Comportements Chaotiques des systèmes dynamiques différentiables

Chaotic properties of differentiable dynamical systems

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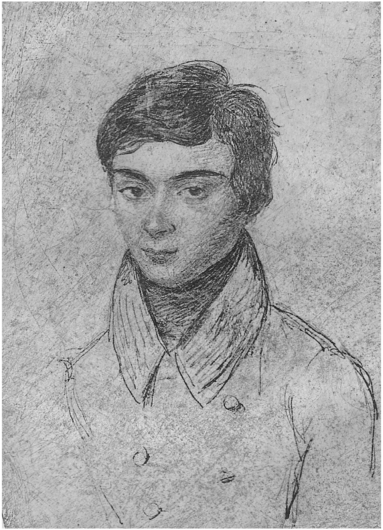
Pr. J. Barral, Pr. F. Béguin, Dr. S. Crovisier, Dr. F. Ledrappier, Pr. E. Pujals,
Pr. M. Shishikura, Pr. D. Turaev.

Au vu des rapports de Dr. S. Crovisier, Dr. F. Ledrappier, Pr. M. Lyubich.

Evariste Galois



Galois' Philosophy

"Sauter à pieds joints sur les calculs, grouper les opérations, les classer suivant leurs difficultés et non suivant leurs formes ; telle est, suivant moi, la mission des géomètres futurs ; telle est la voie où je suis entré dans cet ouvrage."

Evariste Galois, preface *Deux mémoires d'analyse pure*, 1831

"Jump with both feet onto the computations, group the operations, class them following their difficulties and not following their shapes ; this is, in my opinion, the mission of future geometers ; this is the way I entered into this work. "

Plan

Non-chaotic Dynamical Systems

The most simple systems are those which are given by a gradient flow.

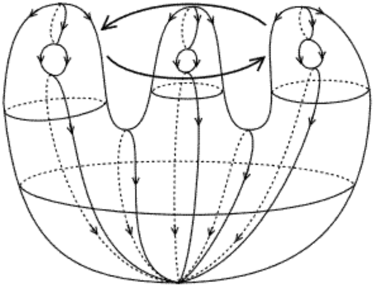


Figure: Morse-Smale dynamics

These systems are not chaotic since a small perturbation of the initial condition does not affect the long time behavior. Furthermore, any perturbation of this dynamics has the same dynamics modulo coordinates change.

In the conservative setting, the dynamics of a pendulum shares the same properties.

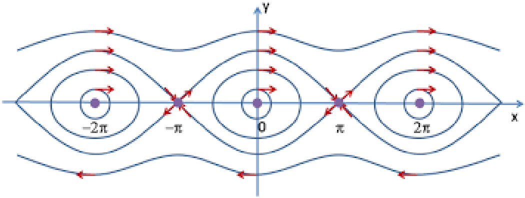
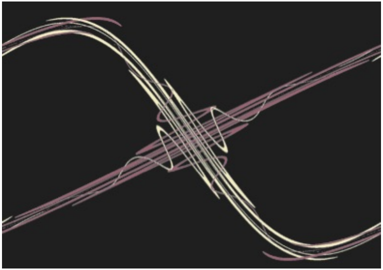


Figure: dynamics of a pendulum

Birth of Chaotic dynamical systems

Figure: Henri Poincaré

He proved in "Les nouvelles méthodes de la mécanique céleste, 1892", that for most perturbations of the time-one-flow map of a pendulum, the stable manifold and unstable manifolds of the hyperbolic fix points form an extremely complicated figure. In particular there are now infinitely many hyperbolic periodic points.



How to describe this Dynamics?

Plan

A first attempt came from the school of Sinai and Anosov in USSR and Smale in USA.

