ, the norm of the vector $d\zeta_q(v)$ for the metric g_ζ is equal to the norm of w in AdS_3. On the other hand, the norm of v in AdS_3 is less than the norm of w, since u is timelike. This completes the proof of the claim.

Claim 2. For every locally isometric immersion $f : \widetilde{\Sigma}_0 \to AdS_3$, the map $\zeta \circ f : \widetilde{\Sigma}_0 \to \mathbb{R}^2$ is an homeomorphism. In particular, the surface $f(\widetilde{\Sigma}_0)$ intersects each fiber of ζ at exactly one point.

By the first claim, the map $\zeta \circ f$ is locally distance increasing (when the surface $\widetilde{\Sigma}_0$ is endowed its riemannian metric, and $\mathbb{R}^2$ is endowed with the metric g_ζ) . Since the riemannian metric of Σ_0 is complete that $\zeta \circ f : \widetilde{\Sigma}_0 \to \mathbb{R}^2$ has the path lifting property, and thus is a covering map. Since H is simply connected, this implies that $\zeta \circ f : \widetilde{\Sigma}_0 \to \mathbb{R}^2$ is an homeomorphism. This completes the proof of claim 2.

Applying claim 2 with f equal to the developping map $\mathcal{D}$, we obtain that $\mathcal{D} : \widetilde{\Sigma}_0 \to \mathbb{A}d\mathbb{S}_3$ is a proper embedding, and that $\widetilde{\Sigma}_0$ is homeomorphic to $\mathbb{R}^2$ (and thus homeomorphic to an open disc). Hence, the surface $S_0 := \mathcal{D}(\widetilde{\Sigma}_0)$ is an open disc properly embedded in AdS_3. Now, let γ be a timelike geodesic of AdS_3. Observe the circle $\zeta^{-1}(0,0)$ is a timelike geodesic of AdS_3. Since $O(2,2)$ acts transitively on the set of timelike geodesic of AdS_3, there exists $\sigma \in O(2,2)$ such that $\sigma(\gamma) = \zeta^{-1}(0,0)$; in particular, $\sigma(\gamma)$ is a fiber of the projection ζ. Applying claim 2 with $f = \sigma^{-1} \circ \mathcal{D}$, we obtain that the surface $\sigma^{-1}(S_0) = \sigma^{-1} \circ \mathcal{D}(\widetilde{\Sigma}_0)$ intersects each fiber of ζ at exactly one point. Hence, the surface S_0 intersects the geodesic γ at exactly point. □

Remark 3.5. *In the proof of proposition 3.4, we have not used any hypothesis on the genus of the surface Σ_0. Hence, proposition 3.4 is still valid without the assumption that the genus of the surface Σ_0 is at least 2. Yet, the proof of proposition 3.4 shows that $\widetilde{\Sigma}_0$ is homeomorphic to a disc. This proves that* there does not exist globally hyperbolic spacetime, locally modelled on AdS_3, with closed orientable Cauchy surfaces of genus 0.

To prove that the closure of S_0 in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ is a closed topological disc, we will use the conformal equivalence between $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ and $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$. Let us start with some remarks:

Remark 3.6. *(i) Let S be a spacelike (resp. nowhere timelike) surface in $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$. Then, every point of S has a neighbourhood in S which is the graph of a contracting*[4] *(resp. 1-Lipschitz) function $f : (U, ds^2) \to (\mathbb{S}^1, dt^2)$, where U is an open subset of $\overline{\mathbb{D}^2}$.*

(ii) Every properly embedded spacelike (resp. nowhere timelike) surface in $(\mathbb{S}^1 \times \mathbb{D}^2, -dt^2 + ds^2)$ is the graph of a contracting (resp. 1-Lipschitz) function $f : (\mathbb{D}^2, ds^2) \to (\mathbb{S}^1, dt^2)$.

(iii) Of course, (i) and (ii) remain true if we replace $\mathbb{S}^1$ by $(-\pi/2, \pi/2)$.

Proof. Item (i) is an immediate consequence of the product structure of $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$. To prove (ii), we consider a properly embedded spacelike (resp. nowhere timelike) surface S in $(\mathbb{S}^1 \times \mathbb{D}^2, -dt^2 + ds^2)$. Let p_2 be the projection of $\mathbb{S}^1 \times \mathbb{D}^2$ on $\mathbb{D}^2$. Using item (i) and the fact that S is properly embedded, it is easy to show that $p_2 : S \to \mathbb{D}^2$ is a covering map. Hence, $p_2 : S \to \mathbb{D}^2$ is a homeomorphism, and the surface S is the graph of a function $f : \mathbb{D}^2 \to \mathbb{S}^1$. By item (i), the function f is contracting (resp. 1-Lipschitz). □

[4] We recall that, given two metric spaces (E, d) and (E', d'), a function $f : (E, d) \to (E', d')$ is said to be *contracting* if $d'(f(x), f(y)) < d(x, y)$ for every $x \neq y$.

Thanks to remark 3.6, we can define a notion of spacelike topological surface in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$:

Definition 3.7. *Let S be a topological surface (with or without boundary) in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. Using the conformal equivalence between $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ and $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$, we can see S as a surface in $\mathbb{S}^1 \times \overline{\mathbb{D}^2}$. We will say that the topological surface S is* spacelike *(resp.* nowhere timelike*) if every point of S has a neighbourhood in S which is the graph of a contracting (resp. 1-Lipschitz) function $f : (U, ds^2) \to (\mathbb{S}^1, dt^2)$, where U is an open subset of $\overline{\mathbb{D}^2}$.*

Proposition 3.8. *The closure $\overline{S_0}$ of the surface S_0 in $AdS_3 \cup \partial AdS_3$ is a nowhere timelike closed topological disc.*

Proof. Using the conformal equivalence between $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ and $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$, we see S_0 as a surface in $\mathbb{S}^1 \times \overline{\mathbb{D}^2}$. By proposition 3.4 and remark 3.6, the surface S_0 is the graph of a contracting function $f : (\mathbb{D}^2, ds^2) \to (\mathbb{S}^1, dt^2)$. Every contracting function from $(\mathbb{D}^2, ds^2)$ to $(\mathbb{S}^1, dt^2)$ can be extended as a 1-Lipschtiz function from $(\overline{\mathbb{D}^2}, ds^2)$ to $(\mathbb{S}^1, dt^2)$. The proposition follows. □

Proposition 3.4 and 3.8 imply that the boundary ∂S_0 of the surface S_0 in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ is a topological simple closed curve included in $\partial\mathbb{A}d\mathbb{S}_3$. Of course, this curve is invariant under the action of the holonomy group $\Gamma = \rho(\pi_1(M))$.

Remark 3.9. *Accroding to the proof of proposition 3.4, the surface S_0 intersects each fiber of the projection $\zeta : AdS_3 \to \mathbb{R}^2$ defined by $\zeta((x_1, x_2, x_3, x_4)) = (x_3, x_4)$. This implies that the curve ∂S_0 intersects each fiber of the projection $\zeta : \partial\mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1$ defined by $\zeta([x_1 : x_2 : x_3 : x_4]) = [x_3 : x_4]$.*

Besides, if we idenfify $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ with $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$, then then the curve ∂S_0 is identified with the graph of a function from $\partial\mathbb{D}^2$ to $\mathbb{S}^1$. This implies, in particular, that the curve ∂S_0 is not null-homotopic in $\partial\mathbb{A}d\mathbb{S}_3$.

Proposition 3.10. *For every $p \in S_0$, the surface $\overline{S_0}$ is included in the affine domain $\mathcal{A}_p \cup \partial\mathcal{A}_p$.*

Proof. Let p be a point on the surface S_0. The point p does not belong to its dual totally geodesic surface p^*. Hence, we can find an open neighbourhood U of p such that, for every q in U, the surface q^* is disjoined from U. Moreover, since S_0 is a spacelike surface without boundary, replacing if necessary U by a smaller neighbourhood, we may assume that every timelike geodesic meeting U intersects $S_0 \cap U$.

Claim. For every $q \in U$, the surface S_0 does not intersects the surface q^*.

Indeed, consider a point $q \in U$, and a point r on the totally geodesic surface q^*. Then, the point r is not in U, and there exists a timelike geodesic γ passing through q and r (see remark 2.7). By construction of U, the geodesic γ intersects $S_0 \cap U$ at some point s. If the point r were on the surface S_0, the geodesic γ would intersect the surface S_0 at two different points (the points r and s); this would contradict proposition 3.4. Hence, the point r is not on S_0. This completes the proof of the claim.

The claim implies in particular that the surface p^* is disjoint from S_0. Moreover, the claim implies that, for every $q \in U$, the surface q^* is disjoined from the surface S_0. When q describes U, the totally geodesic surface q^* describes a neighbourhood of the totally geodesic surface p^*. Hence, the closure $\overline{p^*}$ of p^* in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ is disjoined from the closure $\overline{S_0}$ of S_0. Hence, the closed surface $\overline{S_0}$ is included the connected components of $(\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3) \setminus p^*$ containing p, *i.e.* is included in $\mathcal{A}_p \cup \partial\mathcal{A}_p$. □

Proposition 3.11. *The topological surface $\overline{S_0}$ is spacelike.*

Remark 3.12. *Recall that we are assuming in this section that the the genus of the Cauchy surface Σ_0 is at least 2. This hypothesis is crucial for proposition 3.11 to be true. Indeed, when Σ_0 is torus, the curve ∂S_0 (which can be defined in the same way as in the case where the genus of Σ_0 is at least 2) always contains lightlike geodesics segments, see remark 4.6.*

Proof. We already know that $\overline{S_0}$ is nowhere timelike, and that S_0 is spacelike. Hence, $\overline{S_0}$ is spacelike if and only if the curve ∂S_0 does not contain any non-trivial lightlike arc. Hence, $\overline{S_0}$ is spacelike if and only if ∂S_0 does not contain any non-trivial arc of some circle of the left or the right ruling of $\partial\mathbb{A}d\mathbb{S}_3$.

Let us denote by $\mathbb{RP}^1_L$ (resp. $\mathbb{RP}^1_R$) the space of the circles of the left (resp. right) ruling of $\partial\mathbb{A}d\mathbb{S}_3$. We recall that the action of the holonomy ρ on $\mathbb{RP}^1_L$ reduces to the action of ρ_R (since, ρ_L preserves individually each cicrle of the left ruling). Similarly, the action of ρ on $\mathbb{RP}^1_R$ reduces to the action of ρ_L.

Lemma 3.13. *The actions of the representations ρ_L and ρ_R respectively on $\mathbb{RP}^1_R$ and $\mathbb{RP}^1_L$ are minimal.*

Proof. Let p be a point of the surface S_0, and n the future-pointing unitary normal vector of S_0 at p. If v is a unitary vector tangent to S_0 at p, then $n+v$ is a future pointing lightlike vector. The lightlike geodesic directed by $n+v$ is tangent to ∂AdS_3 at two antipodal points (remark 2.11). These two antipodal points lie on the same circle of the right ruling; denote by $R_{[\lambda:\mu]}$ with circle (with $[\lambda:\mu]\in\mathbb{RP}^1_L$). The map $(p,v)\to(p,R_{[\lambda:\mu]})$ identifies the unitary tangent bundle of the surface Σ_0 with the flat $\mathbb{RR}^1$ bundle over Σ_0 given by $\pi_1(\Sigma_0)\backslash(S_0\times\mathbb{RP}^1)$ where the action of $\pi_1(\Sigma_0)=\pi_1(M)$ is given by $\gamma\,.\,(p\,,\,(\lambda:\mu))=(\rho(\gamma)(p)\,,\,\rho_L(\gamma)([\lambda:\mu]))$. Hence, the Euler class of the representation ρ_L is the Euler class of the unitary tangent bundle of Σ_0. By a theorem of Goldman (see [5])[5], this implies $\rho_L(\pi_1(M))$ is a cocompact Fuchsian subgroup of $SL(2,\mathbb{R})\times Id\simeq SL(2,\mathbb{R})$. In particular, the action of ρ_L on $\mathbb{RP}^1_R$ is minimal. □

End of the proof of proposition 3.11. Denote by U the open subset of ∂S_0, defined as the union of the interiors of all the non-trivial arcs of circles of left ruling included in ∂S_0. Note that the holonomy ρ preserves the open set U invariant (since it preserves ∂S_0, and maps a circle of the left ruling to a circle of the left ruling). Now, let $U_R\subset\mathbb{RP}^1_R$ be the set of all circles of the right ruling that intersect U. Then U_R is an open subset of $\mathcal{C}_R$ which is preserved by ρ_L. Hence, U_R is either empty or equal to $\mathcal{C}_R$. But the equality $U_R=\mathbb{RP}^1_R$ would imply that ∂S_0 is a circle of the left ruling, which is impossible by proposition 3.10. Hence, U_R is empty, *i.e.* the curve ∂S_0 does not contain any non-trivial arc of circle of the left ruling. Similarly, for the right ruling. This completes the proof. □

Remark 3.14. *on the one hand, proposition 3.4 implies that the action of Γ on the surface S_0 free and properly discontinous. On the other hand, lemma 3.13 implies that the action of Γ on ∂S_0 is minimal. As a consequence, the curve ∂S_0 is the limit set of the action of Γ on the surface S_0.*

Proposition 3.15. *For every $p\in\mathbb{A}d\mathbb{S}_3$ such that $\overline{S_0}\subset\mathcal{A}_p\cup\partial\mathcal{A}_p$, the surface $\overline{S_0}$ is an strictly achronal subset of $\mathcal{A}_p\cup\partial\mathcal{A}_p$ (*i.e.* a causal curve included in $\mathcal{A}_p\cup\partial\mathcal{A}_p$ cannot intersect $\overline{S_0}$ at two distinct points).*

To prove this, we need an analog of remark 3.6 for causal curves in $((-\pi/2,\pi/2)\times\overline{\mathbb{D}^2},-dt^2+ds^2)$:

Remark 3.16. *Every timelike (resp. causal) curve in $((-\pi/2,\pi/2)\times\overline{\mathbb{D}^2},-dt^2+ds^2)$ is the graph of a contracting (resp. 1-Lipschitz) function $g:(J,dt^2)\to(\overline{\mathbb{D}^2},ds^2)$, where J is a subinterval of $(-\pi/2,\pi/2)$.*

Proof. Proof of proposition 3.15 Let $p\in\mathbb{A}d\mathbb{S}_3$ such that $\overline{S_0}\subset\mathcal{A}_p\cup\partial\mathcal{A}_p$, and γ be a causal curve in $\mathcal{A}_p\cup\partial\mathcal{A}_p$. Since the result depends only on the causal structure of $\mathcal{A}_p\cup\partial\mathcal{A}_p$, we may identify $\mathcal{A}_p\cup\partial\mathcal{A}_p$ with $((-\pi/2,\pi/2)\times\overline{\mathbb{D}^2},-dt^2+ds^2)$. Then, the surface $\overline{S_0}$ is identified with the graph of a contracting function $f:\overline{\mathbb{D}^2}\to(-\pi/2,\pi/2)$ and the curve γ is identified with the graph of a 1-Lipschitz function $g:J\subset(-\pi/2,\pi/2)\to\overline{\mathbb{D}^2}$. The intersection of two such graphs contains at most one point. □

Remark 3.17. *Let γ be a ligtlike geodesic of $\mathbb{A}d\mathbb{S}_3$. For every $p\in\mathbb{A}d\mathbb{S}_3$, it is easy to verify that $\gamma\cap\mathcal{A}_p$ is connected,* i.e. *that γ induces a single causal curve in $\mathcal{A}_p$. Hence, proposition 3.15 implies that $\gamma\cap S_0$ is a single point. Moreover, let q and $-q$ be the two ends of the geodesic γ in $\partial\mathbb{A}d\mathbb{S}_3$ (see remark 2.11), and let $\overline{\gamma}:=\gamma\cup\{q,-q\}$. For every $p\in\mathbb{A}d\mathbb{S}_3$, $\overline{\gamma}\cap(\mathcal{A}_p\cup\partial\mathcal{A}_p)$ is connected. Hence, proposition 3.15 implies that $\overline{\gamma}\cap\overline{S_0}$ is a single point. In particular, if q or $-q$ is a point of the curve ∂S_0, then the geodesic γ does not intersect the surface S_0.*

Remark 3.18. *Let p be a point such that the surface S_0 is included in $\mathcal{A}_p$. Proposition 3.4 implies that every point of $\mathcal{A}_p$ is either in the past[6] or in the future of the surface S_0. Moreover, proposition 3.15 implies that a point of $\mathcal{A}_p$ cannot be simultaneously in the past and in the future of the surface, except if it is on the surface S_0.*

Remark 3.19. *All the results of subsection 3.2 remain true if one replaces the Cauchy surface Σ_0 by any other Cauchy surface of M.*

3.3 Cauchy development of the surface S_0

In this subsection, we study the Cauchy development of the surface S_0 in $\mathbb{A}d\mathbb{S}_3$. The main goal of the subsection is to prove that M is isometric to the quotient $\Gamma\backslash D(S_0)$.

