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Comportements chaotiques des systèmes dynamiques différentiables

Chaotic properties of differentiable dynamical systems

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Contents

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List of works presented for the habilitation

(1) P. Berger, Structural stability of attractor-repellor endomorphisms with singularities, *Ergodic Theory and Dynamical Systems*, (2012), no. 1, 1-33.

(2) P. Berger -A. Rovella, On the inverse limit of endomorphisms, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 30 (2013), no. 3, 463-475.

(3) P. Berger - A. Kocsard, Structural stability of the inverse limit of endomorphisms, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, accepted in 2016 34p.

(4) P. Berger - P.D. Carasco, Non uniformly hyperbolic attractors derived from the standard map, *Communication in Mathematical Physics*, July (2014) v. 329 1, pp 239-262.

(5) P. Berger, Abundance of one dimensional non-uniformly hyperbolic attractors for surface endomorphisms, *arxiv*, 110 p.

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(11) P. Berger, Generic family with robustly infinitely many sinks, *Inventiones Matematicae*, (2016) 205: 121. 41p. Corrigendum: https://www.math.univ-paris13.fr/ berger/

(12) P. Berger, Emergence and non-typicality of the finiteness of the attractors in many topologies, *to appear in Proc. of the Steklov Institute, the Anosov Volume* (2017) 26 p.

(13) —, Generic family displaying robustly a fast growth of the number of periodic points, *Arxiv* (2016) 38 p.

Introduction

Version française

Dans ce mémoire, nous présentons des travaux développant la théorie non-uniformément hyperbolique, sur les thèmes de la stabilité structurelle, la théorie non-uniformément hyperbolique, et les dynamiques émergentes (systèmes ayant des propriétés pathologiques très complexes).

Avant de les décrire, nous allons rappeler des résultats classiques de la théorie uniformément hyperbolique.

Systèmes dynamiques uniformément hyperboliques[1]

La théorie uniformément hyperbolique a été construite durant les années 1960 sous l'impulsion des écoles de Smale aux États-Unis et d'Anosov et Sinai en Union Soviétique. Elle est aujourd'hui presque complète. Elle dispose de nombreux exemples [Sma70]: application dilatante, fer à cheval, solénoïde de Smale, attracteur de Plykin, application Anosov et dérivée d'Anosov, chacun étant une pièce basique. Rappelons quelques définitions élémentaires. Soit f un difféomorphisme de classe C^1 d'une variété de dimension finie M. Un compact Λ est uniformément hyperbolique s'il est f-invariant et la restriction du fibré tangent TM de M à Λ se décompose en deux sous-espaces Df-invariant $TM|K = E^s \oplus E^u$; tels que E^s soit uniformément contracté et E^u uniformément dilaté.

Alors pour tout $z \in \Lambda$, les ensembles

$$W^s(z) = \{z' \in M : \lim_{n\to+\infty} d(f^n(z), f^n(z')) = 0\},$$

$$W^u(z) = \{z' \in M : \lim_{n\to-\infty} d(f^n(z), f^n(z')) = 0\}$$

sont appelés *variétés stables et instables* de z. Ce sont des variétés immergées tangentes au point z à respectivement $E^s(z)$ et $E^u(z)$.

La *variété ϵ-locale stable* $W^s_\epsilon(z)$ de z est la composante connexe de z dans l'intersection de $W^s(z)$ avec le ϵ-voisinage de z. La *variété ϵ-locale instable* $W^u_\epsilon(z)$ est définie de façon similaire.

Definition 0.1. *Une* pièce basique *est un compact Λ, f-invariant, uniformément hyperbolique qui est transitif et* localement maximal *: il existe un voisinage N de Λ tel que $\Lambda = \cap_{n\in\mathbb{Z}} f^n(N)$. Une pièce basique est un* attracteur *si le voisinage N peut être choisi de sorte que $\Lambda = \cap_{n\geq 0} f^n(N)$. Un attracteur contient les variétés instables de ses points.*

Un difféomorphisme dont l'ensemble non-errant est une union finie de pièces basiques est appelé *uniformément hyperbolique* ou *satisfaisant l'axiome A*.

De tels difféomorphismes jouissent de bonnes propriétés qui sont prouvées dans [Sma67] (ou dans sa bibliographie).

[1] Ce paragraphe est la traduction de la première page de [BY14].

Mesures SRB et physique Soit $\alpha > 0$, et soit Λ un attracteur pour un difféomorphisme f de classe $C^{1+\alpha}$. Alors il existe une unique probabilité invariante supportée par Λ telle que les mesures conditionnelles par rapport à toute partition mesurable en variétés instables locales soient absolument continues par rapport à la mesure de Lebesgue (de ces variétés instables). De telles mesures sont appelées *SRB* (pour Sinai-Ruelle-Bowen). On montre que telles mesures sont *physiques*: la mesure de Lebesgue de son bassin $B(\mu)$

$$(\mathcal{B}) \qquad B(\mu) = \{z \in M : \frac{1}{n}\sum_{i<n} \delta_{f^i(x)} \rightharpoonup \mu\},$$

est positive. En réalité, modulo un ensemble de mesure de Lebesgue nulle, $B(\mu)$ est égale au bassin d'attraction topologique de Λ (ce dernier est formé des points dont l'orbite s'accumule sur Λ).

Stabilité structurelle Une pièce basique Λ d'un difféomorphisme de classe C^1 est *structurellement stable* : toute C^1-perturbation f' de f laisse invariante une pièce basique Λ' qui est homéomorphe à Λ via un homéomorphisme qui conjugue les dynamiques $f|\Lambda$ et $f'|\Lambda'$.

Codage Une pièce basique Λ pour un difféomorphisme f de classe C^1 admet une partition de Markov finie. Cela implique que la dynamique est semi-conjuguée à un décalage de type fini. La semi-conjugaison est bijective sur un sous-ensemble générique. Son manque d'injectivité est codé par un sous-décalage de type fini et d'entropie strictement inférieure. Cela permet d'étudier précisément toutes les mesures invariantes supportées par Λ, la distribution des points périodiques, l'existence et l'unicité de la mesure d'entropie maximale.

Introduction aux développements présentés

Smale désirait décrire le comportement typique d'un système dynamique typique. Pour ce faire, il conjectura la densité des difféomorphismes satisfaisant l'axiome A dans l'espace des difféomorphismes f de classe C^r de variétés compactes M.

En dimension supérieure, des obstructions ont été découvertes rapidement par Abraham-Smale [AS70] puis par Shub [Shu71]. Ce dernier exemple est aujourd'hui un exemple paradigmatique de la théorie partiellement hyperbolique (que nous ne décrirons pas en détail). Pour de tels systèmes, il existe une décomposition de l'espace tangent TM de M en deux sous-fibrés invariants $E^s \oplus E^c \oplus E^u = TM$, telle que E^s soit plus contracté que $E^c \oplus E^u$ et telle que E^u soit plus dilaté que $E^s \oplus E^c$.

Le cas *plutôt dilatant* intervient quand E^c est asymptotiquement dilaté, dans le sens suivant:

$$\exists m > 0 : \quad \limsup_{n\to\infty} \frac{1}{n} \sum \log \|D_{f^n(x)} f^{-1} | E^c\| < -m \quad \text{for Leb. } p.t.\ x \in M\ .$$

Dans ce cas, un théorème de Alves-Bonatti-Viana [ABV00] affirme qu'il existe un nombre fini de mesures SRB dont les bassins recouvrent M modulo un ensemble de mesure de Lebesgue nulle.

Le cas *plutôt contractant* intervient quand le compact maximal invariant K de M est partiellement hyperbolique et tel que pour tout $z \in K$, avec $W^{uu}_{\text{loc}}(z)$ la variété instable locale de z et avec

Leb la mesure de Lebesgue qui lui est associée, on ait :

$$0 < \mathrm{Leb}\{x \in W^{uu}_{loc}(z) : \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^n | E^c(x)\| < 0\} \ .$$

Alors un théorème de Bonatti-Viana [BV00] affirme qu'il existe un nombre fini de mesures SRB dont l'union des bassins recouvre Lebesgue presque tout M.

Ces deux cas s'appliquent à de nombreux exemples et cela robustement pour leurs C^2-perturbations. On peut ainsi se demander s'il existe un ouvert de dynamiques partiellement hyperboliques de classe C^2 pour lesquelles il existe une décomposition mesurable $E^c = E^{cs} \oplus E^{cu}$ pour Lebesgue presque tous les points, et tel que E^{cs} est asymptotiquement contracté et E^{cu} asymptotiquement dilaté. Pour écarter le cas de type produit, on demande aussi que $Df|E^{cs}$ ne soit pas dominé par $Df|E^{cu}$, i.e. il existe des vecteurs dans E^{cu} qui sont plus contractés que certains dans E^{cs}. Dans [BC14] avec Carrasco, nous avons donné un des tous premiers exemples d'un ouvert de difféomorphismes conservatifs de classe C^2 ayant une telle propriété. Celui-ci est obtenu en couplant l'application standard de Chirikov avec un difféomorphisme Anosov. Cet exemple sera décrit dans le chapitre 11.

D'autre part, grâce à des simulations numériques, le climatologue [Lor63] et le mathématicien Hénon [Hén76] ont découvert des systèmes dynamiques ayant des propriétés hyperboliques mais qui ne sont pas uniformément hyperboliques. (voir fig. 8).

Pour inclure ce type d'exemples, la théorie non-uniformément hyperbolique est toujours en construction. Quelques exemples ont été montrés mathématiquement comme tels [Jak81, Ree86, BC85, BC91, PY09] comme nous le verrons dans le chapitre 7. Nous présenterons dans le chapitre 8, notre contribution au programme de régularité forte de Yoccoz [Yoc97] concernant les endomorphismes de type Hénon [Bera]. Le but de ce programme est de formuler une définition combinatoire et topologique des compacts transitifs non-uniformément hyperboliques (s'appuyant sur des généralisations des pièces de puzzle de Yoccoz) et de montrer leur abondance (i.e. que pour toute famille de dynamiques dans un ouvert, pour tout paramètre dans un ensemble de mesure de Lebesgue positif, la dynamique présente un tel compact). Tous les attracteurs mentionnés ci-dessus ont une dimension de Hausdorff (réelle ou complexe) proche de 1. Cependant, nous verrons dans le chapitre 9 qu'il semble possible d'adapter notre preuve pour traiter le cas d'attracteurs de surface ayant une plus grande dimension (peut-être même celle initialement observée par Hénon), en utilisant les techniques de [BM16]. De plus, nous verrons que cette définition combinatoire permet de prouver que les attracteurs de type Hénon fortement réguliers ont des propriétés ergodiques similaires à celles des compacts uniformément hyperboliques [Berd] (voir le chapitre 10): ces attracteurs supportent une unique mesure d'entropie maximale, qui est équidistribuée sur les points périodiques. Aussi nous avons répondu à une question de Carleson en montrant des bornes uniformément hyperboliques sur les exposants de Lyapunov de toutes les mesures de probabilité invariantes.

Une autre obstruction au rêve de Smale a été découverte par son étudiant Newhouse. Pour tout $2 \le r \le \infty$, il a montré l'existence d'un ouvert U de l'espace des difféomorphismes de surface

Figure 1: L'attracteur de Hénon ne peut pas être uniformement hyperbolique car son bassin d'attraction est un disque et il n'y a pas de champ de droites continu sur le disque.

de classe C^r, tel que tout f dans un sous-ensemble topologiquement générique de U présente une infinité de puits [New74, New79]. Pour de telles dynamiques, les puits s'accumulent sur un compact hyperbolique. Aussi la mesure de probabilité invariante de chaque puit est très différente de celles des autres puits. Suivant Yoccoz, une telle coexistence d'un nombre infini de puits est une minoration de la complexité de tels systèmes. Encore aujourd'hui, nous ne comprenons pas un seul exemple de telles dynamiques (par exemple nous ne savons pas si Lebesgue presque tous les points ont leur somme de Birkhoff qui converge). Nous rappelons ce phénomène dans le chapitre 12.

Depuis les années 90, ce phénomène était conjecturé comme improbable/négligeable (au sens de Kolmogorov) dans beaucoup de conjectures [TLY86, Pal00, Pal05, PS96], certaines étant en petite dimension. Au chapitre 13, nous étudierons la direction opposée. D'abord nous présenterons notre travail avec De Simoi [BDS16], qui montre que ce phénomène à codimension au plus 1/2 dans l'espace des difféomorphismes de surface ; un résultat qui ne contredit aucune de ces conjectures. Puis nous présenterons [Ber16b, Ber16a, Ber17a], dans lesquels nous avons montré qu'au sein des endomorphismes de surface ou des difféomorphismes en dimension supérieure, ce phénomène est non négligeable et même important au sens de Kolmogorov.

Cela nous amène à nous demander à quel point une dynamique (localement) typique peut être compliquée. Il y a de nombreuses façons de préciser cette question. Nous avons étudié deux directions:

(i) *Une description d'un système par ses mesures physiques.* Dans le chapitre 14, nous définissons le concept d'émergence [Ber17a] pour un système dynamique. Nous l'avons imaginé pour évaluer la complexité de la description d'un système par ses mesures physiques. Nous énoncerons une conjecture sur celle-ci.

(ii) *La croissance du nombre de points périodiques.* Dans le chapitre 15, nous présentons un complément [Ber17b] aux travaux de Marteens-de Melo-Van Strien, Kaloshin, Turaev, Gonchenko-

Shilnikov-Turaev, Asaoka-Shinohara-Turaev, pour donner une réponse complète à des questions de Smale en 1967, Bowen en 1978 et Arnold en 1989, sur la croissance du nombre de points périodiques. En particulier nous montrerons que cette croissance peut être aussi rapide que désirée en toute dimension ≥ 2 et en toute régularité $\infty \geq r \geq 2$. Aussi, nous avons montré dans [Ber17b] que ce phénomène n'est pas négligeable et même important au sens de Kolmogorov ; ce qui apporte une réponse négative à un problème posé par Arnold 1992).

Les deux résultats ci-dessus sont donnés par une généralisation des mélangeurs de Bonatti-Diaz aux familles à paramètres: les paramélangeurs. Nous rappelons leur définition dans le chapitre 16.

Cependant l'hyperbolité uniforme semble fournir une description satisfaisante des dynamiques structurellement stables. Cette observation remonte à une conjecture de Fatou pour les applications quadratiques (1920) et une conjecture de Smale pour les difféomorphismes lisses (1970). Ces conjectures ont été intensivement étudiées par beaucoup de mathématiciens et sont donc difficiles à attaquer directement.

Malgré cela, à l'interface de la dynamique à une variable complexe et celle de la dynamique différentielle, la dynamique à deux variables complexes s'est développée récemment. Elle permet d'étudier les problèmes de stabilité structurelle grâce aux techniques des deux champs précédents. Aussi les mathématiques développées dans les années 1960 pour les problèmes dynamiques de stabilité structurelle sont très similaires à celles développées par Thom-Mather en théorie des singularités. Cela nous a amené à joindre ces deux champs mathématiques pour étudier la stabilité structurelle des endomorphismes. Dans la partie II, nous présenterons ces deux champs mathématiques, puis nos travaux à leurs interfaces [Ber12, BR13, BK13, BD17], comme il sera détaillé au début de la partie II.

Part I

Introduction

In this work we present developments of the uniformly hyperbolic theory, focused on structural stability, non-uniform hyperbolicity and emergent dynamics (systems displaying a pathological and very complex behavior).

Before defining these topics, we shall first recall the uniformly hyperbolic theory. The following section is an extended version of the first page of [BY14].

1 Uniformly hyperbolic dynamical systems

The theory of *uniformly hyperbolic dynamical systems* was constructed in the 1960's under the dual leadership of Smale in the USA, and Anosov and Sinai in the Soviet Union.

It encompasses various examples that we shall recall: expanding maps, horseshoes, solenoid maps, Plykin attractors, Anosov maps, DA, blenders all of which are *basic pieces.*

1.1 Uniformly hyperbolic diffeomorphisms

Let f be a C^1-diffeomorphism f of a finite dimensional manifold M. A compact f-invariant subset $\Lambda \subset M$ is *uniformly hyperbolic* if the restriction to Λ of the tangent bundle TM splits into two continuous invariant subbundles

$$TM|\Lambda = E^s \oplus E^u,$$

E^s being uniformly contracted and E^u being uniformly expanded: $\exists \lambda < 1$, $\exists C > 0$,

$$\|T_x f^n_{|E^s}\| < C \cdot \lambda^n \quad \text{and} \quad \|T_x f^{-n}_{|E^u}\| < C \cdot \lambda^n, \quad \forall x \in \Lambda, \forall n \geq 0.$$

Example 1.1 (Hyperbolic periodic point). A periodic point at which the differential has no eigenvalue of modulus 1 is called hyperbolic. It is a sink if all the eigenvalues are of modulus less than 1, a source if all of them are of modulus greater than 1, and a saddle otherwise.

Definition 1.2. *A* hyperbolic attractor *is a hyperbolic, transitive compact subset* Λ *such that there exists a neighborhood* N *satisfying* $\Lambda = \cap_{n\geq 0} f^n(N)$.

Example 1.3 (Anosov). If the compact hyperbolic set is equal to the whole compact manifold, then the map is called *Anosov.* For instance if a map $A \in SL_2(\mathbb{Z})$ has both eigenvalues of modulus not equal to 1, then it acts on the torus $\mathbb{R}^2/\mathbb{Z}^2$ as an Anosov diffeomorphism. The following linear map satisfies such a property:

$$A := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$

Example 1.4 (Smale solenoid). We consider a perturbation of the map of the filled torus $\mathbb{T} := \{(\theta, z)\mathbb{R}/\mathbb{Z} \times \mathbb{C} : |z| < 1\}$:

$$(\theta, z) \in \mathbb{T} \mapsto (2\theta, 0) \in \mathbb{T} \ ,$$

which is a diffeomorphism onto its image. This is the case of the following:

$$(\theta, z) \in \mathbb{T} \mapsto (2\theta, \epsilon \cdot z + 2\epsilon \cdot \exp(2\pi i\theta) \in \mathbb{T} \ .$$

This defines a hyperbolic attractor called the *Smale solenoid*.

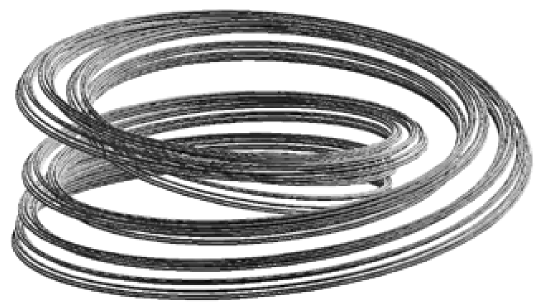


Figure 2: the Smale Solenoid (Credit Steklov institut)

Example 1.5 (Derivated from Anosov (DA) and Plykin attractor). We start with a linear Anosov of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. It fixes the point 0. In local coordinates ϕ of a neighborhood V of 0, it has the form for $0 < \lambda < 1$:

$$(x, y) \mapsto (\lambda x, y/\lambda).$$

For every $\epsilon > 0$, let ρ_ϵ be a smooth function so that:

- it is equal to $x \mapsto \lambda x$ outside of the interval $(-2\epsilon, 2\epsilon)$,
- ρ_ϵ displays exactly three fixed point: $-\epsilon$ and ϵ which are contracting and 0 which is expanding.

Let DA be the map of the two torus equal to A outside of V, and in the coordinate ϕ it has the form:

$$(x, y) \mapsto (\rho_\epsilon(x), y/\lambda).$$

We notice that 0 is an expanding the fixed point of DA. The complement of its repulsion basin is a hyperbolic attractor.

The DA attractor project to a basic set of a surface attractor, the *Plykin attractor*.

Given a hyperbolic compact set Λ, for every $z \in \Lambda$, the sets

$$W^s(z) = \{z' \in M : \lim_{n \to +\infty} d(f^n(z), f^n(z')) = 0\},$$

$$W^u(z) = \{z' \in M : \lim_{n \to -\infty} d(f^n(z), f^n(z')) = 0\}$$

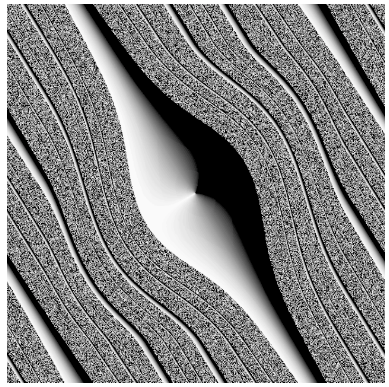


Figure 3: Derivated of Anosov (Credit Y. Coudene)

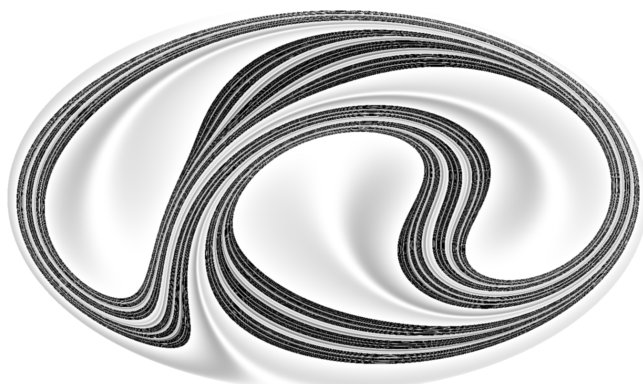


Figure 4: Plykin attractor (Credit S. Crovisier)

are called *the stable and unstable manifolds* of z. They are immersed manifolds tangent at z to respectively $E^s(z)$ and $E^u(z)$.

The ϵ*-local stable manifold* $W^s_\epsilon(z)$ of z is the connected component of z in the intersection of $W^s(z)$ with a ϵ-neighborhood of z. The ϵ*-local unstable manifold* $W^u_\epsilon(z)$ is defined likewise.

Proposition 1.6. *For $\epsilon > 0$ small enough, the subsets $W^s_\epsilon(z)$ and $W^u_\epsilon(z)$ are C^r-embedded manifolds which depend continuously on z and tangent at z to respectively $E^s(z)$ and $E^u(z)$.*

A nice proof of this proposition can be found in [Yoc95].

Definition 1.7. *A* basic set *is a compact, f-invariant, transitive, uniformly hyperbolic set Λ which is* locally maximal*: there exists a neighborhood N of Λ such that $\Lambda = \cap_{n\in\mathbb{Z}} f^n(N)$.*

Example 1.8 (Horseshoe). A *horseshoe* is a basic set which is a Cantor set. For instance take two disjoint sub-intervals $I_+ \sqcup I_- \subset [0,1]$, and let $g\colon I_+ \sqcup I_- \to [0,1]$ be a locally affine map which sends each of the intervals $I_\pm$ onto $[0,1]$. Let g_+ be its inverse branch with value in I_+ and let g_- be the other inverse branch. Let f be a diffeomorphism of the plane whose restriction to $I_\pm \times [0,1]$ is:

$$(x,y) \in (I_+ \sqcup I_-) \times [0,1] \to \begin{cases} (g(x), g_+(y)) & \text{if } x \in I_+ \\ (g(x), g_-(y)) & \text{if } x \in I_- \end{cases}$$

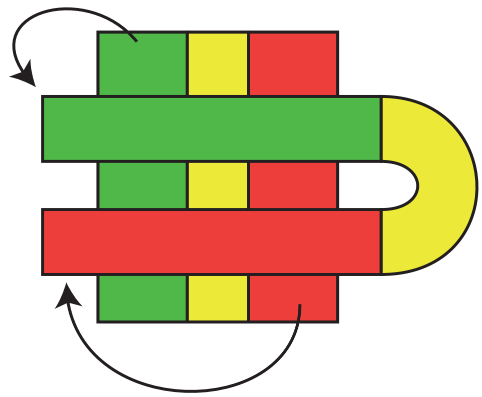


Figure 5: Smale's Horseshoe

Remark 1.9. Usually, one defines a basic piece as a hyperbolic set included in the closure of the set of its periodic points. Actually the three following assertion are equivalent for every uniformly hyperbolic, transitive, compact set K:

- K is locally maximal.
- K has a structure of local product : for $\epsilon > 0$ small enough, and any $x, y \in K$ close enough, the intersection point $W^u_\epsilon(x) \cap W^s_\epsilon(y)$ belongs to K.
- K is included in the closure of the set of periodic points in K: $K = cl(Per(f|K))$.

The equivalence of these conditions is proved in [Shu78].

Definition 1.10 (Axiom A)**.** *A diffeomorphism whose non-wandering set is a finite union of disjoint basic sets is called* axiom A*.*

Example 1.11 (Morse-Smale)*.* A Morse-Smale diffeomorphism is a diffeomorphism of a surface so that its non-wandering set consists of finitely many periodic hyperbolic points, and their stable and unstable manifolds are transverse.

1.2 Uniformly hyperbolic endomorphisms

A *C^r-endomorphism* of a manifold M is a differentiable map of class C^r of M, which is not necessarily injective, nor surjective, and that may possess points at which the differential is not onto (called critical points). The *critical set* is the subset of M of formed by the critical points.

A *local C^r-diffeomorphism* is a C^r-endomorphism without critical point.

A compact subset $\Lambda \subset M$ is *invariant* for an endomorphism f of M if $f^{-1}(\Lambda) = \Lambda$. A compact subset $\Lambda \subset M$ is *stable* for an endomorphism f of M if $f(\Lambda) = \Lambda$.

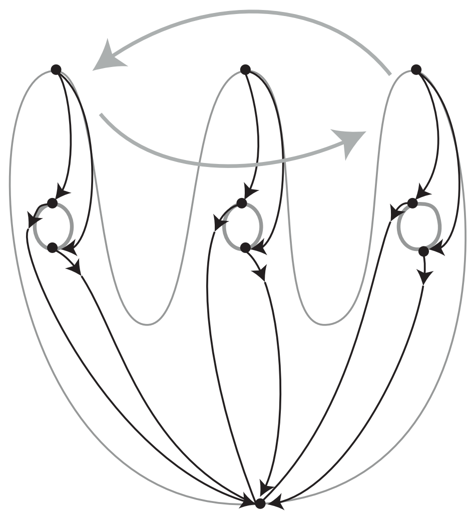


Figure 6: Morse-Smale

An invariant compact set is *hyperbolic* if there exists a subbundle $E^s \subset TM\Lambda$ which is left invariant and uniformly contracted by Df and so that the action of Df on TM/E^s is uniformly expanding.

Example 1.12 (Expanding map). Let $f \in End^1(M)$ and an invariant stable, compact subset K is *expanded* if there exists $n \geq 1$ s.t., for every $x \in K$, $D_x f^n$ is invertible and with contracting inverse. When $K = M$, f is said *expanding.*

Example 1.13 (Anosov endomorphism). If a hyperbolic set is equal to the whole manifold, then the endomorphism is called *Anosov*. For instance this is the case of the dynamics on the torus $\mathbb{R}^2/\mathbb{Z}^2$ induced by a linear maps in $M_2(\mathbb{Z})$ with eigenvalues of modulus not equal to 1. For instance, it the case of the following for every $n \geq 2$:

$$\begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix}$$

The stable manifold of z in a hyperbolic set Λ of an endomorphism is defined likewise:

$$W^s(z) = \{z' \in M : \lim_{n \to +\infty} d(f^n(z), f^n(z')) = 0\}.$$

The unstable manifold depends on the preimages. For every orbit $\underline{z} = (z_n)_{n \in \mathbb{Z}} \in \Lambda^{\mathbb{Z}}$, has an unstable manifold:

$$W^u(\underline{z}) = \{z' \in M : \exists (z'_n)_n \text{ orbit s.t. } \lim_{n \to -\infty} d(z_n, z'_n) = 0\}.$$

If f is a local diffeomorphism then W^s and W^u are immersed, but in general only W^s is injectively immersed.

Definition 1.14. *A hyperbolic set Λ is a* basic piece *if it is locally maximal.*

Example 1.15 (Blender). A blender of surface endomorphism is a basic set so that C^1-robustly its local unstable manifold cover an open subset of the surface.

For instance let I_- and I_+ be two disjoint segments of $[-1,1]$, and let Q be a map which sends affinely each of these segments onto $[-1,1]$. This is the case for instance of the following map:

$$(x,y) \in [-1,1]^2 \mapsto \begin{cases} (Q(x),(2y+1)/3) & x \in I_+ \\ (Q(x),(2y-1)/3) & x \in I_- \end{cases}$$

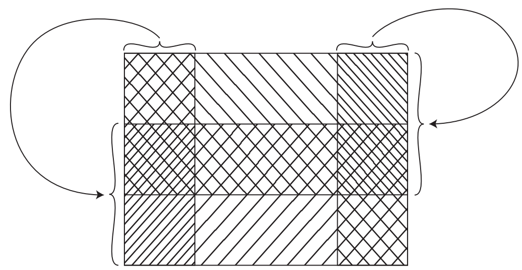


Figure 7: Blender of a surface local diffeomorphism

Definition 1.16. *An endomorphism satisfies* axiom A *if its non-wandering set is a finite union of basic pieces.*

1.3 Basic properties of uniformly hyperbolic dynamics

Hyperbolic attractors enjoy nice properties, which are proved in [Sma67] and the references therein.

SRB and physical measure Let $\alpha > 0$, and let Λ be an attracting basic set for a $C^{1+\alpha}$ -diffeomorphism f. Then there exists a unique invariant, ergodic probability μ supported on Λ such that its conditional measures, with respect to any measurable partition of Λ into plaques of unstable manifolds, are absolutely continuous with respect to the Lebesgue measure class (on unstable manifolds). Such a probability is called *SRB* (for Sinai-Ruelle-Bowen). It turns out that a SRB -measure is *physical*: the Lebesgue measure of its basin $B(\mu)$

$$(\mathcal{B}) \qquad B(\mu) = \{z \in M : \frac{1}{n}\sum_{i<n} \delta_{f^i(x)} \rightharpoonup \mu\},$$

is positive. Actually, up to a set of Lebesgue measure 0, $B(\mu)$ is equal to the topological basin of Λ, i.e the set of points attracted by Λ.

Coding A basic set Λ for a C^1-diffeomorphism f admits a (finite) Markov partition. This implies that its dynamics is semi-conjugated with a subshift of finite type. The semi-conjugacy is 1-1 on a generic set. Its lack of injectivity is itself coded by subshifts of finite type of smaller topological entropy. This enables to study efficiently all the invariant measures of Λ, the distribution of its periodic points, the existence and uniqueness of the maximal entropy measure, and if f is $C^{1+\alpha}$, the Gibbs measures which are related to the geometry of Λ.

Structural stability A basic set Λ for a C^1-diffeomorphism f is *persistent*: every C^1-perturbation f' of f leaves invariant a basic set Λ' which is homeomorphic to Λ, via a homeomorphism which conjugates the dynamics $f|\Lambda$ and $f'|\Lambda'$.

2 Introduction to the presented developments

Smale wished to describe the behavior of typical orbits of a typical dynamical system. For this end, he conjectured the density of axiom A in the space of C^r-diffeomorphisms f of any compact manifold M.

In higher dimensions, obstructions were soon discovered by Abraham-Smale [AS70] and then Shub [Shu71]. The latter became a paradigmatic example of the theory of partially hyperbolic dynamical systems (that we will not describe deeply). For such systems, there exists an invariant splitting $E^s \oplus E^c \oplus E^u = TM$ of the tangent space TM of the manifold M, so that E^s is more contracted than $E^c \oplus E^u$ and E^u is more expanded than $E^s \oplus E^c$.

The *mostly expanded* case occurs when E^c is asymptotically expanded in the following sense:

$$\exists m > 0 : \quad \limsup_{n\to\infty} \frac{1}{n} \sum \log \|D_{f^n(x)} f^{-1}|E^c\| < -m \quad \text{for Leb. } a.e.\ x \in M\ .$$

Then a Theorem of Alves-Bonatti-Viana [ABV00] states that there exist finitely many SRB measures whose basins cover Lebesgue a.e. M.

The *mostly contracting* case is when the maximal invariant set K of M is compact, partially hyperbolic and so that for every $z \in K$, with $W^{uu}_{\text{loc}}(z)$ the local, strong unstable manifold of z and Leb its Lebesgue measure, it holds:

$$0 < \text{Leb}\{x \in W^{uu}_{loc}(z) : \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^n|E^c(x)\| < 0\}\ .$$

Then a Theorem of Bonatti-Viana [BV00] states that there are finitely many SRB probability measures and the union of their basins covers Lebesgue almost every M.

One can wonder about the existence of a C^2-open set of partially hyperbolic diffeomorphisms for which there exists a measurable splitting $E^c = E^{cs} \oplus E^{cu}$ at Lebesgue a.e. point, so that E^{cs} is asymptotically contracted and E^{cu} asymptotically expanded. To avoid product-like examples, one should ask $Df|E^{cs}$ not to be dominated by $Df|E^{cu}$, i.e. there are vectors in E^{cu} which are more contracted than some in E^{cs}. In [BC14] with Carrasco, we gave one of the very first examples of an open set of C^2-conservative dynamics displaying such a property. It is obtained by coupling the Chirikov standard map with an Anosov diffeomorphism of the torus. This example will be described in section 11.

Also, numerical studies by Lorenz [Lor63] and Hénon [Hén76] explored dynamical systems with hyperbolic features that did not fit into the uniformly hyperbolic theory (see fig. 8). In order to include many examples such as the Hénon's attractor, the *non-uniform hyperbolic theory* is still under construction. A few examples of such systems exist [Jak81, Ree86, BC85, BC91, PY09] as related in section 7. We will present in section 8, our contribution to Yoccoz' strong regularity program [Yoc97] on Hénon-like endomorphisms [Bera]. The aim of this program is to find a combinatorial definition of non-uniformly hyperbolic transitive sets (based on developments of Yoccoz' puzzle pieces) and to show their abundances (i.e. their existence for a set of positive Lebesgue measure in some open sets of one-dimensional families of dynamics). Each of the above attractors displays a (real or complex) Hausdorff dimension close to 1. However, we will see in section 9 that it seems possible to adapt our proof for surface attractors of higher Hausdorff dimension (perhaps even the one initially conjectured by Hénon), by using the techniques of [BM16]. Furthermore, this combinatorial definition enables us to prove that Strongly Regular Hénon-like maps display ergodic properties similar to those of uniformly hyperbolic attractors [Berd] (see section 10): their attractors support a unique measure of maximal entropy which is equi-distributed on the periodic points. Moreover we answered a question of Carleson by showing a hyperbolic bound on the Lyapunov exponents which is uniform among every invariant measure.

Figure 8: The Hénon attractor cannot be hyperbolic since its attractor basin is a disk, and there is no continuous line field on a disk.

Another obstruction to Smale's dream was discovered by its student Newhouse. For $2 \leq r \leq \infty$, he showed the existence of an open set U of C^r-surface diffeomorphisms, so that a topologically generic $f \in U$ displays infinitely many sinks [New74, New79]. For such a dynamics f, the sinks accumulate on a hyperbolic set, each of them supporting a very different physical probability measure. Following Yoccoz, such a coexistence of infinitely many sinks is "a lower bound" on the complexity of such dynamics. Even presently, we do not understand a single example of such dynamics (for instance whether Lebesgue almost every point displays a Birkhoff sum which converges). We will recall this phenomenon in section 12.

From the 90's this phenomenon was conjectured to be unprobable/negligible in many conjectures [TLY86, Pal00, Pal05, PS96], some of them in low dimension. In section 13, we study the opposite direction. First we showed in [BDS16] that this phenomenon has codimension at most $1/2$ in the parameter space for surface diffeomorphisms (a result which does not contradict any of these conjectures). Then we showed that for surface endomorphisms or diffeomorphisms of higher dimension, this phenomenon occurs sufficiently to be important in the sens of Kolmogorov [Ber16b, Ber16a, Ber17a].

This leads us to wonder about how complex a (locally) typical dynamics can be. There are many ways to precise this question. We will investigate two directions:

(i) *The description of a system by its physical measures.* In section 14, we will define the concept of Emergence [Ber17a] for a dynamical system, which roughly speaking, quantifies the complexity to describe a dynamical system by means of physical measures. We will state a conjecture about it.

(ii) *The growth of the number of it periodic points.* In section 15, we relate a complement [Ber17b] of the works of Marteens-de Melo-Van Strien, Kaloshin, Turaev, Gonchenko-Shilnikov-Turaev, Asaoka-Shinohara-Turaev, which gives a full answer to questions of Smale in 1967, Bowen in 1978 and Arnold in 1989, about the growth of the number of periodic points. In particular we will show that the growth can be arbitrarily fast for any dimension ≥ 2 and any regularity $\infty \geq r \geq 2$. Furthermore, we showed in [Ber17b] that this occurs at every parameter of a generic family (a negative answer to a question of Arnold in 1992).

The two above results are given by a counterpart of the Bonatti-Diaz Blender for parameter families: the parablender. We will recall its definition in section 16.

Nevertheless, Uniform Hyperbolicity seems to provide a satisfactory way to describe the structurally stable dynamics. This observation goes back to the Fatou conjecture for quadratic maps of the Riemannian sphere in 1920 and the Smale conjecture for smooth diffeomorphisms in 1970. These conjectures have been deeply studied by many mathematicians and so they are difficult to tackle directly.

However at the interface of one-dimensional complex dynamics and differentiable dynamics, the field of two-dimensional complex dynamics grew up recently. It enables to study the structural stability problem thanks to ingredients of both fields. Also the mathematics developed in the 1970's for the structural stability in dynamics is very similar to the one developed for the structural stability in singularity theory. This led us to combine both in the study of the structurally stable endomorphisms. In next part II, we will review some works in these beautiful fields, and we will present our contributions at these interfaces [Ber12, BR13, BK13, BD17], as detailed in the sequel.

Part II
Structural stability

Structural stability is one of the most basic topics in dynamical systems and contains some of the hardest conjectures. We will review some aspects of these problems in the C^r-category, $1 \le r \le \infty$ and in the holomorphic category denoted by $\mathcal{H}$. Let $\mathcal{C}$ be a category in $\{C^r : 1 \le r \le \infty\} \cup \{\mathcal{H}\}$.

Definition 2.1 (Structural stability)**.** *A $\mathcal{C}$-map f is* structurally stable *if every $\mathcal{C}$-perturbation f' of the dynamics is conjugated: there exists a homeomorphism h of the manifold so that $h \circ f = f' \circ h$.*

A weaker notion of structural stability focuses on the non-wandering set Ω_f of the dynamics f.

Definition 2.2 (Ω-stability)**.** *A $\mathcal{C}$-map f is* Ω-stable *if for every $\mathcal{C}$-perturbation f' of f, the dynamics of the restriction of f to Ω_f is conjugated (via a homeomorphism) to the restriction of f' to its non-wandering set $\Omega_{f'}$.*

We recall that an axiom A diffeomorphism f satisfies the *strong transversality condition* if its stable and unstable manifolds intersect transversally. Here is an outstanding conjecture:

Conjecture 2.3 (Palis-Smale structural stability conjecture, 1970 [PS70])**.** *A $\mathcal{C}$-diffeomorphism is structurally stable if and only if it satisfies axiom A and the strong transversality condition.*

For complex rational maps of the sphere, this conjecture takes the form:

Conjecture 2.4 (Fatou Conjecture, 1920)**.** *Structurally stable quadratic map are those which satisfy axiom A and whose critical points are not periodic.*

Actually the initial Fatou conjecture stated the density of axiom A quadratic map. However, in section 3, we will recall the works of Mañé-Sad-Sullivan [MSS83] and Lyubich [Lyu84] showing the existence of an open and dense set of structurally stable rational maps. This implies the equivalence between the original Fatou Conjecture and the above conjecture. Among real quadratic maps, this conjecture[2] has been proved by Graczyk-Swiantek [GŚ97] and Lyubich [Lyu97].

The description of Ω-stable maps involves the no-cycle condition. We recall that any axiom A diffeomorphisms displays a non-wandering set Ω equal to a finite union of basic pieces $\Omega = \sqcup_i \Omega_i$. The family $(\Omega_i)_i$ is called the *spectral decomposition.*

Definition 2.5 (No-cycle condition)**.** *An axiom A diffeomorphism satisfies the* no-cycle condition *if given $\Omega_1, \Omega_2, \dots, \Omega_n$ in the spectral decomposition, if $W^u(\Omega_i)$ intersects $W^s(\Omega_{i+1})$ for every $i < n$ and if $W^u(\Omega_n)$ intersects $W^s(\Omega_1)$, then $\Omega_1 = \Omega_2 = \cdots = \Omega_n$.*

Conjecture 2.6 (Smale Ω-Stability Conjecture, [Sma70])**.** *A $\mathcal{C}$-diffeomorphisms is structurally stable if and only if it satisfies axiom A and the no-cycle condition.*

[2] The Fatou conjecture is implied by the Mandelbrot Locally connected (MLC) conjecture that we will not have the time to recall in this manuscript.

If the above conjectures turn out to be true then they would display a satisfactory description of structurally stable dynamics (for the axiom A diffeomorphisms are very well understood).

In section 3 we will state several theorems and conjectures suggesting the hyperbolicity of structurally stable dynamics. In particular we will recall the seminal work of Mañé [Mañ88] showing this direction in the C^1-category. For holomorphic dynamical systems, we will present the work of Dujardin-Lyubich [DL] and my collaboration with Dujardin [BD17] generalizing some aspects of Mañé-Sad-Sullivan and Lyubich theorems [MSS83, Lyu84] for polynomial automorphisms of $\mathbb{C}^2$.

