Research Statement

Georg Biedermann

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My field of research is algebraic topology with focus in two areas: (non-)realization and moduli problems in mod-p homology; and Goodwillie's calculus of homotopy functors.

In Section 1 I present my past and current work on Goodwillie calculus.

In Section 2 some background to the realization problem along singular mod p homology is described. Then I discuss briefly the main results of my research in this direction contained in [7]. Out of a large array of closely related questions I have picked five concrete projects in 2.3 that I will work on in the next five years.

1 Goodwillie's Calculus of homotopy functors

In a series of landmark papers [27], [29] and [30] Thomas G. Goodwillie developed a "calculus of homotopy functors". One considers homotopy functors – functors that send weak equivalences to weak equivalences – from (pointed) spaces to (pointed) spaces or spectra as analogues of C^∞-functions. Given such a homotopy functor F Goodwillie constructs a tower of functors $\{P_nF\}_{n\in\mathbb{N}}$ under F called the "Taylor tower of F" at a space X that approximates under certain conditions the original F in a certain radius of convergence around X. The tower of the identity functor of pointed spaces interpolates between stable homotopy ($n = 1$) and unstable homotopy ($n = \infty$). Goodwillie's Calculus has had striking applications, e.g. to pseudo-isotopy theory [28] and to homotopy groups of spheres [4].

1.1 Goodwillie calculus and higher topos theory

This is joint work with M. Anel (Paris 7), E. Finster (Polytéchnique), and A. Joyal (UQAM). The general goal is to use higher topos theory to study Goodwillie calculus and vice versa.

1.1.1 The generalized Blakers-Massey Theorem

Consider a homotopy pushout square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & W \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd}$$

There is a canonical map

$$(f,g)\colon\ X \to Y \times^h_Z W$$

into the homotopy pullback of the lower right part of the square that we call the *cartesian gap map*. The classical Blakers-Massey Theorem in topological spaces states that the gap map is $m+n$-connected if f is m-connected and g is n-connected. (Beware that we call a map n-connected if all its fibers are n-connected.) Other fundamental theorems in homotopy theory like the Freudenthal Suspension Theorem or the Hurewicz Theorem are consequences.

Finster and Lumsdaine [23] recently found a new proof of this old theorem in the language of homotopy type theory. Rezk [40] "re-engineered" their proof and formulated it using higher topos theory and homotopical descent.

Using completely different ideas Chacholski, Scherer, and Werndli [17] generalized the Blakers-Massey theorem to the context of acyclic classes of topological spaces in the sense of Dror-Farjoun.

In our first article [2] we combine all these previous efforts. We introduce the notion of a modality: this is a homotopically unique factorization system $(\mathscr{L},\mathscr{R})$ whose left class $\mathscr{L}$ is closed by base change. To each modality in an ∞-topos w associate a generalized Blakers-Massey Theorem.

Theorem: (Anel-Biedermann-Finster-Joyal)
Let $(\mathscr{L},\mathscr{R})$ be a modality on an ∞-topos and let

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & W \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd}$$

be a pushout square. If

$$\Delta(f) \mathbin{\square} \Delta(g) \in \mathscr{L}$$

then

$$(f,g) \in \mathscr{L}.$$

Here, for a map $f\colon\ X \to Y$ the map $\Delta(f)$ is the diagonal map $\Delta(f)\colon\ X \to X \times^h_Y X$ and the box denotes the pushout product ([8, Def. 2.10]). The previous versions of the BMT are direct consequences with one-line proofs.

Note that the pair (n-connected maps, n-truncated maps) forms a modality for each $n \geqslant -2$. Further a map is n-connected if and only if its $\Delta(f)$ is $(n-1)$-conected. Finally, it follows from the equation

$$(S^m \to *) \mathbin{\square} (S^n \to *) = (S^{m+n+1} \to *)$$

that the pushout product of an m-connected map with an n-connected map is $(m+n+1)$-connected. Using our theorem this yields a direct completely formal proof of the classical Blakers-Massey theorem.