Let us first recall the definition of the Cauchy development of a spacelike surface. Given a spacelike surface S in $\mathbb{A}d\mathbb{S}_3$, the *past Cauchy development* $D^-(S)$ of S is the set of all points $p\in\mathbb{A}d\mathbb{S}_3$ such that

[5]The hypothesis that the genus of Σ_0 is at least 2 is crucial for Goldman's theorem

[6]Here, we mean the "past in $\mathcal{A}_p$": a point q in the past of the surface S_0 if there exists a future-directed causal curve including in $\mathcal{A}_p$ going from S_0 to q. Similarly for the future.

every future-inextendable causal curve through p intersects S. The *future Cauchy development* $D^+(S)$ of S is defined similarly. The *Cauchy development* of S is the set $D(S) := D^-(S) \cup D^+(S)$. The following technical lemma provides us with a more practical definition of $D(S)$:

Lemma 3.20. *Let $S \subset \mathbb{A}d\mathbb{S}_3$ be a spacelike surface. The past Cauchy development of S is the set of all points p such that every future-directed lightlike geodesic ray through p intersects S.*

Proof. Let $p \in \mathbb{A}d\mathbb{S}_3$ be a point such that every past-directed lightlike geodesic ray through p intersects the surface S. Then, every past-directed lightlike geodesic ray through p intersects (transversally) the surface S at exactly one point (see remark 3.17). Hence, the set C of all the points of S that can be joined from p by a past-directed lightlike geodesic ray is homeomorphic to a circle. Therefore, C is the boundary of a closed disk $D \subset S$ (recall that S is a properly embedded disc, see proposition 3.4). Let L be the union of all the segments of lightlike geodesics joining p to a point of C. The union of D and L is a non-pathological sphere. By Jordan-Schoenflies theorem, this topological sphere is the boundary of a ball $B \subset \mathbb{A}d\mathbb{S}_3$. A non-spacelike curve cannot escape B through L ; as a consequence, every pat-inextendable non-spacelike curve through p must escape from B through D ; in particular, every past-inextendable non-spacelike curve through p must intersect S. Hence, the point p is in $D^+(S)$. □

Remark 3.21. *If S is a spacelike surface in $\mathbb{A}d\mathbb{S}_3$, then the only points which belong to both $D^-(S)$ and $D^+(S)$ are the points of the surface S itself. Indeed let p be a point which not on S, and let γ be a lightlike geodesic passing through p. By remark 3.17, the geodesic γ cannot intersect the surface S both in the past and in the future of p. Hence, the point p cannot belong to both $D^-(S)$ and $D^+(S)$.*

Remark 3.22. *Since the surface Σ_0 is a Cauchy surface in M, the range $\mathcal{D}(\widetilde{M})$ of the developping map $\mathcal{D}$ must be included in the Cauchy development of the surface $S_0 = \mathcal{D}(\widetilde{\Sigma}_0)$.*

Proposition 3.23. *The developping map $\mathcal{D} : \widetilde{M} \to \mathbb{A}d\mathbb{S}_3$ is one-to-one.*

Lemma 3.24. *This is no timelike geodesic of $\mathbb{A}d\mathbb{S}_3$ entirely included in $D(S_0)$.*

Proof. Let γ be a timelike geodesic of $\mathbb{A}d\mathbb{S}_3$. We recall that γ is a closed geodesic. By proposition 3.4, the geodesic γ intersects the surface S_0 at exactly one point p. We assume that γ is included in $D(S_0)$. We will consider the sets $\gamma^- := \gamma \cap D^-(S_0)$ and $\gamma^+ := \gamma \cap D^+(S_0)$. Since γ is included in $D(S_0)$, we have $\gamma = \gamma^- \cup \gamma^+$. Moreover, by remark 3.21, we have $\gamma^- \cap \gamma^+ = \gamma \cap S_0 = \{p\}$. The points of γ which are close enough to p, and in the past of p, are necessarily in $D^-(S_0)$; in particular, γ^- is not reduced to the point p. Similarly, γ^+ is not reduced to the point p. Since γ is included in $D(S_0)$ and since $D^-(S_0)$ is closed in $D(S_0)$ (by remark 3.21), γ^- is a closed subset of γ. Similarly, γ^+ is closed subset of γ. All these properties are incompatible: a circle cannot be the union of two closed subsets whose intersection is a single point p, except in the case where one of these two sets is the point p itself. □

Proof of proposition 3.23. Let p and q be two different points in $\widetilde{M}$. Let γ be a geodesic segment joining p to q in $\widetilde{M}$. The image of γ under $\mathcal{D}$ is a geodesic curve (since $\mathcal{D}$ is a local isometry) included in $D(S_0)$ (by remark 3.22). If $\mathcal{D}(p) = \mathcal{D}(q)$, then this geodesic curve must be closed. This is forbidden by lemma 3.24 (recall that the only closed geodesic in $\mathbb{A}d\mathbb{S}_3$ are the timelike geodesics). □

Proposition 3.25. *The holonomy group $\Gamma = \rho(\pi_1(M))$ acts freely, and properly discontinuously on the Cauchy development $D(S_0)$ of the surface S_0.*

Proof. First recall that proposition 3.4 implies that the holonomy group Γ acts freely and properly discontinuously on the surface $S_0 = \mathcal{D}(\widetilde{\Sigma}_0)$.

Suppose that the group Γ does not act freely on the Cauchy development $D^+(S_0)$. Then, there exists an element γ of Γ which fixes a point p of $D^-(S_0)$. As in the proof of lemma 3.20, we consider the set C of all the points of S_0 that can be joined from p by a past-directed lightlike geodesic ray. The set C is homeomorphic to a circle, and thus, it is the boundary of a closed disc $D \subset S_0$. The disc D must be invariant under γ (since the surface S_0 is Γ-invariant, and since γ fixes the point p). Hence, by Brouwer theorem, γ fixes a point in D. This contradicts the fact that Γ acts freely on S_0. A a consequence, the group Γ has to act freely on $D^+(S_0)$. The same arguments show that Γ acts freely on $D^-(S_0)$.

Now, let K be a compact subset included in $D^+(S_0)$. All the points of intersection of the past-directed geodesic rays emanating from the points of K with the surface S_0 belong to some compact subset K' of the surface S_0. Since Γ maps lightlike geodesic rays on lightlike geodesic rays, the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is included in the set $\{\gamma \in \Gamma \mid \gamma K' \cap K' \neq \emptyset\}$. Hence, the proper discontinuity of the action of Γ on $D^+(S_0)$ follows from the proper discontinuity of the action of Γ on S_0. The same arguments show that Γ acts properly discontinuously on $D^-(S_0)$. □

Proposition 3.26. *The developping map $\mathcal{D}$ induces a isometry between M and the quotient $\Gamma\backslash D(S_0)$.*

Proof. By proposition 3.25, the quotient $\Gamma\backslash D(S_0)$ is a manifold (which is automatically a globally hyperbolic, since it is the quotient of the Cauchy development $D(S_0)$). By remark 3.22 and proposition 3.23, the developping map $\mathcal{D}$ induces an isometric embedding of M in $\Gamma\backslash D(S_0)$. Since M is assumed to be maximal as a globally hyperbolic manifold, this embedding must be onto. $\square$

According to proposition 3.26, constructing a surface in M with some specified geometrical properties amounts to constructing a Γ-invariant surface in $D(S_0)$. In particular, we will use the following remark several times:

Remark 3.27. *If S is a Γ-invariant spacelike surface included in the Cauchy development $D(S_0)$, then $\Gamma\backslash S$ is a Cauchy surface in $M = \Gamma\backslash D(S_0)$. Indeed, $\Gamma\backslash S$ is a spacelike compact surface in $M = \Gamma\backslash D(S_0)$, and every compact spacelike surface in M is a Cauchy surface.*

3.4 The convex hull and the black domain of the curve ∂S_0

In this subsection, we will consider the convex hull $\mathrm{Conv}(\partial S_0)$ of the curve ∂S_0. The main goal of the subsection is to prove that the set $\mathrm{Conv}(\partial S_0) \setminus \partial S_0$ is included in the Cauchy development of the surface S_0. To prove this result, we will have to define a new set $E(\partial S_0)$, that we call the *black domain* of the curve ∂S_0.

Definition of the set $E(\partial S_0)$. First, for every $q \in \partial S_0$, we consider the set

$$E(\partial S_0) = \{r \in \mathbb{S}^3 \mid B_Q(r,q) < 0 \text{ for every } q \in \partial S_0\}$$

We call this set the *black domain* of the curve ∂S_0 (explanations on this terminology are provided below).

Remark 3.28. *Here are a few observations about the definition of the set $E(\partial S_0)$:*

(i) *We will prove below (proposition 3.32)) that the black domain $E(\partial S_0)$ (which is defined above as a subset of the sphere $\mathbb{S}^3$) is actually included in the anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}_3$. Moreover, we will prove that $E(\partial S_0)$ is included in the affine domain $\mathcal{A}_{p_0}$ for a well-chosen point p_0 (proposition 3.35).*

(ii) *Consider a point $p_0 \in AdS_3$ such that $E(\partial S_0)$ is included in $\mathcal{A}_{p_0}$. According to proposition 2.18, the set $E(\partial S_0)$ is made of the points $r \in \mathcal{A}_{p_0}$ such that there does not exist any causal curve joining r to the curve ∂S_0 within $\mathcal{A}_{p_0}$. In other words, $E(\partial S_0)$ is the set of "all the points of $\mathcal{A}_{p_0}$ that cannot be seen from any point of the curve ∂S_0". This is the reason why we call $E(\partial S_0)$ the* black domain *of the curve ∂S_0.*

(iii) *The black domain $E(\partial S_0)$ is clearly a convex subset of $\mathbb{S}^3$ (by construction, it is an intersection of convex subsets of $\mathbb{S}^3$). In particular, $E(\partial S_0)$ is connected.*

(iv) *Here is a nice way to visualize the $E(\partial S_0)$. Consider a point $p_0 \in \mathbb{A}\mathrm{d}\mathbb{S}_3$ such that $E(\partial S_0)$ is included in the affine domain $\mathcal{A}_{p_0}$. Using the diffeomorphism Φ_{p_0}, we can identify $\mathcal{A}_{p_0}$, $\partial\mathcal{A}_{p_0}$, ∂S_0, $E(\partial S_0)$ with some subsets of $\mathbb{R}^3$; in particular, $\partial\mathcal{A}_{p_0}$ is identified with the hyperboloid of equation $(x^2 + y^2 - z^2 = 1)$. Given $q \in \partial S_0$, the set $T_q = \{r \in \mathcal{A}_p \mid B_Q(q,r) = 0\}$ is the affine plane of $\mathbb{R}^3$ which is tangent to the hyperboloid $\partial\mathcal{A}_{p_0}$ at q. The set $E_q = \{r \in \mathcal{A}_p \mid B_Q(q,r) > 0\}$ is one of the two connected components of $\mathbb{R}^3 \setminus T_q$. Moreover, proposition 3.15 and 2.18 imply that $\partial S_0 \setminus \{q\}$ is included in $E(q)$. As a consequence, the set $E(\partial S_0)$ is the intersection over all $q \in \partial S_0$, of the connected component of $\mathbb{R}^3 \setminus T_q$ containing $\partial S_0 \setminus \{q\}$.*

(v) *Let r be a point of the boundary of $E(\partial S_0)$ in $\mathbb{A}\mathrm{d}\mathbb{S}_3$. The definition of the set $E(\partial S_0)$ and the compacity of the curve ∂S_0 imply that we have $B_Q(p,q) = 0$ for some point q on the curve ∂S_0. Hence, by remark 2.12, there exists a lightlike geodesic γ passing through r, such that one of the two ends of γ is a point of the curve ∂S_0.*

Proposition 3.29. *The surface S_0 is included in $E(\partial S_0)$.*

Proof. The result follows from proposition 3.15 and 2.18. $\square$

Definition of the set C_0. Recall that we denote by $\text{Conv}(\partial S_0)$ the convex hull in $\mathbb{S}^3$ of the curve ∂S_0 (see subsection 2.4). We will consider the set

$$C_0 = \text{Conv}(\partial S_0) \cap \mathbb{A}d\mathbb{S}_3$$

The purpose of the subsection is to prove that this set C_0 is equal to $\text{Conv}(\partial S_0) \setminus \partial S_0$, and that it is included in the Cauchy development $D(S_0)$.

Proposition 3.30. *The set $Conv(\partial S_0) \setminus \partial S_0$ is included in $E(\partial S_0)$.*

Proof. Let q be a point $\text{Conv}(\partial S_0) \setminus \partial S_0$, and let $\widehat{q}$ be any point in $\pi^{-1}(\{q\})$. Let r be a point in ∂S_0, and let $\widehat{r}$ be any point in $\pi^{-1}(\{r\})$. We have to prove that $B_Q(q,r)$ is negative, *i.e.* that $B_Q(\hat{q},\hat{r})$ is negative. Since $\widehat{q}$ is in $\pi^{-1}(\text{Conv}(\partial S_0))$, one can find some points $\widehat{q}_1, \dots, \widehat{q}_n \in \pi^{-1}(\partial S_0)$, and some positive numbers $\alpha_1, \dots, \alpha_n$, such that $\alpha_1 + \dots + \alpha_n = 1$, and such that $\widehat{q} = \alpha_1 \widehat{q}_1 + \dots + \alpha_n \widehat{q}_n$. We denote by $q_1, \dots, q_n$ the projections of the points $\widehat{q}_1, \dots, \widehat{q}_n$. For each $i \in \{1, \dots, n\}$, there are two possibities:
— either $q_i = r$, and then we have $B_Q(\widehat{q}_i, \widehat{r}) = B_Q(\widehat{r}, \widehat{r}) = 0$ (since $\widehat{r}$ is on the quadric $(Q = 0)$),
— or $q_i \neq r$, and then proposition 3.15 and 2.18 imply that $B_Q(\widehat{q}_i, \widehat{r})$ is negative.
Moreover, at least one q_i's is different from r (otherwise, we would have $q_1 = \dots = q_n = r = q$, which is absurd since q is not on ∂S_0). Hence, the quantity $B_Q(\widehat{q}, \widehat{r}) = \alpha_1 B_Q(\widehat{q}_1, \widehat{r}) + \dots + \alpha_n B_Q(\widehat{q}_n, \widehat{r})$ is negative. The proposition follows. □

Lemma 3.31. *For every point $q \in \partial \mathbb{A}d\mathbb{S}_3$, there exists a point $r \in \partial S_0$, such that $B_Q(q,r)$ is non-negative. Moreover, if the point q is not on the curve ∂S_0, then the point r can be choosen such that $B_Q(q,r)$ is positive.*

Proof. Let q be a point in $\partial \mathbb{A}d\mathbb{S}_3$. Denote by $[x_1 : x_2 : x_3 : x_4]$ the coordinates of q in $\mathbb{S}^3$. Remark 3.9 imply that there exist $x_1', x_2' \in \mathbb{R}$ such that the point r of coordinates $[x_1' : x_2' : x_3 : x_4]$ is on the curve ∂S_0. The sign of $B_Q(q,r)$ is the sign $-x_1 x_1' - x_2 x_2' + x_3^2 + x_4^2$. Since the point q and r are both on $\partial \mathbb{A}d\mathbb{S}_3$, we have $Q([x_1 : x_2 : x_3 : x_4]) = Q([x_1' : x_2' : x_3 : x_4]) = 0$. Hence, we have $-x_1 x_1' - x_2 x_2' + x_3^2 + x_4^2 = \frac{1}{2}((x_1 - x_1')^2 + (x_2 - x_2')^2)$. As a consequence, $B_Q(q,r)$ is non-negative. Moreover, if q is not on the curve ∂S_0, then (x_1, x_2) is different from (x_1', x_2'), and thus, $B_Q(q,r)$ is positive. □

Corollary 3.32. *The black domain $E(\partial S_0)$ is included in $\mathbb{A}d\mathbb{S}_3$.*

Proof. Lemma 3.31 says that the intersection of $\partial \mathbb{A}d\mathbb{S}_3$ with $E(\partial S_0)$ is empty. Since $E(\partial S_0)$ is connected, this implies that $E(\partial S_0)$ is either included in $\mathbb{A}d\mathbb{S}_3$, or disjoined from $\mathbb{A}d\mathbb{S}_3$. But, the intersection of $E(\partial S_0)$ with $\mathbb{A}d\mathbb{S}_3$ is non-empty (by proposition 3.30, for example). Hence, $E(\partial S_0)$ is included in $\mathbb{A}d\mathbb{S}_3$. □

Remark 3.33. *Proposition 3.30 and corollary 3.32 imply that the set $Conv(\partial S_0) \setminus \partial S_0$ is included in $\mathbb{A}d\mathbb{S}_3$. Hence, we have $Conv(\partial S_0) \cap \mathbb{A}d\mathbb{S}_3 = Conv(\partial S_0) \setminus \partial S_0$,* i.e. *$C_0) = Conv(\partial S_0) \setminus \partial S_0$.*

We will denote by $\overline{E(\partial S_0)}$ the closure of the black domain $E(\partial S_0)$ in $\mathbb{A}d\mathbb{S}_3 \cup \partial \mathbb{A}d\mathbb{S}_3$.