Definition

An invariant set $\Lambda \subset M$ is uniformly hyperbolic if there exists $N \geq 1$ s.t. for every $x \in \Lambda$, there exists a splitting $T_x M = E^s(x) \oplus E^u(x)$ satisfying:

$$\|Df^N|E^s(x)\| < 1/2 \quad \text{and} \quad \|Df^{-N}|E^u(x)\| < 1/2$$

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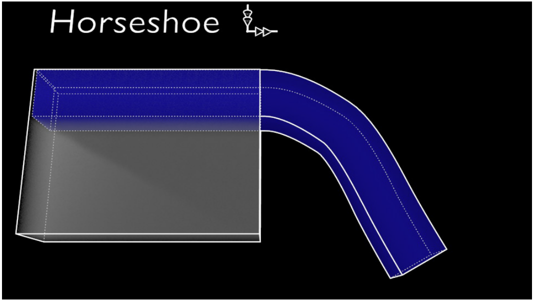

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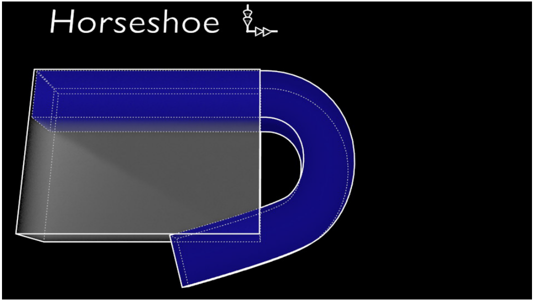

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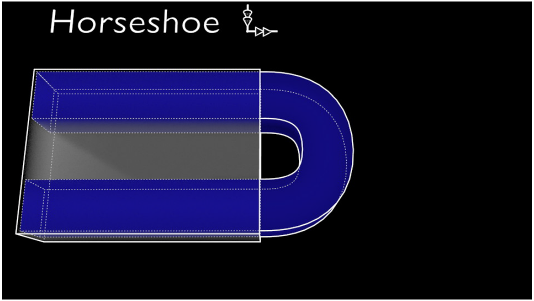

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A hyperbolic set is attracting if the dynamics sends a small neighborhood of it into itself.

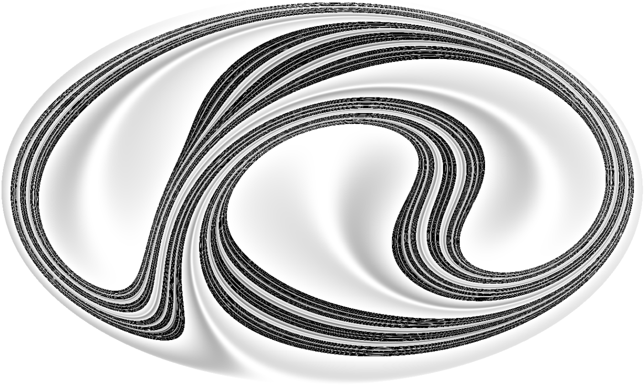

Figure: The Plykin's example of attracting hyperbolic set, credit S. Crovisier.

• Anosov proved that any uniformly hyperbolic set is structurally stable: for every C^1-perturbation $\hat{f}$ of f, there exists a homeomorphism onto its image $h : \Lambda \to M$ so that $\hat{f} \circ h = h \circ f$.

The proof use the fact that the following sets are immersed submanifolds, called unstable and stable manifolds:

$$W^u(x) := \{y \in M : \lim_{n \to -\infty} d(f^n(y), f^n(x)) = 0\}.$$

$$W^s(x) := \{y \in M : \lim_{n \to +\infty} d(f^n(y), f^n(x)) = 0\}.$$

• Sinai-Bowen-Ruelle showed that for every hyperbolic attractor Λ, there exists an invariant measure μ supported by Λ so that Leb a.e. points in U has an orbit with a statistical behavior given by μ:

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{f^k(x)} \rightharpoonup \mu$$

Hence Λ is a Statistical Attractor.

A dynamics whose set of (chain) recurrent points is hyperbolic is called uniformly hyperbolic dynamics.

Such dynamics display excellent dynamical properties. Are they caracterised by these properties?

Question

Is every (typical) statistical attractors uniformly hyperbolic?

Question

Do most of the dynamics behave like them: Lebesgue almost every point converges to a finite number of statistical attractors ?

Question

How are they related to the problem of characterization of the structurally stable dynamical systems?

Plan

Figure: Hénon gave numerical evidence that the map $f(x, y) = (y + 1 - 1.4 \cdot x^2, 0.4 \cdot x)$ displays a statistical attractor. However it cannot be uniformly hyperbolic for topological reasons.