We also proof a “dual”statement. For a square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d, "p"] \\ Y \arrow[r, "q"] & Z \end{tikzcd}$$

one can consider the *cocartesian gap map* or *cogap*

$$p \odot q \colon W \cup^h_X Y \to Z.$$

Theorem: (Anel-Biedermann-Finster-Joyal)
Let $(\mathscr{L}, \mathscr{R})$ be a modality on an ∞-topos and let the previous diagram be a pullback square. If

$$p \mathbin{\square} q \in \mathscr{L}$$

then

$$p \odot q \in \mathscr{L}.$$

1.1.2 A Blakers-Massey theorem for the Goodwillie tower

Our main application though is within the context of Goodwillie calculus and is given in our second joint paper [3]. The observation is that any left exact localization of an ∞-topos yields a modality. Goodwillie's n-excisive approximation P_n is a left exact localization of the topos of functors to spaces. A natural transformation $f \colon A \to B$ of homotopy functors is an n-equivalence if it induces an equivalence $P_n f \colon P_n A \simeq P_n B$ of Goodwillie's n-excisive. These maps form the left class of the n-excisive modality. Its right class is given by n-excisive maps; those are maps of functors all of whose fibers are n-excisive.

Theorem: (Anel-Biedermann-Finster-Joyal)
Let $\mathscr{D}$ be a small category with a terminal object and all finite homotopy colimits. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\ B \arrow[r] & D. \end{tikzcd}$$

be a pushout square of homotopy functors from $\mathscr{D}$ to spaces. If f is an n-equivalence and g is an m-equivalence then (f,g) is an $m+n+1$-equivalence.

Conjecture: (Biedermann) The analogous statement holds for Weiss' orthogonal calculus.

As a consequence we obtain independent and (from my point of view) more basic proofs of Goodwillie's main theorem ([30, Sec. 2], [39]) that homogeneous functors deloop. We can also rederive other known delooping results and show some new delooping theorems for functors in several variables.

Our paper will also contain "dual" results. Given a homotopy pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d, "p"] \\ Y \arrow[r, "q"'] & Z, \end{tikzcd}$$

one can consider the *cocartesian gap map*, ie. the map

$$p \odot q \colon W \sqcup_X^h Y \to Z.$$

Theorem: (Anel-Biedermann-Finster-Joyal)
One has $p \odot q > p \,\square\, q$ in any ∞-topos.

Theorem: (Anel-Biedermann-Finster-Joyal)
In the ∞-topos of homotopy functors from $\mathscr{D}$ to spaces, if p is an n-equivalence and q is an m-equivalence then $p \odot q$ is an $m+n+1$-equivalence.

1.1.3 Higher cubes

Goodwillie [29, Thm. 2.3/2.4] proves higher Blakers-Massey Theorems. These are analogous statements about cubes of higher dimension in which each subsquare is a pushout/pullback.

Conjecture: (Goodwillie) Let $\mathscr{X}$ be a strongly cocartesian m-cube of homotopy functors. Assume that for $1 \leqslant i \leqslant m$ each map

$$\mathscr{X}(\varnothing) \to \mathscr{X}(\{i\})$$

induces an equivalence after applying P_{n_i} for $n_i \geqslant 0$. Then:

$$P_{m+n_1+\cdots+n_m-1}(\mathscr{X}) \text{ is cartesian.}$$

Our theorem above is the case $m = 2$.

Corollary: The category of reduced n-excisive functors is n-excisive in the sense that its identity functor is n-excisive.

This corollary is closely related to Heuts' notion of n-excisive category in [33]. A future goal is to proof Goodwillie's conjecture for higher cubes.