Corollary 3.34. *The intersection of $\overline{E(\partial S_0)}$ with $\partial \mathbb{A}d\mathbb{S}_3$ is the curve ∂S_0.*

Proof. Proposition 3.30 implies that every point of the curve ∂S_0 is in $\overline{E(\partial S_0)}$. Conversely, let q be a point in $\partial \mathbb{A}d\mathbb{S}_3 \setminus \partial S_0$. According to lemma 3.31, there exists a point $r \in \partial S_0$ such that $B_Q(q,r) > 0$. By continuity of the bilinear form B_Q, there exists a neighbourhood U of q in $\mathbb{S}^3$, such that $B_Q(q',r) > 0$ for every $q' \in U$. In particular, there exists a neighbourhood U of q which is disjoint from $E(\partial S_0)$. Hence, q is not in $\overline{E(\partial S_0)}$. □

Proposition 3.35. *There exists a point $p_0 \in \mathbb{A}d\mathbb{S}_3$ such that $E(\partial S_0)$ is included in the domain $\mathcal{A}_{p_0}$.*

Addendum. *If the curve ∂S_0 is not flat*[7]*, then one can choose the point p_0 such that $\overline{E(\partial S_0)}$ is included in $\mathcal{A}_{p_0} \cup \partial \mathcal{A}_{p_0}$.*

Lemma 3.36. *For every point $p \in C_0 = Conv(\partial S_0) \setminus \partial S_0$, the black domain $E(\partial S_0)$ is disjoined from the totally geodesic surface p^* (and thus, it is also disjoined from the closed surface $\overline{p^*}$).*

[7]We say that the curve ∂S_0 is *flat* if it is the boundary of a totally geodesic subspace of $\mathbb{A}d\mathbb{S}_3$, or equivalently, if it is included in a great 2-sphere in $\mathbb{S}^3$.

Proof. Let p be a point in C_0, and $\hat{p}$ be a point in $\mathbb{R}^4 \setminus \{0\}$ such that $\pi(\hat{p}) = p$. Since p is in $\mathrm{Conv}(\partial S_0)$, one can find some points $\widehat{p}_1, \ldots, \widehat{p}_n \in \pi^{-1}(\partial S_0)$ and some positive numbers $\alpha_1, \ldots, \alpha_n$ such that $\widehat{p} = \alpha_1\widehat{p}_1 + \cdots + \alpha_n\widehat{p}_n$. Let q be a point in $E(\partial S_0)$ and $\hat{q}$ be a point in $\mathbb{R}^4 \setminus \{0\}$ such that $\pi(\widehat{q}) = q$. Since q is in $E(\partial S_0)$, the quantity $B_Q(\widehat{p}_i, \widehat{q})$ is negative for every i. Hence, the quantity $B_Q(\widehat{p}, \widehat{q}) = \alpha_1 B_Q(\widehat{p}_1, \widehat{q}) + \cdots + \alpha_n B_Q(\widehat{p}_n, \widehat{q})$ is negative. In particular, the point q is not on the surface $p^* = \{r \in \mathbb{A}d\mathbb{S}_3 \mid B_Q(\widehat{p}, \widehat{r}) = 0\}$. This proves that set $E(\partial S_0)$ is disjoint from the totally surface p^*. Since $E(\partial S_0)$ is included in $\mathbb{A}d\mathbb{S}_3$, it is also disjoint from the closed surface $\overline{p^*}$. □

Proof of proposition 3.35. Let p_0 be a point in C_0. By lemma 3.36, $E(\partial S_0)$ is disjoint from the totally geodesic surface p_0^*. Since $E(\partial S_0)$ is connected, this implies that $E(\partial S_0)$ is included in one of the two connected components of $\mathbb{A}d\mathbb{S}_3 \setminus p_0^*$. By proposition 3.30, the point p_0 is in $E(\partial S_0)$. Hence, $E(\partial S_0)$ is included in the connected component of $\mathbb{A}d\mathbb{S}_3 \setminus p_0^*$ containing p_0, *i.e.* is included in $\mathcal{A}_{p_0}$. □

Proof of the addendum. If ∂S_0 is not flat, then the set C_0 has non-empty interior. Let p_0 be a point in the interior of C_0. On the one hand, the set $E(\partial S_0)$ is disjoint from the closed surface $\overline{p^*}$ for every $p \in C_0$. On the other hand, the union of all the surfaces $\overline{p^*}$ when p ranges over C_0 is a neighbourhood (in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$) of the surface $\overline{p_0^*}$. Hence, $E(\partial S_0)$ is disjoined from a neighbourhood of the surface $\overline{p_0^*}$. Hence, $\overline{E(\partial S_0)}$ is disjoined from the surface $\overline{p_0^*}$. Moreover, by proposition 3.30, the point p_0 is in $\overline{E(\partial S_0)}$. Hence, $\overline{E(\partial S_0)}$ is included in the connected component of $(\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3) \setminus \overline{p_0^*}$ containing p_0, *i.e.* is included in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$. □

From now on, we fix a point $p_0 \in \mathbb{A}d\mathbb{S}_3$, such that $\overline{E(\partial S_0)}$ is included in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$.

Proposition 3.37. *The black domain $E(\partial S_0)$ coincides with the Cauchy development $D(S_0)$.*

Proof. Let us suppose that $D(S_0)$ is not included in $E(\partial S_0)$. Since $D(S_0)$ and $E(\partial S_0)$ have a non-empty intersection (the surface S_0 is included in both $D(S_0)$ and $E(\partial S_0)$), and since $D(S_0)$ is connected, $D(S_0)$ must contain some point r of the boundary of $E(\partial S_0)$. By item (v) of remark 3.28, there exists a lightlike geodesic γ passing through r, such that one of the ends of γ is a point q on the curve ∂S_0 (the other end is necessarily the antipodal point $-q$). Since r is in $D(S_0)$, the lightlike geodesic γ must intersect the surface S_0. But, this is impossible by remark 3.17. This contradiction proves that $D(S_0)$ must be included in $E(\partial S_0)$.

To prove the other inclusion, we work in the affine domain $\mathcal{A}_{p_0}$. Let p be a point in $E(\partial S_0)$. By remark 3.18, every point of $\mathcal{A}_{p_0}$ is either in the past, or in the future of the surface S_0. We assume, for example, that p is in the future of S_0. We will prove that p is in $D^+(S_0)$. For that purpose, we consider a past-directed lightlike geodesic ray γ emanating from r, and we denote by q the past end of γ.

Claim. The geodesic ray γ intersects the boundary of $E(\partial S_0)$ at some point r situated in the past of S_0.

To prove this claim, we will argue by contradiction. First, we suppose that the geodesic ray γ is entirely included in $E(\partial S_0)$. Then, by proposition 3.32 and 3.34, the past end of γ must be a point q of the curve ∂S_0. But then, we have $B_Q(p, q) = 0$, and this contradicts the fact that p is in $E(\partial S_0)$. Now, we suppose that the geodesic ray γ intersects the boundary $E(\partial S_0)$ at some point r situated in the future of the surface S_0. By item (v) of remark 3.28, there exists a lightlike geodesic ray γ' emanating from r, such that the end of γ' is a point q of the curve ∂S_0. The geodesic ray γ' must be past-directed from r to q, since r is in the future of the surface S_0. So, we have a past-directed lightlike geodesic segment going from p to r, and a past-directed geodesic ray going from r to q; concatenating these two curves, we obtain a piecewise C^1 causal curve going from p to $q \in \partial S_0$. This contradicts the fact that p is in $E(\partial S_0)$ (see item (ii) of remark 3.28) and completes the proof of the claim.

Since the point p is in the future of the surface S_0, and since the point r given by the claim is in the past of the surface S_0, the geodesic ray γ must have intersect the surface S_0. So, we have proved that every past-directed geodesic ray emanating from p intersects the surface S_0. Hence, the point p is in $D^+(S_0)$ (lemma 3.20). This proves that $E(\partial S_0)$ is included in $D(S_0)$. □

Remark 3.38. *Proposition 3.37 implies in particular that the Cauchy development $D(S_0)$ depends only on the curve ∂S_0. More precisely, if S is any complete spacelike surface in AdS_3 such that $\partial S = \partial S_0$, then $D(S) = D(S_0)$.*

Remark 3.39. *Let Σ be any Cauchy surface in M, and let $S := \mathcal{D}(\widetilde{\Sigma})$. On the other hand, we have $D(S) = D(S_0) = \mathcal{D}(\widetilde{M})$. On the one hand, propositions 3.34 and 3.37 imply that the curve ∂S_0 is the intersection of the closure in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ of $D(S_0)$ with $\partial\mathbb{A}d\mathbb{S}_3$. Similarly, the curve ∂S is the intersection of the closure in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ of $D(S)$ with $\partial\mathbb{A}d\mathbb{S}_3$. As a consequence, we have $\partial S = \partial S_0$.*

Remark 3.40. *For every point $p \in D(S_0) = \mathcal{D}(\widetilde{M})$, one can find a Cauchy surface Σ in M such that $p \in \mathcal{D}(\widetilde{\Sigma})$. By remark 3.14 and 3.39, the limit set of the action of Γ on the surface S is the curve $\partial S = \partial S_0$. As a consequence, the limit of the action of Γ on $D(S_0)$ is also the curve ∂S_0.*

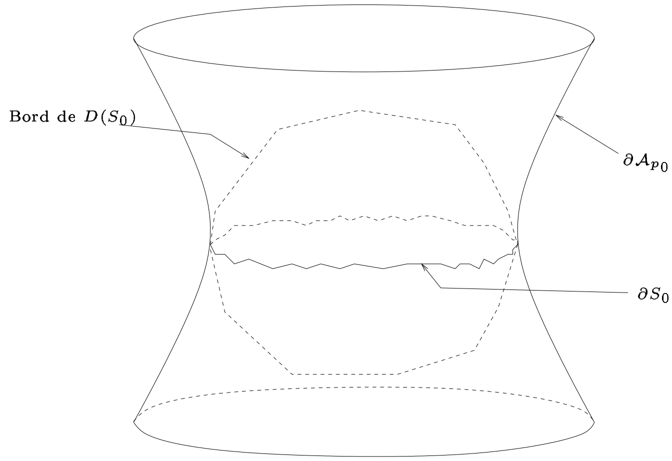

Figure 1: The affine domain $\mathcal{A}_{p_0}$, the curve ∂S_0 and the Cauchy development $D(S_0)$ represented in $\mathbb{R}^3$.

Interlude: proof of theorem 0.2 in the case where ∂S_0 is flat

Our strategy for proving the existence of a pair of barriers in M does not work in the particular case where ∂S_0 is flat, mostly because the addendum of proposition 3.35 is false when ∂S_0 is flat. This is not a big problem, since there is a direct and very short proof of theorem 0.2 in the particular case where ∂S_0 is flat:

Proof of theorem 0.2 in the case where ∂S_0 is flat. Assume that ∂S_0 is flat. Then it is the boundary of a totally geodesic subspace P_0 of $\mathbb{A}d\mathbb{S}_3$. This totally geodesic subspace is necessarily spacelike, since the curve ∂S_0 is spacelike. By construction, P_0 is included in C_0; hence, it is included in the Cauchy development $D(S_0)$ (proposition 3.37 and 3.30). Moreover, the holonomy group $\Gamma = \rho(\pi_1(M))$ preserves P_0 (since it preserves the curve ∂S_0). As a consequence, $\Gamma\backslash P_0$ is a totally geodesic compact spacelike surface in $\Gamma\backslash D(S_0) \simeq M$. In particular, $\Gamma\backslash P_0$ is a Cauchy surface with zero mean curvature in M. Applying theorem 3.1, we obtain theorem 0.2. $\square$

Assumption. From now on, we assume that the curve ∂S_0 is not flat.

3.5 A pair of convex and concave topological Cauchy surfaces

In this subsection, we will first define some notions of convexity and concavity for spacelike surfaces in M. The main interesting feature of this notion for our purpoe is the fact that the mean curvature of a convex (resp. concave) spacelike surface is always non-positive (resp. non-negative). Then, we will exhibit a pair of disjoint topological Cauchy surfaces (Σ_0^-, Σ_0^+) in M, such that Σ_0^- is convex, Σ_0^+ is concave, and Σ_0^+ is in the future of Σ_0^-.

3.5.1 Convex and concave surfaces in $\mathbb{A}d\mathbb{S}_3$

Let S be a topological surface in $\mathcal{A}_{p_0}$, and q be a point of S. A *support plane* of S at q is a (2-dimesional) totally geodesic subspace[8] P of $\mathcal{A}_{p_0}$, such that $q \in P$, and such that S is included in the closure of one of the connected components of $\mathcal{A}_{p_0} \setminus P$.

Remark 3.41. *Let S be a topological surface in $\mathcal{A}_{p_0}$. If S is spacelike (in the sense of definition 3.7), then every support plane of S is spacelike. Conversely, if S admits a spacelike support plane at every point, then S is spacelike.*

[8] By a *totally geodesic subspace of $\mathcal{A}_{p_0}$*, we mean the intersection of a totally geodesic subspace of $\mathbb{A}d\mathbb{S}_3$ with $\mathcal{A}_{p_0}$. Note that there is a subtility: with this definition, the degenerated totally geodesic subspaces of $\mathcal{A}_{p_0}$ are not connected (but their closure in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$ is connected). Do not worry, this subtility does not play any role in the subsequent.

Remark 3.42. *Let S be a topological surface in $\mathcal{A}_{p_0}$ and P be a spacelike support plane of S. Then, S is included in the causal past*[9] *of P, or S is included in the future of P (see remark 2.19).*

Let S be a topological spacelike surface in $\mathcal{A}_{p_0}$. We say that S is *convex*, if it admits a support plane at each of its points, and if it is included in the future of all its support planes. We say that S is *concave*, if it admits a support plane a each of its points, and if it is included in the past of all its support planes.