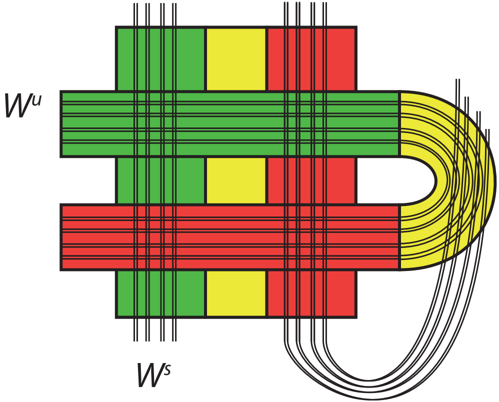

Figure: Newhouse showed the existence of a horseshoe whose stable and unstable manifold are tangent for every C^2-perturbation of the dynamics. Furthermore for topologically generic perturbation, its dynamics displays infinitely many attractors.

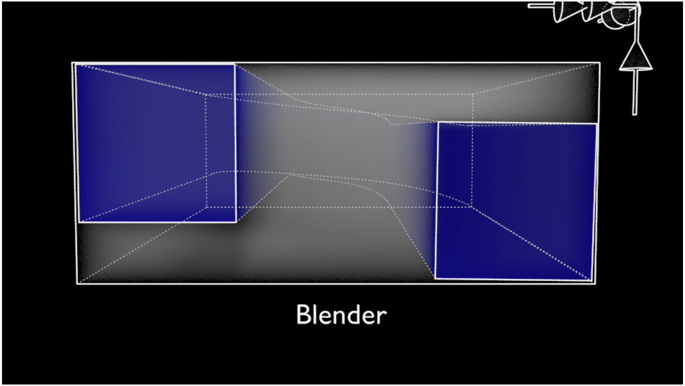

Figure: Bonatti-Diaz found a hyperbolic piece of $\mathbb{R}^3$, called blender which enabled to construct both statistical attractors which are not uniformly hyperbolic, and generic perturbation which displays infinitely many sinks.

Plan

The rough idea of non uniform hyperbolicity is to replace the continuity by the measurability.

An invariant, ergodic, probability measure μ is hyperbolic if its maximal Lyapunov exponent is positive:

$$\lambda^+(x) := \max\{\lambda \in \mathbb{R}^+ :\ \exists u \neq 0 \in T_x M,\ \frac{1}{n} \log \|D_x f^n(u)\| \to \lambda\}\ .$$

By Pesin theory, for μ-a.e. point x the following set is an immersed submanifold:

$$W^u(x) := \{y \in M : \lim_{n\to\infty} -\frac{1}{n} \log d(f^{-n}(y), f^{-n}(x)) > 0\}$$

The measure μ is called SBR if it is absolutely continuous w.r.t. the unstable Pesin manifold.

By Ledrappier-Young's Theorem, every SRB μ for surface $C^{1+\alpha}$-diffeomorphisms f is physical:

$$Leb\{x \in M : \frac{1}{n}\sum_{k=1}^{n} \delta_{f^k(x)} \rightharpoonup \mu\} > 0$$

NUH theory lakes examples (including Hénon initial example).
Here is the list of known paradigmatic examples of NUH maps in low dimension:

Uniformly hyperbolic	Non-uniformly hyperbolic
Expanding maps of the circle	Jakobson's Theorem
Conformal expanding maps of complex tori	Rees' Theorem
Attractors (Solenoid, DA, Plykin...)	Benedicks-Carleson's Theorem
Horseshoes	Palis-Yoccoz' NUH horseshoes
Anosov diffeomorphisms	Standard map ?

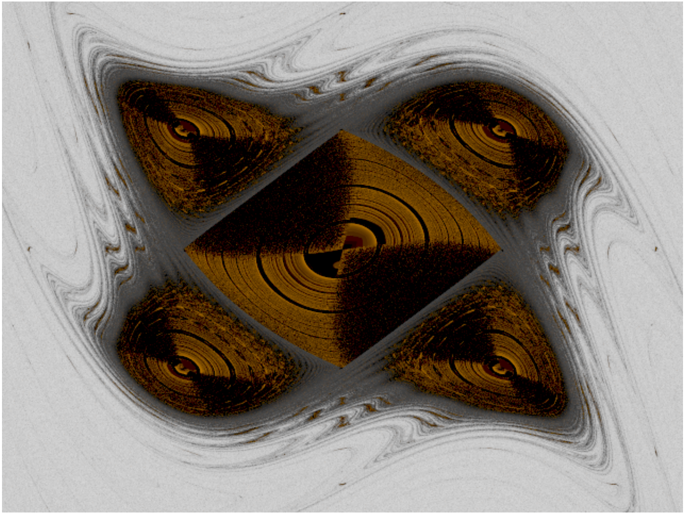

Figure: Standard map $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2 \mapsto (x + a \cdot sin(2\pi y), x + y)$ for a chosen parameter a