1.1.4 The Goodwillie tower of the identity functor of unpointed spaces

Since our techniques from higher topos theory are especially well adapted to the unpointed case we believe that we can eventually understand the Goodwillie tower of the identity functor of the category of unpointed spaces and its layers. One particularly enticing question is: what is the first “derivative” of the identity of unpointed spaces? We think that this subject is well suited for a future research grant application.

1.2 Homotopy nilpotent spaces and groups

Let X be a connected space and let GX denote a simplicial group modelling the loop space ΩX. Let $GX/\Gamma_{n+1}GX$ denote the n-nilpotent quotient in its lower central series. The associated tower $\{\Gamma^n X\}_{n\in\mathbb{N}}$ of spaces under X given by

$$X \mapsto B(GX/\Gamma_{n+1}GX) =: \Gamma^n X$$

was used by Curtis to obtain an unstable Adams spectral sequence. My original observation was that the functor Γ^n is n-excisive and hence, by universality, there is a map from the Taylor tower of the identity into this tower.

1.2.1 The first article

Dwyer and I [6] investigate this relation further. It turns out that on the level of π_1 these towers coincide. We then define the notion of homotopy n-nilpotent groups. Examples of homotopy n-nilpotent spaces include spaces whose homotopy is concentrated in degrees $k+1 \leqslant s \leqslant (n+1)k$ and classifying spaces of n-nilpotent groups. We prove:

- Homotopy n-nilpotent groups interpolate “backwards” between infinite loop spaces ($n = 1$) and loop spaces ($n = \infty$).
- A space has the structure of a homotopy n-nilpotent group if and only if it is the value of an n-excisive functor looped once.
- The homotopy category of homotopy n-nilpotent groups is enriched in n-nilpotent groups generalizing the fact that the homotopy category of infinite loop spaces is enriched in abelian groups.

There are strict inclusions of homotopy categories associated to the following objects:

$$\left\{\begin{array}{c}\text{topological groups,}\\ n-\text{nilpotent}\\ \text{up to homotopy}\end{array}\right\} \subset \left\{\begin{array}{c}\text{homotopy}\\ n-\text{nilpotent groups}\end{array}\right\} \subset \left\{\begin{array}{c}\text{Berstein}-\text{Ganea}\\ n-\text{nilpotent}\\ \text{loop spaces}\end{array}\right\}$$

This is argued in a preprint by Costoya/Scherer/Viruel [18]. They relate the homotopy nilpotence degree of a loop space by inequalities to classical numbers such as the LS cocategory. They apply this to study finite homotopy nilpotent loop spaces, p-compact and p-Noetherian groups.

1.2.2 Functors associated to homotopy nilpotent groups

In [5] we continue to study the category of homotopy n-nilpotent groups. To such an object we can associate an n-excisive functor. This is in fact part of an adjunction between homotopy n-nilpotent groups and the category of looped n-excisive functors. The former category becomes a retract of the latter.

We can further calculate the higher derivatives of the functors associated to homotopy nilpotent groups in terms of their first derivative. The description is similar to how the Poincaré-Birkhoff-Witt theorem presents the free Lie algebra over an abelian group.

Joyal suggests that homotopy n-nilpotent groups form the "connective" part of the theory of n-excisive approximations to the category of spaces given by Heuts [33].

2 Moduli problems in unstable homotopy theory in positive characteristic

2.1 Background

Let p be a prime number and $\mathbb{F}_p$ the field with p elements. Singular cohomology $H^*(X,\mathbb{F}_p)$ is a functor from the category of topological spaces to the category $\mathscr{U}\mathscr{A}$ of unstable algebras over the mod p Steenrod algebra $\mathscr{A}_p$. Dually, singular homology $H_*(X,\mathbb{F}_p)$ can be viewed as a functor to $\mathscr{U}\mathscr{C}$, the category of unstable coalgebras over $\mathscr{A}_p$. Since homology does not involve taking vector space duals it is better behaved. One avoids unnecessary finiteness assumptions. Thus, we prefer to work with homology and coalgebras than with cohomology and algebras.