In section 4, we will present several results in the directions "hyperbolicty $\Rightarrow$ stability". In §4.1, we will recall classical results, including the structural stability theorems of Anosov [Ano67], Moser[Mos69] and Shub [Shu69], and the proof of this direction of the Ω-stability conjecture by Smale[Sma70] and Przytycki [Prz77]. Then in §4.2, we will sketch the proof of this direction of the structural stability theorem by Robbin [Rob71] and Robinson [Rob76] ; and we will relate a few works leading to a generalization of the Przytycki conjecture [Prz77], a description of the structurally stable covering. Finally in §4.3, we will recall our conjecture with Rovella [BR13] stating a description of the endomorphisms (with possibly a non-empty critical set) whose inverse limit is structurally stable, and we will state our theorem with Kocsard [BK13] showing one direction of this conjecture. One surprising fact is that this description does not involve singularity theory, even if they are inverse stable dynamics with non-empty critical set.

In section 5, we will recall several results from singularity theory and we will emphasize on their similarities with those of structural stability.

In section 6, we will present the work [Ber12] which states sufficient conditions for a smooth map with non-empty critical set to be structurally stable. The statement involves developments of Mather's theorem on Singularity Theory of composed mappings.

3 Properties of structurally stable dynamics

Let us define the probabilistic structural stability, which is implied by the Ω-stability. The definition involves the regular subset $\mathcal{R}_f$ of Ω_f. This subset is formed by the points $p \in \Omega_f$ so that for every $a \in \{s, u\}$, there exist $\epsilon > 0$ and a sequence of periodic points $(p_n)_n$ satisfying:

- $(p_n)_n$ converges to p,
- $(W^a_\epsilon(p_n))_n$ is relatively compact in the $\mathcal{C}$-topology.

We showed in [BD17] thanks to Katok's closing Lemma, that the set $\mathcal{R}_f$ has full measure for every ergodic, hyperbolic probability measure.

Definition 3.1. *A $\mathcal{C}$-map f is probabilistically structurally stable if for every $\mathcal{C}$-perturbation f' of f, the restriction of f to $\mathcal{R}_f$ is conjugated to the restriction of f' to its regular set $\mathcal{R}_{f'}$.*

It is rather easy to see that probabilistic structural stability implies weak stability:

Definition 3.2. *A map f is $\mathcal{C}$-weakly stable if every $\mathcal{C}$-perturbation f' of f displays only hyperbolic periodic points.*

To sum it up, the above definitions are related as follows:

$$\Omega\text{-Stability} \Rightarrow \text{Probabilistic Stability} \Rightarrow \text{Weak Stability}$$

The Lambda Lemma Conjecture. This conjecture states that weak stability implies Ω-stability. For the category of rational functions of the Riemannian sphere, this Lemma has been shown independently by Mañé-Sad-Sullivan [MSS83] and Lyubich [Lyu84].

As the space of rational functions is finite dimensional, a neighborhood of a rational function f can be written as an analytic family $(f_\lambda)_{\lambda\in\mathbb{D}^n}$, with $\mathbb{D}$ the complex disk and $f_0 = f$. If $(f_\lambda)_\lambda$ consists of weakly stable maps, then every periodic point p_0 of f_0 persists to as unique periodic point p_λ for f_λ. Moreover the map $\lambda \mapsto p_\lambda$ is holomorphic. The Lambda lemma asks the following question. Given p_0 in closure J_0^* of the set of periodic points of f_0, for every sequence $(p_0^n)_n$ of periodic points converging to p_0, does the family $(\lambda \mapsto p_\lambda^n)_n$ converges? If yes, the *holomorphic motion is said well defined at* p_0.

Lemma 3.3 (Lambda-Lemma, Mañé-Sad-Sullivan [MSS83] and Lyubich [Lyu84])**.** *If* $(f_\lambda)_\lambda$ *is weakly stable, then the holomorphic motion is well defined at every point* $p_0 \in J_0^*$.

We recall that every rational function J^* is equal to the non-wandering set and that any attracting periodic point displays a critical point in its basin. Furthermore if a rational function is not weakly stable, it displays a new attracting periodic point after a perturbation of the rational function. Hence the new critical point belongs to the basin of this attracting periodic point. As the number of critical points is finite, after a finite number of perturbations the rational function turns out to be weakly stable. This shows that weak stability is open and dense among the rational functions. By the Lambda Lemma 3.3, this implies:

Theorem 3.4 (Mañé-Sad-Sulivan [MSS83], Lyubich [Lyu84])**.** *There is an open and dense subset of rational functions of degree* $d \geq 2$ *which are* Ω*-stable.*

This result enables them to deduce a stronger result: the density of the set of structurally stable rational functions.

We recall that a polynomial automorphism of $\mathbb{C}^2$ is a polynomial mapping of $\mathbb{C}^2$ which is invertible and whose inverse is polynomial. Among polynomial automorphisms of $\mathbb{C}^2$, Dujardin and Lyubich [DL] showed that the holomorphic motion is well defined on any uniformly hyperbolic compact set. We improved this result:

Lemma 3.5 (Berger-Dujardin [BD17])**.** *If* $(f_\lambda)_\lambda$ *is a weakly stable family of polynomial automorphisms of* $\mathbb{C}^2$*, the holomorphic motion is uniquely defined on the regular set* $\mathcal{R}_0$ *of* f_0.

An immediate consequence of this result is that weak stability implies probabilistic stability for the category of polynomial automorphisms of $\mathbb{C}^2$.

Unfortunately, there is no hope to get the density of Ω-stable polynomial automorphisms of $\mathbb{C}^2$ because in a non-empty open set [Buz97] of the parameter space is formed by automorphisms displaying a wild horseshoe. However, we will see below that if none pertubations of the dynamics display a homoclinic tangency, then the dynamics is weakly stable (under a mild hypothesis of dissipativeness).

As a corollary of the techniques, we showed that one connected component of the set of weakly stable polynomial automorphisms is formed by those which satisfy axiom A.

The Mañé Conjecture In 1982, Mañe conjectured in [Mañ82] that every C^r-weakly stable diffeomorphism satisfies axiom A for every $1 \le r \le \infty$. He proved this conjecture for $r = 1$ in [Mañ88].

$$\text{Weak Stability} \overset{\text{Mañé Conj.}}{\Longrightarrow} \text{axiom A.}$$

From this Mañé easily deduced that C^1-structurally stable diffeomorphisms satisfy axiom A and the strong transversality condition. This work enabled also Palis to prove the same direction for the C^1-Ω-stability conjecture [Pal88].

After the next section, it will be clear for the reader that the Mañé Conjecture implies the Lambda Lemma Conjecture in any category $\mathcal{C}$.

A Palis Conjecture We recall that a hyperbolic periodic point displays a *homocline tangency* if its stable manifold $W^s(p)$ is tangent to its unstable manifold. Two saddle periodic points p, q display a *heterocline tangency* if $W^s(p)$ intersects transversally $W^u(q)$ whereas $W^s(q)$ is tangent to $W^u(p)$ (or vice versa). It is not hard to show that if a C^r-map is weakly stable then it cannot display a homoclinic nor a heteroclinic tangency, for every $1 \le r \le \infty$. The same is true for one dimensional complex maps. For polynomial automorphisms of $\mathbb{C}^2$, it is a theorem [**?**].

Let us recall also a famous Conjecture of Palis [Pal00] which states that if a dynamics which cannot be perturbed to one which displays a homoclinic nor a heteroclinic tangency, then it satisfies axiom A:

$$\text{Weak Stability} \Rightarrow \text{Far from tangencies} \overset{\text{Palis Conj.}}{\Longrightarrow} \text{axiom A.}$$

In the category of C^1-surface diffeomorphisms, this conjecture has been proved by Pujals-Sambarino [PS00]. In the category of C^1-diffeomorphisms of higher dimensional manifolds, a weaker version has been proved by Crovisier-Pujals [CP15].

We notice that the Palis conjecture implies the Mañé conjecture and so the Lambda lemma conjecture.

A description of structurally stable dynamics as those far from tangencies? This question is widely open in the C^r-category for $r > 1$ (for C^1-surface diffeomorphisms it is a consequence of Mañé's theorem). It is also correct for the category of rational functions. This might be correct for polynomial automorphisms of $\mathbb{C}^2$. Indeed, most of the work of Dujardin-Lyubich was dedicated to prove the following result:

Theorem 3.6 (Dujardin-Lyubich [DL]). *Given a polynomial automorphism f of (dynamical) degree $d \ge 2$ and so that $|det\, Df_0| \cdot d^2 < 1$, either f is weakly stable, either a perturbation of f' admits a homoclinic tangency.*

From Lambda Lemma 3.5 we deduced:

Corollary 3.7 (Berger-Dujardin [DL]). *Given a polynomial automorphism f of (dynamical) degree $d \geq 2$ and so that $|det\, Df_0| \cdot d^2 < 1$, either f is probabilistically stable, either a perturbation of f' displays a homoclinic tangency.*

Let us stress that this direction might be interesting since numerically we can see some local stable and unstable manifolds and observe if they display tangencies. conjecture d Bonatti?

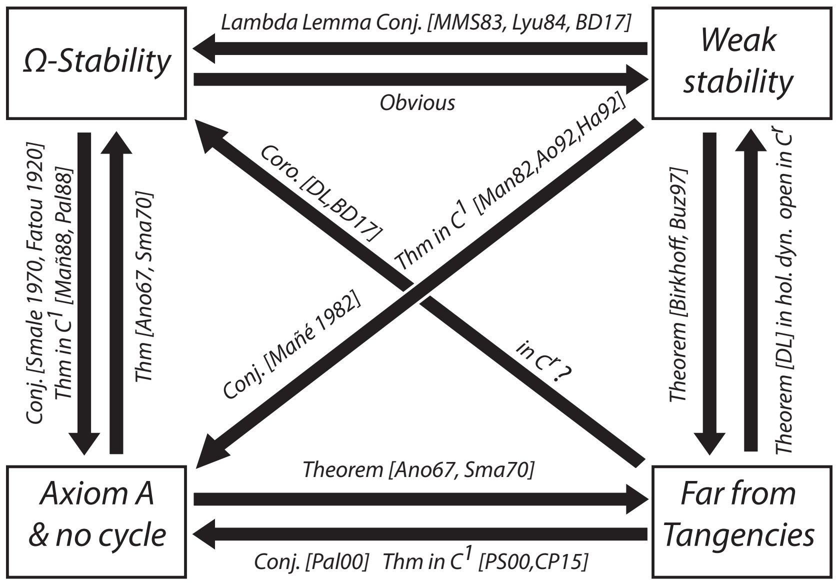


Figure 9: Summary of some Theorems and Conjectures on Structural Stability

4 Hyperbolicity implies structural stability

In the following subsection, we recall the proof ideas of several basic theorems showing the structural stability of subsets from hyperbolic hypotheses.

4.1 Ω-stability of maps satisfying axiom A and the no-cycle condition

First let us recall a generalization of the notion of structural stability for invariant subsets.

Definition 4.1 (Structurally stable subset). *A compact set Λ left invariant by a differentiable map f of a manifold M is* structurally stable *if for every C^r-perturbation f' of f, there exists a continuous injection $i\colon \Lambda \to M$ so that $f' \circ i = i \circ f$.*

We notice that M is structurally stable if and only if f is structurally stable.

Theorem 4.2 (Anosov [Ano67], proof by Moser [Mos69]). *A uniformly hyperbolic compact set Λ for a C^1-diffeomorphisms is structurally stable.*

Proof. We want to solve the following equation:

$$(\star) \qquad f' \circ h \circ f^{-1} = h \ .$$

for f' C^1-close to f and h C^0-close to the canonical inclusion $i\colon \Lambda \hookrightarrow M$. We shall use the implicit function theorem with the map:

$$\Phi\colon (h, f') \in C^0(\Lambda, M) \times C^1(M, M) \to f' \circ h \circ f^{-1} \in C^0(\Lambda, M) \ .$$

We notice that Φ is a C^1-differentiable map of Banachic manifolds. Moreover it satisfies $\Phi(i, f) = i$. Hence to apply the implicit function theorem it suffices to prove that $id - \partial_h \Phi(i, f)$ is an isomorphism.

Note that the tangent space of the Banachic manifold $C^0(\Lambda, M)$ at the canonical inclusion i is the following Banach space:

$$\Gamma := \{\gamma \in C^0(\Lambda, TM) : \forall x \in \Lambda \quad \gamma(x) \in T_x M\}.$$

The partial derivative of $\partial_h \Phi$ at (i, f) is:

$$\Psi := \partial_h \Phi(i, f)\colon \sigma \in \Gamma \mapsto Df \circ \sigma \circ f^{-1} \in \Gamma \ .$$

To compute the inverse of $id - \Psi$, we split Γ into two Ψ-invariant subspaces $\Gamma = \Gamma^u \oplus \Gamma^s$, with:

$$\Gamma^u := \{\gamma \in C^0(\Lambda, TM) : \forall x \in \Lambda \quad \gamma(x) \in E^u_x\} \quad \text{and} \quad \Gamma^s := \{\gamma \in C^0(\Lambda, TM) : \forall x \in \Lambda \quad \gamma(x) \in E^s_x\}.$$

As the norm of $\Psi|\Gamma^s$ is less than 1, the map $(id - \Psi)|\Gamma^s$ is invertible with inverse equal to

$$\sum_{n \geq 0} (\Psi|\Gamma^s)^n \ .$$

As $\Psi|\Gamma^u$ is invertible with contracting inverse, the map $(id - \Psi)|\Gamma^u$ is invertible with inverse:

$$-(\Psi|\Gamma^u) \circ (id - (\Psi|\Gamma^u)^{-1}) = -(\sum_{n \geq 1} (\Psi|\Gamma^u)^{-n}) \ .$$

Hence by the implicit function theorem, for every f' C^1-close to f, there exists a continuous map h C^0-close to i which semi-conjugates the dynamics:

$$f' \circ h = h \circ f \ .$$

As i is injective and close to h, if $h(x) = h(y)$ then x and y are close. Also by semi-conjugacy, $h \circ f^n(x) = h \circ f^n(y)$ for every $n \in \mathbb{Z}$. Hence $f^n(x)$ is close to $f^n(y)$ for every n. By expansiveness (see below), we conclude that $x = y$ and so that h is injective. □

Lemma 4.3 (Expansiveness)**.** *Every hyperbolic compact set Λ for a diffeomorphism is* expansive*: there exists $\epsilon > 0$ so that if two orbits $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ are uniformly ϵ-close, then $x_0 = y_0$.*

Proof. First we notice that for ϵ small enough, given two such orbits, $W^s_{2\epsilon}(y_n)$ intersects $W^u_{2\epsilon}(y_n)$ at a unique point z_n. We observe that $(z_n)_n$ is an orbit. As f is expanding along $W^u_{2\epsilon}(y_n)$ for every n and since $z_n \in W^u_{2\epsilon}(y_n)$, it comes that $z_n = y_n$ for every $n \geq 0$. Using the same argument for f^{-1}, it comes that $z_n = x_n$ for every $n \leq 0$ and so $x_0 = y_0$. $\square$

The image $\Lambda(f') := h(\Lambda)$ is called the *hyperbolic continuation* of Λ. Since the density of periodic points is preserved by conjugacy, it comes:

Corollary 4.4. *If Λ is a basic piece, then its hyperbolic continuation is also a basic piece.*

A similar result has been proved by Shub during his thesis:

Theorem 4.5 (Shub [Shu69]). *An expanding compact set Λ for an endomorphisms f is C^1-structurally stable.*

Proof. First let us notice that f is a local diffeomorphism at a neighborhood of the compact set Λ. Hence there exists $\epsilon > 0$ so that for every f' C^1-close to f, for every $x \in \Lambda$, the restriction $f'|B(x,\epsilon)$ is invertible. This enables us to look for a semi-conjugacy thanks to the map:

$$\Phi\colon (h, f') \in C^0(\Lambda, M) \times C^1(M, M) \to (f'|B(x,\epsilon))^{-1} \circ h \circ f \in C^0(\Lambda, M)$$

The latter is well defined and of class C^1 on the ϵ-neighborhood of the pair of the canonical inclusion $i\colon \Lambda \hookrightarrow M$ with f. Furthermore, it holds $\Phi(i, f) = i$ and the following partial derivative is contracting, with Γ the tangent space of $C^0(\Lambda, M)$ at i.

$$\partial_h \Phi(i, f)\colon \sigma \in \Gamma \to Df^{-1} \circ \sigma \circ f \in \Gamma\ .$$

Thus, by the implicit function Theorem, for f' C^1-close to f, there exists a unique solution with $h \in C^0(\Lambda, M)$ close to i for the semi-conjugacy equation:

$$\Phi(h, f') = h \Leftrightarrow h \circ f = f' \circ h\ .$$

As h is close to the canonical inclusion, if $h(x) = h(x')$ then x and x' must be close. Also by semi-conjugacy, it holds $h(f^n(x)) = h(f^n(x'))$ for every $n \geq 0$. Thus the orbits $(f^n(x))_{n\geq 0}$ and $(f^n(x'))_{n\geq 0}$ are uniformly close. By forward expansiveness (see below), it comes that $x = x'$. $\square$

One easily shows by a similar argument to Lemma 4.3:

Lemma 4.6 (Forward expansiveness). *Every expanding compact set Λ is* forward expansive*: there exists $\epsilon > 0$ so that if two orbits $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are uniformly ϵ-close, then $x_0 = y_0$.*

The two latter theorems enable us to explain the proofs of Smale and Przytycki on Ω-stability. We recall that the local stable and unstable manifolds of the points of a hyperbolic set Λ for an endomorphism f (which might display a non-empty critical set) are uniquely defined, provided that:

- Either $f|\Lambda$ is bijective,

- Either Λ is injective.

On the other hand, the local stable manifold are always uniquely defined. Hence under these assumption, by looking at their images or preimages, the following is uniquely defined for $\epsilon > 0$ small enough:

$$W^s_\epsilon(\Lambda) = \cup_{x\in\Lambda} W^s_\epsilon(x) \quad W^u_\epsilon(\Lambda) = \cup_{x\in\Lambda} W^u_\epsilon(x) \quad W^s(\Lambda) = \cup_{n\geq 0} f^{-n}(W^s_\epsilon(\Lambda)) \ .$$

The following generalizes Smale's definion of axiom A diffeomorphisms:

Definition 4.7 (Axiom A in the sens of Przytycki)**.** *A C^1-endomorphism satisfies* axiom A-Prz*, if its non-wandering set Ω is equal to the closure of the set of periodic points (or equivalently locally maximal), and if it is the disjoint union of an expanding compact set with a bijective, hyperbolic compact set.*

For such maps we can generalize the notion of *spectral decomposition.* Indeed by local maximality and compactness, the non-wandering set Ω of such maps is the finite union of (maximal) transitive subsets Ω_i called *basic pieces*:

$$\Omega = \sqcup_i \Omega_i \ .$$

The family $(\Omega_i)_i$ is called the *spectral decomposition* of the axiom A-Prz endomorphism. Let us generalize the no-cycle condition for such endomorphisms.

Definition 4.8 (No-cycle condition)**.** *An axiom A-Prz, C^1-endomorphism satisfies the* no-cycle condition *if given $\Omega_1, \Omega_2, \dots, \Omega_n$ in the spectral decomposition, if $W^u_\epsilon(\Omega_i)$ intersects $W^s(\Omega_{i+1})$ for every $i < n$ and if $W^u_\epsilon(\Omega_n)$ intersects $W^s(\Omega_1)$, then $\Omega_1 = \Omega_2 = \cdots = \Omega_n$.*

F. Przytycki generalized Smale's Theorem on the Ω-stability of axiom A diffeomorphisms which satisfy the no-cycle condition as follows:

Theorem 4.9 ([Sma70], [Prz77])**.** *If a C^1-endomorphism satifies axiom A-Prz and the no-cycle condition, then it is $C^1 - \Omega$-stable.*

Sketch of proof of the Smale's Ω-stability Theorem. First let us recall that by Anosov Theorem, the non-wandering set Ω is structurally stable, and its hyperbolic continuation is still locally maximal (for a neighborhood uniformly large among an open set of perturbations of the dynamics).

Then the no-cycle condition is useful to construct a filtration $(M_i)_i$:

Proposition 4.10. *If an axiom A, C^1-diffeomorphism f satisfies the no-cycle condition, then there exists a chain of open subsets:*

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_N = M$$

so that $f(M_i) \Subset M_i$ and $\Omega_i \Subset M_i \setminus M_{i-1}$ for every $i \geq 1$.

The proof of this proposition involve Conway Theory and can be find in [Shu78, Thm 2.3 p. 9].

By using this filtration and the (uniform) local maximality of the hyperbolic continuation of the non-wandering set, one easily deduces the Ω-stability. $\square$

4.2 Structural stability of dynamics satisfying axiom A and the strong transversality condition

Structural stability of diffeomorphisms We recall that an axiom A diffeomorphism satisfies the strong transversality condition if for any non-wandering points x and y, the stable manifold of x is transverse to the unstable manifold of y.

Remark 4.11. By using the inclination lemma, one easily shows that the strong transversality condition implies the no-cycle condition.

The following theorem generalizes Anosov Theorems 4.2:

Theorem 4.12 (Robbin [Rob71], Robinson [Rob76])**.** *For every $r \geq 1$, the diffeomorphisms which satisfy axiom A and the strong transversality condition are C^r-structurally stable.*

Let us recall that the Mañe theorem [Mañ88] implies that a C^1-structurally stable diffeomorphism satisfies also axiom A and the strong transversality condition, and so both solve the conjecture of C^1-structural stability.

We will state Conjecture 4.15 generalizing this theorem for local diffeomorphisms. Hopefully the following will help the reader to tackle it.

Sketch of proof of Theorem 4.12. Again we want to solve the following semi-conjugacy equation:

$$(\star) \qquad f' \circ h \circ f^{-1} = h$$

for f' C^1-close to f and h C^0-close to the identity of M.

For $f' = f$ and $h = id$, Equality $(\star)$ is valid. The set of perturbations of the identity is isomorphic to $\Gamma = \{\gamma \in C^0(M, TM) : \forall x \in M \quad \gamma(x) \in T_xM\}$ by using the exponential map (associated to a Riemannian metric on M). Let $\tilde{f} := u \in T_xM \mapsto \exp^{-1}_{f(x)} \circ f \circ \exp_x(u)$.

Then Equation $(\star)$ is equivalent to:

$$(\star\star) \qquad \tilde{f}' \circ \sigma \circ f^{-1} = \sigma, \quad \text{for } \sigma \in \Gamma \quad C^0\text{-small.}$$

As the map $\Phi\colon (\sigma, f') \in \Gamma \times C^1(M, M) \to \Phi_{f'}(\sigma) = \sigma - \tilde{f}' \circ \sigma \circ f^{-1} \in \Gamma$ is of class C^1, and vanishes at $(0, f)$, we shall show that $\partial_h \Phi$ is left-invertible.

Let

$$\Psi := \partial_h \Phi(0, f)\colon \sigma \in \Gamma \mapsto \sigma - Df \circ \sigma \circ f^{-1} \in \Gamma\ .$$

The following is shown in [Rob71]:

Proposition 4.13. *For every i, there exists a neighborhood N_i of Ω_i and continuous extension E^s_i and E^u_i of respectively $E^s|\Omega_i$ and $E^u|\Omega_i$ to N_i, so that:*

- *There exists a filtration $(M_i)_i$ adapted to $(\Omega_i)_i$ so that $N_i = M_i \setminus M_{i-1}$. The subsets $(N_i)_i$ form an open covering of M,*
- *if $x \in N_i \cap f^{-1}(N_j)$, with $j \leq i$, then $Df(E^s_i(x)) \subset E^s_j(f(x))$, and $Df(E^u_i(x)) \supset E^u_j(f(x))$.*

Let $(\gamma_i)_i$ be a partition of the unity adapted to $(N_i)_i$.

For every i let p_i^s and p_i^u be the projections onto respectively E_i^s and E_i^u parallely to E_i^u and E_i^s.

Given $x \in M$ and $v \in T_xM$, we put $v_i^s := \gamma_i \cdot p_i^s(v)$ and $v_i^u := \gamma_i \cdot p_i^u(v)$. We observe that $v = \sum_i v_i^s + v_i^u$. Thus $Df(v) = \sum_i Df(v_i^s) + Df(v_i^u)$. As $(Df^n(v_i^s))_{n\geq 0}$ and $(Df^{-n}(v_i^u))_{n\geq 1}$ converge exponentially fast to 0, we consider:

$$J\colon \sigma \in \Gamma \mapsto \sum_i \sum_{n\geq 0} Df^n(\sigma_i^s \circ f^{-n}(x)) - \sum_{n\geq 1} Df^{-n}(\sigma_i^u \circ f^n(x))\ .$$

We notice that J is a left inverse of Ψ :

$$J \circ \Psi = id$$

.

The following equations are equivalent:

$$\Phi_{f'}(\sigma) = 0 \Leftrightarrow (\Phi_{f'} - \Psi)(\sigma) + \Psi(\sigma) = 0,$$

$$\Leftrightarrow J \circ (\Phi_{f'} - \Psi)(\sigma) + J \circ \Psi(\sigma) = J(0).$$

Now observe that $J(0) = 0$ and $J \circ \Psi(\sigma) = \sigma$. Hence $(\star\star)$ is equivalent to

$$J \circ (\Psi - \Phi_{f'})(\sigma) = \sigma.$$

It is easy to see that whenever f' is C^1-close to f, the map $\Phi_{f'}$ is C^1-close to Ψ at a neighborhood of the 0-section. Hence the map $J \circ (\Psi - \Phi_{f'})$ is contracting and sends a closed ball about the zero section into itself. The contracting mapping theorem implies the existence of a fixed point σ. Hence $(\star)$ displays a solution $h = \exp \circ \sigma$ close to the identity in the space of continuous maps.

It remains to show that the semi-conjugacy h is bijective. Contrarily to Anosov maps, in general axiom A diffeomorphisms are not expansive and the semi-conjugacy is not uniquely defined. Hence Robbin brought a new technique to construct a map h which is bijective. He defined the following metric:

$$d_f(x,y) = \sup_{n\in\mathbb{Z}} d(f^n(x), f^n(y))\ ,$$

where d is the Riemannian metric of the manifold M.

Let us just notice that if the semi-conjugacy $h = \exp \circ \sigma$ satisfies that σ is C^0-small and d_f-Lipschitz with a small constant η, then h is injective.

Indeed if $h(x) = h(y)$, then by $(\star)$, $h(f^n(x)) = h(f^n(y))$ for every n. Since h is close to the identity, the orbits $(f^n(x))_n$ and $(f^n(y))_n$ are uniformely close, and so that $d_f(x,y)$ is small. As σ is η-Lipschitz, it comes:

$$0 = d(h(x), h(y)) \geq d(x,y) - \eta d_f(x,y)$$

The same holds at any n^{th}-iterate:

$$0 = d(h(f^n(x)), h(f^n(y))) \geq d(f^n(x), f^n(y)) - \eta d_f(f^n(x), f^n(y)) = d(f^n(x), f^n(y)) - \eta d_f(x,y)\ .$$

Let n be such that $d(f^n(x), f^n(y)) \geq d_f(x, y)/2$. Then

$$0 = d(h(f^n(x)), h(f^n(y))) \geq (1 - 2\eta) d(f^n(x), f^n(y)) \ .$$

Thus $f^n(x) = f^n(y)$ and so $x = y$.

To obtain the section σ d_f-Lipschitz, Robbin assumed the diffeomorphism f of class C^2. Then in Proposition 4.13, he constructs the section $(E_i^s)_i$ and $(E_i^u)_i$ d_f-Lipschitz, so that the map J preserves the d_f-Lipschitz sections. On the other hand the map $\Psi - \Psi_{f'}$ diminishes the d_f-Lipschitz constant for f' C^1-close to f. Therefore the map $J \circ (\Psi - \Psi_{f'})$ preserves the space of continuous sections with small d_f-Lipschitz constant, and so its fixed point enjoys a small d_f-Lipschitz constant.

The C^1-case was handled by Robinson. His trick was to smooth the map Df to a C^1-map $\tilde{D}f$, and to replace Df by $\tilde{D}f$ in the definition of Ψ to define $\tilde{\Psi}$. Then he defined likewise $\tilde{D}f$-pseudo invariant sections $(\tilde{E}_i^s)_i$ which are d_f-Lipschitz. By replacing $(E_i^s)_i$ by $(\tilde{E}_i^s)_i$ in the definition of J, he defined a left inverse $\tilde{J}$ of $\tilde{\Psi}$. Then he showed likewise that the map $\tilde{J} \circ (\tilde{\Psi} - \Psi_{f'})$ admits a C^0-small, d_f-Lipschitz fixed point, which is a solution of $(\star\star)$. $\square$

Structural stability of covering. We recall that every local diffeomorphism of a compact (connected) manifold is a covering.

F. Przytycki [Prz77] introduced an example of surface covering suggesting the following *strong transversality condition.*

Definition 4.14. *A covering map f satisfies* axiom A *and the strong transversality condition if:*

(i) The non-wandering set is locally maximal.

(ii) The non-wandering set Ω is the union of a hyperbolic set on which f acts bijectively with a repulsive set.

(iii) $\forall x \in \Omega$, $\underline{y}^1, \dots, \underline{y}^k) \in \overleftarrow{\Omega}$, *the following multi-transversality condition holds:*

$$W^s(x) \pitchfork W^u(\underline{y}^1) \pitchfork \cdots \pitchfork W^u(\underline{y}^k) \ .$$

We recall that a finite family of submanifolds $(N_i)_i$ is multi-transverse if N_1 and N_2 are transverse, N_3 is transverse to $N_1 \cap N_2$, ..., and for every $i \geq 3$, N_i is transverse to $N_1 \cap N_2 \cap \cdots \cap N_{i-1}$. We notice that (iii) implies (ii).

Here is a generalization of a conjecture of Przytycki [Prz77]:

Conjecture 4.15. *The C^1-struturally stable coverings are those which satisfy axiom A and the strong transversality condition.*

The fact that structurally stable coverings are axiom A has been proved by Aoki-Moriyasu-Sumi [AMS01], and the strong transversality condition has been proved by Iglesias-Portela-Rovella. The other direction is still open in the general case.

This conjecture has been proved in two special cases. The first one solves the initial Przytycki conjecture for surface coverings:

Theorem 4.16 (Iglesias-Portela-Rovella [IPR12])**.** *If a covering map of a surface satisfies axiom A and the strong transversality condition then it is C^1-structurally stable.*

The other case is for attractor-repellor covering.

Theorem 4.17 (Iglesias-Portela-Rovella [IPR10])**.** *Let M be a compact manifold. If f is a C^1-covering map satisfying axiom A, and so that its basic pieces are either bijective attractors or expanding sets, then f is C^1-structurally stable.*

The strong transversality condition for these maps is certainly satisfied since, the unstable manifolds are either included in the attractor or form open subset of the manifold. They gave the following example:

$$f\colon (z,z') \in \mathbb{S}^1 \times \hat{\mathbb{C}} \mapsto (z^2, z/2 + z'/3),$$

where the non-wandering set consists of an expanding circle and of the Smale solenoid.

In [BK13], we constructed d_f-Lipschitz plane fields for endomorphisms which satisfies axiom A and the strong transversality condition. This might be useful to prove that under the hypothesis of Conjecture 4.15, the following map has a left inverse:

$$\sigma \in \Gamma^0(TM) \mapsto \sigma - Df^{-1} \circ \sigma \circ f \in \Gamma^0(TM)\ .$$

4.3 Structural stability of the inverse limit

Given an endomorphism f of a compact manifold M, the inverse limit $\overleftarrow{M}_f$ of f is the space of orbits :

$$\overleftarrow{M}_f := \left\{\underline{x} = (x_n)_{n\in\mathbb{Z}} \colon x_{n+1} = f(x_n)\right\}\ .$$

It is a closed subset of $M^{\mathbb{Z}}$, which is compact endowed with the product metric:

$$d(\underline{x}, \underline{x}') = \sum_{n\in\mathbb{Z}} 2^{-|n|} d(x_n, x'_n)\ .$$

We notice that the inverse limit is homemorphic to M when f is a homeomorphism of M.

We notice also that the shift dynamics $\overleftarrow{f}$ acts canonically on $\overleftarrow{M}_f$:

$$\overleftarrow{f}\colon (x_n)_n \mapsto (x_{n+1})_n.$$

With $\pi_0\colon (x_n)_n \mapsto x_0$ the zero coordinate projection, it holds:

$$\pi_0 \circ \overleftarrow{f} = f \circ \pi_0.$$

From this one easily deduces that the non-wandering sets $\overleftarrow{\Omega}_f$ and Ω_f of respectively $\overleftarrow{f}$ and f satisfies the following relation:

$$\overleftarrow{\Omega}_f = \Omega_f^{\mathbb{Z}} \cap \overleftarrow{M}_f\ .$$

Definition 4.18. *The endomorphism f is C^r-inverse limit stable if for every C^r-perturbation f' of f, there exists a homeomorphism h from $\overleftarrow{M}_f$ onto $\overleftarrow{M}_{f'}$ so that:*

$$h \circ \overleftarrow{f} = \overleftarrow{f}' \circ h.$$

We can define the unstable manifold of every point $\underline{x} = (x_i)_i \in \overleftarrow{\Omega}_f$:

$$W^u(\underline{x}; \overleftarrow{f}) := \{\underline{y} = (y_i)_i \in \overleftarrow{M}_f \colon d(x_i, y_i) \to 0,\ i \to -\infty\}$$

When f satisfies axiom A, it is an actual manifold embedded in $\overleftarrow{M}_f$. Moreover, the 0-coordinate projection π_0 displays a differentiable restriction $\pi_0|W^u(\underline{x}; \overleftarrow{f})$.

On the other hand, there exists $\epsilon > 0$ so that the following local stable manifold is an embedded submanifold of M, for every $x \in \Omega_f$:

$$W^s_\epsilon(x; f) := \{y \in M \colon d(f^n(x), f^n(y)) \to 0,\ n \to +\infty\}\ .$$

In [BR13], we notice that surprisingly, for certain axiom A endomorphisms, the presence of critical set (made by points with non surjective differential) does not interfere with the C^1-inverse structural stability. This leads us to define:

Definition 4.19. *An axiom A endomorphism f satisfies the weak transversality condition if for every $\underline{x} \in \overleftarrow{\Omega}_f$ and every $y \in \Omega_f$, the map $\pi_0|W^u(\underline{x}; \overleftarrow{f})$ is transverse to $W^s_\epsilon(y)$.*

There are many examples of endomorphisms which satisfy axiom A and the weak transversality condition. For instance:

- any axiom A map of the one point compactification $\hat{\mathbb{R}}$ of $\mathbb{R}$, in particular those of the form $x \mapsto x^2 + c$ and even the constant map $x \mapsto 0$.
- if f_1 and f_2 satisfy axiom A and the weak transversality condition, then the product dynamics (f_1, f_2) do so.
- By the two latter points, note that the map $(x, y, z) \mapsto (x^2, y^2, 0)$ of $\mathbb{R}^3$ satisfies axiom A and the weak transversality condition.

The latter map is not at all structurally stable, for its critical set is not and intersects moreover the non-wandering set. For this reason the following conjecture might sound irrealistic:

Conjecture 4.20 (Berger-Rovella [BR13])**.** *The C^1-inverse limit stable endomorphisms are those which satisfy axiom A and the weak transversality condition.*

However in [BR13], we gave many evidences of veracity of this conjecture. Then in [BK13] we showed one direction of this conjecture ; the other direction is still open.

Theorem 4.21 (Berger-Kocksard [BK13])**.** *If a C^1-endomorphisms of a compact manifold satisfies axiom A and the weak transversality condition, then it is inverse limit stable.*

The proof of this theorem follows the strategy of the Robbin structural stability theorem. The main difficulty is the construction of pseudo-invariant plan fields $(E^s_i)_i$ and $(E^u_i)_i$, for the endomorphisms display in general a non-empty critical set.

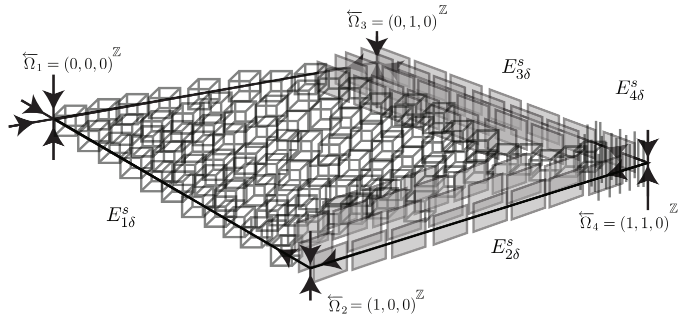


Figure 10: Construction of $(E^s_i)_i$ for the map $(x, y, z) \mapsto (x^2, y^2, 0)$

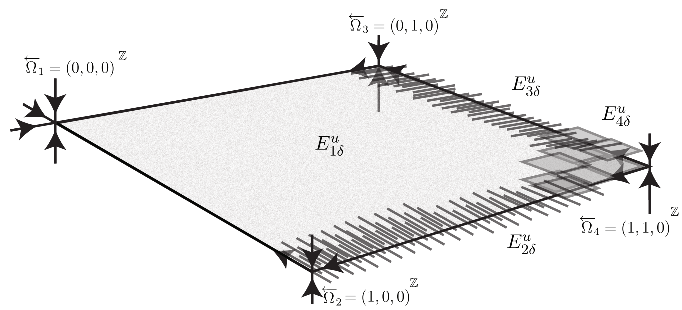


Figure 11: Construction of $(E^u_i)_i$ for the map $(x, y, z) \mapsto (x^2, y^2, 0)$

5 Links between structural stability in dynamical systems and singularity theory

In the last section we saw how the inverse stability does not seem to involve any singularity theory. However let us notice that if a C^∞-endomorphism of a manifold M is structurally stable (that is conjugated to its perturbation via a homeomorphism of M), then its singularities are C^0*-equivalently, structurally stable*:

Definition 5.1. *Let f be a C^∞-map from a manifold M into a possibly different manifold N and $r \in \{0, \infty\}$. The map f is C^r*-equivalently, structurally stable *if for every f' C^∞-close to f, there are $h \in Diff^r(M)$ and $h' \in Diff^r(N)$ which are C^r-close to the identity and such that the following diagram commutes:*

$$\begin{array}{ccccc} & & f' & & \\ & M & \rightarrow & N & \\ h & \uparrow & & \uparrow & h' \\ & M & \rightarrow & N & \\ & & f & & \end{array}.$$

The equivalently, structural stability has been deeply studied, in particular by Whitney, Thom and Mather. We shall recall some of the main results, by emphasizing their similarities with those of structural stability in dynamical systems.

5.1 Infinitesimal stability

Let M, N be compact manifolds. For $r \in \{0, \infty\}$, let $\chi^r(M)$ and $\chi^r(N)$ be the space of C^r-sections of respectively TM and TN.

Definition 5.2. *A Diffeomorphism $f \in Diff^1(M)$ is* C^0-infinitesimally stable *if the following map is surjective:*

$$\sigma \in \chi^0(M) \mapsto Tf \circ \sigma - \sigma \circ f \in \chi^0(f),$$

with $\chi^0(f)$ the space of continuous sections of the pull back bundle f^*TM.

In the Robbin-Robinson proofs of structural stability (Theorem 4.12), we saw the importance of the left-invertibility of $\sigma \mapsto Tf \circ \sigma - \sigma \circ f$. The latter implies the C^0-infinitesimal stability which is equivalent to the C^1-structural stability:

Theorem 5.3 (Robin-Robinson-Mañe [Rob71],[Rob76], [Mañ88])**.** *The C^0-infinitesimally stable diffeomorphisms are the C^1-equivalently stable maps.*

A similar definition exists in Singularity Theory:

Definition 5.4. *Let $f \in C^\infty(M, N)$ is C^∞-* equivalently infinitesimally stable *if the following map is surjective:*

$$(\sigma, \xi) \in \chi^\infty(M) \times \chi^\infty(N) \mapsto Tf \circ \sigma - \xi \circ f \in \chi^\infty(f)$$

with $\chi^\infty(f)$ the space of C^∞-sections of the pull back bundle f^*TM.

It turns out to be equivalent to the C^∞-equivalent stability.

Theorem 5.5 (Mather [Mat68a, Mat69a, Mat68b, Mat69b, Mat70])**.** *The C^∞- infinitesimally equivalently stable maps are the C^∞-equivalently stable maps.*

The latter might sound complicated to verify, but on concrete examples it is rather easy to check. That is why following Mather, it is a satisfactory description of C^∞-equivalently structurally stable maps.

5.2 Density structurally stable maps

Let us point out two similar results on structural stability:

Theorem 5.6 (Thom, Mather [Mat73, Mat76, GWdPL76])**.** *For every manifolds M, N, the C^0-equivalently structural stable maps form an open and dense set in $C^\infty(M, N)$.*

Let us recall:

Theorem 5.7 (Mañe-Sad-Sullivan [MSS83], Lyubich [Lyu84])**.** *For every $d \geq 2$, the set of structurally stable rational functions is open and dense.*

In both cases, we do not know how to describe these structurally stable maps.

Still the axiom A condition is a candidate to describe the structurally stable rational functions, since the famous Fatou conjecture (1920). On the other hand, there is not even a conjecture for the description of the C^0-equivalently structural stable maps.

Following Mather, a nice way to describe the equivalently structural stable maps would be (a similar way to) the C^∞-equivalently infinitesimal stability.

Nevertheless, Mather proved that C^∞-equivalently infinitesimal stable maps are dense if and only if the dimensions of M and N are not "nice" [Mat71]. We define the nice dimensions below. Thus one has to imagine a new criteria (at least of in "not nice" dimensions) to describe the C^0-equivalently structural stable maps.