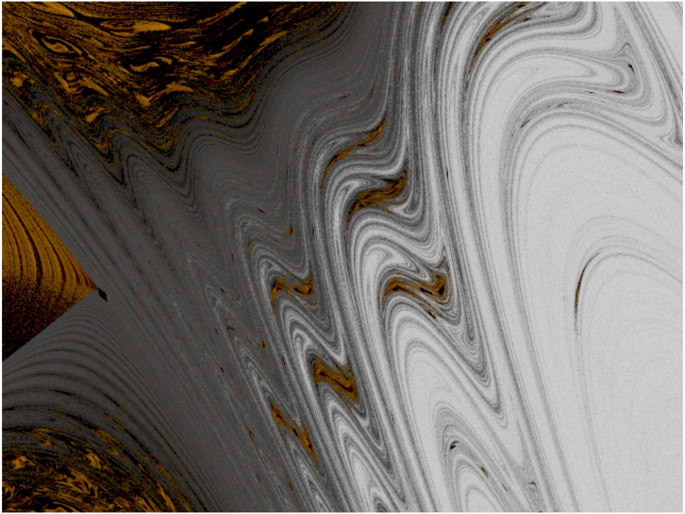

Figure: Detail of the later standard map

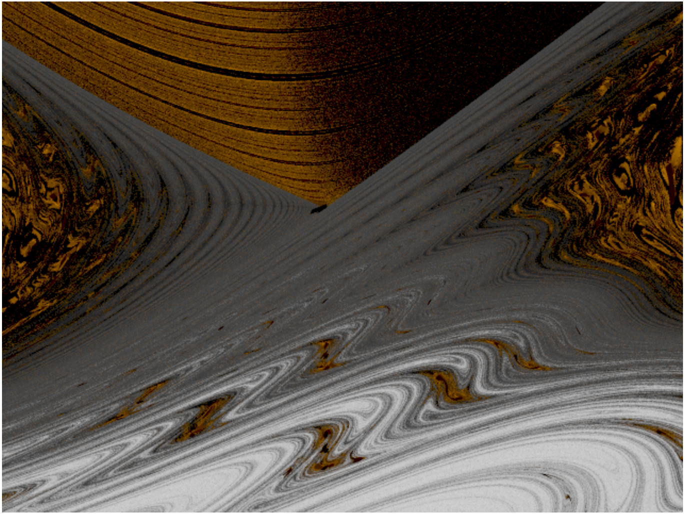

Figure: Another detail of the same standard map

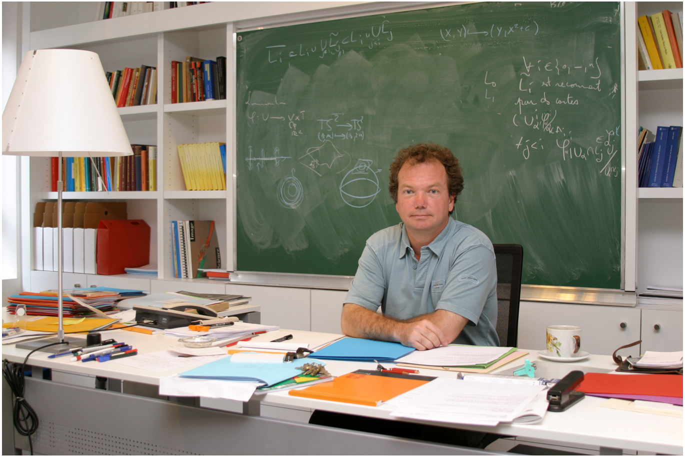

Figure: Jean-Christophe Yoccoz

Yoccoz's program on strong regularity (first and last lecture in College de France) proposed to build a theory (strong regularity) to construct paradigmatic examples.

These examples should be combinatorially and topologically defined, certainly thanks to generalizations of puzzle pieces.