The study of both categories is a classical part of homotopy theory [42]. In particular, the realization problem has attracted continous attention. It asks whether, for a given unstable coalgebra C, there exists a topological space X with an isomorphism $H_*(X,\mathbb{F}_p) \cong C$ in $\mathscr{U}\mathscr{C}$. Similarly, realizations of maps can be studied. This leads to a moduli problem of topological realizations of objects or morphisms in $\mathscr{U}\mathscr{C}$.

2.1.1 Important results

The realization problem has stimulated deep results. For rational coefficients, at least if one restricts to simply connected objects, such realizations always exist by celebrated theorems by Quillen [38] and Sullivan [45]. For mod p coefficients important statements often take the form of non-existence theorems. The first statement in this context is Adams' famous result on the non-existence of elements of Hopf invariant one [1]. Miller [35] proved a version of a conjecture due to Sullivan stating that all continuous maps from a classifying space BG of a finite group G to an iterated loop space $\Omega^n X$ of a finite dimensional CW complex X are constant up to homotopy. Schwartz [43] for $p=2$ and Gaudens/Schwartz [24] for odd p proved that mod p cohomology of a space is either finite dimensional as $\mathbb{F}_p$-vector space or infinitely generated as module over the Steenrod algebra $\mathscr{A}_p$.

2.1.2 Rigidity

A particular variant of the realization problem has inspired a lot of work and lead to the discovery of a curious phenomenon. This line of thought starts with the question posed by Steenrod [44] in the early seventies, which polynomial algebras over $\mathbb{F}_p$ can occur as the cohomology of a space. Dwyer/Miller/Wilkerson [22] and Notbohm [36] demonstrate that good candidates arise from classifying spaces of compact connected Lie groups and their homotopy theoretic analogues, so-called p-compact groups. It was found that many of these examples are p-rigid. A space X is p-rigid if any other space Y, that admits a map $Y \to X$ inducing an isomorphism on mod-p cohomology, is homotopy equivalent to the p-completion of X. Thus, for p-rigid spaces their mod-p homotopy type is completely determined by their mod-p cohomology. An example is the exceptional Lie group G_2 [32]. Further examples for (integral) rigidity, Davis-Januszkiewicz spaces, were found by Notbohm/Ray [37].

2.1.3 Moduli spaces

A moduli problem can be posed asking to describe a space of realizations whose connected components parametrize the non-equivalent realizations of an object or a morphism in $\mathscr{U}\mathscr{C}$. In the rational case a satisfactory deformation theory has been developed by Schlessinger/Stasheff [41] over several decades. The mod p case eluded good answers until our recent joint preprint [7].

2.1.4 Tools

The fact that the category $\mathscr{U}\mathscr{C}$ and the category $\mathscr{S}$ of topological spaces (in fact we use simplicial sets) are not additive, much less abelian, forces a homotopical algebra approach based on (co-)simplicial techniques. A fundamental tool based on cosimplicial resolutions is Bousfield and Kan's ([16] and [15]) unstable Adams spectral sequence *(UASS)*. For spaces X and Y let us write $H_*(X,\mathbb{F}_p) = C$ and $H_*(Y,\mathbb{F}_p) = D$ in $\mathscr{U}\mathscr{C}$. The spectral sequence starts out from the André-Quillen cohomology (AQC) of unstable coalgebras in the E_2-term:

$$E_2^{p,q} = \mathrm{AQ}_C^p(D,\Sigma^q C) \implies \pi_{q-p}\,\mathrm{map}(X,Y_p^\wedge). \quad \text{(UASS)}$$

It converges under favorable circumstances to the homotopy of the mapping space with $Y_p^\wedge$ denoting the p-completion. The AQC groups can be interpreted as derived functors of the Hom functor in the category $\mathscr{U}\mathscr{C}$. A second important spectral sequence is the homology spectral sequence of a cosimplicial space by Bousfield/Kan ([16] and [13])

$$E_2^{p,q} = \pi^p H_q(X^\bullet,\mathbb{F}_p) \implies H_{q-p}(\mathrm{Tot}\,X^\bullet,\mathbb{F}_p) \quad \text{(HSS)}$$

Again, convergence is guaranteed in nice circumstances. Both spectral sequences are obtained by cosimplicial resolutions that can be thought of as injective resolutions in a non-additive setting.