Definition 5.8 (Nice dimensions)**.** *If $m = dim\, M$ and $n = dim\, N$, the pair of dimensions $(m; n)$ is nice if and only if one of the following conditions holds:*

$$\begin{aligned} n - m \geq 4 \quad & \textit{and} \quad m < \tfrac{6}{7}n + \tfrac{8}{7}, \\ 3 \geq n - m \geq 0 \quad & \textit{and} \quad m < \tfrac{6}{7}n + \tfrac{9}{7}, \\ n - m = -1 \quad & \textit{and} \quad n < 8, \\ n - m = -2 \quad & \textit{and} \quad n < 6, \\ n - m = -3 \quad & \textit{and} \quad n < 7. \end{aligned}$$

We notice that if $n := dim\, M = dim\, N$, then the pair of dimensions $(m; n)$ is nice if and only if $n \leq 8$.

Let us finally recall an open question:

Problem 5.9. *In nice dimensions, does a C^0-equivalently structurally stable map is always C^∞-equivalently structurally stable map?*

5.3 Geometries of the structural stability

The proof of the Thom-Mather Theorem 5.6 on the density of C^0-equivalently structurally stable involves the concept of stratification (by analytic or smooth submanifolds).

Similarly, the set of stable and unstable manifolds of an axiom A diffeomorphisms form a stratification of laminations, as defined in [Ber13].

Let us recall these definitions.

5.3.1 Stratifications

A *stratification* is the pair of a locally compact subset A and a locally finite partition Σ by locally compact subsets $X \subset A$, called *s*trata, and satisfying:

$$\forall (X, Y) \in \Sigma^2,\ cl(X) \cap Y \neq \emptyset \Rightarrow cl(X) \supset Y\ .$$

We write then $X \geq Y$.

In practical, the set A will be embedded into a manifold M, and the strata X will be endowed with a structure of analytic manifold, differentiable manifold or even lamination, depending on the context.

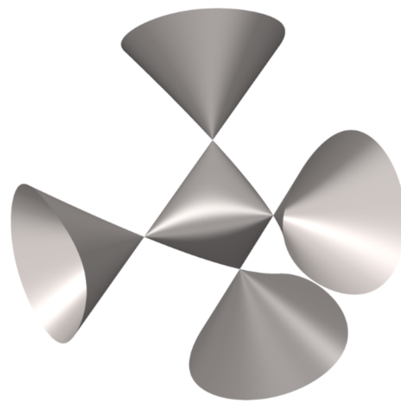


Figure 12: Algebraic variety $x^2 + y^2 + z^2 + 2xyz - 1 = 0$

5.4 Whitney Stratification

The first use of stratification goes back to the work of Whitney to describe the algebraic varieties. Then it has been generalized by Thom and Lojasiewicz for the study of analytic variety and even semi-analytic variety.

Definition 5.10. An analytic variety *of* $\mathbb{R}^n$ *is the zero set of an analytic function on an open subset of* $\mathbb{R}^n$*.* An analytic submanifold *is a submanifold which is also an analytic variety.* A semi-analytic variety *is a subset* A *of* $\mathbb{R}^n$ *which is covered by open subset* U *satisfying:*

$$A \cap U = \cap_{i=1}^{N} \cup_{i=1}^{N} F_{ij}$$

with F_{ij} *of the form* $\{q_{ij} > 0\}$ *or* $\{q_{ij} = 0\}$ *and* q_{ij} *a real analytic function on* U*.*

Theorem 5.11 (Whitney-Lojasiewicz [Łoj71])**.** *Any semi-analytic variety* $S \subset R^n$ *splits into a stratification* Σ *by analytic manifolds.*

One important property of the semi-analytic category is its stability by projection from the Seidenberg Theorem: given any projection p of $\mathbb{R}^n \to \mathbb{R}^p$, the image by p of any semi-analytic variety is a semi-analytic variety.

5.5 Thom-Mather Stratification

The following is a key step in the proof of the Thom-Mather Theorem 5.6.

Theorem 5.12 (Thom-Mather)**.** *For every* C^∞*-generic map from a compact manifold* M *into* N*, there exists a stratification on* Σ_M *of* M *and a stratification* Σ_N *on* N *such that:*

(*i*) *The strata of* Σ_M *and* Σ_N *are smooth submanifolds,*

(*ii*) *the restriction of* f *to each stratum of* Σ_M *is a submersion onto a stratum of* N*,* *(iii) this stratification is structurally stable: for every perturbation f' of f there are stratifications Σ'_M and Σ'_N homeomorphic to respectively Σ_M and Σ_N, so that (ii) holds for f'.*

The proof of this theorem is extremely interesting, it involves in particular the jet space, Thom's transversality theorem and Whitney stratifications in semi-analytics geometry. The work on the jet space influenced some ideas described in section 16.

5.6 Laminar stratification

Analogously to singularity theory, a structurally stable C^1-diffeomorphism displays a stratification.

Definition 5.13. *A* lamination *of M is a locally compact subset $\mathcal{L}$ of M, which is locally homeomorphic to the product of a $\mathbb{R}^d$ with a locally compact set T, so that $(\mathbb{R}^d \times \{t\})_{t\in T}$ corresponds to a continuous family of submanifolds.*

Definition 5.14. *A stratification of laminations is a stratification whose strata are endowed with a structure of lamination.*

Proposition 5.15 ([Ber13])**.** *Let f be a diffeomorphism M which satisfies Axiom A and the strong transversality condition. Then the stable set of every basic piece Λ_i of f has a structure of lamination X_i whose leaves are stable manifolds. Moreover the family $\Sigma_s := (X_i)_i$ forms a stratification of laminations such that $X_i \le X_j$ iff $\Lambda_i \succeq \Lambda_j$ i.e. $W^u(\Lambda_i)\cap W^s(\Lambda_j) \neq \emptyset$.*

6 Structural stability of endomorphisms with singularities

We are now ready to study the structural stability of endomorphisms which display a non empty critical set.

In dimension 2, Iglesias-Portela-Rovella [IPR08] showed the structural stability of C^3-perturbations of the hyperbolic rational functions f which are equivalently stable and whose critical sets do not self-intersect along their orbits, nor intersect the non-wandering set.

In all these examples, the critical set does not self intersect along its orbit. J. Mather suggested me to generalize a study he did about structural stability of graph of maps.

Let $G := (V, A)$ be a finite oriented graph with a manifold M_i associated to each vertex $i \in V$, and with a smooth map $f_{ij} \in C^\infty(M_i, M_j)$ associated to each arrow $[i, j] \in A$ from i to j.

For $k \in \{0, \infty\}$, such a graph is *C^k-structurally stable* if for every C^∞-perturbation $(f'_{ij})_{[i,j]\in A}$ of $(f_{ij})_{[i,j]\in A}$, there exists a family of C^k diffeomorphisms $(h_i)_i \in \prod_{i\in V} Diff^k(M_i, M_i)$ such that the following diagram commutes:

$$\forall [i,j]\in A \qquad \begin{array}{ccccc} & M_i & \xrightarrow{f'_{ij}} & M_j & \\ h_i & \uparrow & f_{ij} & \uparrow & h_j \\ & M_i & \rightarrow & M_j & \end{array}.$$

The graph (V, A) is *convergent* if for every $[i, j], [i', j'] \in A$ if $i = i'$ then $j = j'$. The graph is *without cycle* if for every $n \ge 1$ and every $([i_k, i_{k+1}])_{0\le k<n} \in V^n$ it holds $i_n \neq i_0$.

Theorem 6.1 (Mather). *Let G be a graph of smooth proper maps, convergent and without cycle. The graph is C^∞- structurally stable if the following map is surjective:*
$(\sigma_i)_i \in \prod_{i\in V} \chi^\infty(M_i) \mapsto (Tf_{ij}\circ\sigma_i - \sigma_j\circ f_{ij})_{[ij]} \in \prod_{[i,j]\in A} \chi^\infty(f_{ij})$.

Mather gave me an unpublished manuscript of Baas relating his proof, that I developed to study the structural stability of attractor-repellor endomorphisms with possibly a non-empty critical set.

Definition 6.2. *Let f be a smooth endomorphism of a compact, non necessarily connected manifold. The endomorphism f is* attractor-repellor *if it satisfies axiom A, and its basic pieces are either expanding pieces or attractors which f acts bijectively.*

The following theorem generalizes all the results I know (including [IPR08] and [IPR10]) about structurally stable maps with non-empty critical set.

Theorem 6.3 (Berger [Ber12]). *Let f be an attractor-repellor, smooth endomorphism of a compact, non necessarily connected manifold M. If the following conditions are satisfied, then f is C^∞-structurally stable:*

(i) the singularities S of f have their orbits that do not intersect the non-wandering set Ω,

(ii) the restriction of f to $M\setminus\hat\Omega$ is C^∞-infinitesimally stable, with $\hat\Omega := cl\big(\cup_{n\ge 0} f^{-n}(\Omega)\big)$. In other words, the following map is surjective:

$$\sigma\in\Gamma^\infty(M)\mapsto Df\circ\sigma-\sigma\circ f\in\Gamma^\infty(f)$$

(iii) f is transverse to the stable manifold of A's points: for any $y\in A$, for any point z in a local stable manifold W^s_y of y, for any $n\ge 0$, and for any $x\in f^{-n}(\{z\})$, we have:

$$Tf^n(T_xM)+T_zW^s_y=T_zM.$$

Hypothesis (ii) might seem difficult to verify, but it is not. In [Ber12] we apply it to many example, even for map for which the critical set does self intersect along its orbit.

It would be intersecting to investigate how the attractor-repeller could be relaxed to enjoy a greater generality. However the C^0-equivalently stable singularities are not well classified and so an optimal theorem is today difficult to obtain. Nevertheless, it is not the case in dimension 2. Indeed it is well known that the structurally stable singularities are locally equivalent to one of the following polynomial (called resp. fold and cusp):

$$(x,y)\mapsto(x^2,y)\quad\text{and}\quad(x,y)\mapsto(x^3+xy,y).$$

Hence here is a natural question:

Problem 6.4. *Under which hypothesis an axiom A surface endomorphism with singularity is structurally stable?*

Part III

Non-uniformly hyperbolic dynamics

Non-uniform hyperbolicity is a theory in construction. The aim is to prove that many dynamics can be describe by the ergodic theory.

7 Introduction to non-uniformly hyperbolicity with J.-C. Yoccoz

This section is an article we wrote with J.-C. Yoccoz [BY14] to introduce a book on the notion of strong regularity, a program by Yoccoz from the 90's to prove the non uniform hyperbolicity of low dimensional systems. The book will be formed by two pieces of this program, the Yoccoz proof of the Jakobson Theorem [Yoc97], and then my proof of the abundance of non-uniformly hyperbolic Hénon-like endomorphisms [Bera]. The first page of this introduction has been dropped since already expanded in Section 1. The rest of the text has been kept intact.

7.1 Non-uniformly hyperbolic dynamical systems

7.1.1 Pesin theory

The natural setting for non-uniform hyperbolicity is Pesin theory [BP06, LY85a], from which we recall some basic concepts. We first consider the simpler settings of invertible dynamics.

Let f be a $C^{1+\alpha}$-diffeomorphism (for some $\alpha > 0$) of a compact manifold M and let μ be an ergodic f-invariant probability measure on M. The Oseledets multiplicative ergodic theorem produces Lyapunov exponents (w.r.t. μ) for the tangent cocycle of f, and an associated μ-a.e f-invariant splitting of the tangent bundle into characteristic subbundles.

Denote by $E^s(z)$ (resp. $E^u(z)$) the sum of the characteristic subspaces associated to the negative (resp. positive) Lyapunov exponents.

The *stable and unstable Pesin manifolds* are defined respectively for μ-a.e. z by

$$W^s(z) = \{z' \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n(z), f^n(z')) < 0\},$$

$$W^u(z) = \{z' \in M : \liminf_{n\to-\infty} \frac{1}{n} \log d(f^n(z), f^n(z')) > 0\}.$$

They are immersed manifolds through z tangent respectively at z to $E^s(z)$ and $E^u(z)$.

The measure μ is *hyperbolic* if 0 is not a Lyapunov exponent w.r.t. μ. Every invariant ergodic measure, which is supported on a uniformly hyperbolic compact invariant set, is hyperbolic.

SRB, physical measures An invariant ergodic measure μ is *SRB* if the largest Lyapunov exponent is positive and the conditional measures of μ w.r.t. a measurable partition into plaques of unstable manifolds are μ-a.s. absolutely continuous w.r.t. the Lebesgue class (on unstable manifolds). When μ is SRB and hyperbolic, it is also *physical*: its basin has positive Lebesgue measure.

The paper [You98] provides a general setting where appropriate hyperbolicity hypotheses allow to construct hyperbolic SRB measures with nice statistical properties.

Coding Let μ be a f-invariant ergodic hyperbolic SRB measure. Then there is a partition mod.0 of M into finitely many disjoint subsets $\Lambda_1, \dots, \Lambda_k$, which are cyclically permuted by f and such that the restriction $f^k_{|\Lambda_1}$ is metrically conjugated to a Bernoulli automorphism.

Of a rather different flavor is Sarig's recent work [Sar13]. For a $C^{1+\alpha}$-diffeomorphism of a compact surface of positive topological entropy and any $\chi > 0$, he constructs a countable Markov partition for an invariant set which has full measure w.r.t. any ergodic invariant measure with metric entropy $> \chi$. The semi-conjugacy associated to this Markov partition is finite-to-one.

Non-invertible dynamics One should distinguish between the non-uniformly expanding case and the case of general endomorphisms.

In the first setting, a SRB measure is simply an ergodic invariant measure whose all Lyapunov exponents are positive and which is absolutely continuous.

Defining appropriately unstable manifolds and SRB measures for general endomorphisms is more delicate. One has typically to introduce the inverse limit where the endomorphism becomes invertible.

7.1.2 Case studies

The paradigmatic examples in low dimension can be summarized by the following table:

Uniformly hyperbolic	Non-uniformly hyperbolic
Expanding maps of the circle	Jakobson's Theorem
Conformal expanding maps of complex tori	Rees' Theorem
Attractors (Solenoid, DA, Plykin...)	Benedicks-Carleson's Theorem
Horseshoes	Non-uniformly hyperbolic horseshoes
Anosov diffeomorphisms	Standard map ?

Let us recall what are these theorems, and the correspondence given by the lines of the table.

Expanding maps of the circle may be considered as the simplest case of uniformly hyperbolic dynamics. The Chebychev quadratic polynomial $P_{-2}(x) := x^2 - 2$ on the invariant interval $[-2, 2]$ has a critical point at 0, but it is still semi-conjugated to the doubling map $\theta \mapsto 2\theta$ on the circle (through $x = 2\cos 2\pi\theta$). For $a \in [-2, -1]$, the quadratic polynomial $P_a(x) := x^2 + a$ leaves invariant the interval $[P_a(0), P_a^2(0)]$ which contains the critical point 0.

Theorem 7.1 (Jakobson [Jak81])**.** *There exists a set $\Lambda \subset [-2, -1]$ of positive Lebesgue measure such that for every $a \in \Lambda$ the map $P(x) = x^2 + a$ leaves invariant an ergodic, hyperbolic measure which is equivalent to the Lebesgue measure on $[P_a(0), P_a^2(0)]$.*

Actually the set Λ is nowhere dense. Indeed the set of $a \in \mathbb{R}$ such that P_a is axiom A is open and dense [GŚ97, Lyu97].

Let L be a lattice in $\mathbb{C}$ and let c be a complex number such that $|c| > 1$ and $cL \subset L$. Then the homothety $z \mapsto cz$ induces an expanding map of the complex torus $\mathbb{C}/L$. The Weierstrass function associated to the lattice L defines a ramified covering of degree 2 from $\mathbb{C}/L$ onto the Riemann sphere which is a semi-conjugacy from this expanding map to a rational map of degree $|c|^2$ called a *Lattes map.* For any $d \geq 2$, the set Rat_d of rational maps of degree d is naturally parametrized by an open subset of $\mathbb{P}(\mathbb{C}^{2d+2})$.

Theorem 7.2 (Rees [Ree86])**.** *For every $d \geq 2$, there exists a subset $\Lambda \subset \mathrm{Rat}_d$ of positive Lebesgue measure such that every map $R \in \Lambda$ leaves invariant an ergodic hyperbolic probability measure which is equivalent to the Lebesgue measure on the Riemann sphere.*

For rational maps in Λ, the Julia set is equal to the Riemann sphere. On the other hand, a conjecture of Fatou [Mil06] claims that the set of rational maps which satisfy axiom A is open and dense in Rat_d. The restriction of such maps to their Julia set is uniformly expanding. For such maps, the Hausdorff dimension of the Julia set is smaller than 2.

The (real) Hénon family is the 2-parameter family of polynomial diffeomorphisms of the plane defined for $a, b \in \mathbb{R}$, $b \neq 0$ by

$$h_{a\,b}(x, y) = (x^2 + a + y, -bx)$$

Observe that $h_{a\,b}$ has constant Jacobian equal to b. For small $|b|$, there exists an interval $J(b)$ close to $[-2, -1]$ such that, for $a \in J(b)$, the Hénon map $h_{a\,b}$ has the following properties

- $h_{a\,b}$ has two fixed points; both are hyperbolic saddle points, one, called β with positive unstable eigenvalue, the other , called α, with negative unstable eigenvalue;
- there is a trapping open region B satisfying $h_{a\,b}(B) \Subset B$ which contains α (and therefore also its unstable manifold).

Hénon [Hén76] investigated numerically the behavior of orbits starting in B for $b = -0.3$, $a = -1.4$. Such orbits apparently converged to a "strange attractor ".

Theorem 7.3 (Benedicks-Carleson [BC91])**.** *For every $b < 0$ close enough to 0, there exists a set $\Lambda_b \subset J(b)$ of positive Lebesgue measure, such that for every $a \in \Lambda_b$, the maximal invariant set $\bigcap_{n \geq 0} h^n_{a\,b}(B)$ is equal to the closure of the unstable manifold $W^u(\alpha)$ and contains a dense orbit along which the derivatives of iterates grow exponentially fast.*

An easy topological argument insures that this maximal invariant set is never uniformly hyperbolic. Later Benedicks-Young [BY93] showed that for every such parameters $a \in \Lambda_b$ the Hénon map $h_{a\,b}$ leaves invariant an ergodic hyperbolic SRB measure. Such a measure is physical. Benedicks-Viana [BV01] actually proved that the basin of this measure has full Lebesgue measure in the trapping region B.

From [Ure95], every $a \in \Lambda_b$ is accumulated by parameter intervals exhibiting Newhouse phenomenon: for generic parameters in these intervals, $h_{a\,b}$ has infinitely many periodic sinks in B. In particular, the set Λ_b is nowhere dense.

The starting point in [PY09] is a smooth diffeomorphism of a surface M having a horseshoe [3] K. It is assumed that there exist distinct fixed points $p_s, p_u \in K$ and $q \in M$ such that $W^s(p_s)$ and $W^u(p_u)$ have at q a quadratic heteroclinic tangency which is an isolated point of $W^s(K) \cap W^u(K)$. The authors consider a one-parameter family (f_t) unfolding the tangency and study the maximal f_t-invariant set L_t in a neighborhood of the union of K with the orbit of q. Writing d_s, d_u for the transverse Hausdorff dimensions of $W^s(K)$, $W^u(K)$ respectively, it was shown previously [PT93] that L_t is a horseshoe for most t when $d_s + d_u < 1$. By [MY10] this is no longer true when $d_s + d_u > 1$. However, when $d_s + d_u$ is only slightly larger[4] than 1, some dynamical and geometric information on L_t is obtained in [PY09] for most values of t: in particular, both the stable and unstable sets for L_t have Lebesgue measure 0, and an ergodic hyperbolic f_t-invariant probability measure supported on L_t with geometric content is constructed.

The two papers in this volume are related to these case studies.

In [Yoc97], a proof of Jakobson's theorem is given. The main ingredient is the concept of *strong regularity* (explained below).

In [Berb], a class of endomorphisms of the plane containing the Hénon family is considered. Given any map $B \in C^2(\mathbb{R}^3, \mathbb{R}^2)$ with small C^2-uniform norm, one studies the one-parameter family

$$f_{a,B}(x,y) = (x^2 + a + y, 0) + B(x,y,a).$$

It is shown that there exists a set $\Lambda_B \subset \mathbb{R}$ of positive Lebesgue measure such that, for any $a \in \Lambda_B$, $f_{a,B}$ has an invariant ergodic hyperbolic physical SRB measure. The proof is based on an appropriate generalization of strong regularity.

7.1.3 Open problems

Linear Anosov diffeomorphisms of $\mathbb{T}^2$ are area-preserving and uniformly hyperbolic. In the conservative setting, a very natural case study to consider is the Chirikov-Taylor standard map family. This is a one-parameter family of area-preserving diffeomorphisms of $\mathbb{T}^2$ defined for $a \in \mathbb{R}$ by

$$S_a(x,y) = (2x - y + a \sin 2\pi x, x).$$

One form of a conjecture of Sinai ([Sin94] P.144) about this family is

Conjecture 7.4. *There exists a set $\Lambda \subset \mathbb{R}$ of positive Lebesgue measure such that, for $a \in \Lambda$, the Lebesgue measure on $\mathbb{T}^2$ is ergodic and hyperbolic for S_a.*

[3] A horseshoe is an infinite basic set of saddle type.

[4] The exact condition is $(d_s + d_u)^2 + (\max\{d_s, d_u\})^2 < d_s + d_u + \max\{d_s, d_u\}$

For such parameters, the map S_a cannot have any of the invariant curves produced by KAM-theory. In particular, a cannot be too small.

This conjecture is still completely open despite intense efforts. A weak argument in favor of this conjecture is that, when a is large, the maximal invariant set in the complement of an appropriate neighborhood of the critical lines $\{x = \pm 1/4\}$ is a uniformly hyperbolic horseshoe of dimension close to 2 [Dua94, BC14].

Actually, a large Hausdorff dimension of the invariant sets under consideration appears to be a major difficulty on the way to prove non-uniform hyperbolicity.

For the parameters considered in [BC91] and subsequent papers, the Hausdorff dimension of the Hénon attractor is *a priori* close to 1. On the other hand, numerical studies [RHO80] of the values $a = -1.4$, $b = -0.3$ considered by Hénon indicate an (eventual) attractor of Hausdorff dimension 1.261 ± 0.003.

Problem 7.5. *For every $d < 2$, find an open set of smooth families $(f_t)_t$ of smooth diffeomorphisms of $\mathbb{R}^2$ such that, with positive probability on the parameter, f_t leaves invariant an ergodic hyperbolic SRB probability measure whose support has dimension at least d.*

One should also recall that Carleson conjectured [Car91] that proving non-uniform hyperbolicity (or only the weaker conclusion of [BC91]) for a particular parameter value is in some rigorous sense undecidable.

A similar problem, in the setting of non-uniformly hyperbolic horseshoes, is

Problem 7.6. *Prove the conclusions of [PY09] for an initial horseshoe K of transverse Hausdorff dimensions d_s, d_u satisfying*

$$d_s + d_u > 3/2.$$

Even the non-uniformly expanding case is still incomplete, since it regards only the case of real or complex dimension 1. A positive answer to the following problem would be a 2-dimensional generalization of Jakobson's Theorem for perturbation of the product dynamics:

$$P_a \times P_a \colon (x, y) \mapsto (x^2 + a, y^2 + a).$$

Problem 7.7. *Does there exist an open set of 1-parameter smooth families (f_a) of endomorphisms of the plane, accumulating on $(P_a \times P_a)_a$, with the following property: with positive probability on the parameter, f_a leaves invariant an ergodic absolutely continuous invariant measure with two positive Lyapunov exponents.*

7.2 Proving non-uniform hyperbolicity in low dimension

There are now many proofs of both Jakobson's theorem and Benedicks-Carleson's theorem. Broadly speaking, they rely either on a binding approach, pioneered by Benedicks-Carleson, or on a strong regularity approach, closer to Jakobson's original proof. Both papers in this volume follow the second approach.

In both approaches, the study of the 2-dimensional setting depends very much on the 1-dimensional case.

We now explain some of the differences between the two methods.

7.2.1 The binding approach for quadratic maps

Benedicks-Carleson proved Jakobson's theorem by focusing on the expansion of the post-critical orbit. There are many proofs in this spirit [CE80, BC85, Tsu93a, Tsu93b, Luz00].

One actually proves the existence of a set $\Lambda \subset \mathbb{R}$ of positive Lebesgue measure such that, for $a \in \Lambda$, the quadratic map $P_a(x) = x^2 + a$ satisfies the Collet-Eckmann condition:

$$\liminf_{+\infty} \frac{1}{n} \log \|DP^n(a)\| > 0.$$

This property implies the existence of an absolutely continuous ergodic invariant measure with positive Lyapunov exponent[CE83].

One starts with a parameter a_0 such that the critical value a_0 of P_{a_0} belongs to a repulsive periodic cycle. Then, there exists $\lambda > 1$ so that

- (i) $DP^n_{a_0}(a_0) > \lambda^n$ for every large n,
- (ii) for every $\delta > 0$, the map P_{a_0} is λ-expanding on the complement of $[-\delta, \delta]$ (for an adapted metric).

Then for every large M, for every a close to a_0 the post-critical orbit $(P^n_a(a))_{n \le M}$ is close to $(P^n_{a_0}(a_0))_{n \le M}$ and so has a similar expansion. At the next iterations $N = M + 1$, there are three possibilities:

- (a) either $P^N_a(a)$ is not in $(-\delta, \delta)$ and so the expansion will continue by (i),
- (b) or $P^N_a(a)$ is in $(-\delta, \delta)$ but is not too close to 0; then there exists an integer $k < N$, called *the binding time*, such that the orbits $P^{N+i}_a(a)$ and $P^i_a(0)$ remain close for $i \le k$ and separate for $i = k+1$. The expansion of $(DP^i_a(a))_{i<k}$ is transferred to $(DP^i_a(P^{N+1}_a(a)))_{i<k}$. The logarithmic contraction at time N, equal to $\log |DP_a(P^N_a(a))|$, is only roughly half the logarithmic expansion during the binding period $\log |DP^{k-1}_a(P^{N+1}_a(a))|$.
- (c) or $P^N_a(a)$ is so close to 0 that (b) does not hold.

Cases (a) and (b) are allowed. Case (c) is excluded in the parameter selection by removing the parameter a for which this occurs. Then we can redo the same alternative with $N \leftarrow N+1$ in case (a) and $N \leftarrow N + k$ in case (b).

In case (b), roughly half of the original transferred logarithmic expansion is lost in the binding process. Therefore the Collet-Eckmann condition will not be satisfied if too much time is spent in iterated binding periods. To avoid this, it is asked that:

(H_N) the total length of all the binding periods before N is small with respect to N.

Actually, when appropriately formulated, the condition (H_N) implies that case (c) above does not hold. Hence if (H_N) holds for every N, the map is Collet-Eckmann.

To perform the parameter selection, we look at maximal *critical curves* $\gamma = (P_a^N(a))_{a\in\mathcal{I}}$ so that:

(P_1) Condition (H_n) holds for every $a \in \mathcal{I}$ and for every $n \le N$;

(P_2) the binding periods in $[0, N]$ are the same for every $a \in \mathcal{I}$, and the integer N is not part of a binding period;

(P_3) the length of the curve γ is bounded from below by some uniform constant.

Such a curve is split into different pieces according to which scenario holds at time $N+1$. Pieces corresponding to scenario (a) are iterated once. Pieces corresponding to scenario (c) (or to scenario (b), with a binding time k too long to satisfy (H_{N+k})) are discarded. The other pieces are iterated untill the end of the corresponding binding period. These new critical curves satisfy (P_1) and (P_2). Property $(P3)$ is also satisfied, except for some boundary effects that are easily taken care of.

A large deviation argument, relying on property $(P3)$, shows that the Lebesgue measure of the remaining parameters is positive (actually, a large proportion of the length of the starting parameter interval).

7.2.2 The binding approach for Hénon family

There are many proofs in this spirit [BC91, MV93, WY01, WY08, Tak11].

A major difficulty of the 2-dimensional setting is that critical points are not defined beforehand, and will only be well-defined for good parameters.

Call a curve *flat* if it is C^2-close to a segment of $\mathbb{R} \times \{0\}$. Roughly speaking, given a flat segment $\gamma \subset W^u(\alpha)$ going across the critical strip $\{|x| \le \delta\}$, a critical point on γ should be a point of γ such that the vertical tangent vector is exponentially dilated under positive iteration, while the tangent vector to γ is exponentially contracted.

In the inductive construction of good parameters, only N iterations of the Hénon map are considered at a given stage. Under the appropriate induction hypotheses, one defines an approximate critical set $\mathcal{C}_N$. This is a finite set of cardinality exponentially large with N. Each point of $\mathcal{C}_N$ lies on a flat segment contained in $h_{a\,b}^{\theta N}(W^u_{loc}(\alpha))$, with $\theta \sim |\log |b||^{-1}$.

The main problem of the induction step is to extend the exponential dilation along the finitely many critical orbits beyond time N. As in the 1-dimensional case, this is automatic when the critical orbit at time N lies outside of the critical strip. On the other hand, when the critical orbit at time N returns to a point z_N of the critical strip, one has to find, after excluding inadequate parameters, a *binding* critical point $\tilde{z}_0$ whose initial expansion will be transferred (at some cost) to the orbit of z_N. It is here important that z_N should be in *tangential position*, i.e much closer to the flat segment containing $\tilde{z}_0$ than to $\tilde{z}_0$ itself.

To prove that the set of non-excluded parameters (at the end of the induction process) has positive Lebesgue measure, one has to investigate carefully how the whole structure of approximate critical points, analytical estimates and binding relationships survives through parameter deformation. This is certainly the trickiest part of the method.

7.2.3 Puzzles and parapuzzles

Puzzles and parapuzzles are combinatorial structures which were first introduced in 1-dimensional complex dynamics to study the local connectivity of Julia sets and the Mandelbrot set [Hub93, Mil00]. In real 1-dimensional dynamics, they were instrumental in the proof that almost every quadratic map satisfies either axiom A or the Collet-Eckmann condition [Lyu02, AM03].

For real Julia sets of real quadratic maps, puzzle pieces are defined as follows. Let a be a parameter in $[-2,-1]$. Then the quadratic polynomial P_a has two fixed points α, β, both repelling, denoted so that $-\beta < \alpha < -\alpha < \beta$. The real Julia set is equal to $[-\beta, \beta]$. For $n \geq 0$, the *puzzle pieces of order* n are the closures of the connected components of $[-\beta, \beta] \setminus P_a^{-n}(\{\alpha, -\alpha\})$.

Puzzle pieces of successive orders are related in two fundamental ways: a puzzle piece of order n is contained in a puzzle piece of order $n-1$, and its image is contained in a puzzle piece of order $n-1$. The combinatorics of the partition by puzzle pieces of a given order depend on the sequence of nested puzzle pieces containing the critical value. This leads to a sequence of partitions of parameter space into *parapuzzle* pieces. It is a general rule of thumb that, assuming a mild level of hyperbolicity, the combinatorics and geometry of parapuzzle pieces around a given parameter a are closely related to the combinatorics and geometry of puzzle pieces for P_a around the critical value.

Let a be a parameter in $[-2,-1]$. A *regular interval* is a puzzle piece of some order $n > 0$ which is sent diffeomorphically onto $A := [\alpha, -\alpha]$ by P_a^n. One also asks that the corresponding inverse branch extends to a fixed neighborhood of A, which insures a control of the distortion. The parameter a is *regular* if the measure of the set of points in A which are not contained in a regular interval of order $\leq n$ is exponentially small with n. A classical argument shows that regular parameters satisfy the conclusions of Jakobson's theorem.

To prove that the set of regular parameters has positive Lebesgue measure, one considers a more restrictive condition called *strong regularity*. Assume that the parameter is close to the Chebychev value $a_0 := -2$. Then the return time M of the critical point to A is large. Moreover, the complement in A of a neighborhood of 0 of approximate size 2^{-M} is covered by finitely many regular intervals of order $< M$, which are called *simple*. The parameter a is called *strongly regular* if

- ($\star$) there exists a sequence of regular intervals $(I_j)_{j>0}$ of order $(n_j)_j$ such that $P_a^{M+n_1+\cdots+n_{j-1}}(a) \in I_j$ for all $j > 0$;
- ($\diamondsuit$) most I_j are simple in the sense that $\sum_{i \leq j : I_i \text{ is not simple}} n_i \ll \sum_{i \leq j} n_i$ for all $j > 0$.

The most delicate part of the proof is to establish, through a careful analysis of the puzzle structures, that strongly regular parameters are regular. Then one is able to transfer the exponential regularity estimate from puzzles in phase space to parapuzzles in parameter space. Finally, one concludes through a large deviation argument that the set of strongly regular parameters has positive Lebesgue measure.

7.2.4 The strong regularity approach for Hénon family

The hyperbolic fixed point $(\alpha, 0)$ of $h_{a,0}$ persists as a fixed point P for $h = h_{a\,b}$, with b small. One denotes by $Q \approx (-\alpha, 0)$ the first (transverse) intersection of the stable and unstable manifolds of P. Let $\mathbb{S}$ be the segment of $W^u(P)$ bounded by P and Q. It is a flat curve. Given a segment I of $W^u(P)$ one denotes by $W^s_\theta(\partial I)$ the union of the θ-local stable manifolds of the endpoints of I, with $\theta = 1/|\log b|$.

A flat curve is *stretched* if its end points belong to $W^s_\theta(\partial\mathbb{S})$. A *puzzle piece* of a flat curve S is a pair (I, n_I) of a segment I of S sent by f^{n_I} *onto* a flat *stretched* curve. The puzzle piece is *hyperbolic* if $f^{n_I}|I$ satisfies some hyperbolicity conditions and regular if it satisfies moreover a distortion condition. A *puzzle pseudo-group* is the data of a pair $(\Sigma, \mathcal{Y})$ formed by a family Σ of flat stretched curves (formed in particular by $\mathbb{S}$), and by a set of hyperbolic puzzle pieces $\mathcal{Y}$ associated to the curves of Σ so that, $\forall (I, n_I) \in \mathcal{Y}$, $h^{n_I}(I)$ is a curve of Σ. The puzzle pseudo group is *regular* (and the map is *regular*) if for every curve $S \in \Sigma$, the measure of the set of points in S which are not contained in a regular piece of order $\leq n$ given by $\mathcal{Y}$ is exponentially small with n, and if every puzzle piece in $\mathcal{Y}$ of $S \in \Sigma$ persists as a puzzle puzzle in $\mathcal{Y}$ for $S' \in \Sigma$ nearby S. A classical argument shows that regular maps leave invariant an ergodic, physical SRB measure.

The notion of strong regularity is also generalized to show the abundance regular maps. A Hénon map is *strongly regular* if it preserves a combinatorial and geometrical object called *puzzle algebra*. Such an object does not need the concept of critical point to be defined; it relies basically on the topology of the homoclinic tangle of P.

By hyperbolic continuity, the simple regular intervals persist as puzzle pieces of every flat stretched curve S, their complement in S is denoted by $S_\square$.

A *puzzle algera* is the data of: a puzzle pseudo-group $(\Sigma, \mathcal{Y})$, a family of "semi-artificial" flat stretched curves $\Sigma^\square$, and for every $S \in \Sigma \sqcup \Sigma^\square$ an admissible sequence of puzzle pieces $c(S) = (I_i, n_i)_i \in \mathcal{Y}^\mathbb{N}$ from $\mathbb{S}$ satisfying the condition $(\diamondsuit)$. *Admissibility* means that the intersection $J_k(S) := \cap_{i=1}^k f^{-n_{i-1}-\cdots-n_1}(I_i)$ is a puzzle piece of $\mathbb{S}$ for every $k \geq 1$. One shows that $(\diamondsuit)$ implies that the local stable manifolds $W^s_\theta(\partial J_k)$ have their end points in $\{y > \theta 2^{-M}\}$ and $\{y < -\theta 2^{-M}\}$. Hence one can ask $\forall k > 0$, $S \in \Sigma \sqcup \Sigma^\square$:

$(\star)$ the segment $S_\square$ is folded by f^M between both components of $W^s_\theta(\partial J_k(S))$.

This is the main ingredient of puzzle algebras definition. One notices that $f^M(S_\square)$ is tangent to a local stable manifold of the singleton $\cap_k J_k(S)$. Conversely, from these topological conditions, some combinatorially defined puzzle pieces turn out to be necessarily regular. They form $\mathcal{Y}$. Also some combinatorially defined local unstable manifolds turn out to be necessarily flat. Those which are stretched form Σ, the other are artificially stretched to form $\Sigma^\square$. This combinatorial formalism is certainly the main novelty and difficulty of this proof: pure topological and combinatorial properties imply analytical properties. Then it is rather quick to prove the regularity of $(\Sigma, \mathcal{Y})$ and so the regularity of strongly regular maps.

To handle the parameter selection, by induction on k, for a C^2-open set of dynamics f, we can define combinatorially a finite family of flat stretched curves $\check{\Sigma}_k$. Similar conditions are asked on $\check{\Sigma}_k$. This implies the existence and regularity of many flat stretched curves and puzzle pieces. When the map is strong strongly regular, every curve in $\Sigma \sqcup \Sigma^\square$ can be approximated by a curve of $\check{\Sigma}_k$ for $k \geq 0$. These combinatorial definitions enable one to follow carefully how the whole structure survives by parameter deformation.

8 Strongly regular quadratic maps and Hénon-like maps

In this section we recall Yoccoz' proof of Jakobson's Theorem, and how it has been generalized in [Bera] to prove Benedicks-Carleson's Theorem.

8.1 Strongly regular quadratic maps

For a greater but close to -2, the quadratic map $P\colon x \mapsto x^2 + a$ has two fixed points $-1 \approx A_0 < A_0' \approx 2$ which are hyperbolic. The segment $[-A_0', A_0']$ is sent into itself by P, and its boundary bounds the basin of infinity. All the points of $(-A_0', A_0')$ are sent by an iterate of P_a into $\mathbb{R}_\mathfrak{e} := [A_0, -A_0]$.

Yoccoz' definition of strongly regular maps is based on the position of the critical value a with respect to the preimages of A_0. To formalize this, he used his concept of puzzle pieces.

8.1.1 Puzzle pieces

Definition 8.1 (Piece and puzzle piece)**.** *A* piece $\mathfrak{a} = \{\mathbb{R}_\mathfrak{a}, n_\mathfrak{a}\}$ *is the data of a segment* $\mathbb{R}_\mathfrak{a}$ *of* $\mathbb{R}_\mathfrak{e}$ *and an integer* $n_\mathfrak{a}$ *so that* $P^{n_\mathfrak{a}}|\mathbb{R}_\mathfrak{a}$ *is injective. The piece* $\mathfrak{a}$ *is a* puzzle piece *if* $P^{n_\mathfrak{a}}$ *sends* $\mathbb{R}_\mathfrak{a}$ *bijectively onto* $\mathbb{R}_\mathfrak{e} := [A_0, -A_0]$.

For instance $\mathfrak{e} := \{\mathbb{R}_\mathfrak{e}, 0\}$ is a puzzle piece, called *neutral*.

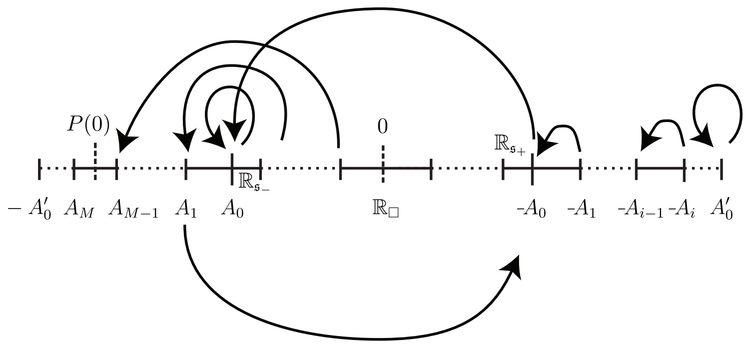
To define the simple puzzle pieces, let us denote by M the minimal integer such that $P^M(a)$ belongs to $[A_0, -A_0]$; M is large since $a > -2$ is close to $-A_0' \approx -2$.

For $i \geq 0$, let $A_i := -(P|\mathbb{R}^+)^{-i}(-A_0)$. Note that $(A_i)_{i\geq 0}$ is decreasing and converges to $-A_0'$. Also $[A_{i+1}, A_i]$ is sent bijectively by P_a^{i+1} onto $\mathbb{R}_\mathfrak{e}$. The same holds for $[-A_i, -A_{i+1}]$.

By definition of M, the critical value a belongs to $[A_M, A_{M-1}]$. Hence for $2 \leq i \leq M$, there is a segment $\mathbb{R}_{\mathfrak{s}_-^i} \subset \mathbb{R}^-$ and a segment $\mathbb{R}_{\mathfrak{s}_+^i} \subset \mathbb{R}^+$ both sent bijectively by P onto $[-A_{i-1}, -A_{i-2}]$.

Definition 8.2 (Simple puzzle piece)**.** *The pairs of the form* $\{\mathbb{R}_{\mathfrak{s}_\pm^i}, i\}$ *for* $2 \leq i \leq M$ *are puzzle pieces called* simple. *There are* $2(M-1)$ *such pairs. The set of simple puzzle pieces is denoted by* $\mathfrak{Y}_0 = \{\mathfrak{s}_\pm^i; 2 \leq i \leq M\}$.