The aim of this program is to encompass more and more abundant examples by increasing the stable and unstable Hausdorff dimensions of the examples.

Theorem (Jakobson, Collet-Eckmann, Benedicks-Carleson, Tsujii, Yoccoz, Shishikura (1980–1995))

There exists a subset $\Lambda \subset \mathbb{R}$ *of Lebesgue positive measure so that for every* $a \in \Lambda$, *the map* $x \mapsto x^2 + a$ *leaves invariant an SRB measure.*

The Hénon map has the following form:

$$f_{a\,b} \colon (x, y) \mapsto (x^2 + a + y, -bx)$$

Theorem (Benedicks-Carleson, Benedicks-Young, Benedicks-Viana, Wang-Young, Takahasi, Berger, Yoccoz (1990–2016))

For b sufficiently small, there exists a subset $\Lambda_b \subset \mathbb{R}$ of Lebesgue positive measure so that for every $a \in \Lambda_b$, the map $f_{a\,b}$ satisfies the following property.
$f_{a\,b}$ has a unique SRB measure whose basin is equal to a neighborhood of Λ modulo a Lebesgue null set.

I proved this theorem by showing the abundance of strongly regular maps. There are several interests in such a formalism:

- The definition of strong regular mapping is basically topological and combinatory. In particular it is a parameter free definition.
- The definition should enable one to prove the abundance of attractors of higher Hausdorff dimension (perhaps like the one of Hénon conjecture).
- the attractor is encoded combinatorially and so many properties can be deduced on its.

Here is one consequence (an answer to questions of L. Carleson):

Theorem (Berger)

For every strongly regular Hénon-like diffeomorphism, there exists $m > 0$, such that for every invariant probability measure μ, the maximal Lyapunov exponent of μ a.e. point is at least m.

Here is one consequence (an answer to questions of Lyubich and Thouvenot):

Theorem (Berger)

For every strongly regular C^2-Hénon-like diffeomorphism, the following sequence converges to a probability measure μ:

$$\frac{1}{\mathit{Card}\ \{z:\ f^n(z) = z\}} \sum_{\{z:\, f^n(z)=z\}} \delta_z \rightharpoonup \mu\ .$$

moreover:

- μ *is ergodic, and is the unique maximal entropy measure,*
- μ *is conjugated modulo a null set to a finite shift,*
- μ *is the exponentially mixing,*
- μ *satisfies the Central Limit Theorem.*

Remark

In the case of C^∞-surface diffeomorphisms, Buzzi-Crovisier-Sarig have shown the uniqueness of the maximal entropy measure for transitive set of positive entropy for C^∞-diffeomorphisms.

Plan

Question

Do most of the dynamics behave like uniformly hyperbolic systems ?

Personal conversation Jean-Christophe Yoccoz 2003

" My personal feeling is that we will not get to understand [typical] dynamical systems in a bounded time. Not everybody agrees with me, some people are optimistic, I respect their belief, but I am not.

I am sure that a scientific revolution is to come, showing that dynamics are much more complicated than we think they are. This should come from the study of a paradigmatic example, which does not need to involve complicated mathematics.

For instance, Newhouse phenomena is [today] rather simple to show."

J.-C. Yoccoz, 2003

Possible interpretations of Yoccoz' quote on complexity:

- Complexity to describe a dynamics by statistics.

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Theorem (Bonatti-Diaz, Asaoka, Diaz-Pujals-Nogueira)

For every $r \geq 1$, for every manifold of dimension at least 3, there exists an open set U in $Diff^r(M)$ so that a generic $f \in U$ displays infinitely many sinks, which accumulate on a non trivial hyperbolic set.

A topologically generic set $\mathcal{R} \subset \mathbb{R}$ can have zero Lebesgue measure (e.g. the set of Liouville numbers). Hence the notion of topologically generic is not considered as typical by many experts.

Kolmogorov suggested a notion of typicality involving parameter families $(f_a)_a$ of dynamics f_a:

> "we shall use the concept of stability in the sense of conservation of a given type of behavior of a dynamical system when there is a slight variation in the functions f, and M or the function H. An arbitrary type of behavior of a dynamical system, for which there exists at least one example of its stable realization, must from this point of view be considered essential and may not be neglected."

Kolmogorov, ICM 1954.