2.2 My research

Although this is a classical subject new tools have emerged. Bousfield's resolution model structures [12] (generalizing Dwyer/Kan/Stover [21]) allow for the freedom and flexibility in the relevant homotopical algebra that one usually enjoys in an abelian setting. Blanc/Dwyer/Goerss [10] devise a new approach to realization problems in algebraic topology.

In my preprint [7], in collaboration with Georgios Raptis (Regensburg) and Manfred Stelzer (Osnabrück), we bring these new ideas to bear on the moduli problem for unstable coalgebras. The main part of our work lies in carefully setting up the homotopy theory in the category $c\mathcal{UC}$ of cosimplicial unstable coalgebras. There we find a homotopy excision theorem. A thorough comparative study of the homotopy theories of cosimplicial unstable algebras and cosimplicial spaces yields an analogous homotopy excision theorem in $c\mathcal{S}$. It is the key ingredient and necessary input to apply the techniques by Blanc et.al. [10] to our setting. We deduce that the skeletal filtration of a cosimplicial space resembles the Postnikov tower of an ordinary space, although in a dual fashion. The inclusion maps from one skeleton to the next are shown to be "principal"; they are obtained as homotopy pushouts of relative Eilenberg-MacLane objects that corepresent André-Quillen cohomology (AQC).

We obtain a decomposition for the moduli spaces associated to the realization problem of unstable coalgebras. As a consequence, a full-fledged obstruction theory arises, that generalizes and sharpens previous obstruction theories due to Bousfield [14], Harper [31], and Blanc [9]. The decomposition is achieved in terms of AQC of unstable coalgebras. The obstruction groups lie in the (-1) and (-2)-line of the UASS, ie. they are of the form $\mathrm{AQ}_C^{n+i}(C, C[n])$ for $i = 1, 2$.

2.3 Current and future projects

A good understanding of the homotopy theory of cosimplicial unstable coalgebras and its relation to cosimplicial spaces is highly desirable – dually, simplicial unstable algebras and cosimplicial spaces. The objective of the following research projects is to provide more infrastructure for the study of cosimplicial unstable coalgebras with a view towards the realization problem and new rigidity results. They might involve collaboration with my coauthors M. Stelzer (Osnabrück, Germany) and G. Raptis (Regensburg, Germany). I have also discussed aspects of these projects with Lionel Schwartz (Paris 13, France), Jean Lannes (Ecole Polytéchnique, Paris, France) and Geoffrey Powell (Angers, France). Concretely:

2.3.1 A generalization of Harper's theorem

The goal is to proof the following conjecture:

Given simply connected, p-complete spaces X and Y where $H_*(X, \mathbb{F}_p) = 0$ for $* > N$ with fixed $N \in \mathbb{N}$. Let $\varphi\colon C = H_*(X, \mathbb{F}_p) \to H_*(Y, \mathbb{F}_p) = D$ be a morphism in $\mathcal{UC}$. A map $f\colon X \to Y$ with $H_*(f) = \varphi$ exists if and only if the obstructions in $\mathrm{AQ}_C^{k+2}(D, C[k])$ for $k = 1, \ldots, N-1$ vanish.

The statement says that all higher obstructions vanish automatically, even though the obstruction groups themselves basically never vanish. The statement is true in the rational case and in the Massey-Peterson case (homology free as modules over the Steenrod algebra) due to Harper[31]. We believe that our new methods for studying cosimplicial spaces can deliver this result in general. The result is especially desirable in connection with the search for new rigidity results described before.