Puzzle pieces enjoy two fundamental properties:



1. Two puzzle pieces $\mathfrak{a}$ and $\mathfrak{b}$ are nested or disjoint:

$$\mathbb{R}_\mathfrak{a} \subset \mathbb{R}_\mathfrak{b} \text{ or } \mathbb{R}_\mathfrak{b} \subset \mathbb{R}_\mathfrak{a} \text{ or } int\, \mathbb{R}_\mathfrak{b} \cap int\, \mathbb{R}_\mathfrak{b} = \varnothing\ .$$

2. For every puzzle piece $\mathfrak{a}$, for every perturbation of the dynamics, the hyperbolic continuities of the relevant preimages of the fixed point A_0 define a puzzle piece for the perturbation.

8.1.2 Building puzzle pieces

The first operation is the so-called *simple product* $\star$:

Definition 8.3 ($\star$-product)**.** *Let* $\mathfrak{a} = \{\mathbb{R}_\mathfrak{a}, n_\mathfrak{a}\}$ *and* $\mathfrak{b} = \{\mathbb{R}_\mathfrak{b}, n_\mathfrak{b}\}$ *be two pieces. The* simple product *of* $\mathfrak{a}$ *with* $\mathfrak{b}$ *is the piece* $\mathfrak{a} \star \mathfrak{b}$ *with* $\mathbb{R}_{\mathfrak{a}\star\mathfrak{b}} = (P^{n_\mathfrak{a}}|\mathbb{R}_\mathfrak{a})^{-1}(\mathbb{R}_\mathfrak{b}) \cap \mathbb{R}_\mathfrak{a}$ *and* $n_{\mathfrak{a}\star\mathfrak{b}} = n_\mathfrak{a} + n_\mathfrak{b}$*. We say that the product is* suitable *if* $\mathbb{R}_{\mathfrak{a}\star\mathfrak{b}}$ *is not empty neither a singleton.*

We notice that if $\mathfrak{a}$ and $\mathfrak{b}$ are puzzle pieces, then $\mathfrak{a} \star \mathfrak{b}$ is a puzzle piece (and suitable).

Note that the simple operation $\star$ is associative. Indeed for any puzzle pieces $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, it holds:

$$\mathfrak{a} \star (\mathfrak{b} \star \mathfrak{c}) = (\mathfrak{a} \star \mathfrak{b}) \star \mathfrak{c} =: \mathfrak{a} \star \mathfrak{b} \star \mathfrak{c}\ .$$

We need another operation to construct pieces in the closure $\mathbb{R}_\square$ of the complement of the simple pieces union in $\mathbb{R}_\mathfrak{e}$:

$$\mathbb{R}_\square := cl(\mathbb{R}_\mathfrak{e} \setminus \cup_{\mathfrak{a}\in\mathfrak{Y}_0} \mathbb{R}_\mathfrak{a}) = P_a^{-1}([-A_M, -A_{M-1}])$$

This is a neighborhood of 0 of length dominated by 2^{-M} when a is close to -2.

This second operation is the so-called *parabolic product* $\square$.

Definition 8.4 ($\square$-product)**.** *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two puzzle pieces so that* $\mathbb{R}_\mathfrak{b} \subsetneq \mathbb{R}_\mathfrak{a}$ *and so that* $P^{M+1}(0)$ *belongs to* $\mathbb{R}_\mathfrak{b}$*. We notice that* $P^{M+1}|\mathbb{R}_\square$ *has two inverse branches, one* g_+ *with image into* $\mathbb{R}^+$ *and the other* g_- *with image into* $\mathbb{R}^-$*.*

We define the parabolic pieces:

$$\square_+(\mathfrak{a} - \mathfrak{b}) := \{g_+(cl(\mathbb{R}_\mathfrak{a} \setminus \mathbb{R}_\mathfrak{b})), M + 1 + n_\mathfrak{a}\} \quad \text{and} \quad \square_-(\mathfrak{a} - \mathfrak{b}) := \{g_-(cl(\mathbb{R}_\mathfrak{a} \setminus \mathbb{R}_\mathfrak{b})), M + 1 + n_\mathfrak{a}\}$$

A parabolic piece $\mathfrak{p} = \square_{\pm}(\mathfrak{a} - \mathfrak{b})$ is never a puzzle piece. Indeed, with $\mathfrak{p} := \{\mathbb{R}_{\mathfrak{p}}, n_{\mathfrak{p}}\}$, the segment $\mathbb{R}_{\mathfrak{p}}$ is sent by $P^{n_{\mathfrak{p}}}$ onto a connected component of $cl(\mathbb{R}_{\mathfrak{e}} \setminus P^{n_{\mathfrak{a}}}(\mathbb{R}_{\mathfrak{b}})) \subsetneq \mathbb{R}_{\mathfrak{e}}$.

We notice that if $\mathfrak{p}$ is a parabolic piece and $\mathfrak{a}$ is a puzzle piece so that $\mathfrak{p} \star \mathfrak{a}$ is suitable, then $\mathfrak{p} \star \mathfrak{a}$ is a puzzle piece.

8.1.3 Yoccoz' definition of strong regularity

The main ingredient of Yoccoz' definition, is to ask for the existence of a sequence of puzzle pieces $\mathfrak{c} = (\mathfrak{a}_i)_{i \geq 1}$ so that the 3 following conditions hold:

(i) with $\mathfrak{c}_k = \mathfrak{a}_1 \star \cdots \star \mathfrak{a}_k$ the first return $P^M(a)$ belongs to a nested intersection of puzzle pieces $\cap_{k \geq 1} \mathbb{R}_{\mathfrak{c}_k}$:

$$(SR_1) \qquad P^{M+1}(0) \in \bigcap_{k \geq 1} \mathbb{R}_{\mathfrak{c}_k} \ ,$$

(ii) the sequence of puzzle pieces $(\mathfrak{a}_i)_{i \geq 1}$ satisfies:

$$(\blacklozenge) \qquad \sum_{j \leq i \ \mathfrak{a}_j \notin \mathfrak{Y}_0} n_{\mathfrak{a}_j} \leq e^{-\sqrt{M}} \sum_{j \leq i-1} n_{\mathfrak{a}_j}, \quad \forall j \leq i.$$

(iii) Every involved segment $\mathbb{R}_{\mathfrak{a}_i}$ has a neighborhood $\hat{\mathbb{R}}_{\mathfrak{a}_i}$ which is sent bijectively by $P^{n_{\mathfrak{a}_i}}$ onto the neighborhood $[A_1, -A_1]$ of $\mathbb{R}_{\mathfrak{e}}$.

The negativity of the Schwarzian derivative of P gives then a distortion bound for $P^{n_{\mathfrak{a}_i}} | \mathbb{R}_{\mathfrak{a}_i}$. Such a hypothesis is assumed in particular for all simple pieces in $\mathfrak{Y}_0$.

Moreover, a computation gives the existence of $c > 0$ such that for every $x \in \mathbb{R}_{\mathfrak{a}}$, $\mathfrak{a} \in \mathfrak{Y}_0$, it holds:

$$\|\partial_x P^{n_{\mathfrak{a}}}\| \geq e^{c n_{\mathfrak{a}}} \ .$$

Then Equation $(\star)$ and the distortion bound implies:

$$(\mathcal{CE}) \qquad \liminf_{n \to \infty} \frac{1}{n} \log \|\partial_x P^n(a)\| \geq c^- := (1 - e^{-\sqrt{M}}) c \ .$$

In particular, strongly regular unimodal maps satisfy the Collet-Eckmann condition.

8.1.4 Alternative definition of strong regularity

The existence of the interval $\hat{\mathbb{R}}_{\mathfrak{a}_i}$ extending $\mathbb{R}_{\mathfrak{a}_i}$ is replaced by an extra condition of $(\lozenge)$ on the post critical orbit, which implies the hyperbolicity of any parabolic pieces and even puzzle pieces.

Definition 8.5. *A puzzle piece or a parabolic piece* $\mathfrak{a} = \{\mathbb{R}_{\mathfrak{a}}, n_{\mathfrak{a}}\}$ *is* hyperbolic *if it satisfies the following condition:*

$$(h - times) \qquad \forall z \in \mathbb{R}_{\mathfrak{a}} \text{ and } l \leq n_{\mathfrak{a}} : |\partial_x P^{n_{\mathfrak{a}}}(z)| \geq e^{\frac{c}{3}(n_{\mathfrak{a}} - l)} |\partial_x P^l(z)| \ ,$$

with $c := \log 2/2$.

It is straight forward to see that a $\star$-product of hyperbolic pieces is hyperbolic.

Suppose that the map P satisfies (SR_1) with $(\mathfrak{c}_k)_{k\geq 1}$. We define the following countable set of symbols $\mathfrak{A} := \mathfrak{Y}_0 \sqcup \{\square_\delta(\mathfrak{c}_k - \mathfrak{c}_{k+1}) :\ k \geq 0,\ \delta \in \{+,-\}\}$.

Proposition 8.6. *Every puzzle piece $\mathfrak{a}$ is a simple product of pieces in $\mathfrak{A}$.*

Proof. We proceed by induction on the order of $\mathfrak{a}$. As the puzzle pieces are nested or disjoint, or $\mathbb{R}_\mathfrak{a}$ is included in a simple piece $\mathbb{R}_\mathfrak{s}$ either it is included in $\mathbb{R}_\square$.

In the first case, $(P^{n_\mathfrak{s}}(R_\mathfrak{a}), n_\mathfrak{a} - n_\mathfrak{s})$ is still a puzzle piece and by induction it is a product of parabolic and simple piece $\mathfrak{a}_1 \star \cdots \star \mathfrak{a}_k$. Hence $\mathfrak{a} = \mathfrak{s} \star \mathfrak{a}_1 \star \cdots \star \mathfrak{a}_k$.

In the second case, $\mathbb{R}_\mathfrak{a}$ is either included in $\mathbb{R}^-$ or in $\mathbb{R}^+$. Also its first return in $\mathbb{R}_\mathfrak{e}$ is $f^{M+1}(\mathbb{R}_\mathfrak{a})$. Note that $\{f^{M+1}(\mathbb{R}_\mathfrak{a}), n_\mathfrak{a} - M - 1\}$ is still a puzzle piece. Let $k \geq 0$ be the greater integer so that $f^{M+1}(\mathbb{R}_\mathfrak{a})$ is included into $\mathbb{R}_{\mathfrak{c}_k}$. Then $\mathbb{R}_\mathfrak{a}$ is included in $\mathbb{R}_{\square_\pm(\mathfrak{c}_k - \mathfrak{c}_{k+1})}$. Also its image by $f^{n_{\square_\pm(\mathfrak{c}_k-\mathfrak{c}_{k+1})}}$ is also a puzzle piece and so we can use the induction hypothesis as above to achieve the proof. $\square$

Definition 8.7. *A puzzle piece $\mathfrak{a}$ is* prime *if it is a simple puzzle piece or if there exist parabolic pieces $\mathfrak{p}_1, \ldots, \mathfrak{p}_k \in \mathfrak{A}$ and a simple puzzle piece $\mathfrak{s} \in \mathfrak{Y}_0$ so that:*

$$\mathfrak{a} = \mathfrak{p}_1 \star \mathfrak{p}_2 \star \cdots \star \mathfrak{p}_k \star \mathfrak{s}.$$

Hence to obtain the hyperbolicity of any puzzle piece, it suffices to give a combinatorial condition on the critical orbit which implies the hyperbolicity of all the simple pieces and all the parabolic pieces in $\mathfrak{A}$. This is the case if P satisfies (SR_1) with a sequence $\mathfrak{c} = (\mathfrak{a}_i)_i$ so that $P^{M+n_{\mathfrak{c}_i}}(a) \in \mathbb{R}_\mathfrak{e}$ does not belong to an exponentially small neighborhood of $\partial\mathbb{R}_\mathfrak{e} = \{A_0, -A_0\}$.

To make the notation less cluttered, we denote $\mathfrak{s}_-^2$ and $\mathfrak{s}_+^2$ by respectively $\mathfrak{s}_-$ and $\mathfrak{s}_+$. These two puzzle pieces have their segment which is a neighborhood of respectively A_0 and $-A_0$ in $\mathbb{R}_\mathfrak{e}$.

Likewise, the segments of the pieces $\mathfrak{s}_-^k := \mathfrak{s}_- \star \cdots \star \mathfrak{s}_-$ and $\mathfrak{s}_+^k := \mathfrak{s}_+ \star \mathfrak{s}_-^k$ are neighborhoods of respectively A_0 and $-A_0$ in $\mathbb{R}_\mathfrak{e}$.

The condition we ask is the following:

$$(\diamondsuit) \qquad P^{M+1+n_{\mathfrak{c}_i}}(0) \notin \mathbb{R}_{s_-^{\aleph(i)}} \sqcup \mathbb{R}_{s_+^{\aleph(i)}}\ ,$$

with $\aleph(0) := \left[\frac{\log M}{6c^+}\right]$ and for $i > 0$, $\aleph(i) := \left[\frac{c}{6c^+}(i + M)\right]$, where $c^+ := \log 5$. Such a condition implies that every parabolic pieces is hyperbolic (see Prop. 8.10 below).

The condition $(\diamondsuit)$ does hold if the sequence $\mathfrak{c} = (\mathfrak{a}_i)$ involved in (SR_1) satisfies $(\blacklozenge\blacklozenge)$:

Definition 8.8. *A* common sequence $\mathfrak{c} = (\mathfrak{a}_i)_i$ *is a sequence of puzzle pieces which satisfies*

$$(\blacklozenge) \qquad \sum_{j\leq i\ \mathfrak{a}_j \notin \mathfrak{Y}_0} n_{\mathfrak{a}_j} \leq e^{-\sqrt{M}} \sum_{j \leq i-1} n_{\mathfrak{a}_j}, \quad \forall j \leq i.$$

Moreover every pieces $\mathfrak{a}_i(S^i)$ is either simple or included in $\mathbb{R}_\square$ and for every $i \geq 0$,

$$(\blacklozenge\blacklozenge) \qquad \mathfrak{a}_i \star \cdots \star \mathfrak{a}_{i+\aleph(i)} \notin \{s_-^{\aleph(i)}, s_+^{\aleph(i)}\}\ .$$

Definition 8.9. *The quadratic map P is* strongly regular *if there exists a common sequence $\mathfrak{c} = (\mathfrak{a}_i)_{i\geq 1}$ so that:*

$$(SR_1) \qquad P^{M+1}(0) \in \cap_{k\geq 0}\mathbb{R}_{\mathfrak{c}_k}, \quad \textit{with } \mathfrak{c}_k = \mathfrak{a}_1 \star \cdots \star \mathfrak{a}_k.$$

(SR_2) *Every puzzle piece $\mathfrak{a}_k$ is prime.*

As announced, we have:

Proposition 8.10 (Prop 1.3 and 4.1 [Bera]). *If P is strongly regular, then every simple piece and parabolic piece is hyperbolic.*

As every puzzle pieces is a $\star$-product of parabolic and simple pieces, it comes:

Corollary 8.11. *If P is strongly regular, then every puzzle piece is hyperbolic.*

As for Yoccoz definition, this implies:

Corollary 8.12. *If P is strongly regular, then it satisfies the Collet-Eckmann Condition $(\mathcal{CE})$.*

8.2 Strongly regular Hénon-like endomorphisms

We now consider a C^2-map $f := f_{a\,B} : (x, y) \mapsto (P(x) + y, 0) + B_a(x, y)$ satisfying that:

- the parameter $a > -2$ is close to -2, so that the first return time M of a by P in $\mathbb{R}_\mathfrak{c}$ is large.
- A real number $b > 0$ small w.r.t. $|a + 2|$ (and is even small w.r.t. $e^{-e^{e^M}}$), which bounds the C^0-norm of $\det Df$ and the C^2-norm of $(x, y, a) \in [-3, 3]^2 \times \mathbb{R} \mapsto B_a(x, y)$.

Put $\theta := |\log b|^{-1}$. We notice that θ is small w.r.t. e^{-e^M}.

We observe that f is b-close to $\hat{P} := (x, y) \mapsto (x^2 + a + y, 0)$ which preserves the line $\mathbb{R} \times \{0\}$ and whose restriction therein is equal to the quadratic map P. Hence, for b small, the fixed point $(A_0, 0)$ for $\hat{P}$ persists as a fixed point A of f.

The strong regularity condition is related to the topology of the homoclinic tangle of $W^s(A; f) \cup W^u(A; f)$.

To formalize this we generalize the definition of puzzle pieces for *flat curves* that we will define in the sequel.

First let us notice that the (compact) local stable manifold $\{(x, y) \in \mathbb{R} \times [-1, \infty) : x^2 + a = A_0\}$ persists as a local stable manifold $W^s_{loc}(A; f)$ for $f_{a\,B}$. With the line $\{y = 2\theta\}$ and the line $\{y = -2\theta\}$, the local stable manifold $W^s_{loc}(A; f)$ bounds a compact set diffeomorphic to a filled square denoted by $Y_\mathfrak{e}$ (see fig. 13).

Let us denote by $\partial^s Y_\mathfrak{e} := Y_\mathfrak{e} \cap W^s_{loc}(A; f)$ and $\partial^u Y_\mathfrak{e} := Y_\mathfrak{e} \cap \{y = \pm 2\theta\}$.

Both sets consists of two connected curves whose union is $\partial Y_\mathfrak{e}$.

Definition 8.13 (flat stretched curve)**.** *A curve $S \subset Y_{\mathfrak{e}}$ is* flat *if it is the graph of a C^{1+Lip}-function ρ over an interval $I \subset \mathbb{R}$, with C^{1+Lip}-norm at most*[5] *θ.*

$$\|\rho\|_{C^0} \le \theta, \quad \|D\rho\|_{C^0} \le \theta, \quad \|Lip(D\rho)\|_{C^0} \le \theta \; .$$

The flat curve S is stretched *if it is included in $Y_{\mathfrak{e}}$ and satisfies that $\partial S \subset \partial^s Y_{\mathfrak{e}}$.*

8.2.1 Puzzle pieces

A puzzle piece is always associated to a flat stretched curve S.

Definition 8.14. *A* puzzle piece $\mathfrak{a}(S)$ *of S is the data of:*

- *an integer $n_{\mathfrak{a}}$ called the* order of the puzzle piece*,*
- *a segment $S_{\mathfrak{a}}$ of S sent bijectively by $f^{n_{\mathfrak{a}}}$ to a flat stretched curve $S^{\mathfrak{a}}$.*

For instance $\mathfrak{e}(S) := \{S, 0\}$ is a puzzle piece called neutral.

A piece $\mathfrak{a}(S) = \{S_{\mathfrak{a}}, n_{\mathfrak{a}}\}$ is hyperbolic *if the following conditions hold:*

h-times *For every $z \in S_{\mathfrak{a}}$, $w \in T_z S_{\mathfrak{a}}$ and every $l \le n_{\mathfrak{a}}$: $\|D_z f^{n_{\mathfrak{a}}}(w)\| \ge e^{\frac{c}{3}(n_{\mathfrak{a}}-l)} \cdot \|D_z f^l(w)\|$.*

We recall that $c = \log 2/2$.

In order to define the simple puzzle piece, we assumed – in the one dimensional case – that $P^{M+1}(0)$ does not belong to $\mathbb{R}_{s_-^{\aleph(0)}} \sqcup \mathbb{R}_{s_+^{\aleph(0)}}$. Hence the following P-forward invariant compact set:

$$K := \{A'_0, -A'_0\} \cup \bigcup_{i \ge 0} \{A_i, -A_i\} \cup \bigcup_{\mathfrak{s} \in \mathfrak{Y}_0} \partial \mathbb{R}_{\mathfrak{s}}$$

is at bounded distance from 0 and so is uniformly expanding for P.

We remark that the set $K \times \{0\}$ is uniformly hyperbolic for $\hat{P}$. For $z_0 = (x_0, 0) \in K \times \{0\}$, put:

$$W^s_{loc}(z_0; \hat{P}) := \{(x, y) \in \mathbb{R} \times [-1, \infty) \colon x^2 + y = x_0^2\} \; .$$

It is a local stable manifold of z_0 and its shape is an arc of parabola.

By hyperbolic continuity, for b sufficiently small, the family of curves $(W^s_{loc}(z_0))_{z_0 \in K \times \{0\}}$ persists as a family $(W^s_{loc}(z_0; f))_{x_0 \in K \times \{0\}}$ so that:

$$f(W^s_{loc}(z_0; f)) \subset W^s_{loc}(\hat{P}(z_0); f) \; . \tag{1}$$

Also for every $\mathfrak{a} \in \mathfrak{Y}_0 \cup \{e, \square\}$, the endpoints (x_-, x_+) of $\mathbb{R}_{\mathfrak{a}}$ belong to K, and the curves $W^s((x_\pm, 0); f)$ are sufficiently close to $W^s((x_\pm, 0); \hat{P})$ so that they cross the strip $\mathbb{R} \times [-2\theta, 2\theta]$ to bound a compact set $Y_{\mathfrak{a}}$ close to $S_{\mathfrak{a}} \times \{0\}$ and diffeomorphic to a filled square (see Fig. 13). The set $Y_{\mathfrak{a}}$ is called the *box*[6] associate to $\mathfrak{a}$.

[5] Actually, in [Bera], we ask the flat stretched curves to be the image by a certain map $y_{\mathfrak{e}}$ of a graph of a function satisfying such bounds. Nevertheless the map $y_{\mathfrak{e}}$ has its C^{1+Lip}-norm bounded and its inverse has its C^{1+Lip}-norm bounded by θ^{-1}. Moreover all bounds on the graph transforms will have sufficiently room so that this does not change the statement of the propositions involving the flat curves.

[6] Also called simple extension in [Bera].

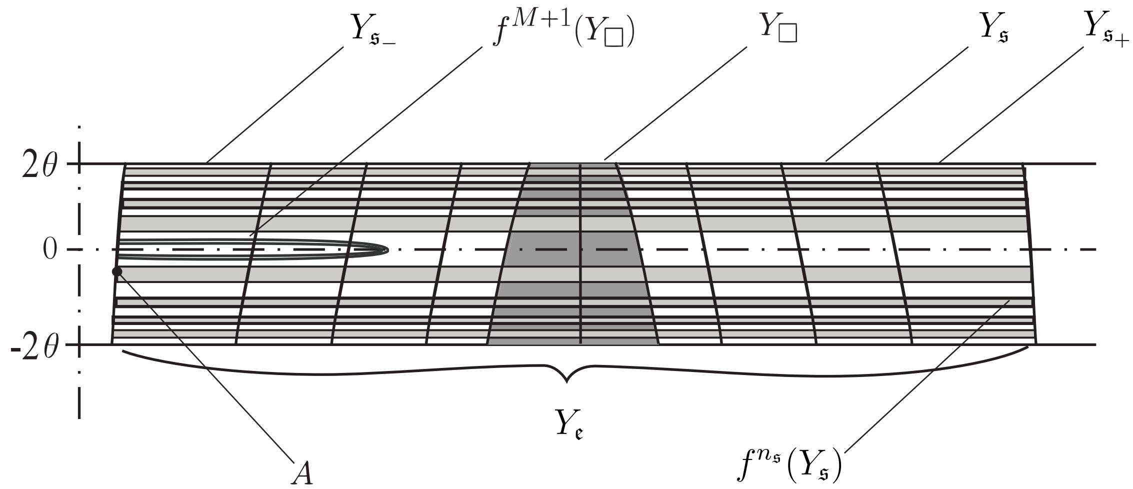


Figure 13: Geometric model for some parameters of the Hénon map.

Let $\partial^u Y_\mathfrak{a} := Y_\mathfrak{a} \cap \{y = \pm 2\theta\}$ and let $\partial^s Y_\mathfrak{a} := Y_\mathfrak{a} \cap \cup_\pm W^s((x_\pm, 0); f)$.

We notice that by (1), it holds that $f^{n_\mathfrak{a}}(\partial^s Y_\mathfrak{a}) \subset \partial^s Y_\mathfrak{e}$, as depicted by Fig. 13.

Definition 8.15 (Simple pieces)**.** *As before, $\mathfrak{s} \in \mathfrak{Y}_0$ denotes a simple symbol. For every flat stretched curve S, $S_\mathfrak{s} := S \cap Y_\mathfrak{s}$. The pair $\mathfrak{s}(S) := \{S_\mathfrak{s}, n_\mathfrak{s}\}$ is called a* simple piece *with image the curve $S^\mathfrak{s} = f^{n_\mathfrak{s}}(S_\mathfrak{s})$.*

In [Bera] Expl. 2.2, we show that $\mathfrak{s}(S)$ is a puzzle piece which is hyperbolic.

Example 8.16 (Curves S^{tt} and $(S^t)_{t \in T_0^{\mathbb{Z}^-}}$). We recall that $\mathfrak{s}_-$ is the simple symbol so that $Y_{\mathfrak{s}_-}$ contains the fixed point A (and is at the right hand side of its stable manifold). The map $S \mapsto S^{\mathfrak{s}_-}$ from the space of flat stretched curves into itself is well defined and C^1-contracting in the space of flat stretched curves. We denote by S^{tt} its fixed point, with tt denoting the constant presequence $(\mathfrak{s}_-)_{i \le -1} \in \mathfrak{Y}_0^{\mathbb{Z}^-}$. It is a half local unstable manifold of the fixed point A. We have also $S^{tt} = \{z_0 \in Y_\mathfrak{e} :\ \exists (z_i)_{i \le -1} \in Y_{\mathfrak{s}_-}^{\mathbb{Z}^-},\ z_{i+1} = f^2(z_i)\}$ since the order of $\mathfrak{s}_-$ is 2.

More generally, for $t = (\mathfrak{a}_i)_{i \le -1} \in \mathfrak{Y}_0^{\mathbb{Z}^-}$, the set:

$$S^t = \{z_0 \in Y_\mathfrak{e} :\ \exists (z_i)_{i \le -1} \in \prod_i Y_{\mathfrak{a}_i},\ z_{i+1} = f^{n_{\mathfrak{a}_i}}(z_i)\},$$

is a flat stretched curve. We put $T_0 := \mathfrak{Y}_0^{\mathbb{Z}^-}$. They define the family of curves $(S^t)_{t \in T_0}$.

8.2.2 Operation $\star$ on puzzle pieces

Similarly to the one-dimensional case, let us define the operation $\star$ on puzzle pieces of flat stretched curves.

Definition 8.17 (Operation $\star$ on puzzle pieces)**.** *Let $\mathfrak{a}(S) := \{S_\mathfrak{a}, n_\mathfrak{a}\}$ and $\mathfrak{b}(S') = \{S'_\mathfrak{b}, n_\mathfrak{b}\}$ be two puzzle pieces of S and S' respectively. If S' is equal to $S^\mathfrak{a} = f^{n_\mathfrak{a}}(S_\mathfrak{a})$, then the pair of puzzle pieces*

$(\mathfrak{a}(S), \mathfrak{b}(S^{\mathfrak{a}}))$ *is called* suitable. *Then we can define the puzzle piece* $\mathfrak{a} \star \mathfrak{b}(S) = \{S_{\alpha\star\beta}, n_{\alpha\star\beta}\}$ *of* S *with:*

$$S_{\alpha\star\beta} = f^{-n_{\mathfrak{a}}}(S'_{\mathfrak{b}}) \cap S_{\mathfrak{a}} \quad \textit{and} \quad n_{\alpha\star\beta} = n_\alpha + n_\beta \ .$$

Indeed the map $f^{n_{\mathfrak{a}\star\mathfrak{b}}}|S_{\mathfrak{a}\star\mathfrak{b}}$ *is a bijection onto* $S^{\mathfrak{a}\star\mathfrak{b}} := f^{n_{\mathfrak{a}\star\mathfrak{b}}}(S_{\mathfrak{a}\star\mathfrak{b}})$.

More generally, a sequence $(\mathfrak{a}^i(S^i))_{1\le i<k}$, for $k \in \mathbb{N}\cup\{\infty\}$ is called suitable if the pair of any two consecutive puzzle pieces is suitable. We can now generalize condition $(\star)$ of Yoccoz' strong regularity definition.

Definition 8.18 (Common sequence)**.** *For* $N \in [1,\infty]$, *a* common sequence $\mathfrak{c}$ *is a suitable sequence of hyperbolic puzzle pieces* $\mathfrak{c} := (\mathfrak{a}_i(S^i))_{i=1}^{N-1}$ *from* $S^1 := S^{tt}$ *which satisfies the following properties:*

$$(\blacklozenge) \qquad \sum_{j\le i\ \mathfrak{a}_j\notin\mathfrak{Y}_0} n_{\mathfrak{a}_j} \le e^{-\sqrt{M}} \sum_{j\le i-1} n_{\mathfrak{a}_j}, \quad i<N.$$

Moreover every pieces $\mathfrak{a}_i(S^i)$ *is either simple or included in* $Y_\square$, *and for every* $i\ge 0$,

$$(\blacklozenge\blacklozenge) \qquad \mathfrak{a}_{i+1}\star\cdots\star\mathfrak{a}_{i+\aleph(i)} \notin \{s_-^{\aleph(i)}, s_+^{\aleph(i)}\}$$

The product $\mathfrak{c}_i := \mathfrak{a}_1\star\mathfrak{a}_2\star\cdots\star\mathfrak{a}_{i-1}\star\mathfrak{a}_i$ is called a *common product of depth* i and it defines a pair $\mathfrak{c}_i(S^{tt}) =: \{S^{tt}_{\mathfrak{c}_i}, n_{\mathfrak{c}_i}\}$ called a *common piece.*

A *common piece of order* 0 is the pair equal to $\mathfrak{c}_0(S^{tt}) := \{S^{tt}, 0\} = \mathfrak{e}(S^{tt})$.

Not all the puzzle pieces have their endpoints with a nice local stable manifold. Nevertheless it is the case for the common piece:

Proposition 8.19 ([Bera] Prop. 3.6)**.** *Each endpoint* $z_\pm$ *of* $S^{tt}_{\mathfrak{c}_i}$ *has a local stable manifold* $W^s_{loc}(z_\pm; f)$ *which stretches across* $Y_{\mathfrak{e}}$ *and is* $\sqrt{b}$-C^2*-close to an arc of curve of the form:*

$$\{(x,y)\colon x^2+y=cst\}.$$

With the lines $\{y=\pm 2\theta\}$, *this bounds a box of* $Y_{\mathfrak{e}}$ *denoted by* $Y_{\mathfrak{c}_i}$. *Moreover, for every* $z\in Y_{\mathfrak{c}_i}$, *the vector* $Df^{n_{\mathfrak{c}_i}}(1,0)$ *is* θ*-close to be horizontal and of norm at least* $e^{c^- n_{\mathfrak{c}_i}}$, *with:*

$$c^- = c - \frac{1}{\sqrt{M}} = \frac{\log 2}{2} - \frac{1}{\sqrt{M}} \ .$$

We put $\partial^s Y_{\mathfrak{c}_i} = \cup_\pm W^s_{loc}(z_\pm; f)\cap Y_{\mathfrak{e}}$ and $\partial^u Y_{\mathfrak{c}_i} = \partial^u Y_{\mathfrak{e}} \cap Y_{\mathfrak{c}_i}$.

By the above Proposition, the width of $Y_{\mathfrak{c}_i}$ is smaller than $2e^{-c^- n_{\mathfrak{c}_i}}$ times the width of $Y_{\mathfrak{e}}$ and so, if $N=\infty$, the following decreasing intersection:

$$W^s_{\mathfrak{c}} := \cap_{i\ge 0} Y_{\mathfrak{c}_i} \ .$$

is a C^{1+Lip}-curve called *common stable manifold*, which is $\sqrt{b}$-C^{1+Lip}-close to an arc of a curve of the form:

$$\{(x,y)\colon x^2+y=cst\}.$$

8.2.3 Tangency condition

Every flat stretched curve S intersects $Y_\square$ at a segment $S_\square = S \cap Y_\square$. This segment is sent by f^{M+1} to a curve $S^\square$ which is C^2-close to a folded curve $\{(-Cst \cdot 4^M t^2 + f_a^M(a), 0) : t \in \mathbb{R}\} \cap Y_\mathfrak{c}$.

The definition of strong regularity for Hénon-like maps supposes the existence of a family of curves $(S^t)_{t\in T^*}$ so that for each $t \in T^*$, there exists a common sequence of puzzle pieces $\mathfrak{c}^t$ so that

(SR_1) $\quad S^{t\square} = f^{M+1}(S^t_\square)$ is tangent to $W^s_{\mathfrak{c}^t}$.

As in dimension 1, conditions are given on the puzzle pieces involved in the common sequences. In this two dimensional case, conditions are moreover given on the flat and stretched curves forming $(S^t)_{t\in T^*}$.

8.2.4 Parabolic operations from tangencies

As in dimension 1, if a flat stretched curve S satisfies that $S^\square$ is tangent to a common stable manifold $W^s_{\mathfrak{c}^t}$, then we can define parabolic pieces.

Indeed, then for every i, $(f^{M+1}|S_\square)^{-1} cl(Y_{\mathfrak{c}_i} \setminus Y_{\mathfrak{c}_{i+1}})$ consists of zero or two segments.

In the latter case, we denote by $S_{\square_-(\mathfrak{c}_i-\mathfrak{c}_{i+1})}$ the left hand side segment and by $S_{\square_+(\mathfrak{c}_i-\mathfrak{c}_{i+1})}$ the right hand side segment. Furthermore, with $\mathfrak{p}$ a symbol in $\{\square_+(\mathfrak{c}_i - \mathfrak{c}_{i+1}), \square_-(\mathfrak{c}_i - \mathfrak{c}_{i+1})\}$ and $n_\mathfrak{p} := M + 1 + n_{\mathfrak{c}_i}$, the pair $\mathfrak{p}(S) := \{S_\mathfrak{p}, n_\mathfrak{p}\}$ is called a *parabolic piece*.

This pair $\mathfrak{p}(S)$ cannot be a puzzle piece since the curve $f^{n_\mathfrak{p}}(S_\mathfrak{p})$ is not stretched (like in the one dimensional model).

However the curve $f^{n_\mathfrak{p}}(S_\mathfrak{p})$ can be extended to a flat stretched curve $S^\mathfrak{p}$ by an algorithm given by Prop. 4.8 and 5.1 in [Bera]. In particular $S^\mathfrak{p} \supsetneq f^{n_\mathfrak{p}}(S_\mathfrak{p})$.

Definition 8.20 (Set of symbols $\mathfrak{A}$)**.** *Let f which satisfies (SR_1) with the flat stretched curves $(S^t)_{t\in T^*}$ and the common sequences $(\mathfrak{c}^t)_{t\in T^*}$.*

Let

$$\mathfrak{A} := \mathfrak{Y}_0 \cup \bigcup_{t\in T^*} \bigcup_{i\geq 0} \{\square_+(\mathfrak{c}^t_i - \mathfrak{c}^t_{i+1}), \square_-(\mathfrak{c}^t_i - \mathfrak{c}^t_{i+1})\} \ .$$

The above union over $t \in T^*$ is not disjoint by the above remark. As they are countably many puzzle pieces of S^{tt}, they are countably many common pieces $\mathfrak{c}_i$ and boxes $Y_{\mathfrak{c}_i}$. Thus $\mathfrak{A}$ is countable.

Proposition 8.21 (Prop. 1.7 and 4.1 of [Bera])**.** *For every $t \in T^*$, every parabolic or simple piece $\mathfrak{a}(S^t)$, with $\mathfrak{a} \in \mathfrak{A}$, is hyperbolic.*

Definition 8.22 (Suitable chain)**.** *Let $(S^i)_{i=1}^n$ be a family of flat stretched curves and let $(W^s_{\mathfrak{c}^i})_{i=1}^n$ be a family of common stable manifolds so that $S^{i\square}$ is tangent to $W^s_{\mathfrak{c}^i}$.*

For each i let $\mathfrak{p}_i$ be a symbol either in $\mathfrak{Y}_0$, or parabolic obtained from $\mathfrak{c}^i$ (that is of the form $\square_\pm(\mathfrak{c}^i_j - \mathfrak{c}^i_{j+1})$).

The chain of symbols $(\mathfrak{p}_i)_{i=1}^n$ is called suitable from S^1 if:

1. *$S^{i+1} = (S^i)^{\mathfrak{p}_i}$ for every $i < n$,*

2. The segment of the pair $\mathfrak{p}_1(S^1) \star \cdots \star \mathfrak{p}_n(S^n)$ is not trivial (it has cardinality > 1).

The chain of symbols is complete if $\mathfrak{p}_n$ belongs to $\mathfrak{Y}_0$, and incomplete otherwise. The chain of symbols $(\mathfrak{p}_i)_i$ is prime *if $\mathfrak{p}_i \notin \mathfrak{Y}_0$ for $i < n$.*

A corollary of Proposition 8.21 is:

Corollary A. *If $(\mathfrak{p}_i)_{i=1}^n$ is suitable, then $\mathfrak{p}_1(S^{t_1}) \star \cdots \star \mathfrak{p}_n(S^{t_n})$ is a hyperbolic piece of S^{t_1}.*

By using that the puzzle pieces are nested or disjoint, one shows:

Proposition 8.23. *If $(\mathfrak{p}_i)_{i=1}^n$ is suitable and complete , then $\mathfrak{p}_1(S^{t_1}) \star \cdots \star \mathfrak{p}_n(S^{t_n})$ is a hyperbolic puzzle piece of S^{t_1}.*

8.2.5 Puzzle algebra and strong regularity definition

In Example 8.16, we defined for every $t \in T_0 := \mathfrak{Y}_0^{\mathbb{Z}^-}$ a flat stretched curve S^t. Every symbol t is a pre-sequence of symbols $t = (\mathfrak{s}_i)_{i \le -1}$. Given a chain of symbols $\mathfrak{a}_{-k}, \ldots, \mathfrak{a}_{-1}$, let $t \cdot \mathfrak{a}_{-k} \cdots \mathfrak{a}_{-1}$ be the pre-sequence of symbols $(\mathfrak{a}'_i)_{i \le -1}$ with $\mathfrak{a}'_{-i} = \mathfrak{a}_{-i}$ for every $0 \le i \le k$ and $(\mathfrak{a}'_i)_{i \le -k-1} = t$.

In [Bera], for a set of parameters $a \in P_B$ of Lebesgue measure positive, we show the existence of a family of curves $(S^t)_{t \in T^*}$ and a family of common sequences $C = (\mathfrak{c}^t)_{t \in T^*}$ which are linked in the following way by the tangency condition and parabolic/simple operations.

(SR_1) $S^{t\square} = f^{M+1}(S^t \cap Y_\square)$ is tangent to $W^s_{\mathfrak{c}^t}$.

$$\text{Put} \quad \mathfrak{A} := \mathfrak{Y}_0 \cup \bigcup_{t \in T^*, \quad i \ge 0} \{\square_+(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t), \square_-(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)\}$$

(SR_2) For every $t \in T^*$, every puzzle piece $\mathfrak{a}_i(S^i)$ involved in $\mathfrak{c}^t = (\mathfrak{a}_i(S^i))_i$ is given by suitable, complete and prime chain of symbols $\underline{\mathfrak{a}}_i$ in $\mathfrak{A}^{(\mathbb{N})}$.

(SR_3) The set T^* is the subset of $\mathfrak{A}^{\mathbb{Z}^-}$ defined by

$$T^* = \{t \cdot \mathfrak{p}_{-n} \cdots \mathfrak{p}_{-1} :\ t \in T_0,\ n \ge 0,\ (\mathfrak{p}_i)_{1 \le i \le n} \in \mathfrak{A}^n \text{ is a suitable chain from } S^t\}.$$

For $t^* = t \cdot \mathfrak{p}_{-n} \cdots \mathfrak{p}_{-1} \in T^*$, we put $S^{t^*} = (\cdots (S^t)^{\mathfrak{p}_{-n}} \cdots)^{\mathfrak{p}_{-1}}$.

Remark 8.24. In (SR_3) the element $t \cdot \mathfrak{p}_{-n} \cdots \mathfrak{p}_{-1}$ is equal to the pre-sequence $(\mathfrak{a}_i)_{i \le -1} \in \mathfrak{A}^{\mathbb{Z}^-}$ defined by $\mathfrak{a}_{-i} := \mathfrak{p}_{-i}$ if $1 \le i \le n$, and, with $t = (\mathfrak{s}_i)_{i \le -1} \in T^0$, $\mathfrak{a}_{-i} : \mathfrak{s}_{-i+n}$ if $i \ge n+1$.

Definition 8.25. *A map f so that there exists a family of flat stretched curves $(S^t)_{t \in T^*}$ and a family of common sequences $(\mathfrak{c}^t)_{t \in T^*}$ satisfying $(SR_1 - SR_2 - SR_3)$ is called* strongly regular.

Definition 8.26. *Let $\mathfrak{G}$ be the set of finite segments of sequences in $T^* \subset \mathfrak{A}^{\mathbb{Z}^-}$. It enjoys a structure of pseudo-monoid structure for the concatenation law $\cdot$. The parabolic operation $\square$ is another law on $\mathfrak{G}$. The triplet $(\mathfrak{G}, \cdot, \square)$ is called a* Puzzle Algebra. [7]

The main result of [Bera] (Theorem 0.1) is the following:

Theorem 8.27. *Every strongly regular map leaves invariant an ergodic, physical SRB measure supported by a non-uniformly hyperbolic attractor. Moreover, strongly regular maps are abundant in the following meaning:*

For every $\epsilon > 0$, there exists $b > 0$, such that for every B of C^2 norm less than b, there exist $\eta > 0$ and a subset $\Pi_B \subset [-2, -2+\eta]$ with $\frac{Leb \Pi_B}{Leb[-2,-2+\eta]} \geq 1-\epsilon$ such that for every $a \in \Pi_B$, the map $f_{a\,B}$ is strongly regular.

Remark 8.28. To fix the idea, we will suppose the following very rough inequalities: $M \geq 1000$ and $-\log b \leq \exp\exp M$. They are sufficient for the new analytic conditions given by this work.