Now, let Σ be a topological spacelike surface in M, let $\widetilde{\Sigma}$ be a lift of Σ in $\widetilde{M}$, and let $S = \mathcal{D}(\widetilde{\Sigma})$. Note that S is a topological spacelike surface included in $\mathcal{D}(\widetilde{M}) \subset \mathcal{A}_{p_0}$ (see section 3.3). We say that Σ is *convex* (resp. *concave*) if S is convex (resp. concave).

Proposition 3.43. *Let Σ be a C^2 spacelike surface in M. If Σ is convex, then Σ has non-positive mean curvature. If Σ is concave, then Σ has non-negative mean curvature.*

Proof. Let $\widetilde{\Sigma}$ be a lift of Σ in $\widetilde{M}$, and let $S = \mathcal{D}(\widetilde{M})$. Assume that Σ is convex. Then S is convex. Hence, for every $q \in S$, the surface S admits a spacelike support plane P_q at q, and is included in the future of P_q. By lemma 1.2, the mean curvature of the surface S at q is smaller or equal than the mean curvature of the support plane P_q. But, since P_q is totally geodesic, it has zero mean curvature. Hence, the surface S has non-positive mean curvature. Hence, the surface Σ also has non-positive mean curvature (since the developping map $\mathcal{D}$ is locally isometric). $\square$

The notions of convexity and concavity defined above can only help us in finding spacelike surfaces with non-positive (resp. non-positive) mean curvature. Yet, to apply Gerhardt's theorem 3.3, we need to find spacelike surfaces with positive (resp. negative) mean curvature. This is the reason why we will define below a notion of *uniformly curved surface* in M.

Let S be a topological surface in $\mathbb{R}^3$, and q be a point on S. We fix an euclidian metric on $\mathbb{R}^3$. We say that the surface S is *more curved than a sphere of radius R at q*, if there exists a closed euclidian ball B of radius R, such that q is on the boundary of B, and such that B contains a neighbourhood of q in S.

Remark 3.44. *Assume that the surface S is C^2. Then, S is more curved than a sphere of radius R at q if and only if the osculating quadric of S at q is an ellipsoid of diameter less than $2R$.*

Now, let Σ be a topological surface in M, let $\widetilde{M}$ be a lift of Σ, and let $S = \mathcal{D}(\widetilde{\Sigma})$. We see S as a surface in $\mathbb{R}^3$ and we fix a euclidian metric on $\mathbb{R}^3$. Let $\Delta \subset \widetilde{\Sigma}$ be a fundamental domain of the covering $\widetilde{\Sigma} \to \Sigma$, and let $D = \mathcal{D}(\Delta)$. We say that the surface Σ is *uniformly curved*, if there exists $R \in (0, +\infty)$ such that the surface S is more curved than a sphere of radius R at each point of D. It is easy to verify that this definition depends neither on the choice of the fundamental domain Δ, nor on the choice the euclidian metric on $\mathbb{R}^3$ (although one has to change the constant R, when changing the fundamental domain Δ or the euclidian metric on $\mathbb{R}^3$).

Proposition 3.45. *Let Σ be a C^2 spacelike surface in M. If Σ is convex and uniformly curved, then Σ has negative mean curvature. If Σ is concave and uniformly curved, then Σ has positive mean curvature.*

Proof. Let $\widetilde{\Sigma}$ be a lift of Σ in $\widetilde{M}$, and let $S := \mathcal{D}(\widetilde{M})$. Assume that Σ is convex and uniformly curved. Then, S is convex. So, for every $q \in S$, the surface S admits a support plane P_q at q, and is included in the future of P_q. Moreover, since Σ is uniformly curved, the surface S and the plane P_q do not have the same osculating quadric (see remark 3.44). By lemma 1.2, this implies that the mean curvature of S at q is strictly smaller than the mean curvature of the plane P_q. Since P_q is totally geodesic, P_q has zero mean curvature. Hence, S has negative mean curvature. Hence, Σ also has negative mean curvature. $\square$

3.5.2 Boundary of Γ-invariant convex sets included in $D(S_0)$

Proposition 3.46. *Let S be a Γ-invariant topological surface in $\mathcal{A}_{p_0}$. Assume that S is included in $D(S_0)$, and that the boundary of S in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$ is equal to the curve ∂S_0. Then every support plane of S is spacelike*[10].

Proof. Using the diffeomorphism Φ_{p_0}, we identify $\mathcal{A}_{p_0}$ with the region of $\mathbb{R}^3$ defined by the inequation $(x^2 + y^2 - z^2 < 1)$, and $\partial\mathcal{A}_{p_0}$ with the one-sheeted hyperboloid $(x^2 + y^2 - z^2 = 1)$. Let q be a point on the surface S and P be a support plane of S at q. The totally geodesic subspace P is the intersection of $\mathcal{A}_{p_0}$ with an affine plane $\widehat{P}$ of $\mathbb{R}^3$.

On the one hand, since P is a support plane of S, the closure of S must be included in the closure of one of the two connected components of $\mathbb{R}^3 \setminus \widehat{P}$. In particular, the curve ∂S_0 must be included in the

[9] By causal past, we mean causal past in $\mathcal{A}_{p_0}$

[10] Note that, for in general, the surface S does not admit any support plane.

closure of one of the two connected components of $\mathbb{R}^3 \setminus \widehat{P}$. On the other hand, ∂S_0 is a simple closed curve drawn on the hyperboloid $\partial\mathcal{A}_{p_0}$, which is not null-homotopic in $\partial\mathcal{A}_{p_0}$ (see remark 3.9). Consequently:

Fact 1. The support plane $P = \widehat{P} \cap \mathcal{A}_{p_0}$ does not contain any affine line of $\mathbb{R}^3$. Indeed, if $\widehat{P} \cap \mathcal{A}_{p_0}$ contains an affine line of $\mathbb{R}^3$, then $\widehat{P} \cap \partial\mathcal{A}_{p_0}$ is a hyperbola, and the two connected components of $\partial\mathcal{A}_{p_0} \setminus \widehat{P}$ are contractible in $\partial\mathcal{A}_{p_0}$ (we recall that $\mathcal{A}_{p_0}$ is the region $(x^2 + y^2 - z^2 < 1)$ in $\mathbb{R}^3$). Hence, every curve included in the closure of a connected component of $\partial\mathcal{A}_{p_0} \setminus \widehat{P}$ is null-homotopic in $\partial\mathcal{A}_{p_0}$.

Fact 2. If the plane $\widehat{P}$ is tangent to the hyperboloid $\partial\mathcal{A}_p$ at some point r, then r is on the curve ∂S_0. Indeed, if $\widehat{P}$ is tangent to the hyperboloid $\partial\mathcal{A}_{p_0}$ at some point r, then every curve included in the closure of one of the two connected components of $\partial\mathcal{A}_{p_0} \setminus \widehat{P}$ which is not null-homotopic in $\partial\mathcal{A}_{p_0}$ pass through r.

Now, we argue by contradiction: we assume that the totally geodesic plane P is not spacelike. Then, P is either hyperbolic (*i.e.* the lorentzian metric restricted to P has signature $(+,-)$), or degenerated (*i.e.* the lorentzian metric restricted to P is degenerated). We will show that the two possibilities lead to a contradiction.

– If P is hyperbolic, then P contains timelike geodesics. By remark 2.14, a timelike geodesic of $\mathcal{A}_{p_0}$ is an affine line of $\mathbb{R}^3$ which is included in $\mathcal{A}_{p_0}$. Hence, $P = \widehat{P} \cap \mathcal{A}_{p_0}$ contains an affine line of $\mathbb{R}^3$. This is absurd according to fact 1 above.

– If P is degenerated then P contains lightlike and spacelike geodesic, but does not contain any timelike geodesic. By remark 2.14, this implies that $\widehat{P}$ is tangent to the hyperboloid $\partial\mathcal{A}_{p_0}$ at some point r. According to fact 2 above, the point r must be on the curve ∂S_0. But then, according to remark 3.28 item (iv), P is disjoined from $E(\partial S_0)$. In particular, the point q is not in $E(\partial S_0)$. This is absurd since, by hypothesis, the surface S is included in $E(\partial S_0) = D(S_0)$. □

Proposition 3.47. *Let C be a non-empty Γ-invariant closed*[11] *convex subset of $\mathbb{A}\mathrm{d}\mathbb{S}_3$, included in $D(S_0)$. Then:*

(i) *The boundary of C in $\mathbb{A}\mathrm{d}\mathbb{S}_3$ is the union of two disjoint Γ-invariant topological surfaces S^- and S^+, such that S^- is convex, S^+ is concave, C is in the future of S^-, and C is in the past of S^+.*

(ii) *$\Sigma^- := \Gamma\backslash S^-$ and $\Sigma^+ := \Gamma\backslash S^+$ are two disjoint Cauchy surfaces in $\Gamma\backslash D(S_0) \simeq M$. Moreover, Σ^- is convex, Σ^+ is concave, and Σ^+ is in the future of Σ^-. Of course, the boundary of the set $\Gamma\backslash C$ in M is the union of the surfaces Σ^- and Σ^+.*

Proof. Since C is included in $D(S_0)$, it is also included in the affine domain $\mathcal{A}_{p_0}$. We denote by ∂C the boundary of C in $\mathcal{A}_{p_0}$, we denote by $\overline{C}$ the closure of C in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$, and we denote by $\overline{\partial C}$ the boundary of $\overline{C}$ in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$. Of course, we have $\partial C = \overline{\partial C} \cap \mathcal{A}_{p_0} = \overline{\partial C} \setminus \partial\mathcal{A}_{p_0}$.

The set $\overline{C}$ is a compact convex subset of $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$. So, the diffeomorphism Φ_{p_0} maps $\overline{C}$ to a compact convex subset of $\mathbb{R}^3$. The boundary of a compact convex subset of $\mathbb{R}^3$ is a topological sphere. Hence, $\overline{\partial C}$ is a Γ-invariant topological sphere. We have to understand the intersection of $\overline{\partial C}$ with $\partial\mathcal{A}_{p_0}$. On the one hand, by hypothesis, C is included in $D(S_0)$; hence, $\overline{C}$ is included in $\overline{D(S_0)}$. The intersection of $\overline{D(S_0)}$ with $\partial\mathcal{A}_{p_0}$ is equal to the curve ∂S_0 (see propositions 3.34 and 3.37). Hence, the intersection of $\overline{\partial C}$ with $\partial\mathcal{A}_{p_0}$ is included in the curve ∂S_0. On the other hand, a non-empty Γ-invariant subset of $D(S_0)$. Hence, the closure of C must contain the curve ∂S_0 (since this curve is the limit set of the action of Γ on $D(S_0)$). As a consequence, we have $\overline{\partial C} \cap \partial\mathbb{A}\mathrm{d}\mathbb{S}_3 = \partial S_0$.

So, we have proved that $\partial C = \overline{\partial C} \setminus \partial\mathcal{A}_{p_0}$ is a Γ-invariant topological sphere minus the Γ-invariant Jordan curve ∂S_0. Hence, ∂C is the union of two disjoint Γ-invariant topological discs S^- and S^+, such that $\partial S^- = \partial S^+ = \partial S_0$. Since the surfaces S^- and S^+ are included in the boundary of a convex set, they admit a support plane at each of their points. Hence, by proposition 3.46 and remark 3.41, the surfaces S^- and S^+ are spacelike. Since S^- is a spacelike disc with $\partial S^- = \partial S_0$, it separates $\mathcal{A}_{p_0}$ into two connected components: the past and the future of S^-. The set C must be included in one of these two connected components, so C is included either in the past or in the future of S^-. Similarly, for S^+. Moreover, C cannot be in the future (resp. the past) of both S^- and S^+. So, up to exchanging $^-$ and S^+, the set C is in the future of S^- and in the past of S^+. In particular, S^+ is in the future of S^-. Since C is in the future of S^-, the surface S^- must be in the future of each of its support planes. Hence, the surface S^- is convex. Similar arguments show that S^+ is concave. This completes the proof of (i).

Now, since S^- and S^+ are Γ-invariant spacelike surfaces in $D(S_0)$, their projections $\Sigma^- := \Gamma\backslash S^-$ and $\Sigma^+ := \Gamma\backslash S^+$ are Cauchy surfaces in $\Gamma\backslash D(S_0) \simeq M$ (we recall that every compact spacelike surface in M is a Cauchy surface). Of course, Σ^+ is in the future of Σ^-, since S^+ is in the future of S^-. Finally,

[11] By such, we mean that C is closed in $\mathbb{A}\mathrm{d}\mathbb{S}_3$, but not necessarily in $\mathbb{A}\mathrm{d}\mathbb{S}_3 \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}_3$. Actually, a non-empty Γ-invariant subset of AdS_3 cannot be closed in $AdS_3 \cup \partial AdS_3$

the convexity of Σ^- and the concavity of Σ^+ follow, by definition, from the convexity of S^- and the concavity of S^-. □

3.5.3 Definition of the topological Cauchy surfaces Σ_0^- and Σ_0^+

The set $C_0 = \mathrm{Conv}(\partial S_0) \setminus \partial S_0$ satisfies the hypothesis of proposition 3.47. Hence, the boundary in AdS_3 of C_0 is made of two disjoint Γ-invariant spacelike topological surfaces S_0^- and S_0^+, such that S_0^- is convex, S_0^+ is concave, and S_0^+ is in the future of S_0^-. Moreover, the surfaces $\Sigma_0^- := \Gamma\backslash S_0^-$ and $\Sigma_0^+ := \Gamma\backslash S_0^+$ are two disjoint topological Cauchy surfaces in $\Gamma\backslash D(S_0) \simeq M$, such that Σ_0^- is convex, Σ_0^+ is concave, and Σ_0^+ is in the future of Σ_0^-.

3.6 A pair of uniformly curved convex and concave topological Cauchy surfaces

Recall that our goal is to find a pair of barriers in M. By proposition 3.45, this goal will be achieved if we find two disjoint smooth Cauchy surfaces Σ^- and Σ^+ in M, such that Σ^- is convex and uniformly curved, such that Σ^+ is concave and uniformly curved, and such that Σ^+ is in the future of Σ^-. The Cauchy surfaces Σ_0^- and Σ_0^+ constructed in the previous subsection satisfy these properties, except for two points: they are not smooth and they are not uniformly curved. The purpose of the present subsection is to prove the following proposition:

Proposition 3.48. *Arbitrarily close to Σ_0^- (resp. Σ_0^+), there exists a topological surface Σ_1^- (resp. Σ_1^+) which is convex (resp. concave) and uniformly curved.*

The idea of the proof of proposition 3.48 is to replace the convex set $C_0 = \mathrm{Conv}(\partial S_0) \setminus \partial S_0$ by its "lorentzian ε-neighbourhood". This idea comes from riemannian geometry. Indeed, it is well-known that the ε-neighbourhood of a convex subset of the hyperbolic space $\mathbb{H}^n$ is uniformly convex. We will prove that a similar phenomenon occurs in $\mathbb{A}d\mathbb{S}_3$ (although many technical problems appear).

We recall that the *length* of a C^1 causal curve $\gamma : [0,1] \to \mathbb{A}d\mathbb{S}_3$ is $l(\gamma) = \int_0^1 (-g(\dot\gamma(t), \dot\gamma(t)))^{1/2} dt$, where g is the lorentzian metric of $\mathbb{A}d\mathbb{S}_3$. Given an achronal subset E of $\mathcal{A}_{p_0}$ and a point p in $\mathcal{A}_{p_0}$, the *distance from p to E* is the supremum of the lengths of all the C^1 causal curves joining p to E in $\mathcal{A}_{p_0}$ (if there is no such curve, then the distance from p to E is not defined)[12] The distance from p to E, where finite, is lower semi-continuous in p. Moreover, the distance from p to E is continuous in p, when p is in the Cauchy development of E (see, e.g., [6, page 215]).