2.3.2 Algebraic desciption of the first lifting obstruction

The first obstruction to realize a given unstable coalgebra is an element in the group $\mathrm{AQ}^3_{\mathcal{C}}(C, C[1])$; the first obstruction for realizing a map $\varphi\colon C \to D$ lives in $\mathrm{AQ}^2_{\mathcal{C}}(D, C[1])$; both results from [7]. It seems likely that these elements have a purely algebraic description in terms of C and D. Another naive conjecture is that the element in $\mathrm{AQ}^3_{\mathcal{C}}(C, C[1])$ is the obstruction to endow C (which comes by definition with all primary operations) with compatible secondary cohomology operations. In particular, it should be strong enough to see the solution to the Hopf invariant one problem [1]. If a good description could be found then several classical results might have easier and more conceptual proofs; new interesting nonrealization results should be possible. This is a hard and open ended subject.

2.3.3 Operations on André-Quillen cohomology of unstable coalgebras

Goerss has determined the algebra of operations on the AQC of $\mathbb{F}_2$-algebras with coefficients in the ground field [26]. He then explores applications to AQC of unstable algebras and the HSS and UASS [25]. One goal of the program is to determine the corresponding operations for odd primes and the coalgebraic versions. A second aim is to construct operations on AQC with varying coefficients other then the ground field. In our work [7] AQC is a corepresentable functor of cosimplicial spaces with some grip on the corepresenting objects. Thus, one should be in a good position to attempt this project. It is my goal to investigate these operations and understand their impact on realization problems.

2.3.4 The Goodwillie tower of the identity of simplicial unstable algebras

My recently published joint paper [8] generalizes Goodwillie calculus to functors between simplicial model categories (satisfying mild technical conditions). This makes the next project sensible.

Singular cohomology with $\mathbb{F}_p$-coefficients is a functor from cosimplicial spaces to simplicial unstable algebras $s\mathcal{U}\mathcal{A}$ over the Steenrod algebra $\mathcal{A}_p$. Endofunctors of the latter category fit into the framework of Goodwillie calculus [30]. The identity functor is the most interesting object to study. Conjecturally, in this situation its linear part is André-Quillen homology. Such statements have been obtained by Kuhn, see [34, Subsection 6.3]. He even obtains descriptions of the higher layers of the Goodwillie tower in terms of André-Quillen homology. Analogous results should be possible for $s\mathcal{U}\mathcal{A}$.

2.3.5 Homotopy operations on simplicial commutative algebras for odd primes

Dwyer [19] computes the algebra of operations on the homotopy of simplicial commutative $\mathbb{F}_2$-algebras. Then their interaction with Steenrod and Dyer-Lashof operations on second quadrant spectral sequence (such as HSS and UASS) are studied [20]. For odd p the existence of such operations is known by Bousfield's unpublished work [11]. However, for odd p, their relations are not known. It seems very likely that a careful study should reveal them. This might also be an excellent topic for a PhD thesis.

2.4 A new construction of the UASS

In joint work with Raptis and Stelzer we are describing a new construction of Bousfield's UASS for unpointed spaces [14, Sec. 9]. Instead of filtering the target by a coskeletal tower we filter the source by our potential n-stages [7, Def. 6.2]. We can prove so far that the associated spectral sequences are isomorphic from the E_2-term on in the positive range which is the part that has been studied almost exclusively in the literature.

The negative part of the UASS has been neglected because Bousfield's definition of the differentials is rather ad hoc. However, this is the part where our obstruction groups lie. It is intuitively clear that the negative part is related to the (not yet defined) cotangent complex of an unstable coalgebra and that it governs its deformation theory and the moduli problem of its realizations.

The ongoing work aims at making these ideas precise, at defining differentials in a more user-friendly and hopefully geometry-related way, and at comparing the two versions of the UASS in this negative part.

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