9 Parameter selection

The main theorem of [Bera] is the first to take place in the C^2-topology ([WY08] works for C^3-mappings at the C^2-neighborhood of the Hénon map). Also it is the first one which gives the SRB construction for the endomorphism case.

I would like to shed light on another aspect of this work. How does the combinatorial formalism enable us to handle the parameter selection, and especially how does it enable us to follow rigorously the structure when the parameter varies.

This will be the occasion to introduce our work with Moreira [BM16] on nested Cantor sets; it might be helpful to study how large the Haudsdorff dimension of an abundant, strongly regular attractor can be.

9.1 k-Strongly regular maps and their combinatorial classes

9.1.1 The quadratic map case

Let us go back to our variation of Yoccoz' definition of strongly regular quadratic maps.

To define a k-strongly regular map, we shall ask for the existence of a common sequence so that $\mathfrak{c} = (\mathfrak{a}_i)_{k \geq i \geq 1}$ $P^{M+1}(0)$ belongs to $\mathbb{R}_{\mathfrak{c}_j}$ for every $j \leq k$, with $\mathfrak{c}_j = \mathfrak{a}_1 \star \cdots \star \mathfrak{a}_k$. Nevertheless this does not imply the hyperbolicity of the parabolic pieces $\square_\pm(\mathfrak{c}_{k-1} - \mathfrak{c}_k)$ (for instance, the point 0 belongs to this piece if the critical value $P^{M+1}(0)$ belongs to the boundary of $\mathbb{R}_{\mathfrak{c}_k}$).

Hence, we shall assume that $P^{M+1}(0)$ does not belong to $\mathbb{R}_{\mathfrak{c}_k \star \mathfrak{s}_-^{\aleph(k)}} \sqcup \mathbb{R}_{\mathfrak{c}_k \star \mathfrak{s}_+^{\aleph(k)}}$. This leads to the following definition:

[7] In [Bera], the presentation of strong regularity is different: The set T^* is presented as the disjoint union of the sets T and $T^\square$, formed by the pre-sequences $t \star \mathfrak{p}_{-n} \star \cdots \star \mathfrak{p}_{-1} \in T^*$ which finish by respectively a simple piece or a parabolic piece. This splits the family of curves $(S^t)_{t \in T^*}$ into two subfamilies $\Sigma = (S^t)_{t\in T}$ and $\Sigma^\square = (S^t)_{t \in T^\square}$. Furthermore, the set of prime puzzle pieces of a curve S^t, with $t \in T$, is denoted therein by $\mathcal{Y}(t)$. We define also $\mathcal{Y} := \sqcup_{t\in T} \mathcal{Y}(t)$. The quadruplet $(\Sigma, \Sigma^\square, C, \mathcal{Y})$ is called a puzzle algebra. This is equivalent to the above definition.

Definition 9.1 (k-strongly regular quadratic maps)**.** *The quadratic map P is* k-strongly regular *if there exists a common sequence $\mathfrak{c} = (\mathfrak{a}_i)_{k\ge i\ge 1}$ so that:*

$$(SR_1) \qquad P^{M+1}(0) \in \cap_{k\ge j\ge 1}(\mathbb{R}_{\mathfrak{c}_j} \setminus (\mathbb{R}_{\mathfrak{c}_j \star \mathfrak{s}_-^{\aleph(j)}} \sqcup \mathbb{R}_{\mathfrak{c}_j \star \mathfrak{s}_+^{\aleph(j)}}), \quad \textit{with } \mathfrak{c}_j = \mathfrak{a}_1 \star \cdots \star \mathfrak{a}_k.$$

We define the set of symbols $\mathfrak{A}_j$ for $j \le k-1$ as follows:

$$\mathfrak{A}_0 := \mathfrak{Y}_0, \quad \mathfrak{A}_{j+1} = \mathfrak{A}_j \cup \{\square_+(\mathfrak{c}_j - \mathfrak{c}_{j+1}), \square_-(\mathfrak{c}_j - \mathfrak{c}_{j+1})\}\ .$$

(SR_2) $\forall 1 \le j \le k$ *the puzzle piece $\mathfrak{a}_j$ is given by a prime, complete sequence of symbols in $\mathfrak{A}_{j-1}$.*

We notice that a k-strongly regular map is k'-strongly regular for every $k' \le k$. Also the strongly regular maps are exactly the ∞-strongly regular maps.

Let $\mathfrak{G}_k$ be the set of finite segments of sequences in $T_k^* \subset \mathfrak{A}_k^{\mathbb{Z}^-}$. It enjoys a structure of pseudo-monoid structure for the concatenation law " $\cdot$ ".

Fact 9.2. *The alphabets $(\mathfrak{A}_j)_{j\le k}$ and the pieces $(\mathfrak{a}_j)_{j\le k}$ are uniquely defined.*

Proof. This is done by induction on $j \le k$. For $j = 0$, we recall that $\mathfrak{a}_0 = \mathfrak{e}$ and $\mathfrak{A}_0 = \mathfrak{Y}_0$. Assume the uniqueness for $j < k$. By (SR_2), $\mathfrak{a}_{j+1}$ is made by pieces in $\mathfrak{A}_j$ and so it is uniquely defined. This implies that $\mathfrak{A}_{j+1}$ is uniquely defined. $\square$

The common sequence definition is a priori not purely combinatorial since such a sequence is asked to be made of hyperbolic pieces. Nevertheless, the hyperbolicity assumption is automatic from (SR_1) and (SR_2). Indeed by (SR_2) it suffices to show that every piece in $\mathfrak{A}_j$ is hyperbolic. This follows from an easy induction.

Fact 9.3. *The hyperbolicity of the puzzle pieces involved in the common sequence of a k-strongly regular quadratic map is "automatic".*

Proof. Note first that $\mathfrak{A}_0 = \mathfrak{Y}_0$ is made by hyperbolic pieces (Prop. 1.3 [Bera]). Let $0 \le j \le k-1$, and assume that $\mathfrak{A}_j$ is made by hyperbolic pieces. To show that the pieces of $\mathfrak{A}_{j+1}$ are hyperbolic, it suffices to show that the pieces $\square_+(\mathfrak{c}_j - \mathfrak{c}_{j+1})$ and $\square_-(\mathfrak{c}_j - \mathfrak{c}_{j+1})$ are hyperbolic as done in Prop. 4.8 [Bera]. $\square$

Hence the k-strongly regular condition on quadratic maps is purely combinatorial and topological. As given a k-strongly regular map P, the k^{th}-common piece $\mathfrak{c}_k$ is uniquely defined by Fact 9.2, we can set:

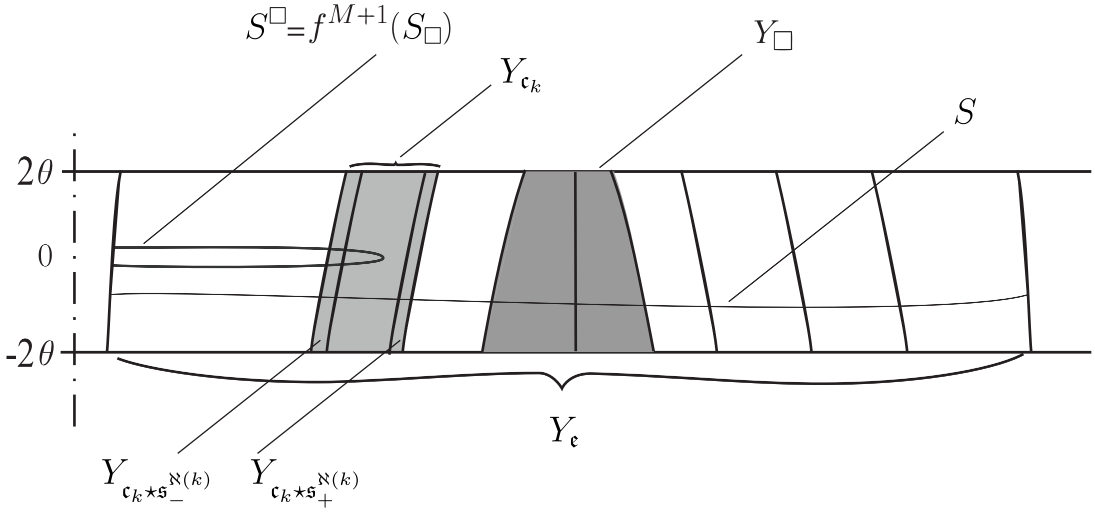
Definition 9.4 (Combinatorial interval)**.** *Let $w \subset \mathbb{R}$ be an interval so that P_a is strongly regular for every $a \in w$, with $(\mathfrak{A}_j(a))_{j\le k}$ as set of symbols and $(\mathfrak{c}_j(a))_j$ as common sequences of depth $j \le k$. The interval w is a combinatorial interval if $a \in w \mapsto \mathbb{R}_{\mathfrak{c}_j(a)}$ is continuous for every $j \le k$.*

Then all the sets $\mathfrak{A}_j(a)$ among $a \in w$ can be identified to a single set $\mathfrak{A}_j(w)$.

9.1.2 The Hénon-like map case

We first need to substitute the notion of tangency between two curves by the notion of tangency between a curve and a box:

Definition 9.5. *A flat stretched curve S is tangent to a common piece $\mathfrak{c}_k$ of finite depth if $S^\square = f^{M+1}(S_\square)$ intersects the interior of the box $Y_{\mathfrak{c}_k}$, and exactly one component of $\partial^s Y_{\mathfrak{c}_k}$.*



A Hénon-like map f is *k-strong regular* if there exists a family of curves $(S^t)_{t\in T_k^*}$ and a family of common sequences $((\mathfrak{a}_j^t)_{1\le j\le k})_{t\in T_k^*}$, so that for every $j \le k$ with $\mathfrak{c}_j^t = \mathfrak{a}_1^t \star \cdots \mathfrak{a}_j^t$ it holds:

(SR_1^k) $S^{t\square} = f^{M+1}(S^t_\square)$ is tangent to $Y_{\mathfrak{c}_j^t}$ but not tangent to $Y_{\mathfrak{c}_j^t \star \mathfrak{s}_+^{\aleph(j)}}$ nor $Y_{\mathfrak{c}_j^t \star \mathfrak{s}_-^{\aleph(j)}}$.

$$\text{Put} \quad \mathfrak{A}_j := \mathfrak{Y}_0 \cup \bigcup_{t\in T^*, \quad j\ge i\ge 1} \{\square_+(\mathfrak{c}_{i-1}^t - \mathfrak{c}_i^t), \square_-(\mathfrak{c}_{i-1}^t - \mathfrak{c}_i^t)\} \ .$$

(SR_2^k) For every $t \in T_k^*$, for every $1 \le j \le k$, the puzzle piece $\mathfrak{a}_j^t$ is given by a suitable, complete and prime chain of symbols in $\mathfrak{A}_{j-1}^{(\mathbb{N})}$.

(SR_3^k) The set T_k^* is the subset of $\mathfrak{A}_k^{\mathbb{Z}^-}$ defined by

$$T_k^* = \{t \cdot \mathfrak{p}_{-n} \cdots \mathfrak{p}_{-1} : \ t \in T_0, \ n \ge 0, \ (\mathfrak{p}_i)_{1\le i\le n} \in \mathfrak{A}_k^n \text{ is a suitable chain from } S^t\}.$$

For $t^* = t \cdot \mathfrak{p}_{-n} \cdots \mathfrak{p}_{-1} \in T_k^*$, we put $S^{t^*} = (\cdots (S^t)^{\mathfrak{p}_{-n}} \cdots)^{\mathfrak{p}_{-1}}$.

The triplet $(\mathfrak{A}_k, \square, \cdot)$ is called a *k-puzzle algebra structure* for the strongly regular map f.

Note that k-strongly regular map is k'-strongly regular for every $k' < k$.

In Proposition 4.8 [Bera] showed that:

Proposition 9.6. *Given a flat stretched curve S, $(\mathfrak{a}_i)_{i\le j}$, with $\mathfrak{c}_{j-1} := \mathfrak{a}_1 \star \cdots \star \mathfrak{a}_{j-1}$ and $\mathfrak{c}_j = \mathfrak{c}_{j-1} \cdot \mathfrak{a}_j$, if the curve $S^\square = f^{M1}(S \cap Y_\square)$ is tangent to $Y_{\mathfrak{c}_j}$ and not $Y_{\mathfrak{c}_j \star \mathfrak{s}_+^{\aleph(j)}}$ nor $Y_{\mathfrak{c}_j \star \mathfrak{s}_-^{\aleph(j)}}$, then for every $\mathfrak{p} \in \{\square_+(\mathfrak{c}_{j-1} - \mathfrak{c}_j), \square_-(\mathfrak{c}_{j-1} - \mathfrak{c}_j)\}$, if $S_\mathfrak{p}$ is not empty, then $\mathfrak{p}(S)$ is a hyperbolic parabolic piece and $S^\mathfrak{p}$ is a well-defined flat, stretched curve.*

This enabled us to show (similarly to the one dimensional case):

Fact 9.7 (Prop. 5.11 [Bera]). *Every k-strongly regular Hénon-like map f, the alphabets $(\mathfrak{A}_j)_{j\le k}$ and the pieces $(\mathfrak{a}_j^t)_{j\le k,\, t\in T_k^*}$ are uniquely defined.*

Proposition 9.6 is very important since it implies that the hyperbolicity and the flatness involved in the strong regularity definition is automatic (among maps in a C^2-neighborhood of $(x^2-2+y,0)$ independent of k).

Hence the k-strongly regular condition on Hénon-like map is purely combinatorial and topological. This is crucial for the parameter selection, since it avoids us to check if the corresponding analytic estimates (flatness and hyperbolicity) are still satisfied for parameter deformations.

Let us now consider the C^2-*family* of maps $(f_a)_a$ of the form $f_a(x,y) = (x^2+a+y,0)+B(a,x,y)$, where B is C^2-small.

Definition 9.8 (Combinatorial interval). *Let $w\subset\mathbb{R}$ be an interval so that f_a is strongly regular for every $a\in w$ with structure $\mathfrak{A}_k(a)$, $T_k^*(a)$, $(\mathfrak{c}_k^t(a))_{t\in T_k^*(a)}$.*

*The interval is k-*combinatorial *if all the $\mathfrak{A}_k(a)$ are in bijection and can be identified to a finite set $\mathfrak{A}_k(w)$ so that:*

- *for every $a\in w$, Condition (SR_3^k) and $\mathfrak{A}_k(a)\approx\mathfrak{A}(w)$ define an inclusion and an identification:*

  $$T_k^*(a)\subset \mathfrak{A}_k^{\mathbb{Z}^-}(a)\approx \mathfrak{A}_k^{\mathbb{Z}^-}(w)\ .$$

  The image $T_k^*(w)\subset\mathfrak{A}_k^{\mathbb{Z}^-}(w)$ of $T_k^*(a)$ must be independent of a.

- *for every $t\in T_k^*(w)$, the curve $S^t(a)$ depends continuously on $a\in w$.*

- *For every $a\in w$, Condition (SR_2^k) states that for every $t\in T_k^*(a)\approx T_k^*(w)$ the puzzle pieces $\mathfrak{a}_i^t(a)$ involved in $\mathfrak{c}_j^t(a)=\mathfrak{a}_1^t(a)\star\cdots\star\mathfrak{a}_k^t(a)$ are in $\mathfrak{A}_k^{(\mathbb{N})}(a)\approx\mathfrak{A}_k^{(\mathbb{N})}(w)$. We ask that the image $\mathfrak{a}_i^t(w)\in\mathfrak{A}_k^{(\mathbb{N})}(w)$ of $\mathfrak{a}_i^t(a)$ does not depend on $a\in w$.*

- *for every $t\in T_k^*(w)$ and $j\le k$, the box $Y_{\mathfrak{c}_j^t}(a)$ depends continuously on $a\in w$ for the Hausdorff distance.*

We remark that a k-combinatorial interval is also a k'-combinatorial interval. This definition enables us to know which pieces and curves are well-defined along a parameter interval.

9.2 Global geometric estimates from combinatory

In this section we show some geometric estimates on the k-puzzle algebras, which are satisfied by any k-strongly regular map and whose statements are combinatorial. They are crucial for the parameter selection.

9.2.1 Lebesgue measure of the complement of the union of puzzle pieces of order $\le j$

The quadratic map case Let P be k-strongly regular. For $M < j \le M - 1 + 2k$, let:

$$\mathcal{E}_k^j := \{x \in \mathbb{R}_\square : x \notin \mathbb{R}_\mathfrak{a}; \forall \mathfrak{a} \in \mathfrak{A}_k^{(\mathbb{N})}, \text{ suitable, complete sequence of symbols s.t. } \; n_\mathfrak{a} \le j\} \, .$$

We notice that every point in $\mathbb{R}_\mathfrak{c} \setminus \mathcal{E}_k^j$ belongs to the segment $\mathbb{R}_\mathfrak{a}$ of a complete, suitable $\mathfrak{A}_k$'s chain $\mathfrak{a}$ of order $\le j$. By Propositions 8.6 and 8.23, $\mathcal{E}_k^j$ consists of the points of $\mathbb{R}_\mathfrak{c}$ which are not in a puzzle piece of order $\le j$ (which is necessarily hyperbolic by Fact 9.3).

Following Yoccoz, the following is a key estimate in his proof of Jakobson's theorem.

Proposition 9.9. *The Lebesgue measure of $\mathcal{E}_k^j$ is smaller than $e^{-\frac{c}{4}j}$.*

Proof. The proof is done by induction on $j \ge M$. For $j = M+1$, it holds $\mathcal{E}_k^j = \mathbb{R}_\square$ whose length is estimated in [Bera, Lemm. 12.2].

Let us assume the bound proved for $M < j < M - 1 + 2k$. Let I be the maximal i so that $M + 1 + n_{\mathfrak{c}_i} \le j - M$ if it exists, and 0 otherwise. An easy computation [Bera, (11) Prop. 6.15] shows that the Lebesgue measure of $\mathbb{R}_{\sigma_I} := (P^{M+1}|\mathbb{R}_\square)^{-1}(\mathbb{R}_{\mathfrak{c}_I})$ is smaller that $\frac{1}{2}e^{-\frac{c}{4}j}$.

We observe that

$$\mathcal{E}_k^{j+1} = \mathbb{R}_{\sigma_I} \cup \bigcup_{i=0}^{I-1} \mathcal{E}_k^j \cap (\mathbb{R}_{\square_+(\mathfrak{c}_i - \mathfrak{c}_{i+1})} \cup \mathbb{R}_{\square_-(\mathfrak{c}_i - \mathfrak{c}_{i+1})}) \, .$$

The set $\mathcal{E}_k^j \cap \mathbb{R}_{\square_+(\mathfrak{c}_i - \mathfrak{c}_{i+1})}$ is equal to the preimage of $\mathcal{E}_k^{j-M-1-n_{\mathfrak{c}_i}}$ by $P^{M+1+n_{\mathfrak{c}_i}}|\mathbb{R}_{\square_+(\mathfrak{c}_i - \mathfrak{c}_{i+1})}$. By Proposition 8.23, the hyperbolicity of the parabolic pieces and the induction hypothesis, it holds:

$$\mathrm{Leb}(\mathcal{E}_k^j \cap \mathbb{R}_{\square_+(\mathfrak{c}_i - \mathfrak{c}_{i+1})}) \le \mathrm{Leb}(\mathcal{E}_k^{j - n_{\square_+(\mathfrak{c}_i - \mathfrak{c}_{i+1})}}) e^{-\frac{c}{3} n_{\square_+(\mathfrak{c}_i - \mathfrak{c}_{i+1})}} \le e^{-\frac{c}{4}j - \frac{c}{12}(i+M+1)} \, .$$

Consequently, since M is assumed large:

$$\mathrm{Leb}(\mathcal{E}_k^{j+1}) \le \frac{1}{2} e^{-\frac{c}{4}j} + e^{-\frac{c}{4}j} \sum_{i=M+1}^{I-1} 2 \cdot e^{-\frac{c}{12}i} \le e^{-\frac{c}{4}j} \, .$$

$\square$

The Hénon-like case Let f be a k-strongly regular Hénon like map.

The above upper bound on the measure of the unpuzzled set holds true as well for the flat stretched curve S^t of f. However, in contrast with the one dimensional case, the hyperbolicity of a puzzle piece does not imply a uniform distortion bound (indeed a curve may be dramatically folded along its orbit, even if it satisfies the hyperbolic time inequality). We will not detail this technical aspect of the proof (and we will not even state the uniform distortion estimate [Bera, Def 2.8]!). Let us just mention that a combinatorial way to obtain uniform distortion bound is to consider the *perfect sequence*:

Definition 9.10. *Let $t \in T_k^*$. A complete, suitable $\mathfrak{A}_k$-sequence $\mathfrak{a}_1 \cdots \mathfrak{a}_m$ from S^t is* perfect *if for every i such that $\mathfrak{a}_i \notin \mathfrak{Y}_0$ it holds:*

$$\sum_{j=i+1}^{m} n_{\mathfrak{a}_j} \ge \lceil \frac{4c^+}{c} \rceil n_{\mathfrak{a}_i} \, .$$

As the perfect sequences are complete, they define puzzle pieces called *perfect pieces.* We notice that the composition of two perfect sequences is a perfect sequence.

For every $t \in T_k^*$, let $\mathcal{E}_k^j(S^t)$ be the set of points which do not belong to a perfect piece of order at most j. A development of Prop 9.9 gives:

Proposition 9.11 (Prop. 6.15 [Bera]). *For every $t \in T_k^*$, and $M \le j \le M-1+2k$, the following estimate holds true:*

$$\frac{1}{j}\log Leb\,\mathcal{E}_k^j(S^t) \le -\delta\ , \quad \text{with } \delta = \frac{c}{4(1+\lceil 4\frac{c^+}{c}\rceil}\ .$$

9.2.2 Hausdorff dimension

Let f be a k-strongly regular Hénon-like map. We recall that $(\mathfrak{G}_k,\cdot)$ is the pseudo-monoid formed by the finite segments of pre-sequences in $T_k^* \subset \mathfrak{A}_k^{\mathbb{Z}^-}$.

It is useful to define combinatorial metric *dist* on T_k^*, so that $t \in T^* \mapsto S^t$ is Lipschitz, where the space of flat stretched curves is endowed with a uniform C^1-distance.

Indeed, when the parameter varies along a k-combinatorial interval, this will enable us to see which curves are close to one another, for every parameter.

For this end, we introduced in [Bera] the right divisibility / on the elements of $\mathfrak{G}_k$.

Definition 9.12. *A word $\mathfrak{a} \in \mathfrak{G}_k$ is (right) divisible by $\mathfrak{a}' \in \mathfrak{G}_k$ and we note $\mathfrak{a}/\mathfrak{a}'$ if one of the following conditions holds:*

$(\mathcal{D}_1)$ $\mathfrak{a} = \mathfrak{a}'$ *or* $\mathfrak{a}' = \mathfrak{e}$.

$(\mathcal{D}_2)$ $\mathfrak{a}$ *is of the form* $\square_\pm(\mathfrak{c}_l - \mathfrak{c}_{l+1})$ *and satisfies* $\mathfrak{c}_l/\mathfrak{a}'$.

$(\mathcal{D}_3)$ *there is a splitting* $\mathfrak{a} = \mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdot \mathfrak{a}_3$ *and* $\mathfrak{a}' = \mathfrak{a}_2' \cdot \mathfrak{a}_3$ *into words of* $\mathfrak{G}_k$ *such that* $\mathfrak{a}_2/\mathfrak{a}_2'$.

The two last conditions are recursive but decrease the order $n_\mathfrak{a}$. Thus the right divisibility is well defined by induction on $n_\mathfrak{a}$.

Proposition 9.13 (Prop 5.14 [Bera]). *The right divisibility relation / is a partial order on $\mathfrak{G}_k$. Moreover for all $\mathfrak{a}, \mathfrak{a}', \mathfrak{a}'' \in \mathfrak{G}_k$ it holds:*

- (i) $\mathfrak{a}/\mathfrak{a}' \Rightarrow n_\mathfrak{a} \ge n_{\mathfrak{a}'}$ *with equality iff* $\mathfrak{a} = \mathfrak{a}'$,
- (ii) $\mathfrak{a}/\mathfrak{a}'$ *and* $\mathfrak{a}/\mathfrak{a}''$ *and* $n_{\mathfrak{a}'} \ge n_{\mathfrak{a}''} \Rightarrow \mathfrak{a}'/\mathfrak{a}''$,
- (iii) $\mathfrak{a}/\mathfrak{a}'$ *and* $\mathfrak{a}\cdot\mathfrak{a}'', \mathfrak{a}'\cdot\mathfrak{a}'' \in \mathfrak{G}_k \Rightarrow \mathfrak{a}\cdot\mathfrak{a}''/\mathfrak{a}'\cdot\mathfrak{a}''$.

By properties (i) and (ii) we can define:

Definition 9.14 (GCD). *The* greatest common divisor *of $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{G}_k$ is the element $\mathfrak{d} \in \mathfrak{G}_k$ dividing both $\mathfrak{a}$ and $\mathfrak{a}'$ with maximal order. We denote $\mathfrak{d}$ by $\mathfrak{a} \wedge \mathfrak{a}'$ and put $\nu(\mathfrak{a}, \mathfrak{a}') = n_{\mathfrak{a}\wedge\mathfrak{a}'}$.*

Given two different $t = \cdots \mathfrak{a}_i \cdots \mathfrak{a}_{-1}, t' = \cdots \mathfrak{a}_i' \cdots \mathfrak{a}_{-1}' \in T_k^*$, let m be minimal such that $\mathfrak{a}_{-m} \neq \mathfrak{a}_{-m}'$. Put $N := \max(n_{\mathfrak{a}_{-m}\cdots\mathfrak{a}_{-1}}, n_{\mathfrak{a}_{-m}'\cdots\mathfrak{a}_{-1}'})$. The greatest common divisor of t, t' is:

$$t \wedge t' := (\mathfrak{a}_{-N}\cdots\mathfrak{a}_{-1} \wedge \mathfrak{a}_{-N}'\cdots\mathfrak{a}_{-1}')\ .$$

We are now able to define the distance $dist$ on T_k^*.

Definition 9.15. *The following is a non-Archimedean metric on T_k^*:*

$$dist\colon (t,t') \mapsto b^{\nu(t,t')/4} .$$

Here is the announced proposition which gives geometrical estimates from this combinatorial distance:

Proposition 9.16 (Prop. 5.17n [Bera]). *The function $t \in (T_k^*, dist) \mapsto S^t$ is 1-Lipschitz for a C^1-uniform distance on the space of flat stretched curves.*

The following is a combinatorial counterpart to the notion of favorable times by Benedicks-Carleson [BC91], although it is used for another purpose.

Proposition 9.17 (Prop. 6.5 [Bera]). *For every $\mathfrak{g} \in \mathfrak{G}_k$, the family $\tau_{\mathfrak{g}}^k := \{\mathfrak{d}_i\}_{i=0}^{t_{\mathfrak{g}}}$ is such that:*

(i) $\mathfrak{g}/\mathfrak{d}_{t_{\mathfrak{g}}}/\mathfrak{d}_{t_{\mathfrak{g}}-1}/\cdots/\mathfrak{d}_1/\mathfrak{d}_0$, *with* $\mathfrak{d}_0 \in \mathfrak{A}_1 = \mathfrak{Y}_0 \sqcup \{\square_\delta(\mathfrak{e}-\mathfrak{s});\ \mathfrak{s} \in \mathfrak{Y}_0,\ \delta \in \{\pm\}\}$,

(ii) $n_{\mathfrak{d}_i} < n_{\mathfrak{d}_{i+1}} \le 2M \cdot n_{\mathfrak{d}_i}$ *for* $i < t_{\mathfrak{d}}$ *and* $n_{\mathfrak{g}} \le 2M \cdot n_{\mathfrak{d}_{t_{\mathfrak{g}}}}$,

(iii) the domain of $\mathfrak{d}_i$ contains all the flat stretched curves of S^t for $t \in T_0$, moreover $t \cdot \mathfrak{d}_i$ belongs to T_k^* for every $t \in T_0$,

(iv) for all $k' < k$ and $\mathfrak{g} \in \mathfrak{G}_k \cap \mathfrak{G}_{k'}$, it holds $\tau_{\mathfrak{g}}^k = \tau_{\mathfrak{g}}^{k'}$,

(v) if $\mathfrak{g} \in \mathfrak{G}_k$ is the $\mathfrak{A}_k$-spelling of a common piece $\mathfrak{c}_i^t$, with $t \in T_k^*$, then $\mathfrak{d}_{t_{\mathfrak{g}}} = \mathfrak{g}$.

(vi) if $\mathfrak{g}, \mathfrak{g}' \in \mathfrak{G}_k$ satisfy $\mathfrak{g}/\mathfrak{g}'$ then $\tau_{\mathfrak{g}'}^k$ is made by the elements of $\tau_{\mathfrak{g}}^k$ of order less than $n_{\mathfrak{g}'}$.

By Property *(iv)* we write $\tau_{\mathfrak{g}}$ instead of $\tau_{\mathfrak{g}}^k$.

To bound from above the Hausdorff dimension of T_k^* we consider the set

$$P_{j\,k} := \{tt \cdot \mathfrak{d} \in T_k^* : \mathfrak{d} \in \mathfrak{G}_k \quad \& \quad n_{\mathfrak{d}} \le j\} .$$

We notice that for $j \le M + k' \le M + k$, it holds $P_{j\,k} = P_{j'\,k}$. In that case we denote $P_{j\,k}$ by P_j.

From the two above propositions it comes:

Fact 9.18. *The set $P_{j\,k}$ is $\sum_{i=[j/2M]}^{\infty} b^{i/4} = \frac{b^{\frac{j}{8M}}}{1-\sqrt[4]{b}}$ dense in T_k^*.*

A last combinatorial computation gives:

Fact 9.19. *The cardinality of $P_{j\,k}$ is at most 2^j.*

Proof. For every $t \in T_k^*$ there are at most two symbols in $\mathfrak{A}_k$ with the same order, and the order is at least 2. Hence the cardinal of $P_{j\,k}$ is bounded by:

$$C_j := \sum_{k\ge 2} Card\{(n_i)_{1\le i\le k} \in (\mathbb{Z} \setminus \{-1,0,1\})^k : \sum |n_i| \le j\} .$$

By induction on j we assume that $C_i \le 2^i$ for every $i < j$. Then it holds:

$$C_j \le 2 + \sum_{k=2}^{j-1} 2 \cdot C_{j-k} \le 2 + \sum_{k=2}^{j-1} 2^{j-k+1} = 2^j \ .$$

□

We recall that the *box dimension* of $(T_k^*, dist)$ is $\limsup_{\epsilon \to 0} -\log N(\epsilon)/\log \epsilon$, where $N(\epsilon)$ is the minimal number of ϵ-balls to cover T_k^*. From Facts 9.18 and 9.19, it comes:

Proposition 9.20. *The box dimension of* $(T_k^*, dist)$ *is at most* $-\frac{8M \log 2}{\log b}$.

Remark 9.21. We recall that first M is assumed large and then b is assumed small in function of M. Hence the box dimension is small in function of b.

Remark 9.22. The above estimate is very coarse. It should be possible to define a better combinatorial distance on T_k^*, so that $t \mapsto S^t$ is 1-Lipshitz and whose Hausdorff dimension is nearly $\log 2/\log b$. When Yoccoz was preparing his last lecture at Collège de France with this work, he suggested to state that the GCD of two different simple pieces $\mathfrak{s}_\pm^i$ and $\mathfrak{s}_\pm^j$ of orders i and j, the order of the GCD should not be 0 but $\min(i,j)-1$. This should help to prove a better estimate.

9.3 Ideas of proof in the Parameter selection

9.3.1 Transversality

The one-dimensional case An important fact in the prove of Jakobson's Theorem states that the motion w.r.t a of the common puzzle pieces is "slower" than the one of $P_a^{M+1}(0)$.

Let $\mathfrak{c} = (\mathbb{R}_\mathfrak{c}(a), n_\mathfrak{c})$ be a common piece of P_a. By hyperbolicity this piece persists to a puzzle piece $\mathbb{R}_\mathfrak{c}(a')$ for a' close to a.

To express this difference of speed, it is easier to work in $\mathbb{R}_\mathfrak{r}(a) := [-A_M, -A_{M-1}]$ which contains the critical value a. Indeed in this interval, the derivative of critical of value a w.r.t. a is obviously 1. We recall that $\mathbb{R}_\mathfrak{r}(a)$ is sent by P_a^M onto $\mathbb{R}_\mathfrak{c}(a)$, and we put $n_\mathfrak{r} := M$, and we consider the piece $\mathfrak{r} := (\mathbb{R}_\mathfrak{r}(a), n_\mathfrak{r})$.

For every common piece $\mathfrak{c} := (\mathbb{R}_\mathfrak{c}(a), n_\mathfrak{c})$ for P_a, let $\mathbb{R}_{\mathfrak{r}\star\mathfrak{c}}(a) := (P_a|\mathbb{R}_\mathfrak{r}(a))^{-M}(\mathbb{R}_\mathfrak{c}(a))$.

Proposition 9.23 (Prop. 9.3[Bera])**.** *For every common piece* $\mathfrak{c}$ *for* P_a*, if* $\mathbb{R}_{\mathfrak{r}\star\mathfrak{c}}(a) =: [x^-(a), x^+(a)]$*, then for* M *large, it holds:*

$$\partial_a x^\pm(a) = \frac{1}{3} + o(e^{-\sqrt{M}}) \ .$$

The Hénon like case Similarly, we consider the box:

$$Y_\mathfrak{r}(\hat{P}_a) := \{(x,y) \in \mathbb{R}^- \times [-2\theta, 2\theta] : x^2 + y \in [A_{M-1}^2, A_M^2]\} \ .$$

It is bounded by the lines $\{y = \pm 2\theta\}$ and by the following segments of the local stable manifold of the fixed point $(A_0, 0)$ of $\hat{P}_a : (x,y) \mapsto (x^2 + a + y, 0)$.

$$W^s_{\mathrm{loc}}(-A_M; \hat{P}_a) \sqcup W^s_{\mathrm{loc}}(-A_{M-1}; \hat{P}_a) := \{(x,y) \in \mathbb{R}^- \times [-2\theta, 2\theta) : x^2 + y \in \{A_M^2, A_{M-1}^2\}\} \ .$$

For $(f_a)_a$ C^2-close to $(\hat{P}_a)_a$, the local stable manifolds $W^s_{\text{loc}}(-A_M;\hat{P}_a)\sqcup W^s_{\text{loc}}(-A_{M-1};\hat{P}_a)$ persists to $W^s_{\text{loc}}(-A_M;f_a)\sqcup W^s_{\text{loc}}(-A_{M-1};f_a)$, and bounds with the line $\{y=\pm 2\theta\}$ a box $Y_{\mathfrak{r}}(a)$.

Let w be a k-combinatorial interval for $(f_a)_a$ and let $a\in w$.

Given a common sequence $\mathfrak{c}$ we define the box $Y_{\mathfrak{r}\star\mathfrak{c}}(a) := f_a^{-M}(Y_{\mathfrak{c}}(a))\cap Y_{\mathfrak{r}}(a)$. It is bounded by the lines $\{y=\pm 2\theta\}$ and by the arcs $\partial^s Y_{\mathfrak{r}\star\mathfrak{c}}(a)$ of local stable manifolds of A.

Proposition 9.24 (Prop. 9.3 [Bera])**.** *For every $a_0\in w$, there exists a neighborhood V_a of a_0 in w and a C^2-function $\rho_{\mathfrak{c}}\colon V_A\times \partial^s Y_{\mathfrak{r}\star\mathfrak{c}}(a_0)\to\mathbb{R}^2$ such that for every $z\in\partial^s Y_{\mathfrak{r}\star\mathfrak{c}}(a_0)$:*

(i) $\rho_{\mathfrak{c}}(a_0,z)=z$,

(ii) $|\partial_a\rho_{\mathfrak{c}}(a,z)|=\frac{1}{3}+o(e^{-\sqrt{M}})$ *for M large and $\partial_a\rho_{\mathfrak{c}}(a,z)$ is horizontal,*

(iii) $\rho_{\mathfrak{c}}(a,\partial^s Y_{\mathfrak{r}\star\mathfrak{c}}(a_0))$ *is equal to $\partial^s Y_{\mathfrak{r}\star\mathfrak{c}}(a)$ for every $a\in V_a$.*

On the other hand, the flat stretched curves $(S^t(a))_{t\in T_k^*(w)}$ display a *non-artificial* segment which is C^1-close to depend horizontally on $a\in w$. For $t=t_0\cdot\mathfrak{a}\in T_k^*$ with $t_0\in T_0$, the non-artificial segment of $S^t(a)$ is $f_a^{n_{\mathfrak{a}}}(S^{t_0}_{\mathfrak{a}}(a))$.

Proposition 9.25 (Prop. 10.3 [Bera])**.** *For every $a_0\in w$, for every $t=t_0\cdot\mathfrak{a}\in T_k^*(w)$ with $t_0\in T_0$, the following surface of $\mathbb{R}^3$ displays a tangent space which makes an angle less than 2θ with the plane $\mathbb{R}^2\times\{0\}$.*

$$\bigcup_{a\in w}\{a\}\times f_a^{n_{\mathfrak{a}}}(S^{t_0}_{\mathfrak{a}}(a))\ .$$

Hence they are folded by f_a in a fashion way which is C^2-close to $\{(x^2+a,0): x\in\mathbb{R}_e(a)\}$.

9.3.2 Cantors sets of positive measure and rough idea of the parameter selection

The one dimensional case A necessary (and actually sufficient) condition for a quadratic map $P(a)$ to be strongly regular is that there exists a common sequence $\mathfrak{c}=(\mathfrak{a}_i)_{i\ge 1}$ so that the critical value a of P_a belongs to the intersection point $x_{\mathfrak{c}}(a)$ of $\cap_{i\ge1}\mathbb{R}_{\mathfrak{a}_1\star\cdots\star\mathfrak{a}_i}(a)$.

By using a large deviation argument (as in the parameter selection of [Yoc97]), we show:

Proposition 9.26 (Prop 3.10 [Bera])**.** *Let P_a be strongly regular, and let $\mathbb{R}_{\mathcal{L}}(a)\subset\mathbb{R}_{\mathfrak{e}}$ be the set of intersection points $x_{\mathfrak{c}}(a)=\cap_{i\ge1}\mathbb{R}_{\mathfrak{a}_1\star\cdots\star\mathfrak{a}_i}(a)$ of common sequence $\mathfrak{c}:=(\mathfrak{a}_i)_i$. Then it holds for M large:*

$$\frac{Leb\,\mathbb{R}_{\mathcal{L}}(a)}{Leb\,\mathbb{R}_{\mathfrak{e}}}=1+o(1)\ .$$

By transversality and the latter estimate we can believe that when the critical value $P_a^{M+1}(0)$ varies with a, it belongs to $\mathcal{L}(a)$ for a positive set of parameters a.

It is a coarse idea since when a varies, the geometry of $\mathcal{L}(a)$ varies as well...

The Hénon-like case A necessary condition for a Hénon-like map f_a is that each of the curves $(S^t)^\square := f_a^{M+1}(S^t \cap Y_\square)$ is tangent to a common stable manifold $W^s_{\mathfrak{c}}(a)$.

Actually the union $\mathcal{L}(a) := \cup_{\mathfrak{c}} W^s_{\mathfrak{c}}(a)$ of the local stable manifolds defined by common sequences is a Lipschitz lamination [Bera, Lemma 13.9]. Hence for C^1-coordinate on $Y_{\mathfrak{c}} \approx \mathbb{R}_{\mathfrak{c}} \times [-2\theta, 2\theta]$, the set $\mathcal{L}$ corresponds to the product of a Cantor set $\mathbb{R}_{\mathcal{L}}(a)$ with $[-2\theta, 2\theta]$:

$$\mathcal{L}(a) \approx \mathbb{R}_{\mathcal{L}}(a) \times [-2\theta, 2\theta]$$

By Proposition 3.10 [Bera], the Lebesgue measure of the Cantor set $\mathbb{R}_{\mathcal{L}}(a)$ is positive.

Also for every $t \in T^*(a)$, the curve $(S^t)^\square$ is tangent to a unique fiber $\{x_t(a)\} \times [-2\theta, 2\theta]$ of $Y_{\mathfrak{c}} \approx \mathbb{R}_{\mathfrak{c}} \times [-2\theta, 2\theta]$.

Hence the tangency condition is equivalent to ask that the Cantor set $T^*(a) \approx \{x_t(a) : t \in T^*(a)\}$ is included in the Cantor set of positive measure $\mathbb{R}_{\mathcal{L}}(a)$.

From the transversality condition, a rough idea of the parameter selection would be to show that for a positive set of translation $\tau \in \mathbb{R}$, it holds that $T^*(a) + \tau \subset \mathbb{R}_{\mathcal{L}}(a)$.