Arnold – student of Kolmogorov – formalized this notion as follows

Definition (Arnold-Kolmogorov Typicallity)

A property $(\mathcal{P})$ *is* C^r*-k-typical if for generic* C^r*-families* $(f_a)_{a\in\mathbb{R}^k}$ *of* C^r*-dynamics* f_a*, for Lebesgue almost every* $a \in \mathbb{R}^k$ *sufficiently small, the map* f_a *satisfies* $(\mathcal{P})$*.*

Definition (Arnold Topology)

A family $(f_a)_{a\in\mathbb{R}^k}$ *is of class* C^r*, if the map* $(a, x) \mapsto f_a(x)$ *is of class* C^r*.*

Theorem (Berger 2015-2016)

For every $r \geq 1$, there exists an open set of $\mathcal{U}$ of C^r-families of C^r-dynamics, and a topologically generic set $\mathcal{R} \subset \mathcal{U}$ so that for every $(f_a)_a \in \mathcal{R}$, for every $\|a\| \leq 1$, the map f_a displays infinitely many sinks.

This theorem is in opposition with many conjectures from the 90's.

This theorem invites us to study two directions:

1. How to measure the complexity of such mapping? For this end I introduced the concept of Emergence.
2. How to find new examples of dynamics with infinitely many sinks and typical in a stronger sens.

Plan

Let $Per_n\, f$ be the set of n-periodic points of f.
Let $Per_n^0 \subset Per_n\, f$ be those which are isolated.

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The number $Card\ Per_n{}^{(0)}$ growths at most exponentially fast if:

$$\limsup_{n\to\infty} \frac{1}{n} \log Card\ Per_n{}^{(0)}\ f < \infty\ .$$

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Theorem (M. Martens, W. de Melo and S. van Strien 1992)

There exists an open and dense set of C^r-maps of a segment, whose number of periodic points growths at most exponentially fast.

Question (Smale 1957, Bowen 1978, Arnold 1989)

Does a topologically generic infinitely smooth map f display always a number of periodic points which grows at most exponentially fast?

Problem (Arnold 1992)

Show that a typical map f does not display a fast growth of the number of periodic points.

Theorem (Gochenko-Shilnikov-Turaev, Kaloshin)

For every manifold M of dimension at least 2, for every $2 \leq r < \infty$, there exists an open set $U \subset Diff^r(M)$ so that a generic dynamics in U displays a fast growth of the number of periodic points:
Given any $(a_n)_n \in \mathbb{N}^{\mathbb{N}}$ a topologically generic $f \in U$ satisfies:

$$\limsup_{n \to \infty} \frac{1}{a_n} Per_n^0(f) = \infty$$

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For $r = 1$ and $dim\, M \geq 3$, the above theorem have been shown by Bonatti-Diaz-Ficher.

Here is full negative answer to the questions of Smale, Bowen and Arnold.

Theorem (Berger)

For every manifold M of dimension at least 2, for every $2 \leq r \leq \infty$, for every $(a_n)_n \in \mathbb{N}^{\mathbb{N}}$ there exists an open set $U \subset C^r(M, M)$ so that a generic $f \in U$ displays a fast growth of the number of periodic points. Moreover U consists of diffeomorphisms if $dim M \geq 3$.

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Plan

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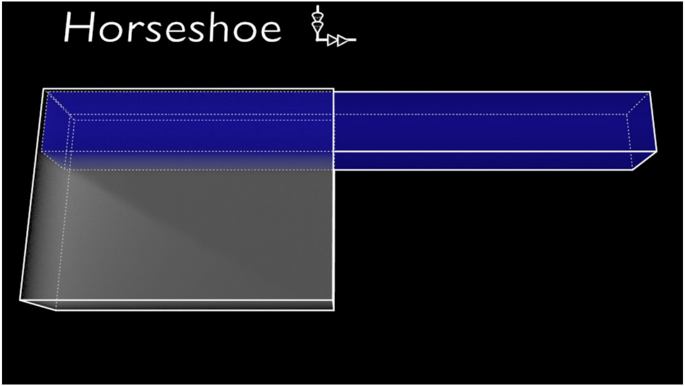