Given an achronal subset E of $\mathcal{A}_{p_0}$ and $\varepsilon > 0$, the *ε-future* of E is the set made of the points $p \in \mathcal{A}_{p_0}$, such that p is in the future of E, and such that the distance from p to E is at most ε. We define similarly the *ε-past* of E. We denote by $I_\varepsilon^-(E)$ and $I_\varepsilon^+(E)$ the ε-past and the ε-future of the set E.

Lemma 3.49. *For ε small enough, the ε-past and the ε-future of the surface S_0^+ is included in $D(S_0)$.*

Proof. Since the set $D(S_0)$ is a neighbourhood of the surface S_0^+, and since the surface $\Sigma_0^+ = \Gamma\backslash S_0^+$ is compact, one can find a Γ-invariant neighbourhood U_0^- of the surface S_0^+, such that U_0^+ is included in $D(S_0)$, and such that $\Gamma\backslash U_0^+$ is compact.

Claim. There exists $\varepsilon > 0$ such that the distance from any point $p \notin U_0^+$ to the surface S_0^+ is at least ε.

By contradiction, suppose that, for every $n \in \mathbb{N}$, there exists a point $x_n \in \mathcal{A}_{p_0} \setminus U_0^+$ such that the distance from x_n to the surface S_0^+ is less than $1/n$. Then, for each n, we consider a causal curve γ_n joining the point x_n to the surface S_0^+. This curve γ_n must intersect the boundary of U_0^+; let z_n be a point in $\gamma_n \cap \partial U_0^+$. Since z_n is on a causal curve joining x_n to the surface S_0^+, the distance from z_n to S_0^+ must be smaller than $1/n$. Now, recall that $\Gamma\backslash U_0^+$ is compact. Hence, up to replacing each z_n by its image under some element of Γ (this operation does not change the distance from z_n to S_0^+, since Γ acts by isometries), we may assume that all the z_n's are in a compact subset of the boundary of U_0^+. Then, we consider a limit point z of the sequence $(z_n)_{n\in\mathbb{N}}$. By lower semi-continuity of the distance, the distance from z to the surface S_0^+ is equal to zero (note that the distance from z to the surface S_0^+ is well-defined, since every point of $\mathcal{A}_{p_0}$ can be joined from the surface S_0^+ by a timelike curve, see remarks 3.18 and 3.19). Hence, the point z is on the surface S_0^+. This is absurd, since z must be on the boundary of U_0^+, and since U_0^+ is a neighbourhood of S_0^+. This completes the proof of the claim. The lemma follows immediately. □

[12] The same definition work in the case where the set E is not achronal. But then, the distance from p to E might be positive even if $p \in E$!

Definition of the set C_1. From now on, we fix a number $\varepsilon > 0$ such that the ε-pasts and ε-futures of the surfaces S_0^- and S_0^+ are included in $D(S_0)$. We consider the set

$$C_1 := C_0 \cup I_\varepsilon^-(S_0^-) \cup I_\varepsilon^+(S_0^+)$$

By construction, C_1 is included in $D(S_0)$. Moreover, it is easy to show that C_1 is a Γ-neighbourhood of C_0. Actually, the set C_1 should be thought as a "lorentzian ε-neighbourhood" of C_0.

Our aim is to prove that the boundary of the set $\Gamma\backslash C_1$ is made of two topological Cauchy surfaces which are convex/concave and uniformly curved. For that purpose, we first need to prove that C_1 is convex. Let us introduce some notations. We denote by $\mathcal{P}(S_0^-)$ (resp. by $\mathcal{P}(S_0^+)$) the set of the support planes of the surface S_0^- (resp. the surface S_0^+).

Lemma 3.50. *The set C_1 is made of the points $p \in \mathcal{A}_{p_0}$ such that:*
- for every plane P in $\mathcal{P}(S_0^+)$, the point p is in the past or in the ε-future of P,
- for every plane P in $\mathcal{P}(S_0^-)$, the point p is in the future or in the ε-past of P.
In other words:

$$C_1 = \left(\bigcap_{P \in \mathcal{P}(S_0^-)} I_\varepsilon^-(P) \cup I^+(P) \right) \cap \left(\bigcap_{P \in \mathcal{P}(S_0^+)} I^-(P) \cup I_\varepsilon^+(P) \right) \tag{1}$$

Remark 3.51. *Lemma 3.50 is a "lorentzian analog" of the following elementary fact of euclidian geometry: the (euclidian) ε-neighbourhood of a convex set $C \subset \mathbb{R}^n$ is the intersection of the ε-neighbourhoods of all the affine half spaces containing C.*

Proof of lemma 3.50. We denote by C_1' the right-hand term of equality (1). Let p be a point of $\mathcal{A}_{p_0}$ which is not in C_1'. Assume for instance that there exists a plane $P \in \mathcal{P}(S_0^+)$, such that p is in the future of P, and the distance from p to P is bigger than ε. Since the surface S_0^+ is in the past of P, this implies that p is in the future of S_0^+ and that the distance from p to Σ_0^+ is bigger than ε. Hence, p is not in C_1.

Conversely, let p be a point of $\mathcal{A}_{p_0}$ which is not in C_1. Assume for instance that p is in the future of the surface S_0^+ and the distance from p to S_0^+ is bigger than ε. Then there exists a timelike curve γ joining p to a point $q \in \Sigma_0^+$, such that the length of γ is bigger than ε. Let P be a support of C_0 such that $q \in P \cap C_0$. By definition, P is an element of $\mathcal{P}(S_0^+)$, the point p is in the future of P, and the distance from p to P is bigger than the length of γ. Hence, p is not in C_1'. $\square$

Using the diffeomorphism Φ_{p_0} (see subsection 2.3), we identify the domain $\mathcal{A}_{p_0}$ with the region of $\mathbb{R}^3$ defined by the inequation $(x^2 + y^2 - z^2 < 1)$. Let P_0 be the totally geodesic subspace of $\mathcal{A}_{p_0}$ defined as the intersection of $\mathcal{A}_{p_0}$ with the affine plane $(z = 0)$ in $\mathbb{R}^3$. It is easy to check that P_0 is spacelike.

Lemma 3.52. *The set $I^-(P_0) \cup I_\varepsilon^+(P_0)$ is the region of $\mathcal{A}_{p_0}$ defined by the inequation*

$$z \leq \tan\varepsilon.\sqrt{1 - x^2 - y^2}$$

Sketch of the proof. All the calculations have to be made in the linear model of the anti-de Sitter space, using the coordinates x_1, x_2, x_3, x_4 (because this is the model where we know the exact form of the lorentzian metric). The equation of P_0 in this system of coordinates is $(x_1 = 0)$. The equation $(z = \tan\varepsilon.\sqrt{1 - x^2 - y^2})$ corresponds to the equation $(x_1 = \sin\varepsilon)$. On the one hand, since P_0 is a smooth spacelike surface, the distance from a $q \in D(P_0)$ to the plane P_0 is realized as the length of a geodesic segment joining q to P_0 and orthogonal to P_0 (see, for instance, [6,]). On the other hand, proposition 2.2 implies every point q on the surface $(x_1 = \sin\varepsilon)$ belongs to a unique geodesic which is orthogonal to P_0. So, we are left to prove that, for every point p on P_0, the length of the unique segment of geodesic orthogonal to P_0 and joining p to the surface $(x_1 = \sin\varepsilon)$ is equal to ε. This follows from proposition 2.2 and from an elementary calculation. $\square$

Remark 3.53. *Lemma 3.52 shows that $I^-(P_0) \cup I_\varepsilon^+(P_0)$ is a relatively convex subset of AdS_3. Moreover, it shows that there exists R such that the boundary of the set $I^-(P_0) \cup I_\varepsilon^+(P_0)$ is more curved than a sphere of radius R at every point: if we consider the euclidian metric on $\mathbb{R}^3$ for which (x, y, z) is an orthonormal system of coordinates, then we can take $R = (\tan\varepsilon)^{-1}$. Although this does not clearly appear in the proof of lemma 3.52, this phenomenon is related with the fact that the curvature of $\mathbb{A}\mathrm{d}\mathbb{S}_3$ is curvature.*

Corollary 3.54. *The set C_1 is convex.*

Proof. Consider a totally geodesic subspace $P \in \mathcal{P}^+(S_0^+)$. There exists $\sigma_P \in O_0(2,2)$, such that $\gamma_P(P_0) = P$. Of course, σ_P maps the set $I^-(P_0) \cup I^+_\varepsilon(P_0)$ to the set $I^-(P) \cup I^+_\varepsilon(P)$. By remark 3.53, the set $I^-(P_0) \cup I^+_\varepsilon(P_0)$ is relatively convex. Hence, the set $I^-(P) \cup I^+_\varepsilon(P)$ is also relatively convex. The same arguments show that, for every $P \in \mathcal{P}^-(C_0)$, the set $I^-_\varepsilon(P) \cup I^+(P)$ is relatively convex. Together with lemma 3.50, this shows that the set C_1 is a relatively convex subset of $\mathbb{A}d\mathbb{S}_3$. Moreover, C_1 is included in $D(S_0)$, which is a convex subset of $\mathbb{A}d\mathbb{S}_3$ (see item (iii) of remark 3.28 and proposition 3.37). As a consequence, C_1 is a convex subset of $\mathbb{A}d\mathbb{S}_3$. □

Definition of the surfaces S_1^-, S_1^+, Σ_1^- and Σ_1^+. The set C_1 is a Γ-invariant closed convex subset of $\mathbb{A}d\mathbb{S}_3$, containing C_0, and included in $D(S_0)$. By proposition 3.47, the boundary of C_1 in $\mathbb{A}d\mathbb{S}_3$ is the union of two Γ-invariant spacelike topological surfaces S_1^- and S_1^+, such that S_1^- is convex, such that S_1^+ is concave, and such that S_1^+ is in the future of S_1^-. Alo by proposition 3.47, $\Sigma_1^- := \Gamma\backslash S_1^-$ and $\Sigma_1^+ := \Gamma\backslash S_1^+$ are two disjoint topological Cauchy surfaces in $M = \Gamma\backslash D(S_0)$, respectively convex and concave, and such that Σ_1^+ is in the future of Σ_1^-.

Remark 3.55. *The surface S_1^- (resp. S_1^+) is the set made of all the points of $\mathcal{A}_{p_0}$ which are in the past of the surface S_0^- (resp. S_0^+), at distance exactly ε of S_0^- (resp. S_0^+): this follows from the definition of the set C_1, and from the continuity of the distance from a point p to the surface S_0^- (resp. S_0^+) when p ranges in $D(S_0) = D(S_0^-) = D(S_0^+)$. Thus, the surface Σ_1^- (resp. Σ_1^+) is the set made of all the points of M which are in the past of the surface Σ_0^- (resp. Σ_0^+), at distance exactly ε of Σ_0^- (resp. Σ_0^+).*

Proposition 3.56. *The surfaces Σ_1^- and Σ_1^+ are uniformly curved.*

Proof. Fix a euclidian metric on $\mathbb{R}^3$, and let $\Delta_1^+ \subset S_1^+$ be a compact fundamental neighbourhood of the action of Γ on S_1^+. Let Δ_0^+ be the intersection of the past of Δ_1^+ with the surface S_0^+. Note that Δ_0^+ is compact (since Δ_1^+ is compact, and since Δ_1^+ and S_0^+ are included in a globally hyperbolic subset of $\mathbb{A}d\mathbb{S}_3$). Let $\mathcal{P}(\Delta_0^+)$ be the set of all the support planes of S_0^+ that meet S_0^+ at some point of Δ_0^+.

Claim 1. There exists R such that, for every $P \in \mathcal{P}(\Delta_0^+)$, the boundary of the set $I^-(P) \cup I^+_\varepsilon(P)$ is more curved than a sphere of radius R.

On the one hand, $\mathcal{P}(\Delta_0^+)$ is a compact subset of the set of all spacelike totally geodesic subspaces of $\mathbb{A}d\mathbb{S}_3$. As a consequence, there exists a compact subset $\mathcal{K}$ of $O_0(2,2)$ such that $\mathcal{P}(\Delta_0^+) \subset \mathcal{K}.P_0$. On the other hand, there exists R_0 such that the boundary of the set $I^-(P_0) \cup I^+_\varepsilon(P_0)$ is more curved than a sphere of radius R_0 (see remark 3.53). The claim follows.

Claim 2. Every $q \in \Delta_1^+$ is on the boundary of the set $I^-(P) \cup I^+_\varepsilon(P)$ for some P in $\mathcal{P}(\Delta_0^+)$.

Let $q \in \Delta_1^+ \subset S_1^+$. By definition of S_1^+, the point q is in the future of the surface S_0^+ and the distance from q to S_0^+ is equal to ε. Moreover, since q and S_0^+ are included in a globally hyperbolic region of $\mathbb{A}d\mathbb{S}_3$, the distance between q and S_0^+ is realized: there exists a causal curve γ of length ε joining q to a point $p \in S_0^+$. By construction, the point p is in Δ_0^+. Let P be any support plane of S_0^+ at p. Of course, P is in $\mathcal{P}(\Delta_0^+)$. On the one hand, since γ is a causal arc of length ε joining q to a point of P, the distance from p to P is at least ε. On the other hand, lemma 3.50 implies that the distance from p to P must be at most ε. The claim follows.

Let q be a point of Δ_1^+. By claim 2, there exists $P \in \mathcal{P}(\Delta_0^+)$ such that q is on the boundary of the set $I^-(P) \cup I^+_\varepsilon(P)$. By lemma 3.50, the surface S_1^+ is included in $I^-(P) \cup I^+_\varepsilon(P)$. Putting these together with claim 1, we obtain that the surface S_1^+ is more curved than a sphere of radius R at q. Hence, the surface Σ_1^+ is uniformly curved. □

This completes the proof of proposition 3.48.

Remark 3.57. *All the results of this subsection are still valid if one replaces Σ_0^- and Σ_0^+ by any two Cauchy surfaces Σ^- and Σ^+ in M, such that Σ^- is convex, Σ^+ is concave, and Σ^+ is in the future of Σ^- (in the proofs, the set C_0 has to be replaced by the image under $\mathcal{D}$ of the lift of the region of M situated between the Cauchy surfaces Σ^- and Σ^+).*

Remark 3.58. *It is well-known that the boundary of the ε-neighbourhood of any geodesically convex subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ is a C^1 hypersurface. Unfortunately, this phenomenon does not generalise to lorentzian geometry. In particular, the surfaces S_1^-, S_1^+, Σ_1^- and Σ_1^+ are not C^1 in general.*

3.7 Smoothing the Cauchy surfaces Σ_1^- and Σ_1^+

Recall that, to apply Gerhard's theorem, we need two disjoint C^2 Cauchy surfaces Σ^- and Σ^+, such that Σ^- is convex and uniformly curved, Σ^+ is concave and uniformly curved, and Σ^+ is in the future of Σ^-. The topological surfaces Σ_1^- and Σ_1^+ constructed in the previous section satisfy these properties, except that they are not smooth. The purpose of this subsection is to prove the following proposition:

Proposition 3.59. *Arbitriraly close to Σ_1^- and Σ_1^+, there exist some C^∞ Cauchy surfaces Σ^- and Σ^+, such that Σ^- is convex and uniformly curved and Σ^+ is concave and uniformly curved.*

Unfortunatly, we could not find any simple proof of proposition 3.59 (see remark 3.60). Our proof is divided in three steps. In 3.7.1, we approximate the surfaces Σ_1^- and Σ_1^+ by some *polyhedral* Cauchy surfaces Σ_2^- and Σ_2^+ (respectively convex and concave). Then, in 3.7.2, we describe a method for smoothing convex and concave polyhedral Cauchy surfaces. Using this method, we obtain two disjoint C^∞ Cauchy surfaces Σ_3^- and Σ_3^+, respectively convex and concave. Finally, in 3.7.3, using the same trick as in subsection 3.6, we get a pair of smooth Cauchy surfaces Σ_4^- and Σ_4^+, such that Σ_4^- is convex and uniformly curved, and Σ_4^+ is concave and uniformly curved.