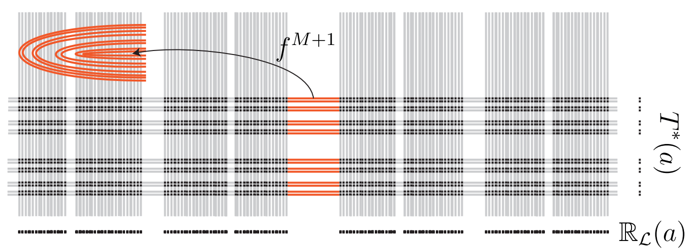


Figure 14: Rough idea of the parameter selection for Hénon-like endomorphisms

For many people this condition is anti-intuitive, in the sense that they did not believe that one can find two Cantor sets K and $\tilde{K}$ so that

$$\text{Leb}\,\{\tau \in \mathbb{R} : K + \tau \subset \tilde{K}\}\,.$$

However our main result with Moreira states a sufficient condition (which is also necessary in many cases) for this to happen. More specifically, we showed the following:

Theorem 9.27 (Thm 2.1 [BM16])**.** *Let $K \subset \mathbb{R}$ be a Cantor set of box dimension smaller than d: there exists $C_K > 0$ so that K can be covered by $C_K \epsilon^{-d}$ ϵ-balls for every $\epsilon > 0$.*

Let $\tilde{K} := [0, diam\,\tilde{K}] \setminus \sqcup_n (a_n, b_n)$ be a Cantor set such that with $l_n = b_n - a_n$ it holds:

$$(C_{1-d}) \qquad \sum_n l_n^{1-d} < \infty\,.$$

Then the set of parameters $t \in \mathbb{R}$ so that $K + t \subset \tilde{K}$ has positive measure if it holds:

$$\sum_{n: l_n > diam\,K} (diam\,K + l_n) + 2C_K \sum_{n: l_n \leq diam\,K} (l_n)^{1-d} < diam\,\tilde{K} - diam\,K$$

Remark 9.28. This result enjoys many applications in Diophantine approximation, see [BM16].

We recall that in Proposition 9.20 we showed that there exists d small when b is small s.t. $T^*(a)$ is covered by a $C_k \epsilon^{-d}$ ϵ-balls for every $\epsilon > 0$.

It is reminiscent in the proof of [Bera] that the set $\mathbb{R}_{\mathcal{L}}(a)$ satisfies Condition (C_{1-d}) for d sufficiently small. This leads us to ask:

Problem 9.29. *What is the minimal p so that $\mathbb{R}_{\mathcal{L}}(a)$ satisfies Condition (C_p)?*

In [BM16] we studied several toy models for $\mathbb{R}_{\mathcal{L}}(a)$. This let us dream that the case of a strongly regular theory could even contain an example of attractor of the same dimension as the one of the initial Hénon conjecture.

Here again it is a rough idea of the parameter selection: when a varies, the geometries of both $\{x_t(a) : t \in T^*(a)\}$ and $\mathbb{R}_{\mathcal{L}}(a)$ vary.

To overcome this difficulty we cover the $j \ll k$-combinatorial intervals by finitely many parameter intervals, each of which is associated to point $t \in P_j$. Then we do the parameter selection at the level k. By working on an induction at two scales, we deal with a (locally) constant geometry (for the relevant pieces). Then we are able to evaluate the measure of the parameters removed by a large deviation argument along a tree (defined thanks to the right division).

10 Ergodic properties of strongly regular maps

In this section we present [Berd] which showed that every strongly regular Hénon-like diffeomorphism displays ergodic properties very similar to those of uniformly hyperbolic attractors.

10.1 Topological Collet-Eckmann condition

It is well known that the Collet-Eckmann condition for unimodal map of the interval (which is implied by the Yoccoz' strong regularity condition) implies the topological Collet-Eckmann condition. The latter implies (see [PRLS03]) the existence of $m > 0$ so that the positive Lyapunov exponent of every invariant ergodic probability measure is bounded from below by m.

L. Carleson (as related by S. Newhouse during the first Palis-Balzan conference) asked if some non-uniformly hyperbolic Hénon-like diffeomorphism displays such a property. In [Berd], we answered to his question:

Theorem 10.1 ([Berd])**.** *For every strongly regular Hénon-like map f, ergodic probability measure μ has a Lyapunov exponent greater than $\frac{c}{3} > 0$.*

The same conclusion has been recently proved for non-uniformly hyperbolic horseshoes which appear as perturbations of the first bifurcation of Hénon-like maps [Tak13].

Let us share a problem we discussed several times with R. Dujardin and M. Lyubich:

Problem 10.2. *Define the notion of topological Collet-Eckmann complex Hénon maps, prove the existence of new examples and show similar properties to those of [PRLS03].*

10.2 Maximal entropy measure and equi-distribution on the periodic points

Definitions Let us recall the definitions of *entropy*. For two covers $\mathcal{O}$ and $\mathcal{O}'$ of M, the family of intersections of a set from $\mathcal{O}$ with a set from $\mathcal{O}'$ forms a covering $\mathcal{O} \vee \mathcal{O}'$, and similarly for multiple covers. For any finite open cover $\mathcal{O}$ of M, let $H(\mathcal{O})$ be the logarithm of the smallest number of elements of $\mathcal{O}$ that cover M. The following limit exists:

$$H(\mathcal{O}, f) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{O} \vee f^{-1}\mathcal{O} \vee \cdots \vee f^{-n}\mathcal{O}).$$

Definition 10.3 (Topological entropy)**.** *The* topological entropy $h(f)$ of f *is the supremum of* $H(\mathcal{O}, f)$ *over all finite covers* $\mathcal{O}$ *of* M*.*

Given a measure μ, the *entropy of* μ is defined similarly. For a finite partition $\mathcal{O}$, put:

$$H_\mu(\mathcal{O}, f) = \lim_{n\to\infty} \frac{1}{n} \sum_{E \in \mathcal{O} \vee f^{-1}\mathcal{O} \vee \cdots \vee f^{-n}\mathcal{O}} -\mu(E) \log \mu(E) \ .$$

Definition 10.4 (Metric entropy)**.** *The* entropy h_μ of μ *is the supremum of* $H_\mu(\mathcal{O}, f)$ *over all possible finite partitions* $\mathcal{O}$ *of* M*.*

From the Variational Principle, the topological entropy is the supremum of entropies of invariant probability measures:

$$h(f) = \sup\{h_\mu(f) : \mu \text{ probability } f\text{-invariant}\}.$$

Therefore the topological entropy is an *ergodic invariant*, *i.e.* it is invariant by bi-measurable conjugacy.

Definition 10.5 (Maximal entropy measure)**.** *A probability* μ *has* maximal entropy *if* $h(f) = h_\mu(f)$*. The measure* μ *is equidistributed on the set of periodic points if* μ *is the limit of the following sequence:*

$$\frac{1}{Card\ Fix\ f^n} \sum_{z \in Fix\ f^n} \delta_z \rightharpoonup \mu \ .$$

Example 10.6 (Uniformly hyperbolic maps)*.* We recall that every uniformly hyperbolic set Λ admits a (finite) Markov partition. This implies that its dynamics is semi-conjugated with a subshift of finite type. The semi-conjugacy is 1-1 on a generic set. Its lack of injectivity is itself coded by subshifts of finite type of smaller topological entropy. This enables one to study efficiently all the invariant measures of Λ, to show the existence and uniqueness of the maximal entropy measure ν, and to show the equidistribution of the set of periodic points w.r.t. ν.

Example 10.7 (Unimodal map)*.* In [Hof81], a coding is given to prove the existence and the uniqueness of the maximal entropy measure for unimodal maps of positive entropy. This measure does not need to be equidistributed on the set of periodic points [KK12] if the critical point is flat and the map finitely smooth.

We recall that Yoccoz' strongly regular map displays a positive entropy.

Example 10.8 (Complex map). For every rational function of the Riemannian sphere, the existence and uniqueness of a maximal entropy measure is known, since the works of Lyubich [Lju83] and [Mañ83]. They showed also its equidistribution on the set of periodic points.

For every complex polynomial automorphism of $\mathbb{C}^2$, the existence, uniqueness, and equidistribution on the set of periodic points of a maximal entropy measure has been shown in the work of Bedford-Lyubich-Smillie [BLS93].

Example 10.9 (C^∞-case). In finite regularity, a measure of maximal entropy needs not exist [Gur69]. Nevertheless, a famous theorem of Newhouse states the existence of a maximal entropy measure for every C^∞-diffeomorphism [New89] of any compact manifold.

Recently Buzzi-Crovisier-Sarig announced the proof of the uniqueness of the maximal entropy measure for every surface C^∞-diffeomorphism whose non-wandering set is transitive and whose entropy is positive. This result uses the finite to one Markovian coding of Sarig [Sar13] of the union of the support of invariant measures with Lyapunov exponents uniformly far from 0.

We recall that the unique abundant examples of surface diffeomorphisms which displays a transitive set with topological positive entropy are:

- The non-uniformly hyperbolic horseshoe. In [PY09], a certain Markovian coding is given on the maximal invariant set, but it is not easy to see if this implies the uniqueness of the maximal entropy measure, neither if all the Lyapunov exponents of every measure are uniformly far from 0.

- The Hénon-like attractor. In [YW01], a certain coding is given in order to prove the existence of a maximal entropy measure for Hénon attractors of Benedicks-Carleson type, but the formalism does not seem to imply easily its uniqueness.

In [Berd], we showed the existence and uniqueness of a measure of maximal entropy (this answers a question of Lyubich and Pesin) for every strongly regular Hénon-like diffeomorphism (with sufficiently small determinant). Let us point out that the dynamics is only of class C^2 and so the existence of such a measure is not implied by Newhouse Theorem [New89]. Furthermore we proved that the maximal entropy measure is equi-distributed on the periodic points and is finitarily Bernoulli (an answer to a question of Thouvenot).

Theorem 10.10. *Every strongly regular Hénon-like diffeomorphisms f leaves invariant a unique probability of maximal entropy ν. Moreover ν is equi-distributed on the periodic points of f, finitarily Bernoulli, exponentially mixing and it satisfies the central limit Theorem.*

A *Bernouilli shift* is the shift dynamics of $\Sigma_N := \{1, \ldots, N\}^{\mathbb{Z}}$ endowed with the product probability $p^{\mathbb{Z}}$ spanned by a probability $p = (p_i)_{i=1}^N$ on $\{1, \ldots, N\}$. The entropy of the probability $p^{\mathbb{Z}}$ is $h_p = -\sum_i p_i \log p_i$. By Ornstein and Kean-Smorodinsky isomorphism Theorems, any two Bernouilli shifts $(\Sigma_N, p^{\mathbb{Z}})$ and $(\Sigma_{N'}, p'^{\mathbb{Z}})$ with the same entropy $h_{p^{\mathbb{Z}}} = h_{p'^{\mathbb{Z}}}$ are *finitarily isomorphic* [KS79]. A bi-measurable isomorphism is *finitary* if it and its inverse send open sets to open sets, modulo null sets.

To be *finitarily Bernoulli* means that the dynamics, with respect to the maximal entropy measure, is finitarily isomorphic to a Bernouilli shift.

The *central limit Theorem* is that for every Hölder function ψ of ν-mean 0, such that $\psi \neq \phi - \phi \circ f$ for any ϕ continuous, there exists $\sigma > 0$ such that $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Psi \circ f^i$ converges in distribution (w.r.t. ν) to the normal distribution with mean zero and standard deviation σ.

The measure ν is *exponentially mixing* if there exists $0 < \kappa < 1$ such that for every pair of functions of the plane $g \in L^\infty(\nu)$ and h Hölder continuous, there is $C(g,h) > 0$ satisfying for every $n \geq 0$:

$$Cov_\nu(g, h \circ f^n) < C(g,h)\kappa^n, \quad \text{with } Cov \text{ the covariance.}$$

The proof is based on a new construction of Young's tower for which the first return time coincides with the symbolic return time, and whose orbit is conjugated to a strongly positive recurrent Markov shift.

10.3 Idea of proofs

10.3.1 Idea of proof of Theorem 10.1

We recall that the strong regularity definition involves a countable alphabet $\mathfrak{A}$ and a set of pre-sequences $T^* \subset \mathfrak{A}^{\mathbb{Z}^-}$. Every $t \in T^*$ is associated to a curve S^t and a common sequence $\mathfrak{c}^t$. The common sequence defines a nested sequence of puzzle pieces $(\mathfrak{c}_i^t(S^{tt}))_{i \geq 0}$ of S^{tt}. Local stable manifolds of the endpoints of $\mathfrak{c}_i^t(S^{tt})$ cross the domain $Y_\mathfrak{e}$ to define a domain $Y_{\mathfrak{c}_i^t} \subset Y_\mathfrak{e}$. The intersection $W^s_{\mathfrak{c}^t} = \cap_i Y_{\mathfrak{c}_i^t}$ is a local stable manifolds. By (SR_1), the curve $f^{M+1}(S^t \cap Y_\square)$ is tangent to $W^s_{\mathfrak{c}^t}$.

Let us denote $Y_{\square(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)} := (f^{M+1}|Y_\square)^{-1}(cl(Y_{\mathfrak{c}_i^t} \setminus Y_{\mathfrak{c}_{i+1}^t}))$ and $Y_{\square\mathfrak{c}^t} := (f^{M+1}|Y_\square)^{-1}(W^s_{\mathfrak{c}^t})$.

We observe that $\{Y_\mathfrak{s} : \mathfrak{s} \in \mathfrak{Y}_0\} \sqcup \{Y_{\square(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)} : i \geq 0\} \cup \{Y_{\square\mathfrak{c}^t}\}$ is a partition of $Y_\mathfrak{e}$ (modulo the stable manifold of the fixed point A).

In figure 15, we depict that the set $Y_{\square(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)}$ is formed by 1, 2 or 3 connected components:

- a (below) component $Y_{\square_b(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)}$ which may exist,
- either a component $Y_{\square_a(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)}$ or two components $Y_{\square_+(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)}$ and $Y_{\square_-(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)}$.

If a component $Y_{\square_a(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)}$ occurs, then we split it into two domains $Y_{\square_+(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)}$ and $Y_{\square_-(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)}$ (following a certain algorithm), as depicted in figure 16.

Then we observe that the following set is a partition of $Y_\mathfrak{e}$ (modulo $W^s(A)$):

$$\mathcal{P}(t) := \{Y_\mathfrak{s} : \mathfrak{s} \in Y_0\} \cup \{Y_{\square_\pm(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)} : i \geq 0\} \cup \{Y_{\square_b(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t)} : i \geq 0\} \cup \{Y_{\square\mathfrak{c}^t}\} .$$

We denote by $\mathfrak{P}(t) := Y_0 \cup \{\square_\pm(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t) : i \geq 0\} \cup \{\square_b(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t) : i \geq 0\} \cup \{\square\mathfrak{c}^t\}$ the set of symbols associated.

We observe that $\mathfrak{P}(t)$ depends on t, and is included in $\mathfrak{A} \sqcup \{\square_b(\mathfrak{c}_i^t - \mathfrak{c}_{i+1}^t) : i \geq 0\} \cup \{\square\mathfrak{c}^t\}$

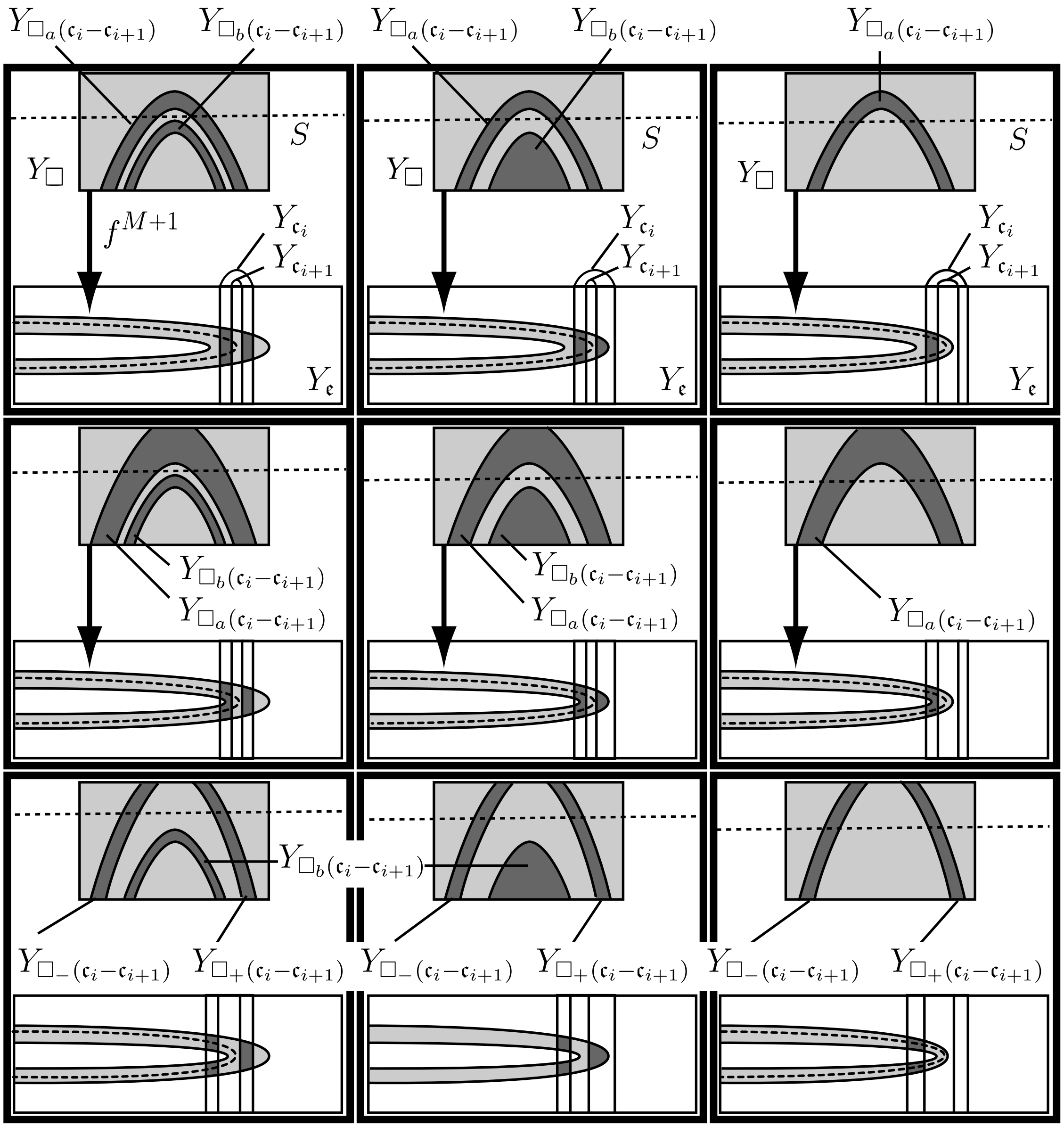


Figure 15: Possible shapes for $Y_{\Box(\mathfrak{c}_i-\mathfrak{c}_{i+1})}$.

A fibered encoding

Definition 10.11. *A sequence of symbols $\mathfrak{g} = (\mathfrak{b}_i)_{i=0}^m \in \mathfrak{A}^m$ is* regular *if $\mathfrak{g}$ is suitable from $S^{\mathtt{tt}}$ and the following inequality holds for every $i \le m$:*

$$n_{\mathfrak{b}_i} \le M + \Xi \sum_{1 \le j < i} n_{\mathfrak{b}_j} \ ,$$

with $\Xi := e^{\sqrt{M}}$.

We denote by $\tilde{\mathfrak{R}} \subset \mathfrak{A}^\infty$ the set of regular sequence of infinite length.

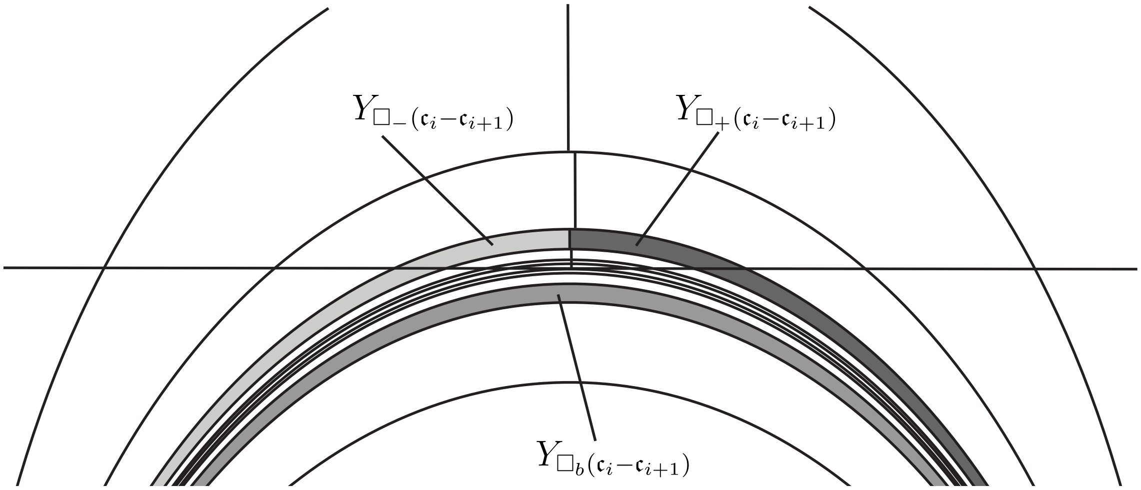


Figure 16: Partition of $Y_\Box$.

For every regular sequence $\mathfrak{g} = \mathfrak{b}_0 \cdots \mathfrak{b}_m$, we define the set

$$Y_\mathfrak{g} := \{z \in Y_\mathfrak{e} : \ f^{n_{\mathfrak{b}_0 \cdots \mathfrak{b}_i}}(z) \in Y_{\mathfrak{b}_{i+1}}, \ \forall i < m\}$$

In proposition 2.12 [Berd], we showed that $Y_\mathfrak{g}$ displays a very nice geometry (in the same way as a common piece $\mathfrak{c}$ displays a box $Y_\mathfrak{c}$ with a nice geometry). Moreover, we show that for every $z \in Y_\mathfrak{g}$, for every unit vector u θ-close to the horizontal, it holds for every $i \le m$:

$$(D) \qquad \|D_z f^{n_{\mathfrak{b}_0}+\cdots+n_{\mathfrak{b}_i}}(u)\| \ge e^{\frac{c}{3}(n_{\mathfrak{b}_0}+\cdots+n_{\mathfrak{b}_i})} .$$

In particular when $m = \infty$, the set $Y_\mathfrak{g}$ is a stable curve which is asymptotically normally expanded by a factor $e^{c/3}$. The idea of the proof is to show that given any ergodic, probability measure μ (not supported by a certain uniformly compact set K^* with small topological entropy), it holds that μ-a.e. point z is sent by a certain iterate into the union of curves:

$$\tilde{\mathcal{R}} = \bigcup_{\mathfrak{g} \in \tilde{\mathfrak{R}}} Y_\mathfrak{g} .$$

Then the uniform bound from below of the Lyapunov exponent follows from (D).

To show this given $z \in Y_\mathfrak{e} \setminus W^s(A)$, we look at the maximal regular length p of a regular sequence $\mathfrak{g} = \mathfrak{a}_0 \cdots \mathfrak{a}_{p-1}$ so that z belongs to $Y_\mathfrak{g}$. Note that p may be equal to 0.

If $p = \infty$ we are done.

Otherwise we show that $z_1 := f^{n_\mathfrak{g}+M+1}(z)$ belongs to $Y_{\mathfrak{c}_i} \subset Y_\mathfrak{e}$ with $n_{\mathfrak{c}_i} \ge \Xi n_\mathfrak{g}$. Hence we can define again a maximal regular sequence $\mathfrak{g}_1$ so that $z_1 \in Y_{\mathfrak{g}_1}$. And so on we continue by defining $\mathfrak{g}_2, ..., \mathfrak{g}_k, ...$ and $z_2, ..., z_k, ...$ until we may fall into a $\tilde{\mathcal{R}}$.

We recall that if z_k belongs to $\tilde{\mathcal{R}}$ for some k, then we are done. Otherwise, we recall that $n_{\mathfrak{g}_{i+1}} \ge \Xi n_{\mathfrak{g}_i}$. Then:

- Either $n_{g_i} = 0$ for every i, and so z belongs to the (finite) orbit of the compact set

$$K_\square := \cap_{i\geq 0} f^{-i(M+1)}(Y_\square) \ ,$$

- or $(n_{\mathfrak{g}_i})_i$ grows eventually exponentially fast (with factor Ξ) to infinity.

To solve the first case we show that $K^* := \cup_{0\leq j\leq M} f^j(K)$ is a uniformly hyperbolic set [Bera, Prop. 2.19] with expansion at least $e^{c/3}$.

In the second case, we recall that μ-a.e. point z displays defined Lyapunov exponent.

If they are both negative, then by Katok's closing Lemma, z is a periodic sink. Hence an iterate of z belongs to infinitely many $Y_{\mathfrak{g}_i}$. We chose one $\mathfrak{g}_i$ so that $n_{\mathfrak{g}_i}$ is larger than the period of z. This contradicts (D).

If one Lyapunov exponent is non-negative, then a non-contracted direction E^{cu} is well defined at z. However z_{i+1} is exponentially close to the tangency point between $f^{M+1}(S^{\mathfrak{t}\cdot\mathfrak{g}_i} \cap Y_\square)$ with $W^s_{c^{\mathfrak{t}}\cdot\mathfrak{g}_i}$, and we show that the image $D_z f^{n_{\mathfrak{g}_0}+M+1+n_{\mathfrak{g}_1}\dots+M+1+n_{\mathfrak{g}_i}}(E^{cu})$ of the E^{cu} is exponentially close to the tangent space at this tangency [Berd, Prop. 2.7]. By b-contraction of $W^s_{c^{\mathfrak{t}}\cdot\mathfrak{g}_i}$, it follows that E^{cu} is sufficiently contracted during a sufficiently long time. This implies that the sequence $(\frac{1}{n}\log\|D_z f^k|E^{cu}\|)_k$ does not converge. Hence the contradiction. $\square$

10.3.2 Idea of proof of Theorem 10.10

In the previous section we saw that any invariant measure is supported by the orbit of $\tilde{\mathcal{R}}$ and $K_\square$.

It is rather easy to bound from above the topological entropy of $\hat{K}_\square = \cup_{i=0}^M f^i(K_\square)$ by $\log 2/M$. In particular this set does not support an invariant measure with high entropy.

Hence we shall study the measure supported by the orbit of $\tilde{\mathcal{R}} = \cup_{\mathfrak{g}\in\tilde{\mathfrak{R}}} Y_{\mathfrak{g}}$. The set $\tilde{\mathfrak{R}}$ is included in $\mathfrak{A}^{\mathbb{N}}$ on which the shift dynamics acts. We consider the set $\mathfrak{R}$ formed by the sequence $\mathfrak{g}\in\tilde{\mathfrak{R}}$ which come back infinitely often in $\tilde{\mathfrak{R}}$ by the shift dynamics. We put $\mathfrak{E} := \tilde{\mathfrak{R}}\setminus\mathfrak{R}$. This splits $\mathcal{R}$ into two sets:

$$\mathcal{E} := \cup_{\mathfrak{g}\in\mathfrak{E}} Y_g \quad \text{and} \quad \mathcal{R} := \cup_{\mathfrak{g}\in\mathfrak{R}} Y_{\mathfrak{g}} \ .$$

In section 4.2 of [Bera] we use an argument based on the Hausdorff dimension (following a study close to [Sen03]) and the Ledrappier-Young entropy Formula [LY85b] to show that any measure supported by $\mathcal{E}$ displays a small entropy.

Hence all invariant measures with substantial entropy are supported by the orbit of $\mathcal{R}$. In order to study them thanks to the combinatory of $\mathfrak{R}$, we consider for every $\mathfrak{g}$ its first return $\mathfrak{g}_1$ in $\mathfrak{R}$ by the shift. This defines $\mathfrak{a}_1\in\mathfrak{A}^{(\mathbb{N})}$ so that $\mathfrak{a}_1\cdot\mathfrak{g}_1 = \mathfrak{g}$ and for $x\in Y_{\mathfrak{g}}$, an integer $N(x) := n_{\mathfrak{a}_1}$. We put:

$$f^{\mathcal{R}} := x\in\mathcal{R}\mapsto f^{N(x)}(x)\in\mathcal{R}.$$

We notice that $f^{\mathcal{R}}$ is semi-conjugated to the first return map induced by the shift on $\mathfrak{R}$. Nevertheless, $f^{\mathcal{R}}$ is in general not the first return map of $\mathcal{R}$ into $\mathcal{R}$. To obtain such a property, we consider $R := \cap_{n\geq 0}(f^{\mathcal{R}})^n(\mathcal{R})$.

It turns out that the orbit of R supports the same measures as the orbit of $\mathcal{R}$ (see Prop. 3.2 [Berd]), and so we can indeed focus on the dynamics $f^{\mathcal{R}}|\mathcal{R}$.

Furthermore, a surprising property is that the first return of a point x of R in R is exactly $N(x)$ (see Prop. 3.3. [Bera]). This implies that the inverse limit $\overleftarrow{\mathfrak{R}}$ of $\mathfrak{R}$ for the shift dynamics is canonically conjugated to $f^{\mathcal{R}}|R$ (see Prop 3.4 [Bera]).

Hence it suffices to study the combinatory of $\overleftarrow{\mathfrak{R}}$ to deduce the ergodic properties of $f^{\mathcal{R}}|R$ therefore any measure of f with substantial entropy. This part of the proof is rather straightforward. As a matter of fact we obtain a Young tower, whose first return time coincides with the combinatorial return time, and whose associated Markov chain is strongly positive recurrent.

11 Dynamics derivated from the standard map

A wished candidate in the list of paradigmatic examples of non-uniformly hyperbolic dynamics is the Chirikov-Taylor Standard map family $(S_r)_{r\in\mathbb{R}}$. It is formed by conservative maps of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$S_r(x, y) = (2x - y + r\sin(x), x).$$

Sinai conjectured ([Sin94] P.144) that for every non-zero parameter r, the Lyapunov exponents of S_r are non-zero at a set of points $z \in \mathbb{T}^2$ of positive Lebesgue measure. Equivalently, this conjecture states that the metric entropy of S_r is positive for every $r \neq 0$.

This conjecture is very hard: there is not a single r for which we know that the metric entropy of S_r is positive.

On the other hand, the important negative result of P. Duarte [Dua94] states that there exists $r_0 \geq 0$ so that for a topologically generic $r \geq r_0$, the map S_r displays infinitely many elliptic islands (see fig. 17). J. De Simoi [DS13] showed that this generic set of parameters has Hausdorff dimension at least 1/4. A. Gorodetski gave a description of a "stochastic sea" for a generic set of these parameters [Gor12]: his impressive construction showed that for such parameters there exists an increasing sequence of hyperbolic sets with Hausdorff dimension converging to 2, and such that elliptic islands accumulate on them.

Another way to construct examples of non-uniformly hyperbolic dynamics is to work with those which (locally) fiber over a uniformly hyperbolic one. Such techniques have been used notably by [Shu71], [Via97], and [SW00] to produce new examples which are robustly non-uniformly hyperbolic.

Definition 11.1. *A map f of a compact manifold M is C^s-*robustly non-uniformly hyperbolic *if there exists a C^s-neighborhood U of f such that every map $g \in U$ the following property holds true: For Lebesgue almost every point z of M, there are two subbundle $E^s_z \oplus E^u_z = T_zM$ so that:*

$$\limsup \frac{1}{n}\log\|D_zf^n|E^s_z\| < 0 \quad \textit{and} \quad \liminf \frac{1}{n}\log\|(D_zf^n)^{-1}|E^u_z\| > 0\ .$$

The general expectation is that (with possibly a few more hypotheses) if f is robustly non-uniformly hyperbolic (in the sense above) then f leaves invariant at most countably many physical, probability measures, whose union of the basins covers M modulo a Lebesgue null set. Many intermediate results exist in that direction, generalizing the initial works of Alvez-Bonatti-Viana and Bonatti-Viana [ABV00, BV00]. When the dynamics preserve the Lebesgue measure of M,

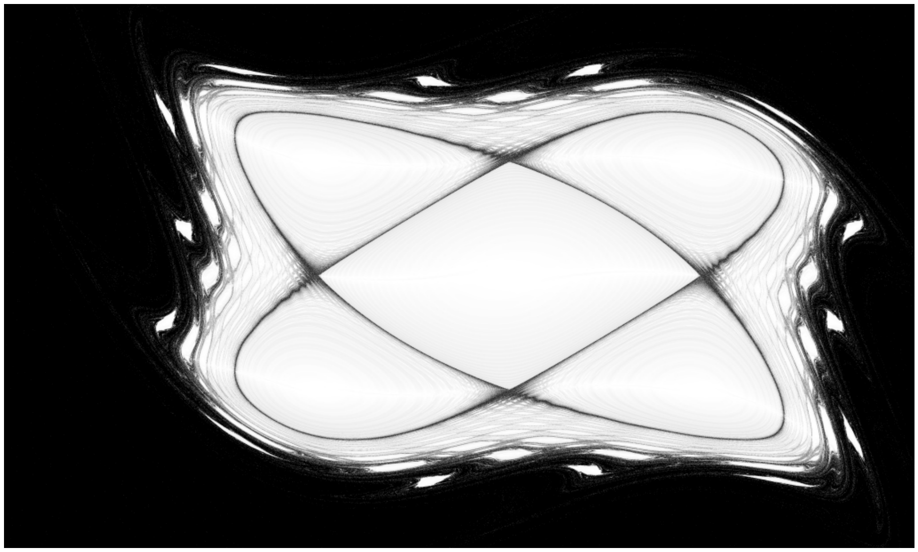


Figure 17: Lyapunov exponent for the parameter $r = -0.364$ of the standard map. White regions are expected to be elliptic islands whereas the black regions are possibly the homoclinic web of a NUH attractor.

this expectation is a theorem of Pesin [Pes77]: if Lebesgue a.e. point displays non-zero Lyapunov exponents, then M is covered by countably many measures, modulo a Lebesgue null set.

In [AV10], another example has been given, namely a non-hyperbolic ergodic toral automorphism for which most symplectic perturbations are non-uniformly hyperbolic. The techniques developed there are also suitable for dealing with some conservative cases, pushing forward the method developed in [SW00] for volume preserving diffeomorphisms.

In [BC14] we showed the existence of a non-hyperbolic robust conservative non-uniformly hyperbolic diffeomorphism, adding a different type of example to the above list.

Let $A \in SL_2(\mathbb{Z})$ be a hyperbolic matrix with eigenvalues $\lambda < 1 < 1/\lambda$. Consider the manifold $M = \mathbb{T} \times \mathbb{T}$ with coordinates $m = (x, y, z, w)$, and the analytic diffeomorphism $f_N : M \to M$ given by

$$f_N(m) = (\mathbf{s}_N(x, y) + P_x \circ A^N(z, w), A^{2N}(z, w))$$

where $N \geq 0$, and P_x is the projection of $\mathbb{R}^2$ to the x-axis $\mathbb{R} \cdot (1, 0)$.

Theorem 11.2. *There exist N_0 and $c > 0$ such that for every $N \geq N_0$, the map f_N satisfies for Lebesgue a.e. $z \in \mathbb{T} \times \mathbb{T}$ and every unit vector $v \in \mathbb{R}^4$:*

$$\lim_{n\to\infty} \left|\frac{1}{n} \log \|D_z f_N^n(v)\|\right| > c \log N.$$

Moreover the same holds for every conservative diffeomorphism in a C^2-neighborhood of f_N.

Blumenthal-Xue-Young [BXY17] used a similar argument to our proof to prove that random perturbations of every large parameter of the standard map displays a positive metric entropy .

11.1 Idea of proof

The initial idea goes back to a theorem of M. Viana:

Theorem 11.3 (Viana [Via97]). *For N, s large and $\epsilon > 0$ small, the following map is C^s-robustly non-uniformly hyperbolic:*

$$V_\epsilon \colon (\theta, x) \in \mathbb{R}/\mathbb{Z} \times [-2, 2] \mapsto (N\theta, x^2 - 2 + \epsilon(2 + \sin(N\theta))) \in \mathbb{R}/\mathbb{Z} \times [-2, 2] \,.$$

Actually, this Theorem holds true for $s = 2$. Let us describe an idea of proof of this theorem, and how it has been modified to proved Theorem 11.2.

Sketch of proof of Viana's Theorem.
Adapted metric For every $\epsilon > 0$ small, one can show the existence of an adapted metric g on $[-2, 2]$ so that for every critical value a which is 3ϵ-close to -2, the derivative of $P_a(x) = x^2 + a$ is greater than $3/2$ at every point but those in $[-\epsilon^2, \epsilon^2]$ at which the derivative is bounded from below by $|x|$.
Admissible curve Let Γ be the set of curves of the form:

$$\gamma_0 := \{(\theta, x) \in [-\frac{1}{2}, +\frac{1}{2}) \times \mathbb{R} : x = x_0 + \epsilon \sin\theta\} \text{ among } x_0 \in [a, a^2 + a].$$

When N is large it is easy to show the existence of a C^2-neighborhood N_Γ of Γ, so that for every $\gamma_0 \in N_\Gamma$, the image of $V_\epsilon(\gamma_0)$ is the union of N-curves γ_i in N_Γ.

For every $\gamma \in N_\Gamma$, we notice that a very small proportion of γ intersects the strip $\mathbb{R}/\mathbb{Z} \times [-\epsilon^2, \epsilon^2]$.

Furthermore the Lebesgue measure of points which are less that $\eta > 0$-expanded is smaller than $\sqrt[3]{\eta}$. This enables to prove the following inequality:

$$\int_{z\in\gamma} -\log \|V_\epsilon^{-1}(z)\| d\mathrm{Leb}(z) > \log\frac{4}{3} \,.$$

Iterations We recall that every $\gamma_0 \in N_\Gamma$ is sent by V_ϵ to the union of N curves $(\gamma_i)_{1\le i\le N}$ of N_Γ. By reapplying the above inequality, it comes:

$$\int_{z\in\gamma} -\log \|(DV_\epsilon^2)^{-1}(z)\| d\mathrm{Leb}(z) > 2\log\frac{4}{3} \,.$$

And so on, for every $k \ge 0$, it comes:

$$\frac{1}{k}\int_{z\in\gamma} \log \|(DV_\epsilon^k)^{-1}(z)\|^{-1} d\mathrm{Leb}(z) > \log\frac{4}{3} =: \sigma \,.$$

As $-\log \|V_\epsilon^{-1}(z)\|$ is bounded by $M = \log 2$, it holds:

$$\sigma \le M \cdot \mathrm{Leb}\{z \in \gamma : \frac{1}{k}\log \|V_\epsilon^{-k}(z)\|^{-1} \ge \frac{\sigma}{2}\} + \frac{\sigma}{2} \cdot \mathrm{Leb}\{z \in \gamma : \frac{1}{k}\log \|V_\epsilon^{-k}(z)\|^{-1} \le \frac{\sigma}{2}\} \,.$$

This implies the existence of a subset $A \subset \gamma$ of measure $\ge \sigma/2M$, so that every point $z \in A$ satisfies:

$$\limsup_{k\ge 0} \frac{1}{k}\log \|(DV_\epsilon^k)^{-1}(z)\|^{-1} d\mathrm{Leb}(z) \ge \frac{\sigma}{2} \,.$$

Conclusion Then it is easy to see that A is equal to γ modulo a Lebesgue null set. Indeed, otherwise we take a density point of $\gamma \setminus A$, and by Lebesgue density Theorem, there exists a segment $S \subset \gamma$ which intersects A in proposition much smaller than $\sigma/2M$ and which is send by an iterate V_ϵ^k to an admissible curve γ_k in V_Γ. Then a proportion at least $\sigma/2M$ of γ_k belongs to A. As the segment S of γ is uniformly expanded to γ_k, a distortion bound holds true and implies a contradiction.

To conclude it suffices to show that any translation $\gamma + (0, y)$ of γ is still in N_Γ and so by Fubini's Theorem we showed that A covers Lebesgue a.e. $\mathbb{T} \times [-2, 2]$.

Finally let us observe that all the above arguments are valid for C^2-perturbation of V_ϵ. □

In the same article of M. Viana, the skew product of the Hénon map is considered. Actually the same approach works by using the strong dissipativeness.

In both case the proof is much more easier than Jakobson's or Benedicks-Carleson's. That is why we developed this technique to explore the non-uniformly hyperbolic properties of the standard map in [BC14].

Sketch of Proof of Theorem 11.2. Likewise, for every large parameter, the standard map displays a horizontal cone field χ which is expanded and invariant, at every point but those located in two small strips centered at $x = 0$ and $x = \pi$. These strips are called critical and denoted by C.

We chose the skew product f_N of the standard map with the Anosov map so that the strong unstable manifolds cross C in a set of proportional Lebesgue measure very small. Here the strong local unstable manifolds play the role of admissible curves.

We needed to develop the above argument by considering the pairs (γ, X) of a local unstable manifold γ of length 1 supporting a $\frac{1}{2}$-holder vector field X on it.

We showed that after a few iterations by Df_N, these vector fields are eventually $\frac{1}{2}$-holder with small constant. This enabled us to prove thanks to a simple probabilistic argument (which uses a Markov chain like in [BXY17]) that for every k large, for most of the points $z \in \gamma$, $D_z f_N^k(X(z))$ is in χ. By integrating $\log \|D_z f_N^{k+1}(X(z))\| / \|D_z f_N^k(X(z))\|$ along γ we obtained a positive lower bound which is independent of γ and k. Then we concluded similarly as in the above proof. □

Remark 11.4. J. Bochi and M. Viana [BV05] showed that for any closed symplectic manifold (M, ω) there exists a C^1-generic set $\mathcal{R} \subset Sym^1_\omega(M)$ such that for $g \in \mathcal{R}$ then either (a) at least two Lyapunov exponents of g are zero Lebesgue almost everywhere, or (b) g is Anosov.

We observed in [BC14, Cor.7] that f_N is symplectic and not uniformly hyperbolic. Hence this implies that the statement of Bochi-Viana theorem does not hold true in the C^2 setting.