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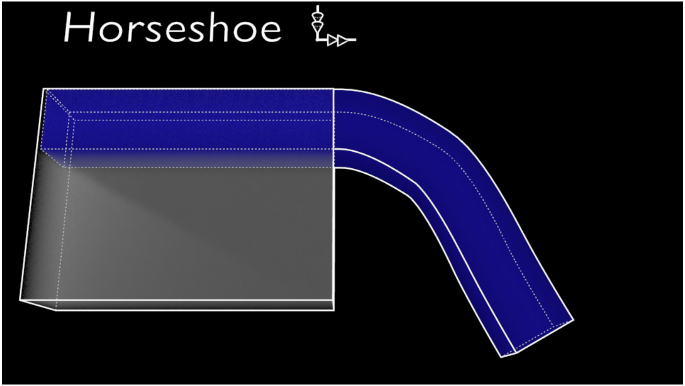

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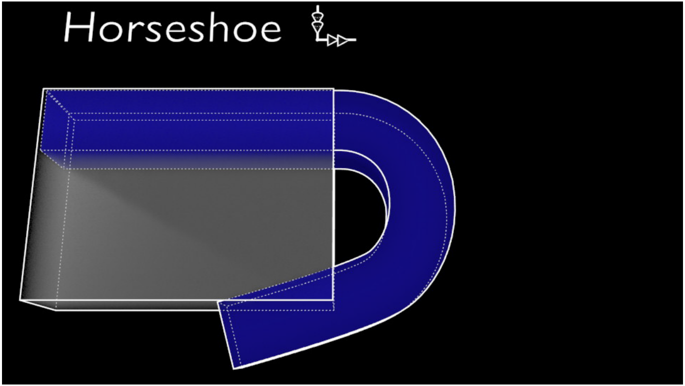

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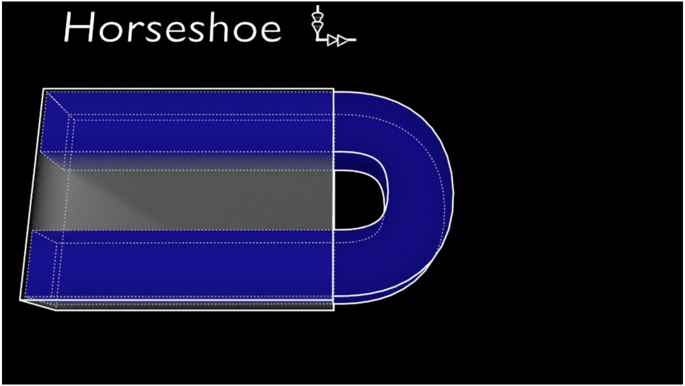

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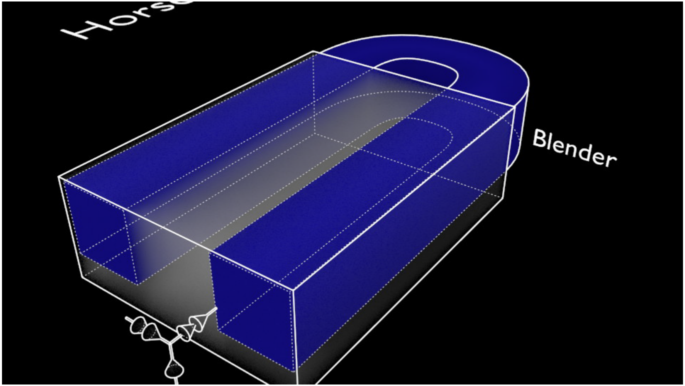

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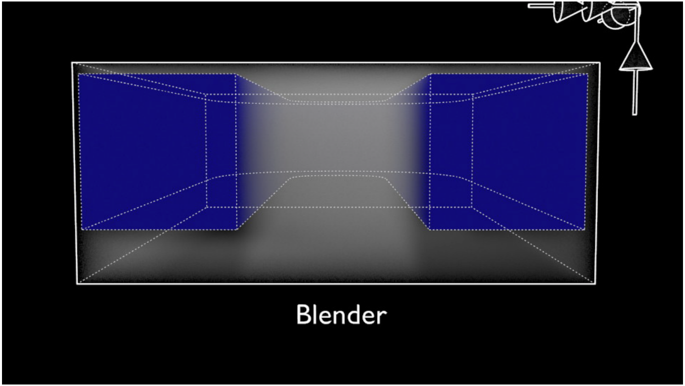