Remark 3.60. *The first idea which comes to mind for smoothing a convex surface is to use some convolution process. Unfortunately, to make this idea work, one needs a locally euclidian structure*[13]*. This is the reason why this idea does not fit our situation (there is no locally euclidian structure on the manifold M). We think that our proof of proposition 3.59 can be generalized in order to get a method for smoothing boundaries of convex sets in locally affine manifolds (that are not locally euclidian).*

3.7.1 Polyhedral convex and concave Cauchy surfaces

In this subsubsection, we will define a notion of *polyhedral surface* in M. Then, we will two polyhedral Cauchy surfaces Σ_2^- and Σ_2^+ in M, such that Σ_2^- is convex, Σ_2^+ is concave, and Σ_2^+ is in the future of Σ_2^-.

A subset Δ of M is a *2-simplex*, if there exists an affine chart $\Phi : U \subset M \to \mathbb{R}^3$, such that $\Delta \subset U$ and such that $\Phi(\Delta)$ is a 2-simplex in $\mathbb{R}^3$. A compact surface Σ in M is called *polyhedral* if it can be decomposed as a finite union of 2-simplices.

Remark 3.61. *Let Σ be a compact spacelike surface in M, let $\widetilde{\Sigma}$ be a lift of Σ in $\widetilde{M}$, and let $S := \mathcal{D}(\widetilde{\Sigma})$. Using the embedding $\Phi_{p_0} : \mathcal{A}_{p_0} \to \mathbb{R}^3$, we can see S as a surface in $\mathbb{R}^3$. Then, Σ is a polyhedral surface if and only if S can be decomposed as a finite union of orbits (for Γ) of 2-simplices of $\mathbb{R}^3$.*

Remark 3.62. *Let Σ be a compact convex spacelike polyhedral surface in M. Then, one can decomposed Σ as a finite union of subsets $\Sigma := \Delta_1 \cup \cdots \cup \Delta_n$, where each Δ_i is the intersection of Σ with one of its support planes, and each Δ_i has non-empty interior (as a subset of Σ). The decomposition is unique (provided that the Δ_i's are pariwise distinct). The Δ_i's called the* sides *of Σ. Each side of Σ is a finite union of 2-simplices, but is not necessarily a topological disc (e.g. in the case where Σ is totally geodesic).*

Definition of the set C_2, of the surfaces S_2^-, S_2^+, Σ_2^- and Σ_2^+ We consider a Γ-invariant set E of points of $\partial C_1 = S_1^- \cup S_1^+$, such that $\Gamma \backslash E$ is finite (in particular, E is discrete). We denote by C_2 the convex hull of E. By construction, C_2 is a Γ-invariant convex subset of C_1. In particular, C_2 is a Γ-invariant convex subset of $D(S_0)$. So, by proposition 3.47, the boundary of C_2 in AdS_3 is made of two disjoint Γ-invariant spacelike achronal surfaces S_2^- and S_2^+, such that S_2^- is convex, S_2^+ is concave, and S_2^+ is in the future of S_2^+. Also by proposition 3.47, $\Sigma_2^- := \Gamma\backslash S_2^-$ and $\Sigma_2^+ := \Gamma\backslash S_2^+$ are two disjoint Cauchy surfaces in M, respectively convex and concave, and Σ_2^+ is in the future of Σ_2^-.

Given $\delta > 0$, we say that the set E is *δ-dense in the surfaces S_1^- and S_1^+*, if every euclidian ball of radius δ centered at some point of S_1^- (resp. S_1^+) contains some points of E. The remainder of the subsection is devoted to the proof of the following proposition:

Proposition 3.63. *There exists $\delta > 0$ such that, if the set E is δ-dense in the surfaces S_1^- and S_1^+, then the surfaces Σ_2^- and Σ_2^+ are polyhedral.*

Remark 3.64. *The proof of proposition 3.63 is quite technical. The reader who is not interested in technical details can skip the proof of proposition 3.63. Nevertheless, it should be noticed that the boundary of the convex hull of a discrete set of points is* not *a polyhedral surface in general. In particular, proposition 3.63 would be false if the surfaces Σ_1^- and Σ_1^+ were not uniformly curved.*

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[13] For example, any convex function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated by a smooth convex function $\widehat{f}$, obtained as the convolution of f with an approximation of the unity. Yet, the proof of the convexity of $\widehat{f}$ uses the euclidian structure of $\mathbb{R}^{n+1}$.

Given a set $F \subset \mathbb{R}^3$, we say that an affine plane P of $\mathbb{R}^3$ *splits* the set F, if F intersects the two connected components of $\mathbb{R}^3 \setminus P$. The starting point of the proof of proposition 3.63 is the following well-known fact (which follows from very basic arguments of affine geometry):

Fact 3.65. *If F is a finite set of points in $\mathbb{R}^3$, then the boundary of Conv(F) is a compact polyhedral surface; more precisely, the boundary of Conv(F) is the union of all the 2-simplices Conv(p, q, r), such that the points p, q, r are in F, and such that the plane (p, q, r) does not split F.*

Let γ be a continuous curve in a euclidian plane, and p be a point on γ. We say that the curve γ is *more curved than a circle of radius R at p* if there exists a euclidian disc Δ of radius R, such that p is on the boundary of Δ, and such that Δ contains a neighbourhood of p in γ. The proof of the following lemma uses only elementary planar geometry; we leave it to the reader:

Lemma 3.66. *Given two positive numbers ρ and R, there exists a positive number $\delta = \delta(\rho, R)$ such that: for every convex set D in an euclidian plane, if there exists a subarc α of the boundary of D, such that the boundary of D is more curved than a circle of radius R at each point of α, and such that the diameter of α is bigger than ρ, then D contains a euclidian ball of radius δ.*

Proof of proposition 3.63. Consider a compact fundamental domain U for the action of Γ on C_1. Then, consider a compact neighbourhood V of U in C_1, and a compact neighbourhood W of V in C_1. One can find a positive number ρ such every euclidian ball of radius ρ centered in U (resp. V) is included in V (resp. W). Moreover, since V is compact, one can find a positive number R, such that the surface S_1^- (resp. S_1^+) is more curved than a sphere of radius R at every point of $S_1^- \cap V$ (resp. $S_1^- \cap V$).

From now on, we assume that the set E is δ-dense in the surfaces S_1^- and S_1^+, where $\delta = \delta(\rho, R)$ is the positive number given by lemma 3.66. Up to replacing δ by $\min(\delta, \rho)$, we can assume that δ is smaller than ρ. Under these assumptions, we shall prove that the surfaces S_2^- and S_2^+ are polyhedral.

Claim 1. If p, q, r are three points of E, such that the 2-simplex Conv(p, q, r) intersects U, and such that the affine plane $P := (p, q, r)$ does not split the set E, then the three points p, q, r are in W.

To prove this claim, we argue by contradiction: we suppose that there exists three points p, q, r in E, such that the 2-simplex Conv(p, q, r) intersects U, such that the affine plane $P := (p, q, r)$ does not split the set E, and such that one of the three points p, q, r is not in W. We shall show that these suppositions contradict the δ-density of the set E.

Since P does not split the set E, one of the two connected components of $\mathcal{A}_{p_0} \setminus P$ is disjoint from E. We denote by H_P this connected component. First of all, we observe that H_P does not intersect the curve ∂S_0, since H_P does not contain any point of E, since E is a non-empty Γ-invariant subset of $D(S_0)$, and since the curve ∂S_0 is the limit set of the action of Γ on $D(S_0)$. Hence, the intersection of H_P with the boundary of C_1 is included in one of the two connected components S_1^- and S_1^+ of $\partial C_1 \setminus \partial S_0$. Without loss of generality, we assume that $H_P \cap \partial C_1$ is included in S_1^+, and we consider the set $D^+ := H_P \cap S_1^+$ (see figure 2).

We shall prove that there exists an euclidian ball B of radius δ centered at some point of D^+, such that $B \cap S_1^+ \subset D^+$. Since D^+ must be disjoint from E (because $D^+ \subset H_P$), this will contradict the fact that E is δ-dense in S_1^+. For that purpose, we consider the curve $\gamma := P \cap S_1^+$. Observe this curve γ is the boundary of the topological disc D^+. Moreover, the curve γ is also the boundary of the convex subset $D := P \cap C_1$ of the plane P. The curve γ passes through the points p, q and r, and the 2-simplex Conv(p, q, r) is included in the convex set D. We shall distinguish two cases (and get a contradiction in each case):

First case: the curve γ does not intersect the neighbourhood V. We consider a point m in $D \cap U$ (such a point does exist, since Conv$(p, q, r) \cap U \neq \emptyset$, and since Conv$(p, q, r) \subset D$), and we denote by m' the unique point of intersection of D^+ with the line passing through m and orthogonal to the plane P. The point m is in U, and the curve γ does not intersect V; so, by definition of ρ, the euclidian distance between m and γ must be bigger than ρ, and thus, bigger than δ. Moreover, the euclidian distance between the point m' and the curve γ is bigger than the distance between m and γ. So, we have proved that the euclidian ball B of radius δ centered at m' does not intersect the curve γ. Hence, the connected component of $B \cap S_1^+$ containing the point m' is included in D^+. Since D^+ is disjoint from E, this contradicts the δ-density of E in S_1^+.

Second case: the curve γ does intersect the neighbourhood V. Then, by definition of ρ, we can find an subarc α of the curve γ, such that the diameter of α is bigger than ρ, and such that α is included in W. Since S_1^+ is more curved than a sphere of radius R at every point of V, the curve γ is more curved than a circle of radius R at each point of α. So, applying lemma 3.66, we find a point $m \in D$ such that the euclidian distance between the point m and the curve γ is bigger than δ. The same argument as above shows that this contradicts the δ-density of E in the surface S_1^+.

In both case, we have obtained a contradiction. So, we have completed the proof of claim 1.

Claim 2. If W' is a compact subset of $\mathcal{A}_{p_0}$ such that $W \subset W'$, then the sets $Conv(E \cap W') \cap U$ and $Conv(E \cap W) \cap U$ coincide.

This claim is a consequence of Claim 1 and from fact 3.65. Since W' is a compact subset of $\mathbb{A}d\mathbb{S}_3$, the set $E \cap W'$ is finite. Hence, the boundary of the set $\mathrm{Conv}(E \cap W')$ is the union of the 2-simplices $[p, q, r]$, such that the three points p, q, r are in $E \cap W'$, and such that the affine plane (p, q, r) does not split $E \cap W'$. By claim 1, such a 2-simplex can intersect U only if the three points p, q and r are in W. Using once again fact 3.65, this implies that the boundary of $\mathrm{Conv}(E \cap W')$ intersected with U is included in the boundary of $\mathrm{Conv}(E \cap W)$ intersected with U. But, if the boundary of a convex set is included in the boundary of another convex set, then these two convex sets must be equal. The claim follows.

End of the proof. Let us consider a increasing sequence $(W_n)_{n\in\mathbb{N}}$ of compacts subsets of $\mathbb{A}d\mathbb{S}_3$, such that $\bigcup_{n\in\mathbb{N}} W_n = AdS$. On the one hand, we clearly have $\mathrm{Conv}(E) = \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} \mathrm{Conv}(E \cap W_n))$. On the other hand, according to Claim 2, there exists an integer n_0 such that $\mathrm{Conv}(E \cap W_n) \cap U = \mathrm{Conv}(E \cap W) \cap U$ for every $n \geq n_0$. As a consequence, we have $\mathrm{Conv}(E) \cap U = \mathrm{Conv}(E \cap W) \cap U$. Now, since $E \cap W$ is a finite set, the boundary of $\mathrm{Conv}(E \cap W)$ is a compact polyhedral surface. Thus, we have proved that the boundary of the set $C_2 = \mathrm{Conv}(E)$ coincides, in U, with a polyhedral surface. Since U contains a fundamental domain for the action of Γ on C_2, this implies each of the surfaces S_2^- and S_2^+ can be decomposed as a finite union of orbits of 2-simplices. Hence, the surfaces Σ_2^- and Σ_2^+ are polyhedral (see remark 3.61). □

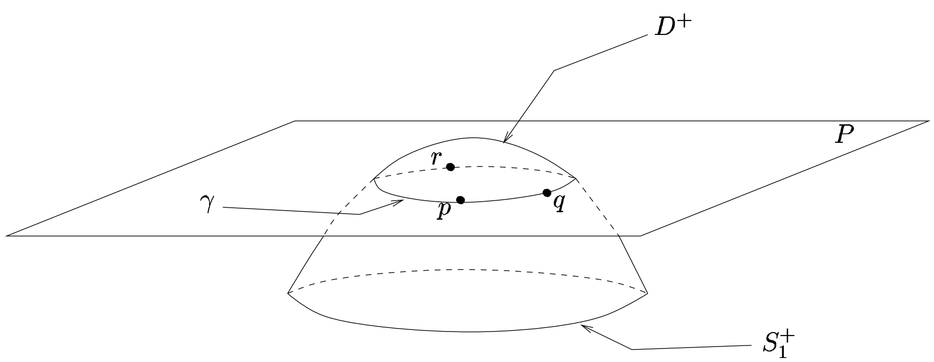

Figure 2: The situation in the proof of proposition 3.63

Addendum. *There exists $\delta > 0$ such that, if the set E is δ-dense in the surfaces S_1^+, S_1^+, then each side of the polyhedral surfaces Σ_2^-, Σ_2^+ is included in the domain of an chart of M.*

Proof. From the proof of proposition 3.63, one can extract the following statement: for every $\rho > 0$, there exists $\delta > 0$ such that, if the set E is δ-dense in the surface S_1^-, then, for every support plane P of the surface S_2^-, the diameter of the set $P \cap S_2^-$ is less than ρ. Of course, there is a imilar statement for the surface S_2^+. The addendum follows immediately. □

3.7.2 Smooth convex and concave Cauchy surfaces

In this subsubsection, we describe a process for smoothing the polyhedral Cauchy surfaces Σ_2^- and Σ_2^+. More precisely, we prove the following:

Proposition 3.67. *Let Σ be a convex polyhedral Cauchy surface in M. Assume that each side of Σ is included an affine domain of M. Then, arbitrarily close to Σ, there exists a C^∞ convex Cauchy surface.*

Of course, the analogous statement dealing with concave Cauchy surfaces is also true. The proof of proposition 3.67 relies on the following technical lemma:

Lemma 3.68. *Let U be some subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a continuous convex function. Then, for every $\eta > 0$, there exists a continuous convex function $\widehat{f} : U \to \mathbb{R}$ satisfying the following properties:*

- *$\widehat{f} \geq f$, the distance between f and $\widehat{f}$ is less than 2η, and $\widehat{f}$ coincides with f on the set $f^{-1}([2\eta, +\infty[)$;*
- *$\widehat{f}$ is constant on the set $f^{-1}([0, \eta])$; in particular, $\widehat{f}$ is C^∞ on the set $f^{-1}([0, \eta[)$;*
- *if f is C^∞ on some subset U of $Dom(f)$, then $\widehat{f}$ is also C^∞ on U.*

Proof. We consider a C^∞ function $\varphi : [0,+\infty[\to [0,+\infty[$ such that: φ is non-decreasing and convex, $\varphi(t) = \frac{3}{2}\eta$ for every $t \in [0,\eta]$, and $\varphi(t) = t$ for every $t \in [2\eta, +\infty[$. Then, we consider the function $\widehat{f} : U \to [0,+\infty[$ defined by $\widehat{f} := \varphi \circ f$. This function satisfies all the desired properties. □

We endow M with a riemannian metric; this allows us to speak of the (riemannian) ε-neighbourhood of any subset of M for any $\varepsilon > 0$. We say that a surface Σ_1 is *ε-close* to another surface Σ_2 if there exists a homeomorphism $\Psi : \Sigma_1 \to \Sigma_2$ which is ε-close to the identity. The following remark will allow us to see a polyhedral surface as a collection of graphs:

Remark 3.69. *Let Σ be a convex compact surface in M, let Π be a support plane of Σ and let $\Delta := \Sigma\cap\Pi$. We assume that Δ is included in an affine domain of M. Then, we can find a neighbourhood V of F in M, and some local affine coordinates (x, y, z) on V, such that:*

– $\Pi \cap V$ is the plane of equation $(z = 0)$, and $\Sigma \cap V$ is the graph $(z = f(x, y))$ of a non-negative convex function $f : U \to [0,+\infty[$ (where U is some convex subset of $\mathbb{R}^2$).