Part IV

Emergence

In 2003, during one of my very first talks with my adviser, J.-C. Yoccoz told me the following about my thesis subject:

> "*My personal feeling is that we will not get to understand [typical] dynamical systems in a bounded time. Not everybody agrees with me, some people are optimistic, I respect their belief, but I am not. I am sure that a scientific revolution will come, showing that dynamics are much more complicated than we think they are.*
> *This should come from the study of a paradigmatic example, which does not need to involve complicated mathematics. For instance, Newhouse phenomena is rather simple to show.*"

At first I was unhappy to hear this: I was wishing to prove positive results exhibiting the strength of mathematics. But years after years, after exploring a few branches of dynamical systems, I became more and more convinced by this vision. I am now convinced that this point of view is already revolutionary, even if there are still a lot of works to show mathematically such a faith.

There are many ways to interpret Yoccoz' quote, from logic mathematics (indecidability), topology of a phenomenon and the one of parameters for which it occurs (e.g., sets which are not locally connected), the growth of the number of periodic points, etc.

We will first recall Newhouse phenomenon in section 12 which provides examples of very complex dynamics. In section 13 we will study the typicality of such a phenomenon. In particular we will see that this phenomenon is sufficiently typical to do not be neglected. In section 14, we will define the concept of Emergence for a dynamical system f, which roughly speaking, quantifies the complexity to describe a dynamical system by means of physical, probability measures.

In section 15, we will see that the growth of the number of periodic points can be faster than exponential (as contrarily to what was expected by Smale in 1967, Bowen 1978, Arnold 1989-92) and this in any dimension ≥ 2 and any regularity $2 \leq r \leq \infty$.

The two main results presented are given by a counterpart of the Bonatti-Diaz Blender for parameter families: the parablender. We will recall its definition in section 16.

12 Newhouse phenomenon

12.1 Newhouse's discovery of the co-existence of infinitely many sinks

Beside Smale asked to his student Newhouse to work on the genericity of axiom A condition, among surface diffeomorphisms (as he had conjectured), Newhouse discovered an open set of C^2-surface diffeomorphisms which do not satisfy axiom A condition. His counter example involves a *wild horseshoe*.

Let K be a basic set for a surface diffeomorphism f. Let us recall that a point $z \in K$ displays a *homoclinic tangency* if its stable manifold $W^s(z; f)$ is tangent to its unstable manifold $W^u(z; f)$.

If such a point exists, we say that *the horseshoe displays a homoclinic tangency.* We notice that an axiom A diffeomorphism cannot exhibit a horseshoe displaying a homoclinic tangency, for the tangency point is in the non-wandering set, but its stable and unstable directions are not transverse.

Definition 12.1 (Wild horseshoe)**.** *A basic piece is* wild *if it displays a C^2-robust homoclinic tangency: for every C^2-perturbation f' of f, the hyperbolic continuation of K displays a homoclinic tangency.*

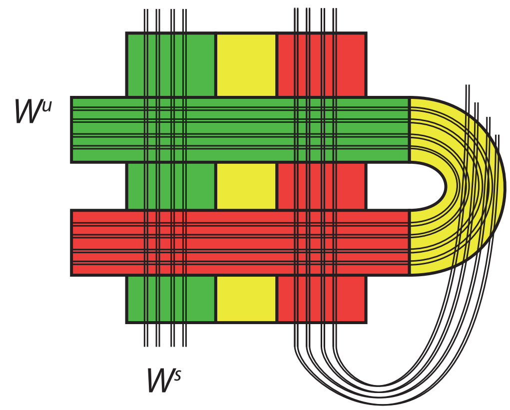


Figure 18: Wild horseshoe

To show the existence of such wild sets, Newhouse defined the concept of stable and unstable thickness for horseshoes (that we will not recall). Then he showed that given a horseshoe K, if the product of the thicknesses of K is greater than one and if K displays a homoclinic tangency, then K is wild.

Although the above sufficient condition is simple to check on many examples, it is not a necessary condition. Up to perturbation, a satisfactory description of wild horseshoe involves the Hausdorff dimension.

First, Palis-Takens showed that if the Hausdorff dimension of a horseshoe is less than 1, then it cannot be wild. Then Moreira-Yoccoz proved the following conjecture of Palis:

Theorem 12.2 (Palis, Moreira-Yoccoz)**.** *For every $r \geq 2$, let K be a horseshoe for a C^r-diffeomorphism f which displays a homoclinic tangency. Then there exists a C^r-perturbation f' of f so that the hyperbolic continuation of K is wild if and only if the Hausdorff dimension of K is ≥ 1.*

12.1.1 Sinks Creation

Since at least the work of Birkhoff, the following is well known:

Proposition 12.3. *Let $r \geq 1$ and let f be a C^r-surface diffeomorphism ($r \geq 1$) and let P be a saddle fixed point which displays a homoclinic tangency and which is area contracting ($|\det D_P f| < 1$), then for every $M > 0$, there exists a small perturbation f' of f which has a sink S of period at least M.*

It will be important to notice that for the original proof of this proposition, the invariant probability measure supported by S is close to the one of P when M is large.

Definition 12.4. *A* topologically generic set *of a metric set* (E, d) *is a countable intersection of open and dense sets of* E*. A* locally, topologically generic set *of* (E, d) *is a set which is topologically generic set in a non-empty open set of* E*.*

Theorem 12.5 (Newhouse)**.** *Let* $r \geq 2$ *and let* f *be a* C^r*-surface diffeomorphism wich displays a wild horseshoe* K*. Suppose that* $|detDf|K| < 1$*. Then, there exists a neighborhood* N *of* f*, and a topologically generic set* $\mathcal{R} \subset N$*, so that every* $f' \in \mathcal{R}$ *displays infinitely many sinks.*

Proof. Let N be the open set of perturbations f' for which the hyperbolic continuation K' of K remains wild and the restriction of f' to K' contracts the volume.

Let $M \geq 0$. By density of the periodic points in this horseshoe, for every $f' \in N$, there exists a C^r-perturbation f'' of f' so that there exists a periodic point P_M which displays a homoclinic tangency.

By Proposition 12.3, there exists a perturbation f''' of f'' which displays an attracting sink S_M of period $\geq M$. Hence the following set is dense:

$$O_M := \{f' \in N : \ f' \text{ has a sink of period } \geq M\} ,$$

and it is also open by hyperbolic continuation. Hence, the following is topologically generic in N and made by diffeomorphisms displaying infinitely many sinks:

$$\mathcal{R} := \bigcap_{M \geq 0} O_M .$$

□

In the latter proof, we can choose homoclinic periodic point P_M supporting a *rather* different invariant, probability measure to the previous steps $(P_k)_{k<M}$. Then the sink S_M supports an invariant, probability measure close to the one of P_M and *rather* different to the ones of $(S_k)_k$.

Consequently a diffeomorphism in $\mathcal{R}$ displays infinitely many sinks, and each of which supports a *rather* different invariant probability measure. In this sense, the dynamics of these maps is very complex.

That is why these dynamics seem to me difficult to be described by the uniformly hyperbolic or non-uniformly hyperbolic theory.

To measure our misunderstanding of the dynamics exhibiting Newhouse phenomena, let me point out that we do not know if there is a single example of such a map for which Lebesgue almost every point display a Birkhoff sum which converge.

Remark 12.6. Actually the above proof works if we relax the area contracting hypothesis on the horseshoe K to a weaker one: the existence of an area contracting periodic point in K. Then it is easy to show the existence of a dense set of area contracting periodic points in the horseshoe (by using the semi-conjugacy of the horseshoe dynamics with a finite shift for instance).

12.2 Newhouse phenomenon from the Bonatti-Diaz blender

Another way to obtain robust homoclinic tangencies involves the blender. This approach has been used by Bonatti-Diaz [BD99] and then by Diaz-Nogueira-Pujals [DNP06] to exhibit a locally, topologically generic set formed by C^r-diffeomorphisms of an n-manifold which display infinitely, for any $\infty \ge r \ge 1$ and $n \ge 3$.

Let us present a variation of the argument of [DNP06], in the surface, local diffeomorphism case.

For local diffeomorphisms, we can define the same concepts as Newhouse's, where a blender is considered instead of the horseshoe. We recall that a blender of a surface local diffeomorphism is a basic set for an endomorphism so that the union of its unstable manifolds contains robustly a non-empty open set O of the manifold. The set O is called a *covered domain of the blender.*

Let $r \ge 1$. A blender K of a C^r-local diffeomorphism f displays a *homoclinic* tangency if there exists a point $z \in K$ so that its unstable manifold $W^u(z; f)$ is tangent to its stable manifold $W^s(z; f)$. The blender K is *wild* if every C^1-perturbation f' of f exhibits a homoclinic tangency.

From remark 12.6, the following is straight forward:

Theorem 12.7. *Let K is a blender for a C^r-local diffeomorphism f. Assume that K contains a periodic point Ω which is area contracting and that K is wild, then there exists a C^r-neighborhood N of the dynamics so that a C^r-generic diffeomorphism f in N displays infinitely many sinks.*

To accomplish the proof, it remains to prove the existence of such a local diffeomorphism f. Let us cook an example of such dynamics.

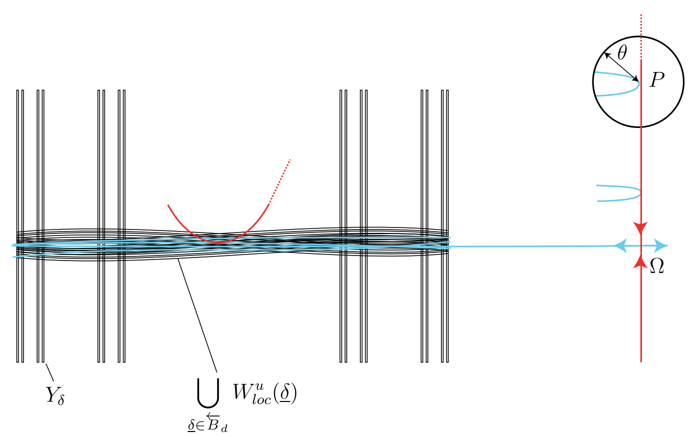


Figure 19: Diaz-Nogueira-Pujals construction

Example 12.8. Let $I_+ \sqcup I_- \sqcup I_c$ be three non trivial intervals of $(-1,1) \setminus \{0\}$ and let $Q\colon I_+ \sqcup I_- \sqcup I_c \to [-1,1]$ be a locally affine map which sends each of the intervals I_+, I_-, I_c onto $[-1,1]$. Let $\lambda := \frac{1}{3}\min(\mathrm{Leb}(I_+), \mathrm{Leb}(I_c), \mathrm{Leb}(I_-))$. Let K be the maximal invariant set of the following map:

$$f\colon (x,y) \in I_+ \sqcup I_- \sqcup I_c \times [-1,1] \mapsto \begin{cases} (Q(x), (2y+1)/3) & x \in I_+ \\ (Q(x), (2y-1)/3) & x \in I_- \\ (Q(x), \lambda y) & x \in I_c \end{cases}$$

We notice that it is a hyperbolic set (with vertical stable direction) and transitive. Hence, it is a basic set. Moreover, we saw in Example 1.15, that the maximal invariant set of the restriction $f|I_- \sqcup I_+ \times [-1,1]$ is a blender with 0 in its covered domain. Hence K is a blender.

Furthermore it contains an area contracting fixed point Ω in $I_c \times \{0\}$.

To create a homoclinic tangency, it suffices to extend f at a neighborhood N of 0 so that a local stable manifold of Ω intersects N at $N \cap \{(x, x^2) : x \in \mathbb{R}\}$. Then the blender K displays a homoclinc tangency. In [Ber16b, prop. 2.1] we showed that K is wild.

□

The following counterpart of the Palis-Moreira-Yoccoz theorem is an open problem:

Problem 12.9. *For every $r \geq 1$, let K be a blender which displays a homoclinic tangency. Show the existence of a C^r-perturbation f' of f so that the hyperbolic continuation of K is wild.*

13 On typicality of the Newhouse phenomenon

In the last section we saw the existence of a locally, topologically generic set of dynamics which exhibit infinitely many sinks accumulating on a basic set. We saw also that we do not know how to describe these dynamics.

However do they appear typically? Do they form a subset of dynamics which is negligible?

Indeed locally topologically generic does not mean typical in the probabilistic sense. For instance, there are topologically generic subsets of the real line whose Lebesgue measure is null (e.g. the set of Liouville numbers).

That is why, until recently, most of the community was rather optimistic, and believed that the set of maps exhibiting the Newhouse phenomenon should be negligible in some sense.

However there is no canonical measure such as the Lebesgue measure for the Banach spaces. But, we recall that many results of abundance in non-uniformly hyperbolic dynamical system have been formulated thanks to families of dynamics. This way is somehow similar to the concept of typicality sketched by Kolmogorov during his famous plenary talk in the ICM 1954. Here is a version of typicality which appears in many conjectures:

Definition 13.1 (Arnold-Kolmogorov typicality)**.** *A property $\mathcal{P}$ on dynamics of a manifold M is typical if there exists a Baire generic set of C^d-families $(f_a)_{a\in\mathbb{R}^k}$ of C^r-dynamics so that $\mathcal{P}$ is satisfied by Lebesgue almost every small parameter a.*

Hence this definition of typicality involved integers k, d, r.

Definition 13.2 (C^d-families of C^r-self-mappings)**.** *A family $(f_a)_{a\in\mathbb{R}^k}$ of C^r-self-mappings f_a of M is* of class C^d*, if the following derivatives are well defined and continuous for every $i \leq d$ and $i + j \leq r$:*

$$\partial_a^i \partial_z^j f_a \colon (a, z) \in \mathbb{R}^k \times M \mapsto \partial_a^i \partial_z^j f_a(z)$$

The space of C^d-families of C^r-self-mappings is endowed with the Whitney topology with respect to these derivatives. We recall that the following is an elementary open subset, among $\epsilon \in C^0(\mathbb{R}^k \times M, (0,\infty))$.

$$O((f_a)_a, \epsilon) := \{(g_a)_a : \quad d(\partial_a^i \partial_z^j f_a, \partial_a^i \partial_z^j g_a) < \epsilon(a,z) \ , \quad \forall a \in \mathbb{R}^k, z \in M, i \leq d, i+j \leq r\}$$

To take into account the aforementioned examples and counter examples, there are several conjectures from Tedeschini-Lalli & Yorke [TLY86], Palis & Takens [PT93], Palis himself [Pal00, Pal05, Pal08] – most of them for low dimensional dynamical systems – claiming the typicality of the finiteness of the number of attractors. Let us recall the following:

Conjecture 13.3 (Pugh-Shub [PS96])**.** *Typically (in the sens of Arnold-Kolmogorov) a diffeomorphism of a compact manifold displays at most a finite number of topological attractors (and so sinks).*

All these conjectures aim to describe typical dynamics thanks to finitely many attractors. In one-dimensional dynamics or "skew-product" of one-dimensional dynamics, the seminal works of Lyubich [Lyu02] , Tsujii [Tsu05] and Kozlovsk-Shen-van Strien [KSvS07] gave evidence that this should be true. In higher dimension, the general strategy proposed to prove them was to study the unfolding of stable and unstable manifolds (in analogy with Thom-Mather works in singularity Theory).

13.1 On typically of Newhouse phenomenon for surface diffeomorphisms

These conjectures are open for surface[8] diffeomorphisms.

Adapting some of Newhouse's results, Robinson (see [Rob83]) later proved that the phenomenon takes place for a residual set of parameters in a one-parameter family of diffeomorphisms which non-degenerately unfold a homoclinic tangency.

The first attempt towards a measure theoretic understanding of Newhouse phenomenon is due to Tedeschini-Lalli and Yorke (see [TLY86]): they considered a one-parameter unfolding of a homoclinic tangency involving a linear horseshoe, and showed that the set of parameters whose corresponding diffeomorphism admits infinitely many periodic *simple sinks* (i.e. sinks obtained with the Newhouse construction) is a null set for Lebesgue measure. In the same setting (one-parameter unfolding of a homoclinic tangency involving a linear horseshoe), Wang (see [Wan90]) proved that the Hausdorff dimension of the parameter set of diffeomorphisms admitting an infinite number of periodic simple sinks is strictly positive and smaller than $\frac{1}{2}$.

More recently, Gorodetski and Kaloshin (see [GK07]) obtained the measure-zero result in a much broader setting: they introduced a quantitative notion of combinatorial complexity of periodic orbit visiting a neighborhood of a homoclinic tangency, which they call *cyclicity*[9]. Their result shows

[8] The topics of surface dynamics was especially important for Smale since he believed that they would reflect already many important behavior of higher dimensional dynamical systems[Sma70]

[9] In their terminology, *simple sinks* which were considered above correspond to *cyclicity one sinks*.

that a *prevalent* dissipative surface diffeomorphism in a neighborhood of one exhibiting a non-degenerate homoclinic tangency has only finitely many sinks of cyclicity which is either bounded or negligible with respect to the period of the orbit. We will see in section 15 that prevalence does not imply nor implied by the Arnold-Kolmogorov typicality.

The techniques of [New79, Rob83] do not apply to conservative surface diffeomorphisms; on the other hand, a clear analog of the Newhouse phenomenon still occurs in a vicinity of conservative diffeomorphisms exhibiting non-degenerate homoclinic tangencies, with elliptic islands filling in for the rôle of sinks. This result was finally established by Duarte and Gonchenko–Shilnikov (see [Dua99, GS03] and [Dua08] for the one-parameter version). In [DS13] de-Simoi proved an analog to Tedeschini-Lalli–Yorke and Wang result for the Standard Family of conservative diffeomorphisms in the large parameters regime: the set of (sufficiently large) parameters for which the Standard Family admits infinitely many simple sinks has zero Lebesgue measure and its Hausdorff dimension is not smaller than $1/4$.

In [BDS16], we obtained a similar lower bound on the Hausdorff dimension for dissipative surface diffeomorphisms. We proved that the Newhouse parameter set for a generic family of sufficiently smooth diffeomorphisms which nondegenerately unfolds a homoclinic tangency has Hausdorff dimension not smaller than $1/2$:

Theorem 13.4 ([BDS16]). *Let $r \geq 2$ and let $(f_a)_a$ be a Cr-non-degenerate unfolding of a homoclinic tangency for an area contracting saddle point. Then the Hausdorff dimension of the following set is at least $1/2$:*

$$\mathcal{N} := \{a \in \mathbb{R} : \ f_a \textit{ display infinitely many sinks}\}.$$

It is important to stress that our lower bound takes into account non-simple sinks (and thus does not contradict Wang's result) and moreover does not assume linearity of the horseshoe. The proof of our result hinges on two crucial ingredients: the first one is an improved version of Newhouse construction of a wild hyperbolic set; the second one, proved in [Berc], provides precise estimates on the length of the *stability range* of a sink which is created by unfolding a homoclinic tangency via the Newhouse construction. As a consequence of these results, we obtain that the Hausdorff dimension of the simple Newhouse parameter set for strongly dissipative Hénon-like families is close to one $1/2$. We conclude by using Palis-Takens renormalization.

13.2 On typically of Newhouse phenomenon for surface self-mappings and diffeomorphisms of higher dimension

Recently, in [Ber16b], a mechanism has been found to stop the unfolding for an open set of self-mappings' families. This mechanism is given by the *parablender*, a counterpart of the Bonatti-Diaz Blender for parameter families. It will be explained in section 16. This enabled to prove:

Theorem 13.5 ([Ber16b, Ber16a, Ber17a]). *For every manifold M of dimension at least 2, for every $k \geq 0$, for every $1 \leq d \leq r \leq \infty$ and $d < \infty$, there exists an open set $\hat{\mathcal{U}}$ of C^d-families $(f_a)_a$ of C^r-self-mappings of M, so that for a generic $(f_a)_a \in \hat{\mathcal{U}}$,* for every *parameter $a \in \mathbb{R}^k$, the map f_a displays infinitely many sinks.*

Moreover if dim $M \geq 3$, *then the family of self-mappings is made by diffeomorphisms.*

In [Ber16b, Ber16a] the result was proved thanks to a parametric counterpart of the wild-blender, after correction we needed $\infty \geq r > d \geq 1$. In a work in progress with Crovisier and Pujals we investigated the case where a source is put in the covered domain of a wild horseshoe. This setting motivated [Ber17a] where the case $\infty > r = d \geq 1$ was added in the above theorem. This enables to show in the work in progress that the Newhouse phenomenon is actually locally typical in the sense of Arnold-Kolmogorov among surface, finitely smooth, self-mappings.

Such results are very disturbing since the general trend was to use the bifurcation theory to show the finiteness of attractors. Here the bifurcation theory enables one to stop the bifurcation and shows the non-typicality of the finiteness of attractors.

Let us end this section by remembering that during his plenary talk at ICM 1954, Kolmogorov said about phenomena which persist for small perturbations along 1-parameter families of dynamical system (that he called stable realization):

An arbitrary type of behavior of a dynamical system, for which there exists at least one example of its stable realization, must from this point of view be considered essential and may not be neglected.

From this point of view, Newhouse phenomenon is essential and may not be neglected. In the next section we propose a way to quantify the complexity of such dynamics.

14 Quantifying the complexity of dynamics

I would like to propose a statistical interpretation of Yoccoz' quote:

Problem 14.1. *Show the existence of an open set of deterministic dynamical systems which typically cannot be described by means of statistics.*

In particular, this problem wonders if the statistical mechanics tools, introduced by Sinai in his seminal works (and those of Ruelle, Bowen ...), may describe all typical dynamical systems.

The aim is not to prove that statistics never apply (they do for many systems!), but that they do not apply for many typical systems, even among the finite dimensional, deterministic differentiable dynamical systems. We shall formalize this problem. For this end, we are going to define the Emergence of dynamical systems. This concept evaluates the complexity to approximate a system by statistics.

In statistic and computer science, it is standard to use the Wasserstein distance d_{W_1} on the space of probability measures $\mathbb{P}(M)$ of a compact manifold M:

$$d_{W_1}(\nu, \mu) = \sup_{\phi \in Lip^1(M,[-1,1])} \int_M \phi(x)\, d(\mu - \nu)(x) , \quad \forall \nu, \mu \in \mathbb{P}(M)$$

where $Lip^1(M, [-1, 1])$ is the space of 1-Lipschitz functions with values in $[-1, 1]$.

Given a differentiable map f of M, $x \in M$ and $n \geq 0$, we denote by $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)}$ the probability measure which associates to an observable $\phi \in C^0(M, \mathbb{R})$ the mean $\frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k(x))$.

Proposition 14.2. *Given a probability measure μ, the following functions are continuous:*

$$x \in M \mapsto d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \mu) \in \mathbb{R}\ , \quad \forall n$$

Proof. We notice that it suffices to show that for every $\delta > 0$, there exists $\eta > 0$ such that if x and x' are η distant, then

$$d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x')}, \mu) \geq d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \mu) - \delta\ .$$

We recall that $Lip^1(M, [-1, 1])$ endowed with C^0-uniform norm is compact, by Arzelà-Ascoli Theorem. Hence, there exists $\phi \in L^1(M, [-1, 1])$ such that:

$$d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \mu) = \frac{1}{n}\sum_{k=0}^{n-1}\phi(f^k(x)) - \int_M \phi d\mu\ .$$

As ϕ and $(f^k)_{k\leq n}$ are Lipschitz, there exists $\eta > 0$ so that for x' η-close to x, it holds:

$$\frac{1}{n}\sum_{k=0}^{n-1}\phi(f^k(x')) \geq \frac{1}{n}\sum_{k=0}^{n-1}\phi(f^k(x)) - \delta \Rightarrow d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x')}, \mu) \geq d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \mu) - \delta\ .$$

□

We recall that the space of probabilities over a compact manifold and endowed with the metric d_{W^1} is relatively compact.

Hence, given a differentiable map f of M, we can define the *Emergence* $\mathcal{E}(f, \epsilon)$ *of* f *at scale* $\epsilon > 0$ as the minimal number N of probability measures $\{\mu_i\}_{1\leq i\leq N}$ so that

$$\limsup_{n\to\infty} \int_{x\in M} \min_{1\leq i\leq N} d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \mu_i)\, d\text{Leb} \leq \epsilon\ .$$

Definition 14.3 (Emergence)**.** *The Emergence is* **F** *if* $\mathcal{E}(f, \epsilon) = O(1)$ *when* $\epsilon \to 0$.

The Emergence is at most **P** *if there exists* $k > 1$ *so that* $\mathcal{E}(f, \epsilon) = O(\epsilon^{-k})$.

The Emergence is **Sup-P** *if* $\limsup \frac{\log \mathcal{E}(f,\epsilon)}{-\log \epsilon} = +\infty$.

We notice that the Emergence is a lower bound on the complexity (in space[10] and in time) to approximate numerically a dynamical system by statistics with precision ϵ. Following, the celebrated Cobham's thesis an algorithm in Sup-P is – in practical – not feasible [Cob65].

Examples with F-Emergence If a dynamical system f admits finitely many ergodic attractors $(\Lambda_i, \mu_i)_{1\leq i\leq N}$ whose basins $(B_i)_i$ cover Lebesgue almost all the manifold, then the Emergence is bounded by N (and so it is of type F).

[10] The number of data to store.

Proof. By the dominated function theorem, it suffices to show that for every $i \le N$ and every $x \in B_i$, $d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \mu_i) \to 0$. By compacity of $Lip^1(M, [-1,1])$, for every n, there exists $\phi_n \in Lip^1(M, [-1,1])$ so that:

$$\Delta_n := d_{W^1}(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}, \mu_i) = \int_M \frac{1}{n}\sum_{k=0}^{n-1}\phi_n(f^k(x))d\mathrm{Leb} - \int_M \phi_n d\mu_i \ .$$

Let $\phi \in Lip^1(M, [-1,1])$ be a cluster value of $(\phi_n)_n$ and let $(n_j)_{j\ge 0}$ be an increasing sequence so that $\phi_{n_j} \to \phi$. Then

$$\Delta_{n_j} \le 2\int_M \|\phi_{n_j} - \phi\|_{C^0} d\mu_i + \int_M \frac{1}{n_j}\sum_{k=0}^{n_j-1}\phi(f^k(x))d\mathrm{Leb} - \int_M \phi d\mu_i \to 0 \ .$$

Thus every cluster value of $(\Delta_n)_n$ is zero, and so this sequence converges to zero. □

Remark 14.4. We recall that a diffeomorphism satisfying Axiom A, an irrational rotation or a Hénon map for Benedicks-Carleson parameters have finitely many ergodic attractors whose basins cover Lebesgue almost all the phase space M. Hence their Emergences are finite.

Example with P-Emergence. Let f be the identity. Observe that $\mathcal{E}(f, \epsilon) = O(\epsilon^{-n})$ with n the dimension of M. Hence its Emergence is polynomial. Also the Emergence of an irrational rotation on a cylinder, which is the product of systems with Emergences 1 and $O(\epsilon^{-1})$, is $O(\epsilon^{-1})$.

It seems also possible to prove that the Emergence of the so-called Bowen eyes dynamics is $O(\epsilon^{-1})$.

Hence, it seems that all the well understood dynamical systems have an Emergence at most P. However, the main conjecture of this work states that those of Sup-P Emergence should not be neglected:

Conjecture 14.5. *There exists an open set $U \subset Diff(M)$ so that a typical $f \in U$ has Emergence Sup-P.*

Let us explain why a proof of this conjecture would solve Problem 14.1 from the computational view point. Given a typical $f \in U$, to describe by means of statistics with precision ϵ, all of its orbit, but a proportion Lebesgue measure $1-\epsilon$, we would need at least a super-polynomial number of invariant probabilities w.r.t. $\frac{1}{\epsilon}$. To find them by means of statistics, we need *at least* one data for each of them, and so to do a super polynomial number of operations. By Cobham's thesis this is not feasible by a computer.

Also we notice that when the Emergence is Sup-P, the Hausdorff dimension of the set of probabilities which would model our system is infinite.

Hence to find these invariant probabilities, we would not be able to use the (finite dimensional) parametric statistics, but only the non-parametric ones, whose computational cost is higher (and much more than 1 as in the above lower bound).

Furthermore let us notice that the product of $f \in Diff(M)$ with the identify of compact manifold N of dimension d displays an Emergence $\mathcal{E}(f \times id_N, \epsilon)$ dominated by $\epsilon^{-d} \times \mathcal{E}(f, \epsilon)$. Indeed the Birkhoff sum of the product of the dynamics on $M \times N$ are the product of a Birkhoff sum of M with a Dirac of a point in N. Hence it is a product of an invariant probability measure of id_N with an invariant probability measure of M. Thus

$$\mathcal{E}(f \times id_N, \epsilon) = \mathcal{E}(id_N, \epsilon) \times \mathcal{E}(f, \epsilon) = O(\epsilon^{-d}) \times \mathcal{E}(f, \epsilon)$$

During a seminar I gave, Ledrappier asked the following question:

Question 14.6. *Supppose that G is a compact, Lie group of symmetries acting smoothly on M. Suppose that $f \in Diff(M)$ is equivariant by f: There exists $x \in M \mapsto \sigma_x \in Aut(G)$ such that $f(g \cdot x) = \sigma_x(g) \cdot f(x)$. Then f projects on the quotient M/G as a map $\check{f}\colon M/G \to M/G$. Is that true that $\mathcal{E}(f, \epsilon)$ is dominated by $\mathcal{E}(\check{f}, \epsilon) \times \epsilon^{-\dim G}$?*

Note that Newhouse phenomenon is an example of infinite Emergence, but we do not know if we can exhibit one example of at most polynomial Emergence. Similarly KAM theory provides examples of at least P-Emergence in the conservative setting.

Candidates for Sup-P-Emergence. It is perhaps possible to construct a unimodal map with Sup-P Emergence from [HK90], or a locally C^r-dense set of surface diffeomorphisms with Sup-P Emergence from [KS15]. It would be very challenging to derivate from these systems one which is moreover locally typical.

It is perhaps possible to make a variation of Newhouse's construction to produce a generic dynamics with Sup-P Emergence.

Let me mention also the concept of universal dynamics of Bonatti-Diaz [BD02] and Turaev [Tur15a] which might produce locally Baire generic sets of diffeomorphisms with high Emergence.

It would be interesting to study Conjecture 14.5 w.r.t. different notions of typicality [HK10] and smoothness. Also it might be interesting to investigate the concept of Emergence for other metrics than W_1 on the space of invariant probability measures.

Also it would be interesting to provide numerical evidences for such a program (from big data?). The following problem remains open.

Problem 14.7. *Show numerical simulations depicting a (typical) dynamical systems which displays infinitely many sinks.*

Let us point out that by definition, a Sup-P Emergent dynamical system is very complex to describe, and so the non-existence of such pictures is consistent with their conjectured local typicality.

15 Growth of the number of periodic points

As we introduced, in some extends, the growth of the number of periodic points depicts the complexity of the dynamics studied. Let us develop this topic.

Let f be a diffeomorphism (or a local diffeomorphism) of a compact manifold M. We denote by $Per_n f := \{x \in M : f^n(x) = x\}$ the set of its n-periodic points. To study its cardinality, we consider also the subset $Per_n^0 f \subset Per_n f$ of isolated n-periodic points. We notice that the cardinality of $Per_n^0 f$ is an invariant by conjugacy. Hence it is natural to study the growth of this cardinality with n.

Clearly, if f is a polynomial map, the cardinality of $Per_n^0 f$ is bounded by the degree of f^n, which grows at most exponentially [DNT16].

The first study for the C^∞-case goes back to Artin and Mazur [AM65] who proved that there exists a dense set $\mathcal{D}$ in $Diff^r(M)$, $r \le \infty$, so that for every $f \in \mathcal{D}$, the number $Card\, Per_n^0 f$ grows at most exponentially, i.e. , there exists $K(f) > 0$ so that:

$$\text{(A-M)} \qquad \frac{1}{n} \log Card\, Per_n^0 f \le K(f) \, .$$

This leads Smale [Sma67] and Bowen [Bow78] to wonder about the relationship between rate of growth of the number of periodic points on one hand and dynamical ζ-function or topological entropy on the other hand for (topologically) generic diffeomorphisms. In particular, these questions asked whether (A-M) diffeomorphisms are generic. Finally Arnold asked the following problem:

Problem 15.1 (Smale 1967, Bowen 1978, Arnold Pb. 1989-2 [Arn04])**.** *Can the number of fixed points of the n^{th} iteration of a topologically generic infinitely smooth self-mapping of a compact manifold grows, as n increases, faster than any prescribed sequence $(a_n)_n$ (for some subsequence of time values n)?*

We recall that a property is *topologically generic* if it holds for a countable intersection of open and dense sets. The topology on the space of C^∞-maps is the union of the ones induced by the C^r-topologies $C^r(M,M)$ among $r \ge 0$ finite.

Another notion to quantify the abundance of a phenomenon was sketched by Kolmogorov during its plenary talk at the ICM 1954. Then his student Arnold formalized it as follows [IL99, KH07]:

Definition 15.2 (Arnold's Typicality)**.** *A property $(\mathcal{P})$ on the set of C^r-mappings $C^r(M,M)$ of M is* typical *if for every $k \ge 1$, for a topologically generic C^r-family of $(f_a)_{a\in\mathbb{R}^k}$ of C^r-maps f_a, for Lebesgue almost every $a \in \mathbb{R}^k$, the map f_a satisfies the property $(\mathcal{P})$.*

We recall that $(f_a)_{a\in\mathbb{R}^k}$ is of class C^r if the map $(a,z) \mapsto f_a(z)$ is of class C^r. When $r < \infty$, the topology on this space is equal to the compact-open C^r-topology of $C^r(\mathbb{R}^k \times M, M)$. When $r = \infty$, the topology on the space of C^∞-families is the one given by the union of those induced by the $C^{r'}$-topologies among $r' \ge 0$ finite.

Problem 15.3 (Arnold 1992-13 [Arn04])**.** *Prove that a typical, smooth, self-map f of a compact manifold satisfies that $(Card\, Per_n f)_n$ grows at most exponentially fast.*

Remark 15.4. Many other Arnold's problems are related to this question [Arn04, 1994-47, 1994-48, 1992-14].

These problems enjoy a long tradition.

In dimension 1, Martens-de Melo-van Strien [MdMvS92] showed that for every $\infty \geq r \geq 2$, for an open and dense set[11] of C^r-maps the number of periodic points grows at most exponentially.

Kaloshin [Kal99] answered to a question of Artin and Mazur (in the finitely smooth case) by proving that for a dense set $\mathcal{D}$ in $Diff^r(M)$, $r < \infty$, the set $Per_n f$ is finite for every n (and so equal to $Per_n^0 f$) and its cardinality grows at most exponentially fast.

However, in [Kal00] Kaloshin proved that for $2 \leq r < \infty$ and $\dim M \geq 2$, *a locally topologically generic diffeomorphism displays a fast growth of the number of periodic points*: there exists an open set $U \subset Diff^r(M)$, so that for any sequence of integers $(a_n)_n$, a generic $f \in U$ satisfies:

$$(\star) \qquad \limsup_{n\to\infty} \frac{Card\, P_n^0(f)}{a_n} = \infty\ .$$

Furthermore, Bonatti-Diaz-Ficher [BDF08] extended this result to the C^1-case in dimension ≥ 3. The counterpart of this result in the conservative case has been proved by Kaloshin-Saprykina in [KS06]. Kaloshin theorem is based on a result by Gonchenko-Shilnikov-Turaev [GST93b, GST99] that surface diffeomorphisms with degenerate parabolic points form a localy dense subset of $Diff^r(M^2)$; since their proof in [GST99, GST07] is valid in the C^∞-case, Kaloshin theorem immediately extends to the C^∞-case as well. Recent seminal work of Turaev [Tur15b] also implies that among C^∞-surface diffeomorphisms the fast growth of the number of periodic points is locally a topologically generic property.

However the C^∞-case in dimension ≥ 3 remained open[12]. In this term, our first result accomplishes the study of problem 15.1, in any smoothness ≥ 2 and any dimension:

Theorem 15.5 ([Ber17b]). *Let $\infty \geq r \geq 2$ and let M be a compact manifold of dimension d.*

If $d = 1$, Property (AM) is satisfied by an open and dense set of C^r-self-mappings.

If $d \geq 2$, there exists a (non empty) open set $U \subset Diff^r(M)$ so that given any sequence $(a_n)_n$ of integers, a topologically generic f in U satisfies $(\star)$.

Actually, the proof of this theorem will be done in dimension ≥ 3 for the diffeomorphism's case, and dimension 2 for self-mappings. For the one dimensional case has been proved in [MdMvS92] whereas the surface diffeomophism's case is proved as aforementioned.

This result is proved following a method which contains one aspect related to the work Asaoka-Shinohara-Turaev [ASTar] on the fast growth of the number of periodic points for a locally generic free group action of the interval. In our proof, basically a free group of diffeomorphisms of the circle is embedded into the manifold as a normally hyperbolic fibration by circles. As in Asaoka-Shinohara-Turaev's approach, we consider a robust hetero-dimensional cycle given by a Bonatti-Diaz Blender [BD96]. Thanks to a new renormalization trick, we exibit a dense set of perturbations which display a parabolic dynamics on an invariant, finite union of circles. We perturb it to a rotation thanks to Herman-Yoccoz' development of KAM-theorem. Then it is easy to construct a topologically generic perturbation which exhibits a fast growth of the number of periodic points.

[11] whose complement is the infinite codimentional manifold formed by maps with at least one flat critical point.

[12] in [GST93a], Theorem 7, Gonchenko-Shilnikov-Turaev theorem was claimed to be true for any dimension but the proof has never been published.

As there are topologically generic sets of the real line whose Lebesgue measure is null, a negative answer to Problem 15.1 does not need to suggest a negative answer to Problem 15.3.

To provide a positive answer to Arnold Problem 15.3, Hunt and Kaloshin [KH07] used a method described in [GHK06] to show that for $\infty \geq r > 1$, a *prevalent* C^r-diffeomorphism satisfies:

$$(\Diamond) \qquad \limsup_{n\to\infty} \frac{\log P_n(f)}{n^{1+\delta}} = 0, \quad \forall \delta > 0 \, .$$

The notion of prevalence was introduced by Hunt, Sauer and York [HSY92]. A property is *prevalent* if *roughly speaking* almost all perturbations in the embedding of a Hilbert cube at every point of a Banach space (like $C^r(M, M)$), the property holds true. We notice that ($\Diamond$) is satisfied for a prevalent diffeomorphism but not for a topologically generic diffeomorphism (see other examples of mixed outcome in [HK10]).

However the latter did not completely solve Arnold's problem 15.3 in particular because the notion of prevalence is *a priori* independent to the notion of typicality initially meant by Arnold. Indeed his problem was formulated for typicality in the sense of definition 15.2 (see explanation below problem 1.1.5 in [KH07]).

In this term the second and main result of this work is surprising since it provides a negative answer to Arnold's problem 15.3 in the finitly smooth case:

Theorem 15.6 ([Ber17b])**.** *Let $\infty > r \geq 1$ and $0 \leq k < \infty$, let M be a manifold of dimension ≥ 2, and let $(a_n)_n$ be any sequence of integers.*

Then there exists a (non-empty) open set $\hat{U}$ of C^r-families $(f_a)_a$ of C^r-self-mappings f_a of M so that a topologically generic $(f_a)_a \in \hat{U}$ consists of maps f_a satisfying ($\star$), for every $\|a\| \leq 1$. Moreover if $\dim M \geq 3$, we can choose $\hat{U}$ to be formed by families of diffeomorphisms.

Remark 15.7. Actually, the same proof shows that the statement of Theorem 15.6 holds true in the category of C^r-families of C^∞-self-mappings.

The proof of this theorem follows the same scheme as for Theorem 15.5, beside the fact that the blender is replaced by a new object: the λ-parablender (which generalizes both the blender and the parablender as introduced in [Ber16b]). A generalization to the parameter case of the renormalization trick enables us to display a dense set of families of self-mappings which leave invariant a finite union of normally hyperbolic circles on which the restrictions are constantly parabolic. Then a careful study of the parabolic bifurcation and renormalization together with KAM-Herman-Yoccoz' Theorem enables us to perturb these families by one which exhibits a constant family of rotations. Finally it is easy to perturb the family so that it displays a fast growth of the number of periodic point at every parameter $\|a\| \leq 1$.

This last step has been implemented recently by Asaoka [Asa16] who showed that for every $r \in \{\infty, w\}$, there exists a C^r-open set of *conservative* surface diffeomorphisms in which typically in the sense of Arnold, a map displays a fast growth of the number of periodic points. Indeed, he observed that, in the conservative case, a mere application of KAM theory implies the existence of diffeomorphisms which leave invariant and persistent circles and act on them as irrational rotations (whose angles are constant for conservative perturbations).

To conclude the presentation of these results on the growth of the number of periodic points, let me recall that Arnold's philosophy was not to propose *Problems of binary type admitting a "yes-no" answer*, but rather to propose *wide-scope programs of explorations of new mathematical (and not only mathematical) continents, where reaching new peaks reveals new perspectives, and where a preconceived formulation of problems would substantially restrict the field of investigations that have been caused by these perspectives. [...] Evolution is more important than achieving records,* as he explained in his preface [Arn04].

Let us remark that in this sense the contrast between the result of Kaloshin-Hunt and Theorem 15.6 is interesting since they shed light how an answer to a question might depend on the definition of typicality.