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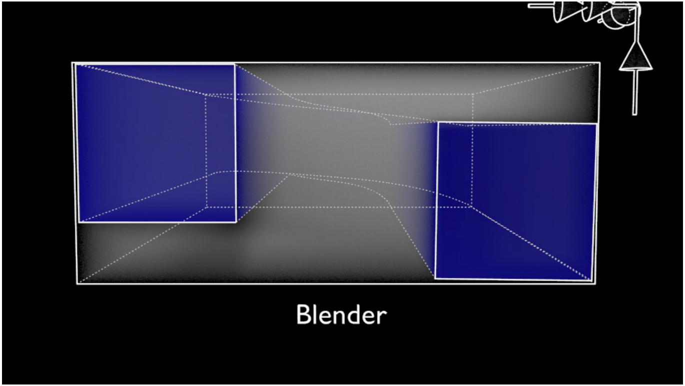


Figure: Bonatti-Diaz found a hyperbolic piece of $\mathbb{R}^3$, called blender which enabled to construct both statistical attractors which are not uniformly hyperbolic, and generic perturbation which displays infinitely many sinks.

***

An IFS of contracting maps $(f_i)_i$ of $\mathbb{R}^n$ is a C^r-blender if its limit set

$$\Lambda = \lim_{n\to\infty} \bigcup_{i_1,\dots,i_n} \{f_{i_n} \circ \dots \circ f_{i_1}(0)\}$$

contains robustly a non-empty open set U of $\mathbb{R}^n$ (among C^r-perturbation of $(f_i)_i$).

Parablenders are designed to stop some bifurcations. In particular those which are responsible for the existence of sinks or of parabolic points.

They are a counterpart of the blender to parameter families, and more precisely to Jets of parameter families.

***

Given a C^r-family of contracting maps $\hat{f}_i = (f_{i\,a})_{a\in\mathbb{R}^k}$, a C^r-family of IFS $(\hat{f}_i)_i$ acts on C^r-family of points $(z_a)_a$ as:

$$(z_a)_a \mapsto (f_{i\,a}(z_i))_a$$

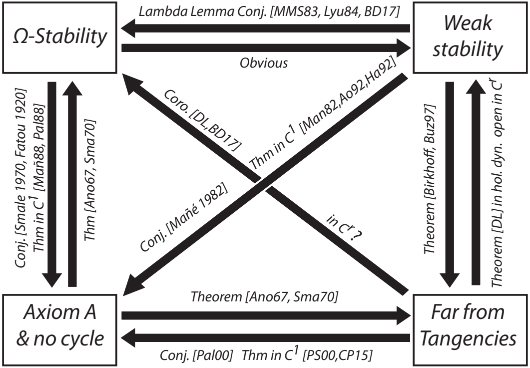
Hence it acts on the space $J_0^r M$ on C^r-jet $J_0^r \hat{z}$ of family $\hat{z} = (z_a)_a$ of points $(z_a)_a$ at $a = 0$:

$$J_0^r \hat{z} \mapsto J_0^r \hat{f}_i \circ J_0^r \hat{z} = J_0^r(f_{a\,i}(z_a))$$

It is a C^r-para-blender if the IFS $(J_0^r \hat{f}_i)_i$ is a blender.

Plan

Uniform hyperbolicity remains nonetheless a wished candidate to describe structurally stable maps, from outstanding conjectures of Fatou and Smale.



Weak stability implies Ω stability

If a map f is structurally stable or Ω-stable, then its periodic point does not bifurcate. This means that for every periodic point p of f, for every perturbation f' of f, a saddle (resp. sink, resp. source) periodic point p persists as a saddle p' (resp. sink, resp. source) for f'.

Weak stability implies Ω stability

Theorem (Mañe, Hayashi, Aoki)

C^1-weakly stable maps are structurally stable.

Lemma (Lambda-Lemma, Mañe-Sad-Sulivan, Lyubich (84))

If a rational function f of the Riemannian sphere is weakly stable (for the family of holomorphic perturbation) then f is Ω-structurally stable.

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Lemma (Berger-Dujardin (2015))

For every weakly stable polynomial automorphism f, there exists a set $\Omega' \subset \Omega$ which is structurally stable and Ω' is of full measure for every hyperbolic probability measure.