– if Σ' is a convex Cauchy surface close enough to Σ, then $\Sigma' \cap V$ is the graph $z = f'(x, y)$ of a convex function $f' : U \to \mathbb{R}$. The function f' depends continuously of the surface Σ'. Moreover, if Σ' is in the future of Σ, then $f' \geq f$ (and thus, $f' \geq 0$).

We denote by $\Delta_1, \ldots, \Delta_n$ the sides of the polyhedral surface Σ. To prove proposition 3.67, we will construct a sequence of convex Cauchy surfaces $\Sigma_0, \ldots, \Sigma_n$, where $\Sigma_0 = \Sigma$, and where Σ_{k+1} is obtained by smoothing Σ_k in the neighbourhood of Δ_{k+1}. More precisely, we will prove the prove the following:

Proposition 3.70. *For every $k \in \{0, \ldots, n\}$, for every $\varepsilon > 0$ small enough, there exists a convex Cauchy surface $\Sigma_{k,\varepsilon}$ in M such that:*

– the surface $\Sigma_{k,\varepsilon}$ is in the future of the surface Σ.

– the surface $\Sigma_{k,\varepsilon}$ is ε-close to the surface Σ,

– the surface $\Sigma_{k,\varepsilon}$ is smooth except maybe in the ε-neighbourhoods of the sides $\Delta_{k+1}, \ldots, \Delta_n$.

Notice that proposition 3.70 implies proposition 3.67 (for $k = n$, the surface $\Sigma_{k,\varepsilon}$ is a smooth convex Cauchy surface, ε-close to the initial surface Σ). So, we are left to prove proposition 3.70.

Proof of proposition 3.70. First of all, we set $\Sigma_{0,\varepsilon} := \Sigma$ for every $\varepsilon > 0$. Now, let $k \in \{0, \ldots, n-1\}$, and let us suppose that we have constructed the surface $\Sigma_{k,\varepsilon}$ for every $\varepsilon > 0$ small enough. We will construct the surface $\Sigma_{k+1,\varepsilon}$ for every $\varepsilon > 0$ small enough.

Since Δ_{k+1} is a side of Σ, there exists a support plane Π_{k+1} of Σ such that $\Pi_{k+1} \cap \Sigma = \Delta_{k+1}$. Using remark 3.69, we find a compact neighbourhood V of Δ_{k+1} in M, and some local affine coordinates (x, y, z) on V, such that in these coordinates, $\Pi_{k+1} \cap V$ is the plane of equation $(z = 0)$, and the surface $\Sigma \cap V$ is the graph $(z = f(x, y))$ of a non-negative convex function $f : \mathrm{Dom}(f) \subset \mathbb{R}^2 \to \mathbb{R}$. Moreover, the function f is positive in restriction to $\partial\mathrm{Dom}(f)$, and thus, the quantity $\delta := \inf\{f(x, y) \mid (x, y) \in \partial\mathrm{Dom}(f)\}$ is positive ($\partial\mathrm{Dom}(f)$ is compact).

Now, we fix some $\varepsilon > 0$ such that $\varepsilon/3 < \delta/2$. According the second item of remark 3.69, we can find $\varepsilon' > 0$, such that $\varepsilon' < \varepsilon/3$, and such that the surface $\Sigma_{k,\varepsilon'} \cap V$ is the graph of a convex function $g : \mathrm{Dom}(g) = \mathrm{Dom}(f) \to \mathbb{R}$. Moreover, since $\Sigma_{k,\varepsilon'}$ is in the future of Σ, the function g is bigger than f; in particular, g is non-negative, and we have $g(x, y) > \delta$ for every $(x, y) \in \partial\mathrm{Dom}(g)$.

Applying lemma 3.68 to the function g with $\eta := \varepsilon/3$, we obtain a convex function $\hat{g} : \mathrm{Dom}(g) \to [0,+\infty[$ satisfying the following properties:

(a) $\widehat{g} \geq g$ and the distance between g and $\widehat{g}$ is less than $2\varepsilon/3$,

(b) $\widehat{g}$ is C^∞ on $g^{-1}([0, \varepsilon/3])$,

(c) if g is smooth on some open subset of $\mathrm{Dom}(g) = \mathrm{Dom}(\widehat{g})$, then $\widehat{g}$ is also smooth on U,

(d) $\widehat{g}$ coincides with g on $g^{-1}([2\varepsilon/3, +\infty[)$; in particular, $\widehat{g}$ coincide with g on $\partial\mathrm{Dom}(\widehat{g}) = \partial\mathrm{Dom}(g)$.

We construct the surface $\Sigma_{k+1,\varepsilon}$ as follows: starting from the surface $\Sigma_{k,\varepsilon'}$, we cut off $\Sigma_{k,\varepsilon} \cap V$ (*i.e.* we cut off the graph of g), and we paste the graph of $\widehat{g}$. This is possible since the graphs of the functions g and $\widehat{g}$ coincide near the boundary of V (property (d)). There is natural a diffeomorphism Ψ between the surfaces $\Sigma_{k,\varepsilon'}$ and $\Sigma_{k+1,\varepsilon}$ defined as follows: Ψ coincides with the identity outside V, and maps the point of coordinates $(x, y, g(x, y))$ to the point of coordinates $(x, y, \widehat{g}(x, y))$. By property (a), Ψ is $(2\varepsilon/3)$-close to the identity; hence, the surface $\Sigma_{k+1,\varepsilon}$ is $(2\varepsilon/3)$-close to the surface $\Sigma_{k,\varepsilon'}$. Since $\Sigma_{k,\varepsilon'}$ is ε'-close to Σ, and since $\varepsilon' < \varepsilon/3$, we get that $\Sigma_{k+1,\varepsilon}$ is ε-close to Σ.

The inequality $\widehat{g} \geq g$ implies that $\Sigma_{k+1,\varepsilon}$ is in the future of $\Sigma_{k,\varepsilon'}$, and *a fortiori* in the future of Σ. The convexity of the function $\widehat{g}$ implies that $\Sigma_{k+1,\varepsilon}$ admits a support plane at each of its points.

By proposition 3.46 and remark 3.41, this implies that $\Sigma_{k+1,\varepsilon}$ is a spacelike surface. Hence, $\Sigma_{k+1,\varepsilon}$ is a Cauchy surface (every compact spacelike surface embedded in M is a Cauchy surface). Now, since $\Sigma_{k+1,\varepsilon}$ is a spacelike surface admitting a support plane at each point, it is either convex or concave; and since it coincides with $\Sigma_{k,\varepsilon}$ outside V, it cannot be concave. So, $\Sigma_{k+1,\varepsilon}$ is a convex Cauchy surface.

It remains to study the smoothness of $\Sigma_{k+1,\varepsilon}$. Let q be a point on the surface $\Sigma_{k+1,\varepsilon}$, which is not in the union of the ε-neighbourhoods of the sides $\Delta_{k+2},\dots,\Delta_n$, and let $p := \Psi^{-1}(q) \in \Sigma_{k,\varepsilon'}$. Since the distance between the points p and q is less than $2\varepsilon/3$, the point p cannot be in the union of the $\varepsilon/3$-neighbourhoods of the sides $\Delta_{k+2},\dots,\Delta_n$. There are two cases:

– *if the point p is in the $\varepsilon/3$-neighbourhood of the side Δ_{k+1}*, then the distance between p and the plane Π_{k+1} is less than $\varepsilon/3$, and thus, property (b) implies that the surface $\Sigma_{k+1,\varepsilon}$ is smooth in the neighbourhood of $\Psi(p) = q$;

– *if the point p is not in the $\varepsilon/3$-neighbourhood of the side Δ_{k+1}*, then the surface $\Sigma_{k,\varepsilon'}$ is smooth in the neighbourhood of p (here, we use the inequality $\varepsilon' < \varepsilon/3$); hence, property (c) implies that the surface $\widehat{\Sigma}_{k,\varepsilon}$ is smooth in the nieghbourhood of $\Psi(p) = q$.

As a consequence, the surface $\Sigma_{k+1,\varepsilon}$ is smooth except maybe in the union of the ε-neighbourhoods of the sides $\Delta_{k+2},\dots,\Delta_n$. So, the surface $\Sigma_{k+1,\varepsilon}$ satisfies all the desired by properties. □

Applying proposition 3.67 to the polyhedral Cauchy surfaces Σ_2^- and Σ_2^+, we get two disjoint C^∞ Cauchy surfaces Σ_3^- and Σ_3^+, respectively convex and concave, such that Σ_3^+ is in the future of Σ_3^-.

3.7.3 Smooth uniformly curved convex and concave Cauchy surfaces

The Cauchy surfaces Σ_3^- and Σ_3^+ are smooth, respectively convex and concave, but not uniformly curved. Using the same trick as in subsection 3.6, we will aproximate Σ_3^- and Σ_3^+ by some smooth uniformly curved Cauchy surfaces Σ_4^- and Σ_4^+.

Definition of the Cauchy surfaces Σ_4^- and Σ_4^+. Let ε be a positive number. Let Σ_4^+ be the set made of the points $p \in M$, such that p is in the past of the surface Σ_3^+ and such that the distance from p to Σ_3^+ is exactly ε. If ε is small enough, then Σ_4^+ is a topological Cauchy surface which is convex and uniformly curved (see remark 3.57 and remark 3.55). We construct similarly a topological Cauchy surface Σ_4^- which is concave, uniformly curved, and included in the past of Σ_3^-. By construction, Σ_4^+ is in the future of Σ_4^-.

Proposition 3.71. *If ε is small enough, the Cauchy surfaces Σ_4^- and Σ_4^+ are smooth (of class C^∞).*

Proof. We denote by TM the tangent bundle of M, by π the canonical projection of TM on M, and by $(\varphi^t)_{t\in\mathbb{R}}$ the geodesic flow on TM. We consider the subset $T_N\Sigma_3^+$ of TM made of the couples (p,ν) such that p is a point of the surface Σ_3^+ and ν is the future-pointing unit normal vector of Σ_3^+ at p.

Let p be a point on the surface Σ_4^+. By construction of Σ_4^+, the distance from p to Σ_3^+ is exactly ε. Since M is globally hyperbolic, and since Σ_3^+ is a smooth spacelike surface, this implies that there exists a timelike geodesic segment of length exactly ε, orthogonal to Σ_3^+, joining Σ_3^+ to p (see, for example, [6, page 217]). As a consequence, the surface Σ_4^+ is included in the set $\pi(\varphi^\varepsilon(T_N\Sigma_3^+))$.

We are left to prove that the set $\pi(\varphi^\varepsilon(T_N\Sigma_3^+))$ is a smooth surface. Since Σ_3^+ is a smooth compact spacelike surface in M, $T_N\Sigma_3^+$ is a smooth compact surface in TM, nowhere tangent to the fibers of the projection π. Hence, for ε small enough, $\varphi^\varepsilon(T_N\Sigma_3^+)$ is a smooth compact surface in TM, nowhere tangent to the fibers of π. Hence, $\pi(\varphi^\varepsilon(T_N\Sigma_3^+))$ is a smooth surface in M. □

3.8 End of the proof of theorem 0.2 in the case $g \geq 2$

In the previous paragraph, we have constructed a pair of smooth Cauchy surfaces Σ_4^-,Σ_4^+ in M, such that Σ_4^- is convex and uniformly curved, such that Σ_4^+ is concave and uniformly curved, and such that Σ_4^- is in the past of Σ_4^+. By proposition 3.45, the surface Σ_4^- have negative curvature and the surface Σ_4^+ have positive curvature. As a consequence, (Σ_4^-,Σ_4^+) is a pair of barriers in M. By theorem 3.1 and 3.3, the existence of a pair of barriers implies the existence of a CMC time function. This completes the proof of theorem 0.2 in the case where the genus of the Cauchy surfaces is at least 2.

4 Proof of theorem 0.2 in the case $g = 1$

The purpose of this section is to prove theorem 0.2 in the case where the genus of the Cauchy surfaces of the spacetime under consideration is equal to 1. The proof is very from those of the case $g \geq 2$. In subsection 4.1, we will define a class of spacetimes, called Torus Universes[14], and we will prove that Torus Universe admits a CMC time function (actually, we will construct explicitly a CMC time function on any such spacetime). Then, in subsection 4.2, we will prove that every maximal globally hyperbolic spacetime, locally modelled on AdS_3, whose Cauchy surfaces are two-tori, is isometric to a Torus Universe.

4.1 Torus Universes

Consider the 1-parameter subgroup of $SL(2,\mathbb{R})$ of diagonal matrices $(g^t)_{t\in\mathbb{R}}$ where:

$$g^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = e^{t\Delta} \quad \text{where:} \quad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We denote by A the set of elements of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ for which both left and right components belongs to the one-parameter subgroup $(g^t)_{t\in\mathbb{R}}$. Of course, A is a free abelian Lie subgroup of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. This group acts isometrically on $\mathbb{A}d\mathbb{S}_3$ (recall that the isometry group of $\mathbb{A}d\mathbb{S}_3$ can be identified with $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, see subsection 2.2). We denote by Ω the union of spacelike A-orbits in $\mathbb{A}d\mathbb{S}_3$.

We will see below that Ω has four connected components which are open convex domains of $\mathbb{A}d\mathbb{S}_3$. For any lattices $\Gamma \in A$, the action of Γ on Ω is obviously free and properly discontinuous, and preserves each of the four connected components of Ω.

Definition 4.1. *A Torus Universe is the quotient $\Gamma\backslash U$ of a connected component U of Ω by a lattice Γ of A.*

Theorem 4.2. *Every Torus Universe is a globally hyperbolic spacetime, admitting a CMC time function.*

To prove proposition 4.2, we will use the $SL(2,\mathbb{R})$-model of $\mathbb{A}d\mathbb{S}_3$ (see subsection 2.5). We recall that $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ acts on $SL(2,\mathbb{R})$ by $(g_L, g_R).g = g_L g g_R^{-1}$.

Lemma 4.3. *For every element $g \in \Omega$, the A-orbit contains a unique element of the form*

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \textit{with} \qquad \theta \in [0, 2\pi[$$

When g ranges over Ω, the angle θ varies continously with g, and ranges over the set of all real numbers in $[0, 2\pi[$ which are not multiples of $\frac{\pi}{2}$.

Proof. Consider an element g in $\mathbb{A}d\mathbb{S}_3 \simeq SL(2,\mathbb{R})$ and write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1$$

Then, the elements of the A-orbit of g are the matrices

$$g^t g g^{-s} = \begin{pmatrix} ae^{t-s} & be^{t+s} \\ ce^{-(t+s)} & de^{s-t} \end{pmatrix}$$

where s and t range over $\mathbb{R}$. Thus, the A-orbit of g is spacelike if and only if, for every $p, q \in \mathbb{R}$, the determinant of:

$$\begin{pmatrix} (p-q)a & (p+q)b \\ -(p+q)c & (q-p)d \end{pmatrix}$$

is negative, *i.e.* if and only if the the quadratic form $(p-q)^2 ad - (p+q)cd$ is positive definite. Since $ad - bc = 1$, it follows that the A-orbit of g is spacelike if and only if:

$$\begin{array}{rcl} 0 < & ad & < 1 \\ -1 < & bc & < 0 \end{array}$$

In particular, if the A-orbit of g is spacelike, then $abcd \neq 0$. It follows that, if the A-orbit of g is spacelike, then the A-orbit of contains a element of the form

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

[14] These spacetimes were already considered by several authors, see remark 4.7

(take s, t such that $e^{2(t-s)} = d/a$ and $e^{2(s+t)} = c/b$). The angle θ is obviously unique, it is not a multiple of $\frac{\pi}{2}$ (since $d \neq 0$ and $c \neq 0$), it varies continuously with g, and it takes any value in $[0, 2\pi[$ that is not a multiple of $\frac{\pi}{2}$ when g ranges over Ω. □

Remark 4.4. *If* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Omega$, *then the unique number* $\theta \in [0, 2\pi[$ *such that the rotation* R_θ *is in the* A*-orbit of* g *is characterized by the equalities* $\cos^2\theta = ad$ *and* $-\sin^2\theta = bc$ *(see the proof of lemma 4.3).*

Lemma 4.3 implies that Ω has four connected components (corresponding to $\theta \in]0, \frac{\pi}{2}[$, $\theta \in]\frac{\pi}{2}, \pi[$, $\theta \in]\pi, \frac{3\pi}{2}[$, and $\theta \in]\frac{3\pi}{2}, 2\pi[$).