Furthermore, the proofs of this work do not only answer questions, it also develops new tools which will certainly be useful for our program on emergence [Ber17a]. Let us notice that the C^∞-case of problem 15.3 (or conjecture 1994-47 [Arn04]) remains open, although in view of Remark 15.7, I would bit for a negative answer; I would even dare to propose:

Conjecture 15.8. *For every $r \in \{1, ..., \infty, \omega\}$, there exists an open set of diffeomorphisms $U \in Diff^r(M)$, so that given any $k \geq 0$, for any C^r-generic family $(f_a)_{a\in\mathbb{R}^k}$ with $f_a \in U$, for every a small, the growth of the number of periodic points of f_a is fast.*

15.1 A circle mapping based proof

As in Asaoka works[Asa16], the theorems are proved by exhibiting an invariant circle whose rotation number is Diophantine. We will see below why it is sufficient. In the surface conservative case (which is the setting of Asaoka), a non-degenerated elliptic fixed point exhibits a robust invariant circle with constant, Diophantine rotation number (by KAM theorem). In our dissipative case, there is no invariant circle with rotation number robustly Diophantine. To show the existence of a dense set of such dynamics we shall introduce the λ-blenders and their para-version in the next section. They enable to exhibit invariant circles with parabolic dynamics, that we perturb to diophantine rotations.

Let us recall the definitions related to one dimensional dynamics.

Rotation number Given a homeomorphism $g \in Diff^0(\mathbb{R}/\mathbb{Z})$ of the circle $\mathbb{R}/\mathbb{Z}$, one defines its rotation number ρ_g as follows. We fix $G \in Diff^0(\mathbb{R})$ a lifting of g for the canonical projection $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$:

$$\pi \circ G = g \circ \pi \ .$$

Then Poincaré proved that $\rho_G = \lim_{n\to\infty} G^n(0)/n$ is uniquely defined. Furthermore, $\rho_g = \pi(\rho_G)$ does not depend on the lifting G of g. The *rotation number* of g is ρ_g.

It is easy to show that the rotation number depends continuously on g.

Maps with Diophantine rotation number A number $\rho \in \mathbb{R}$ is *Diophantine*, if there exist $\tau > 0$ and $C > 0$ so that for every $p, q \in \mathbb{N} \setminus \{0\}$ it holds:

$$|q\rho - p| \geq Cq^{-\tau} .$$

Let us recall that the set of Diophantine numbers is of full Lebesgue measure.

Here is a Yoccoz' development of Herman's theorem on Arnold Conjecture:

Theorem 15.9 (Arnold-Herman-Yoccoz [Her79, Yoc84])**.** *If the rotation number of $g \in Diff^\infty(\mathbb{R}/\mathbb{Z})$ is a Diophantine number ρ, then there exists $h \in Diff^\infty(\mathbb{R}/\mathbb{Z})$ which conjugates g with the rotation R_ρ of angle ρ:*

$$h \circ g \circ h^{-1} = R_\rho .$$

Moreover, if $(g_a)_a$ is a C^∞-family of diffeomorphisms with constant rotation number ρ which is Diophantine, then there exists a C^∞-family $(h_a)_a$ of diffeomorphisms h_a which conjugates $(g_a)_a$ with R_ρ:

$$h_a \circ g_a \circ h_a^{-1} = R_\rho .$$

Hence once we exhibit an invariant C^∞-circle on which the dynamics displays a diophantine rotation number ρ, by the above theorem, we exhibit an invariant circle on which f acts as a rotation of angle ρ. Then it is easy to perturb this dynamics of the circle to a root of the identity (by moving ρ to a rational number). In particular an iterate of f is the identity on an interval. We can perturb to create an arbitrarily large number of periodic points.

Parabolic maps of the circle A key new idea in this work is to exhibit circle diffeomorphisms with Diophantine rotation number by creating first *parabolic diffeomorphisms of the circle* (they are indeed easier to exhibit densely thanks to geometrical arguments).

Definition 15.10. *A C^2-diffeomorphism g of a circle $\mathbb{T}$ is* parabolic *if there exists $p \in \mathbb{T}$ so that*

- *The point p is the unique fixed point of g,*
- *The point p is a non-degenerated parabolic fixed point of g:*

$$g(p) = p, \quad D_p g = 1 \quad , \quad D_p^2 g \neq 0 \quad .$$

This idea might sound anti-intuitive since the rotation number of a parabolic map is zero.

The interest of parabolic maps of the circle is that they have a geometric definition and produce irrational rotations after perturbations. Indeed if g is a C^r-parabolic circle map, with $r \geq 2$, then its rotation number is 0. Also, one sees immediately that the composition $R_\epsilon \circ g$, with R_ϵ a rotation of angle $\epsilon > 0$ small, has non-zero rotation number. Hence, by continuity of the rotation number and density of Diophantine number in $\mathbb{R}$, we can chose $\epsilon > 0$ arbitrarily small so that the rotation number $\rho(\epsilon)$ of $R_\epsilon \circ g$ is Diophantine. This proves:

Proposition 15.11. *For every $r \geq 2$, the set D^r of C^r-circle maps with Diophantine rotation number accumulates on the set P^r of C^r-parabolic maps:*

$$cl(D^r) \supset P^r \; .$$

The above argument is topological. Hence the following is a non trivial extension of the latter proposition for parameter families.

Theorem 15.12. *Let $k \in \mathbb{N}$ and let $V \subset \mathbb{R}^k$ be an open subset and $V' \Subset V$. Given any C^∞-family $(g_a)_{a\in V}$ of circle maps so that for every $a \in V$ the map g_a is parabolic, there exists an arbitrarily small Diophantine number $\alpha > 0$, there exists a small C^∞-perturbation $(g'_a)_a$ of $(g_a)_a$ so that:*

- *the rotation number of g'_a is α for every $a \in V'$.*
- *the family $(g'_a)_{a\in V}$ is of class C^∞.*

The proof involves the parabolic renormalization for an unfolding of $(g_a)_a$, and the Arnold-Herman-Yoccoz Theorem.

A trick on cocycles To produce parabolic maps of the circle, we consider a normally hyperbolic fibration by circles, so that the system is roughly speaking given by an iterated function systems by finitely many circle mappings.

We wish to recover densely the following situation which produces a parabolic map h:

Fact 15.13. *Let $f_1(x) = \frac{2}{3}x$ and $f_2(x) = \frac{x}{2x+1}$. It holds:*

$$f_2^n \circ f_1^n(x) \underset{n\to\infty}{\to} \frac{x}{2x+1} =: h(x) \; .$$

Proof. Indeed this follows from the fact that $f_2 = h \circ f_1^{-1} \circ h^{-1}$, and so $f_2^n = h \circ f_1^{-n} \circ h^{-1}$. Restricted to any compact subset of $\mathbb{R}$, the map $h^{-1} \circ f_1^n$ is equivalent to f_1^n and so the limit follows. $\square$

This observation is useful since h is parabolic. We observe that if f'_1 and f'_2 are C^2-perturbation of f_1 and f_2, so that they fix the same point with whose eigenvalue are reciprocal, it holds also that $f_2'^n \circ f_1'^n$ converges to a parabolic map.

Therefore, it suffices to construct an IFS whose finite compositions construct a dense set of fixed points and eigenvalues nearby 0 and 2/3. This is given by a λ-blender.

16 Blenders, parablenders and their λ-versions

In this section we recall the notion of blender, λ-blender and parablenders which enable to prove the results of [Ber16b, Ber17a, Ber17b].

16.0.1 Blender

A hyperbolic set K of a surface local diffeomorphism f is a Bonatti-Diaz' *blender* [BD96] if $\dim E^u = 1$ and a continuous union of local unstable manifolds $\cup_{\overleftarrow{z} \in \overleftarrow{K}} W^u_{loc}(\overleftarrow{z}; f)$ contains robustly a non-empty open set O of M:

$$\bigcup_{\overleftarrow{z} \in \overleftarrow{K}} W^u_{loc}(\overleftarrow{z}; f') \supset O \quad , \quad \forall f' \; C^1\text{-close to } f \; .$$

The set O is called a *covered domain* of the blender K.

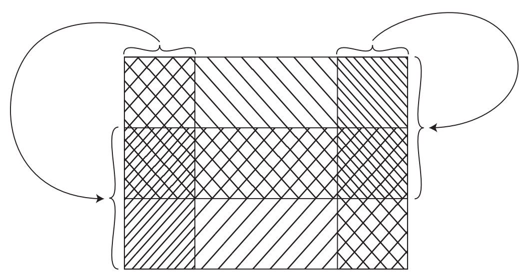


Figure 20: A blender of a surface.

Example 16.1. Let $I_{-1} \sqcup I_{+1}$ be a disjoint union of non-trivial segments in $(-1, 1)$. Let σ be a map which sends affinely each $I_{\pm 1}$ onto $[-1, 1]$. Put:

$$f : (x, y) \in [-2, 2] \times I_{-1} \sqcup I_{+1} \mapsto \begin{cases} f_{+1}(x, y) = (\frac{2}{3}(x-1) + 1, \sigma(y)) & \text{if } y \in I_{+1} \\ f_{-1}(x, y) = (\frac{2}{3}(x+1) - 1, \sigma(y)) & \text{if } y \in I_{-1} \end{cases}$$

Let $K := \cap_{n \geq 0} \sigma^{-n}(I_{-1} \sqcup I_{+1})$ be the maximal invariant of σ, and let $B := [-1, 1] \times K$. We notice that B is a hyperbolic set for f with vertical stable direction and horizontal unstable direction. Given a pre-sequence $\underline{\mathfrak{b}} = (\mathfrak{b}_i)_{i \leq -1} \in \{-1, 1\}^{\mathbb{Z}^-}$ we define the local unstable manifold:

$$W^u_{loc}(\underline{\mathfrak{b}}; f) := \bigcap_{i \geq 1} f^i([-2, 2] \times I_{\mathfrak{b}_{-i}}) \; .$$

We notice that for any C^1-perturbation f' of f, the following is a hyperbolic continuation of $W^u_{loc}(\underline{\mathfrak{b}}; f)$:

$$W^u_{loc}(\underline{\mathfrak{b}}; f') := \bigcap_{i \geq 1} f'^i([-2, 2] \times I_{\mathfrak{b}_{-i}}) \; .$$

Fact 16.2. *B is a blender for f, and its covered domain contains $O := (-2/3, 2/3) \times (-1, 1)$.*

Proof. Let us notice that B satisfies the following *covering property*. With $O_{+1} := [0, 2/3) \times (-1, 1)$ and $O_{-1} := (-2/3, 0) \times (-1, 1)$, it holds:

$$O = O_{+1} \cup O_{-1}, \quad cl(f_{-1}^{-1}(O_{-1}) \cup f_{+1}^{-1}(O_{+1})) \subset O \; .$$

Hence, for any perturbation of the dynamics, for any $z \in O$, there exists a preorbit $(z_i)_{i \leq 0}$ so that z_i belongs to $O_{\mathfrak{b}_i}$ for $\mathfrak{b}_i \in \{\pm 1\}$. With $\underline{\mathfrak{b}} = (\mathfrak{b}_i)_{i \leq -1}$ we note that by shadowing $z \in W^u_{loc}(\underline{\mathfrak{b}}; f')$. $\square$

In higher dimension $n \geq 2$, a hyperbolic compact set K with one dimensional unstable direction is a *blender* for a C^1-dynamics F, if there exists a continuous family of local unstable manifolds $(W^u_{loc}(\overleftarrow{z}; F))_{\overleftarrow{z} \in \overleftarrow{K}}$ whose union intersects robustly a C^1-neighborhood N of an $n-2$-dimensional sub-manifold S:

$$\bigcup_{\overleftarrow{z} \in K} W^u_{loc}(\overleftarrow{z}; F') \cap S' \neq \emptyset \quad , \quad \forall S' \in N \quad \forall F' \ C^1 \text{ close to } F \ .$$

Example 16.3. Let $F : (t, x, y) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R} \mapsto (0, f(x, y))$. We notice that $\{0\} \times B$ is a hyperbolic set of F with one-dimensional unstable direction.

Fact 16.4. *The hyperbolic set $\{0\} \times B$ of F is a blender.*

Proof. We notice that $\mathbb{R}^{n-2} \times \{0\}$ is the strong stable direction. Hence DF^{-1} leaves invariant the constant cone field $\chi = \{(u, v) \in \mathbb{R}^{n-2} \times \mathbb{R}^2 : \|v\| \leq \|u\|\}$.

Let V be the set of C^1-submanifolds S of the form $S = Graph\, \phi$ where $\phi \in C^1((-1, 1)^{n-2}, \mathbb{R}^2)$ so that $\phi(0) \in O$ and $TS \subset S \times \chi$.

We notice that any small C^1-perturbation F' of F satisfies that DF'^{-1} leaves invariant χ. Hence for every $S \in V$, if $S \cap \{0\} \times \mathbb{R}^2 \subset \{0\} \times O_{-1}$ (resp. $\{0\} \times O_{+1}$) then a connected component S' of $F'^{-1}(S) \cap (-1, 1)^{n-2} \times O$ is in V.

Hence, by induction, for every F' C^1-close to F, for every $S \in V$, we can define a sequence of preimages $(S^i)_i$ associated to symbols $\underline{\mathfrak{b}} = (\mathfrak{b}_i)_{i \leq 0}$.

We notice that $(F'^i(S^i))_i$ is a nested sequence of subsets in S, whose intersection $\cap_{i \geq 1} F'^i(S^i)$ consists of a single point z. By shadowing, it comes that z belongs to $W^u_{loc}(\underline{\mathfrak{b}}; F')$ and so S intersects $\bigcup_{z \in K} W^u_{loc}(z; F')$. $\square$

16.0.2 λ-blender

In this subsection let us introduce a blender with a special property.

Let M be a manifold, f a self mapping of M which leaves invariant a hyperbolic compact set K with one dimensional unstable direction. Let N_G be an open neighborhood of $E^s|K$ in the Grassmannian bundle GM of TM, which projects onto a neighborhood N of K and satisfies:

$$\forall z \in N \cap f^{-1}(N) \quad D_z f^{-1} N_G(f(z)) \Subset N_G(z) \text{ , with } N_G(z) = N_G \cap GM_z \ .$$

Definition 16.5 (λ-Blender when $\dim M = 2$)**.** *The hyperbolic set K is a* λ-blender *if the following condition is satisfied. There exist a continuous family of local unstable manifolds $(W^u_{loc}(\overleftarrow{z}; f))_{\overleftarrow{z} \in \overleftarrow{K}}$ and a non-empty open set O of $M \times \mathbb{R}$ so that for every f' C^1-close to f, for every $(Q, \lambda) \in O$, there exists $\overleftarrow{z} \in \overleftarrow{K}$, so that:*

- *$Q \in W^u_{loc}(\overleftarrow{z}; f')$, and with $(Q_{-n})_n$ the preorbit of Q associated to $\overleftarrow{z}$,*
- *for every line L in $N_G(Q)$, with $L_n := (D_{Q_{-n}} f^n)^{-1}(L)$, the sequence $(\frac{1}{n} \log \|D_{Q_{-n}} f^n | L_n\|)_n$ converges to λ.*

The open set O is called a covered domain *for the λ-blender K.*

Example 16.6. Let $\mathfrak{B} = \{-1,+1\}^2$ and let $(I_\mathfrak{b})_{\mathfrak{b}\in\mathfrak{B}}$ be four disjoint, non trivial segments in $(-1,1)$. Let σ be a map which sends affinely each $I_\mathfrak{b}$ onto $[-1,1]$. For $\epsilon > 0$ small put:

$$f : (x,y) \in [-2,2] \times \sqcup_\mathfrak{B} I_\mathfrak{b} \mapsto \Big((\tfrac{2}{3})^{1+\epsilon\cdot\delta'}(x-\delta) + \delta, \sigma(y) \Big) \quad \text{if } y \in I_\mathfrak{b} \text{ and } \mathfrak{b} = (\delta,\delta') \quad .$$

Let $K := \cap_{n\geq 0}\sigma^{-n}(\cup_{\mathfrak{b}\in\mathfrak{B}} I_\mathfrak{b})$ be the maximal invariant of σ, and let $B := [-1,1]\times K$. We notice that B is a hyperbolic set for f with vertical stable direction and horizontal unstable direction. Let $N_G := \{(u,v) \in \mathbb{R}^2 : \|u\| \geq \|v\|\}$.

Given a pre-sequence $\underline{\mathfrak{b}} = (\mathfrak{b}_i)_{i\leq -1}$ and f' C^1-close to f, we define the local unstable manifold:

$$W^u_{loc}(\underline{\mathfrak{b}}; f') := \bigcap_{i\geq 1} f'^i([-2,2]\times I_{\mathfrak{b}_{-i}}) \ .$$

Fact 16.7. *The uniformly hyperbolic B is a λ-blender for f, and its covered domain contains:*

$$O := ((-2/3,2/3)\times(-1,1)) \times (\log 2/3 - 2\epsilon; \log 2/3 + 2\epsilon) \ .$$

Proof. Given $\mathfrak{b} = (\delta,\delta') \in \{-1,1\}^2 = \mathfrak{B}$, let

$$O_\mathfrak{b} = \{((x,y),\lambda) \in O : \ x\cdot\delta \geq 0, (\lambda - \log 2/3)\cdot\delta' \geq 0\} \ .$$

Given a C^1-perturbation f' of the dynamics f, for every $(Q,\lambda) \in O$, given any unit vector $u \in N_G(Q)$, we define inductively a Df'-preorbit $(Q_n,u_n)_n$ associated to a pre-sequence of symbols $(\mathfrak{b}_n)_n$ as follows. Put $Q_0 = Q$ and $u_0 = u$. For $n \leq 0$, we define $\mathfrak{b}_{n-1} = (\delta_{n-1},\delta'_{n-1})$ such that δ_{n-1} is the sign of first coordinate of Q_n, and δ'_{n-1} is the sign of $\log\|u_n\| - n\lambda$. By decreasing induction one easily verifies that $(Q_n, \log\|u_n\| - n\lambda + \log 2/3) \in O_{\mathfrak{b}_n}$ for every n. Hence $\frac{1}{n}\log\|u_n\| \to \lambda$ as asked.

As for the proof of Fact 16.7, this implies that $z \in W^u_{loc}(\underline{\mathfrak{b}}; f')$, with $\underline{\mathfrak{b}} = (\mathfrak{b}_{-n})_{n\leq -1}$. $\square$

Definition 16.8 (λ-blender when $\dim M \geq 3$)**.** *The hyperbolic set K is a* λ-parablender *if the following condition is satisfied.*

There exist a continuous family of local unstable manifolds $(W^u_{loc}(\overleftarrow{z}; f))_{\overleftarrow{z}\in\overleftarrow{K}}$ and a neighborhood O of a pair (S,λ_0) of a number $\lambda_0 \in \mathbb{R}$ with an $n-2$-dimensional C^1-submanifold S so that, for every $(S',\lambda) \in O$, there exists $\overleftarrow{z} \in \overleftarrow{K}$ satisfying:

- *$W^u_{loc}(\overleftarrow{z}; f')$ intersects S' at a point Q, and with $(Q_{-n})_n$ the preorbit of Q associated to $\overleftarrow{z}$,*
- *for any $(n-1)$-plane E in $N_G(Q)$, the sequence $(\frac{1}{n}\log\|D_{Q_{-n}}f^n|E_n\|)_n$ converges to λ, with $E_n := (D_{Q_{-n}}f^n)^{-1}(E)$.*

Example 16.9. Let $F := (t,x,y) \in \mathbb{R}^{n-2}\times\mathbb{R}\times\mathbb{R} \mapsto (0, f(x,y))$ with f as in example 16.6. We notice that $\{0\}\times B$ is a hyperbolic set of F with one-dimensional unstable direction.

In the proof of Fact 16.4, we define a C^1-open set V of $(n-1)$-submanifolds.

Fact 16.10. *The hyperbolic set $\{0\}\times B$ of F is a λ-blender with $O = V\times(-\log 2/3 - 2\epsilon, +\log 2/3 + 2\epsilon)$ in its covered domain.*

The proof is done by merging the one of Facts 16.4 and 16.7, and so it is left as an exercise to the reader.

16.1 Hyperbolic theory for families of dynamics

Let us fix $k \geq 0$, $1 \leq r < \infty$, and a C^r-family $\hat{f} = (f_a)_a$ of C^r-maps of M.

16.1.1 Hyperbolic continuation for parameter family

It is well known that if f_0 has a hyperbolic fixed point P_0, then it persists as a hyperbolic fixed point P_a of f_a for every a small. Moreover, the map $a \mapsto P_a$ is of class C^r.

More generally, if K is a hyperbolic set for f_0 (with possibly $f_0|K$ not injective), it persists in a sense involving its inverse limit $\overleftarrow{K}$. Let $\overleftarrow{f}_0$ be the shift map on $\overleftarrow{K}$.

Theorem 16.11 (Th. 14 [Ber17a]). *For every a in a neighborhood V of 0, there exists a map $h_a \in C^0(\overleftarrow{K}; M)$ so that:*

- *h_0 is the zero-coordinate projection $(z_i)_i \mapsto z_0$.*
- *$f_a \circ h_a = h_a \circ \overleftarrow{f_0}$ for every $a \in V$.*
- *For every $\underline{z} \in \overleftarrow{K}$, the map $a \in V \mapsto h_a(\underline{z})$ is of class C^r.*

The point $h_a(\underline{z})$ is called the *hyperbolic continuation* of $\underline{z}$ for f_a. We denote $\underline{z}_a \in M$ the zero-coordinate of $h_a(\underline{z})$. The family of sets $(K_a)_a$, with $K_a := \{\underline{z}_a : \underline{z} \in \overleftarrow{K}\}$, is called the hyperbolic continuation of K.

The local stable and unstable manifolds $W^s_{loc}(z; f_a)$ and $W^u_{loc}(\underline{z}; f_a)$ are canonically chosen so that they depend continuously on a, z and $\underline{z}$. They are called the *hyperbolic continuations* of $W^s_{loc}(z; f)$ and $W^u_{loc}(\underline{z}; f)$ for f_a. Let us recall:

Proposition 16.12 (Prop 15 [Ber17a]). *For every $z \in K$, the family $(W^s_{loc}(z; f_a))_{a\in V}$ is of class C^r. For every $\underline{z} \in \overleftarrow{K}$, the family $(W^u_{loc}(\underline{z}; f_a))_{a\in V}$ is of class C^r. Both vary continuously with $z \in K$ and $\underline{z} \in \overleftarrow{K}$.*

16.1.2 Parablender

The bifurcation theory studies the hyperbolic continuation of hyperbolic sets and their local stable and unstable manifolds, to find dynamical properties. Hence we shall study the action of C^r-families $\hat{f} = (f_a)_a$ on C^r*-jets.*

Given a C^r-family of points $\hat{z} = (z_a)_{a\in\mathbb{R}^k}$, its C^r*-jet at* $a_0 \in \mathbb{R}^k$ is $J^r_{a_0}\hat{z} = \sum_{j=0}^r \frac{\partial^j_a z_{a_0}}{j!} a^{\otimes j}$. Let $J^r_{a_0}M$ be the space of C^r-jets of k-parameters, C^r-families of points in M.

We notice that any C^r-family $\hat{f} = (f_a)_a$ of C^r-maps f_a of M acts canonically on $J^r_{a_0}M$ as the map:

$$J^r_{a_0}\hat{f} \colon J^r_{a_0}(z_a)_a \in J^r_{a_0}M \mapsto J^r_{a_0}(f_a(z_a))_a \in J^r_{a_0}M \ .$$

The first example of parablender was given in [Ber16b]; in [BCP16] a new example of parablender was given. Let us give for the first time a definition for a dynamics on a manifold M of any dimension n.

Definition 16.13 (C^r-Parablender when $\dim M = 2$)**.** *A family $(K_a)_a$ of blenders K_a endowed with a continuous family of local (one dimensional) unstable manifolds $(W^u_{loc}(\underline{z}; f_a))_{\overleftarrow{z}}$ is a C^r-*parablender *at $a = a_0$ if the following condition is satisfied. There exists a non-empty open set O of C^r-families of 2-codimensional C^r-submanifolds so that for every $(f'_a)_a$ C^r-close to $(f_a)_a$, for every $(S_a)_{a\in\mathbb{R}^k} \in O$, there exist $\underline{z} \in \overleftarrow{K}$ and a C^r-curve of points $\hat{Q} = (Q_a)_a$ in $(W^u_{loc}(\overleftarrow{z}; f'_a))_a$ and a C^r-curve of points $\hat{P} = (P_a)_a$ in $(S_a)_a$ satisfying:*

$$J^r_{a_0}\hat{Q} = J^r_{a_0}\hat{P} \ .$$

The open set O is called a covered domain *for the C^r-parablender $(K_a)_a$.*

Remark 16.14. When $n = 2$, the set O is an open subset of family of points $(S_a)_a \in C^r(\mathbb{R}^k, M)$.

Remark 16.15. We notice that if $J^r_{a_0}(K_a)_a := \{J^r_a(\underline{z}_a)_a : \underline{z} \in \overleftarrow{K}\}$ is a blender for $J^r_{a_0}(f_a)_a$ then $(K_a)_a$ is a C^r-parablender at a_0 for $(f_a)_a$. We do not know if it is a necessary condition.

Example 16.16 (C^r-Parablender in dimension 2)*.* Let $\Delta_r := \{-1, 1\}^{E_r}$ with $E_r := \{i = (i_1, \dots, i_k) \in \{0, \dots, r\}^k : i_1 + \cdots + i_k \le r\}$. For $\delta \in \Delta_r$ we put:

$$P_\delta(a) = \sum_{i\in E_r} \delta(i) \cdot a_1^{i_1} \cdots a_k^{i_k} \ .$$

Consider $Card\,\Delta_r$ disjoint segments $D_r := \sqcup_{\mathfrak{a}\in\Delta_r} I_\delta$ of $(-1, 1)$. Let $\sigma\colon \sqcup_{\delta\in\Delta_r} I_\delta \to [-1, 1]$ be a locally affine, orientation preserving map which sends each I_δ onto $[-1, 1]$. Let $(f_a)_a$ be the k-parameters family defined by:

$$f_a(x, y)\colon (x, y) \in [-3, 3] \times D_r \longmapsto \ \left(\tfrac{2}{3}(x - P_\delta(a)) + P_\delta(a), \sigma(y)\right) \quad \text{if } y \in I_\delta \ .$$

We notice that the maximal invariant set of f_0 is a blender K.

Let us define the following subsets of the space $C^r_0(\mathbb{R}^k, M)$ of germs at 0 of C^r-functions:

$$\hat{O}_r := \{\hat{z} \in C^r_0(\mathbb{R}^k, M) : J^r_0\hat{z} = \sum_{i\in E_r} (x_i, y_i) \cdot a_1^{i_1} \cdots a_k^{i_k} \text{ and } |x_i| < 1, |y_i| < 2/3\} \ .$$

$$\hat{O}_\delta := \{\hat{z} \in C^r_0(\mathbb{R}^k, M) : J^r_0\hat{z} = \sum_{i\in E_r} (x_i, y_i) \cdot a_1^{i_1} \cdots a_k^{i_k} : |x_i| < 1, 0 \le \delta(i) \cdot y_i < 2/3\} \ .$$

We observe that $\hat{O}_r = \cup_{\delta\in\Delta_r}\hat{O}_\delta$. Also for every $\delta \in \Delta_r$, the inverse of $J^r_0(f_a)_a$ maps $cl(\hat{O}_\delta)$ into the interior of $\hat{O}_r$. Hence by proceeding as in [Ber17a, Example 19], we prove that the hyperbolic continuation $(K_a)_a$ of K is a C^r-parablender at $a_0 = 0$ with $\hat{O}$ included in its covered domain.

Example 16.17 (C^r-Parablender in dimension $n \ge 2$)*.* Let $(f_a)_a$ be given by previous example 16.16 with parablender $(K_a)_a$ and covered domain $\hat{O}_r = \cup_{\delta\in\Delta_r}\hat{O}_\delta$. Let $\hat{F} = (F_a)_a$ be defined by:

$$F_a(x, y)\colon (t, x, y) \in (-1, 1)^{n-2} \times [-3, 3] \times D_r \longmapsto (0, f_a(x, y)) \ .$$

We notice that $(\{0\} \times K_a)_a$ is a family of hyperbolic sets for $(F_a)_a$. Let us show that it is a C^r-parablender.

Let $\hat{V}_r$ be the space of germs at $a = 0$ of C^r-family $\hat{\phi} = (\phi_a)_a$ of C^r-maps $\phi_a \in C^r((-1, 1)^{n-2}, \mathbb{R}^2)$ so that:

- $(\phi_a(0))_a \in \hat{O}_r$,
- the map $(a,t) \mapsto \phi_a(t) - \phi_a(0)$ has C^r-norm smaller than 1.

We identify $\hat{V}_r$ with an open set O of germs at $a=0$ of C^r- family of C^r-$(n-2)$-submanifolds of $\mathbb{R}^n$ by associating to $\hat{\phi}$ its family of graphs $(Graph\,\phi_a)_a$. Let us show that for every $\hat{F}'$ C^r-close to $\hat{F}$, for every $(S_a)_a \in O$, there exists $\underline{\delta} \in \Delta^{\mathbb{Z}^-}$, $(P_a)_a \in (S_a)_a$ and $(Q_a)_a \in (W^u_{loc}(\underline{\delta}; f_a))_a$ so that $J^r_0(P_a)_a = J^r_0(Q_a)_a$.

For every $\delta \in \Delta$, we define $\hat{V}_\delta$ as the subset of $\hat{\phi} \in \hat{V}_r$ so that $J^r_0(\phi_a(0))_a$ belongs to $\hat{O}_\delta$. Note that $\hat{V}_r = \cup_\delta \hat{V}_\delta$. Given any C^r-perturbation $\hat{F}'$ of the family $\hat{F}$ and every $\hat{\phi} \in V_\delta$ we define:

$$\hat{\phi}_\delta = (\phi_{a\,\delta})_a\ , \quad \text{with } Graph\,\phi_{a\,\delta} = (F'_a|(-1,1)^{n-2} \times [-3,3] \times I_\delta)^{-1} Graph\,\phi_a \quad \forall a \text{ small.}$$

Fact 16.18. *For every $\hat{F}'$ C^r-close to $\hat{F}$, for every $\hat{\phi} \in \hat{V}_\delta$, the family $\hat{\phi}_\delta$ is well defined and in $\hat{V}_r$.*

Proof. If $\hat{F}' = \hat{F}$, for every $\hat{\phi}$ in $\hat{V}_\delta$, the family $\hat{\phi}_\delta$ is well defined by transversality of the map $(a,t,x,y) \mapsto (a, F_a(t,x,y))$ with the submanifold $\cup_{a\in(-1,1)^{n-2}}\{a\} \times Graph\,\phi_a$. Furthermore, the family $\hat{\phi}_\delta$ is equal to $(t \mapsto g^\delta_a \circ \phi_a(0))_{a\in(-1,1)^{n-2}}$, where g^δ_a is the inverse of $f_a|[-3,3] \times I_\delta$. As $J^d_0(\phi_a(0))_a$ is in $\hat{O}_\delta$, it comes that $J^d_0(g^\delta_a \circ \phi_a(0))_a$ is in $J^d_0(g^\delta_a)_a(\hat{O}_\delta) \Subset \hat{O}_r$. Note that $\hat{\phi}_\delta$ is in a subset of $\hat{V}_r$ at positive distance to the complement of $\hat{V}_r$.

Hence by transversality, for every $\hat{F}'$ in a C^r-small neighborhood of $\hat{F}$, for every $\hat{\phi} \in \hat{V}_\delta$, the family $\hat{\phi}_\delta$ is in $\hat{V}_r$. $\square$

From the latter fact, for every $\hat{F}'$ in a C^r-small neighborhood of $\hat{F}$, for every $\hat{\phi} \in \hat{V}_r$, we can define a sequence $\underline{\delta} \in \Delta^{\mathbb{Z}^-}$ and preimages $(\hat{\phi}^n)_{n\le -1} \in V_r^{\mathbb{Z}^-}$ with $\hat{\phi}^n = \hat{\phi}^{n+1}_{\delta_n}$ and $\hat{\phi}^0 = \hat{\phi}$.

Let $S^n_a = Graph\,\phi^n_a$ and observe that S^n_a is mapped into $S^{n+1}_a = Graph\,\phi^{n+1}_a$ for every $n \le -1$.

Hence the point $P^n_a := (0, \phi^n_a(0))$ is well defined for a small, and by contraction of $F'^n_a|S^n_a$, $F'^n_a(P^{-n}_a)$ belongs to S^0_a and the jets $J^r_0(F^{n-k}_a(P^{-n}_a))_a$ are bounded for every $n \ge k \ge 0$.

Hence $(J^r_0 F'^n_a(P^{-n}_a))_n$ converges to the C^r-jet at $a=0$ of a C^r-curve of points $(P_a)_a \in (S_a)_a$, and displays a preorbit $((P_{k\,a})_a)_{k\le -1}$ associated to $\underline{\delta}$ with bounded C^r-jet at 0 for every $k \le -1$.

By the shadowing property of the hyperbolic set $J^r_0(\{0\} \times K_a)_a$ for $J^r_0\hat{F}'$, the point $J^r_0(P_a)_a$ belongs to the unstable manifold of $J^r_0(\{0\} \times K_a)_a$ associated to $\underline{\delta}$. In other words, there exists a C^r-curve $(Q_a)_a \in (W^u_{loc}(\underline{\delta}; F'_a))_a$ so that $J^r_0(Q_a)_a = J^r_0(P_a)_a$.

16.1.3 λ-parablender

Let us generalize the notion of λ-blender to its parametric version the λ-parablender for a C^r-family of dynamics $(f_a)_a$ of a manifold M of dimension n.

For this end, given $(\hat{z}, \hat{u}) \in C^{r-1}(\mathbb{R}^k, TM)$ so that $\hat{z} \in C^r(\mathbb{R}^k, M)$, we consider:

$$\check{J}^r_{a_0}(\hat{z}, \hat{u}) := (J^r_{a_0}\hat{z}, J^{r-1}_{a_0}\hat{u}) \quad \text{and} \quad \check{J}^r_{a_0}TM := \{\check{J}^r_{a_0}(\hat{z}, \hat{u}) :\ (\hat{z}, \hat{u}) \in C^r(\mathbb{R}^k, TM)\}$$

We note that $D\hat{f}$ acts canonically on $\check{J}^r_{a_0}TM$ as:

$$\check{J}^r_{a_0} D\hat{f} \circ \check{J}^r_{a_0}(\hat{z}, \hat{u}) = \check{J}^r_{a_0}(\hat{f} \circ \hat{z}, D_{\hat{z}}\hat{f}(\hat{u}))$$

Let $(K_a)_a$ be a family of C^r-parablenders for a C^r-family of dynamics $(f_a)_a$. Let N_G be a neighborhood of the stable direction of K_0 in the Grassmanian bundle GM of TM, which projects onto a neighborhood N of K_0 and satisfies:

$$\forall z \in N \cap f_0^{-1}(N) \quad D_z f_0^{-1} N_G(f_0(z)) \Subset N_G(z) \; .$$

Let $\check{J}_0^r N_G \subset \check{J}_0^r TM$ be the subset of jets $(J_0^r(z_a)_a, J_0^{r-1}(u_a)_a) \in \check{J}_0^r TM \setminus \{0\}$ so that $u_a \in N_G(z_a)$ for every a.

Let $\hat{\mathcal{V}}$ be the space of C^r-families $\hat{S} = (S_a)_a$ of 2-codimentional submanifolds S_a. We notice that $\hat{\mathcal{V}}$ is equal to $C^r(\mathbb{R}^k, M)$ if $n = 2$.

Definition 16.19 (C^r-λ-Parablender)**.** *A family $(K_a)_a$ of blenders for $(f_a)_a$ is a C^r-λ-*parablender *at $a = a_0$ if the following condition is satisfied.*

There exists a continuous family of local unstable manifolds $(W^u_{loc}(\overleftarrow{z}; f_a))_{\overleftarrow{z}\, a}$, a non-empty open set O of $\hat{\mathcal{V}} \times C^{r-1}(\mathbb{R}^k, (-\infty, 0))$ so that for every $(f'_a)_a$ C^r-close to $(f_a)_a$, for every $(\hat{S}, \hat{\lambda}) \in O$, there exist $\overleftarrow{z} \in \overleftarrow{K}$ and a C^r-curve of points $\hat{Q} = (Q_a)_a \in (W^u_{loc}(\overleftarrow{z}; f'_a))_a$ and $\hat{P} = (P_a)_a \in (S_a)_a$ satisfying:

$$J^r_{a_0} \hat{P} = J^r_{a_0} \hat{Q} \; .$$

Furthermore, for every C^{r-1}-family of $(n-1)$-planes $(E_a)_a$ in $(N_G(Q_a))_a$ and $(Q_{-n\, a}, E_{-n\, a})_a$ the preimage by $(Df_a^n)_a$ of $(Q_a, E_a)_a$ associated to the preorbit $\overleftarrow{z}$, it holds:

$$\lim_{n \to \infty} \frac{1}{n} J^{r-1}_{a_0} (\log \| D_{Q_{-n\, a}} f'^n_a | E_{-n\, a} \|)_a = J^{r-1}_{a_0} \hat{\lambda} \; .$$

The open set O is called a covered domain *for the C^r-λ-parablender $(K_a)_a$.*

Remark 16.20. In particular $(K_a)_a$ is a C^r-parablender and K_0 is a λ-blender.

Example 16.21 (C^r-λ-Parablender in dimension 2). Let E_r, Δ_r and P_δ be defined as in Example 16.16 and put $\mathfrak{B} := \Delta_r \times \Delta_{r-1}$. Let $\hat{O}_r$, $\hat{O}_\delta$ be the subset of $C^r(\mathbb{R}^k, M)$ defined therein.

Consider $Card\,(\mathfrak{B})$ disjoint segments $D := \sqcup_{\mathfrak{a} \in \mathfrak{B}} I_\mathfrak{a}$ of $(-1, 1)$. Let $\sigma \colon \sqcup_{\mathfrak{a} \in \mathfrak{B}} I_\mathfrak{a} \to [-1, 1]$ be a locally affine map which sends each $I_\mathfrak{a}$ onto $[-1, 1]$. For $\epsilon > 0$ small, let $(\tilde{f}_a)_a$ be the k-parameters family defined by:

$$\tilde{f}_a \colon (x, y) \in D \times [-3, 3] \longmapsto \left(\tfrac{2}{3} \cdot \exp(\epsilon \cdot P_{\delta'}(a)) \cdot x + \tfrac{P_\delta(a)}{3}, \sigma(y)\right) \quad \text{if } y \in I_\mathfrak{a},\ \mathfrak{a} = (\delta, \delta') \; .$$

We notice that the maximal invariant set $\tilde{K}_a$ of $\tilde{f}_a$ in $[-3, 3] \times D$ is hyperbolic and for $\underline{\mathfrak{b}} = (\mathfrak{b}_i)_{i \le -1} \in \mathfrak{B}^{\mathbb{Z}^-}$, with local unstable manifold $W^u_{loc}(\underline{\mathfrak{b}}, \tilde{f}_a) := \bigcap_{i \ge 1} \tilde{f}^i_a([-2, 2] \times I_{\mathfrak{b}_{-i}})$.

Let N_G be the cone field constantly equal $\{(u, v) : \|u\| \le \|v\|\}$. We notice that it is backward invariant. Let:

$$\tilde{O} = \hat{O}_r \times \{\hat{\lambda} \in C_0^{r-1}(\mathbb{R}^k, \mathbb{R}) : J_0^{r-1} \hat{\lambda} = \log \frac{2}{3} + \sum_{i \in E_{r-1}} \lambda_i a^i : \ \lambda_i \in [-2\epsilon, 2\epsilon]\} \; .$$

Fact 16.22. *$(\tilde{K}_a)_a$ is a λ- C^r-parablender for $(\tilde{f}_a)_a$, and its covered domain contains $\tilde{O}$.*

Proof. We consider the covering $(\tilde{O}_{\mathfrak{b}})_{\mathfrak{b}\in\mathfrak{B}}$ of $\tilde{O}$ with for every $\mathfrak{b} = (\delta, \delta') \in \mathfrak{B}$:

$$\tilde{O}_{\mathfrak{b}} := \hat{O}_\delta \times \{\hat{\lambda} \in C_0^{r-1}(\mathbb{R}^k, \mathbb{R}) : J_0^{r-1}\hat{\lambda} = \log\frac{2}{3} + \sum_{i\in E_{r-1}} \lambda_i a^i : \ \delta'_i \cdot \lambda_i \in [0, 2\epsilon]\} ,$$

and proceed by merging the proofs of Fact 16.7 and Example 16.16. □

Example 16.23 (C^r-λ-Parablender in dimension $n \geq 2$). Let $(\tilde{f}_a)_a$ be given by previous example 16.21 with λ-parablender $(\tilde{K}_a)_a$ and covered domain $\tilde{O}_r = \cup_{\mathfrak{b}\in\mathfrak{B}}\tilde{O}_{\mathfrak{b}}$. Let $(\tilde{F}_a)_a$ be defined by:

$$\tilde{F}_a(x, y)\colon (t, x, y) \in (-1, 1)^{n-2} \times [-3, 3] \times D \longmapsto (0, \tilde{f}_a(x, y)) .$$

Let $\tilde{B}_a := \{0\} \times \tilde{K}_a$ and note that $(\tilde{B}_a)_a$ is a family of hyperbolic sets for $(F_a)_a$. Let $\hat{V}_r$ be the subspace of C^r-families of $(n-2)$-dimensional submanifolds defined in Example 16.17.

Fact 16.24. *The family of hyperbolic sets $(\tilde{B}_a)_a$ for $(\tilde{F}_a)_a$ is a λ-C^r-parablender with covered domain:*

$$\hat{V}_r \times \{\hat{\lambda} : J_0^{r-1}\hat{\lambda} = \log\frac{2}{3} + \sum_{i\in E_{r-1}} \lambda_i a^i : \ \lambda_i \in [-2\epsilon, 2\epsilon]\} .$$

Proof. Similarly to Fact 16.22, the proof is done by merging those of Fact 16.10 and Example 16.17. □

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