Remark 4.5. *The four connected components of* Ω *are all isometric one to the other by isometries centralizing the group* A*. Hence, with no loss of generality, we may restrict ourselves to Torus Universes that are obtained as quotients of the connected component corresponding to* $0 < \theta < \pi/2$.

Proof. Proof of theorem 4.2 Denote by U the connected component of Ω correspoding to $0 < \theta < \frac{\pi}{2}$. Consider a lattice Γ in A, and consider the associated Torus Universe $M = \Gamma\backslash U$. Lemma 4.3 provides us with a continuous function $\theta : U \mapsto]0, \frac{\pi}{2}[$. By construction, this function is increasing with time and Γ-invariant: it follows that the quotient manifold $M = \Gamma\backslash U$ is equipped with a time function $\bar{\theta}$.

The equalities $\cos^2\theta = ad$ and $-\sin\theta = bc$ (see lemma 4.3) imply that the connected component U is exactly

$$\left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \text{ such that } 0 < a\ ,\ 0 < b\ ,\ 0 > c \text{ and } 0 < d \right\}$$

Thus, in the spherical model of $\mathbb{A}d\mathbb{S}_3$, the connected component U is the interior of a simplex which is the convex hull of four points in $\partial\mathbb{A}d\mathbb{S}_3$ (these points are nothing but the fixed points of A) (see figure 3). The main information we extract from this observation is that U is a convex domain in $\mathbb{A}d\mathbb{S}_3$, in particular, its intersection with any geodesic - in particular, nonspacelike geodesics - is connected. Moreover, geodesics joining two points of ∂U satisfying both $bc = 0$ (respectively $ad = 0$) are spacelike. Hence, nonspacelike segments in U admits two extremities in ∂U, one satisfying $bc = 0$, and the other $ad = 0$. The equalities $ad = \cos^2\theta$, $bc = -\sin^2\theta$ imply that θ restricted to such a nonspacelike segment takes all values between 0, and $\frac{\pi}{2}$. In other words, every nonspacelike geodesic in U intersects every fiber of θ. Hence, every nonspacelike geodesic in M intersects every fiber of $\bar{\theta}$: these fibers are thus Cauchy surfaces, and M is globally hyperbolic.

Since every fiber of $\bar{\theta}$ is a A-orbit, it obviously admits constant mean curvature $\kappa(\bar{\theta})$. Let us calculate this mean curvature value at R_θ. We will need to take covariant derivatives, and here, the situation is similar to the familiar situation concerning riemannian embeddings in euclidean spaces: if X, Y are vector fields in $M(2, \mathbf{R})$ both tangent to G, then the covariant derivative $\overline{\nabla}_X Y$ in G is the orthogonal projection on the tangent space to G of the natural affine covariant derivative $\nabla_X Y$ for the affine connection on the ambient linear space.

A straightforward calculation shows that the curve $\theta \mapsto R_\theta$ is orthogonal to the A-orbits, hence, the unit normal vector to AR_θ at R_θ is:

$$n(\theta) = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}$$

Moreover, this unit normal vector is indeed future oriented if we consider the orientation of U for which θ increases with time. Now, for any p, q, consider the curve $t \mapsto c(t) = g^{pt} n(\theta) g^{-qt}$. Its tangent vector at $t = 0$ is:

$$X_{p,q} = \begin{pmatrix} (p-q)\cos\theta & (q+p)\sin\theta \\ (q+p)\sin\theta & (q-p)\cos\theta \end{pmatrix}$$

The unit normal vector $n(t)$ to the A-orbit at $c(t) = g^{pt} R_\theta g^{-qt}$ is

$$g^{pt} n(\theta) g^{-qt} = \begin{pmatrix} -e^{t(p-q)}\sin\theta & e^{t(q+p)}\cos\theta \\ -e^{-t(q+p)}\cos\theta & -e^{t(q-p)}\sin\theta \end{pmatrix}$$

Hence, the derivative at $t = 0$ is:

$$\begin{pmatrix} (q-p)\sin\theta & (q+p)\cos\theta \\ (q+p)\cos\theta & (p-q)\sin\theta \end{pmatrix}$$

The orthogonal projection of this vector tangent vector to AR_θ at R_θ is the covariant derivative of the unit normal vector along the curve $t \mapsto c(t)$. It follows that the second fundamental form is:

$$II(X_{p,q}, X_{p,q}) = -\langle X_{p,q} \mid \overline{\nabla}_{X_{p,q}} n(t) \rangle = ((p-q)^2 - (p+q)^2)\sin(2\theta)$$

Whereas the first fundamental form, i.e., the metric itself, is:

$$\langle X_{p,q} \mid X_{p,q} \rangle = (p-q)^2 \cos^2\theta + (p+q)^2 \sin^2\theta$$

Therefore, the principal eigenvalues are $-2\mathrm{cotan}\theta$ and $2\tan\theta$. It follows that the mean curvature value is $\kappa(\theta) = -4\mathrm{cotan}(2\theta)$. The function $\kappa \circ \bar{\theta}$ is then increasing with time: this is the required CMC time function. $\square$

Remark 4.6. *The closure of the domain U meets the conformal boundary at infinity $\partial\mathbb{A}d\mathbb{S}_3$ on a topological nontimelike circle, but it is not a spacelike curve. Actually, the intersection of the closure of U with $\partial\mathbb{A}d\mathbb{S}_3$ is the union of four lightlike geodesic segments (see figure 3).*

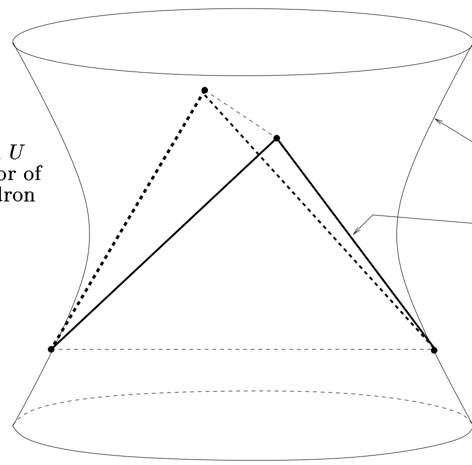

Figure 3: The domain U represented in the projective model of $\mathbb{A}d\mathbb{S}_3$ (more precisely, here we use an affine chart mapping some domain of $\mathbb{A}d\mathbb{S}_3$ in $\mathbb{R}^3$).

Remark 4.7. *The Torus Universes as defined above are the same as those described in [3] in the case of negative cosmological constant (this follows immediatly from the results of subsection 4.2 below). Observe that the expression of the metric on the A-orbit just above enables to recover easily the features discussed in [3]: the volume of the slices $\bar{\theta} = Cte$ are proportionnal to* $\sin 2\theta$*, and the conformal classes of these toroidal metrics describe geodesics in the modular space $Mod(T)$ of the torus. More precisely: on the slice $\bar{\theta} = Cte$, the conformal class and the second differential form define naturally a point in the cotangent bundle of $Mod(T)$, and when the Cte is evolving, these data describe an orbit of the geodesic flow on $T^*Mod(T)$. Inversely, every orbit of the geodesic flow on $T^*Mod(T)$ correspond to a Torus Universe.*

4.2 Every maximal globally hyperbolic spacetime, locally modelled on $\mathbb{A}d\mathbb{S}_3$, with closed Cauchy surfaces of genus 1 is a Torus Universe

In this section, we consider a maximal globally hyperbolic lorentzian manifold M, locally modelled on $\mathbb{A}d\mathbb{S}_3$, whose Cauchy surfaces are 2-tori. We will prove that such a spacetime M is isometric to a Torus Universe (as defined in subsection 4.1). Together with theorem 4.2, this will imply that M admits a CMC time function.

As in section 3, we consider a Cauchy surface Σ_0 in M, and the lift $\widetilde{\Sigma}_0$ of Σ_0 in the universal covering $\widetilde{M}$ of M. We have a locally isometric developping map $\mathcal{D} : \widetilde{M} \to \mathbb{A}d\mathbb{S}_3$, and a holonomy representation ρ of $\pi_1(M) = \pi_1(\Sigma_0)$ in the isometry group of $\mathbb{A}d\mathbb{S}_3$. We denote $\Gamma = \rho(\pi_1(M)) \subset SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ (here, we prefer to see the isometry group of $\mathbb{A}d\mathbb{S}_3$ as $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ rather than $O(2,2)$), and we denote $S_0 = \mathcal{D}(\widetilde{\Sigma}_0)$. Observe that proposition 3.4 is still valid in the present context (the proof of proposition 3.4 does not depend on the genus of Σ_0) ; in particular, S_0 is properly embedded in $\mathbb{A}d\mathbb{S}_3$.

The surface σ_0 is a two-torus : hence, the fundamental group of Σ_0 is isomorphic to $\mathbb{Z}^2$. Moreover, according to proposition 3.4, $\Gamma = \rho(\pi_1(M)$ is a discrete subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Hence, Γ is a lattice in some abelian group $A = H_L \times H_R$, where $H_L = \{e^{th_L}\}_{t\in\mathbb{R}}$ (resp. $H_R = \{e^{sh_R}\}_{s\in\mathbb{R}}$ is a one parameter subgroup of $SL(2,\mathbb{R}) \times \{id\}$ (resp. $\{id\} \times SL(2,\mathbb{R})$). Since A is isomorphic to $\mathbb{R}^2$, these one-parameter groups are either parabolic or hyperbolic. In other words, up to factor switching and conjugacy, there are only three cases to consider:

- *Hyperbolic - hyperbolic*:

$$h_L = h_R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- *Parabolic - parabolic*:

$$h_L = h_R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- *Hyperbolic - parabolic*:

$$h_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad h_R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Let us consider an orbit O of A. The restriction to O of the ambient lorentzian metric defines a field of quadratic forms on O. Since A is a group of isometries, the quadratic forms appearing in this field have a well-defined type: each of them is either spacelike, timelike, lorentzian or degenerate. We call such a field of quadratic forms a degenerate pseudometric. The following lemma describes all the "isometry" type of degenerate pseudometrics which can arise by this construction:

Lemma 4.8. *Every orbit O of A has dimension 1 or 2. Moreover:*

- If O has dimension 1, then it is isometric to an euclidean line, or to an isotropic line (i.e. equipped with the trivial degenerate pseudometric).

- If O has dimension 2, then it is isometric to the euclidean plane, to the Minkowski plane, or to the degenerate plane, i.e. the plane with coordinates (x, y) equipped with the quadratic form dx^2.

Proof. An element (e^{th_L}, e^{-sh_R}) fixes a point g in $SL(2,\mathbb{R})$, then $e^{th_L} = ge^{sh_R}g^{-1}$. Observe that in the hyperbolic-parabolic case, it implies $s = t = 0$: in this case, every orbit of A is a 2-dimensional plane. In the hyperbolic-hyperbolic and parabolic-parabolic cases, it implies $s = t$ and $g = e^{th_L}$: hence, there is no 0-dimensional orbits, 1-dimensional orbits are lines, and 2-dimensional orbits are planes.

We parametrize the A-orbit O of an element g_0 of $\mathbb{A}d\mathbb{S}_3 \approx SL(2,\mathbf{R})$ by $(s,t) \mapsto e^{th_L}g_0e^{-sh_R}$. The differential of this parametrization is:

$$(h_L e^{th_L} g_0 e^{-sh_R})ds - (e^{th_L} g_0 e^{-sh_R} h_R)dt$$

Since h_R and h_L commute respectively with their exponential, and since these exponentials have determinant 1, the determinant of this expression reduces to the determinant of:

$$(h_L g_0)ds - (g_0 h_R)dt$$

The quadratic form induced on the tangent space of O at (s,t) is $-\det$ of this expression.

If O has dimension 1, then $g_0 h_R g_0^{-1} = h_L = h_R$, thus this determinant is equal to the determinant of $h_L ds - h_L dt$. In the parabolic-parabolic case, we obtain identically 0: O is an isotropic line. In the hyperbolic-hyperbolic case, we obtain $(d(s-t))^2$: O is an euclidean line.

When O has dimension 2, it is diffeomorphic to the plane. Observe that in the expression above, s and t appears only by their differentials: it means that the degenerate pseudometric is actually a parallel field of quadratic forms. In other words, it is given by the quadratic form $-\det(h_L g_0 ds - g_0 h_R dt)$ on the 2-plane O with linear coordinates (s,t). The lemma follows from the classification of quadratic forms on the plane (the negative definite case and the case $-(dx)^2$ are excluded since the quadratic form is obtain by the restriction of a lorentzian quadratic form). □

Lemma 4.9. *The surface S_0 intersects only 2-dimensional spacelike orbits of A.*

Proof. Let O be the A-orbit of an element x_0 of S_0. Assume first that O has dimension 1: according to lemma 4.8, O is a line. Observe that O is preserved by the action of Γ. Since Γ acts freely on S_0, x_0 is not fixed by any element of Γ. Hence, every Γ-orbit in O is dense. It follows that x_0 admits Γ-iterates of itself arbitrarly near to itself. This is impossible, since Γ acts properly in a neighbourhood of S_0.

Hence, O has dimension 2. Assume that O is not spacelike. According lemma 4.8, it is isometric to the Minkowski plane or the degenerate plane. Since S_0 is spacelike, S_0 and O are transverse. Their intersection is a closed 1-manifold L. Moreover, the ambient lorentzian metric restricts as an euclidean metric on L which is complete. The argument used in proposition 3.4 can then be applied once more: if O is a Minkowski plane, L intersects every timelike line in O in one and only one point, and if O is degenerate, the same argument proves that L must intersect every degenerate line $y = Cte$ in one and only one point (in this situation, the projection of L on the coordinate x is an isometry!).

It follows that in both cases, L is connected. Therefore, it is isometric to $\mathbb{R}$. But since O and S_0 are both preserved by Γ, the same is true for L: we obtain that $L \approx \mathbb{R}$ admits a free and properly continuous isometric action by $\Gamma \approx \mathbb{Z}^2$. Contradiction. □

According to the lemma, A must admit spacelike orbits, and it excludes all the cases except the hyperbolic - hyperbolic case. Hence, A is precisely the abelian group of isometries studied in subsection 4.1 for the definition of the Torus Universes. Moreover, lemma 4.9 states precisely that S_0 is contained in a connected component U of the domain Ω. Since this is true for any Cauchy surface Σ, and since M is globally hyperbolic, the image of the developping map is contained in U. Hence, M embedds isometrically in the Torus Universe $\Gamma \backslash U$. Since M is maximal as a globally hyperbolic spacetime, M is actually isometric to this quotient.

Thus, we have proved:

Theorem 4.10. *Every maximal globally hyperbolic lorentzian manifold, locally modelled on $\mathbb{A}d\mathbb{S}_3$, with closed oriented Cauchy surfaces of genus* 1 *is isometric to a Torus Universe.*

Corollary 4.11. *Torus Universes are maximal as globally hyperbolic spacetimes.*

Proof. Proof of theorem 0.2 in the case where the genus of the Cauchy surfaces is equal to 1 The result follows from theorem 4.10 and 4.2. □

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