A Generalized Blakers-Massey Theorem

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History/Acknowledgments

New proof of classical BMT in Homotopy Type Theory
Favonia-Finster-Licata-Lumsdaine
"A Mechanization of the Blakers-Massey Connectivity Theorem in Homotopy Type Theory"
`arXiv:1605.03227`

"Reverse engineered version"
Rezk
"Proof of the Blakers-Massey Theorem", homepage

Different approach
Chachólski-Scherer-Werndli
"Homotopy Excision and Cellularity"
`arXiv:1408.3252`

Generalized Blakers-Massey Theorem

Theorem (Anel-B-Finster-Joyal)

Let $(\mathcal{L}, \mathcal{R})$ be a modality in an ∞-topos. Consider a homotopy pushout

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & D. \end{array}$$

If

$$\Delta(f) \square \Delta(g) \in \mathcal{L}$$

then the cartesian gap map

$$(f, g)\colon A \to B \times^h_D C$$

is in $\mathcal{L}$. In symbols: $(f, g) > \Delta(f) \square \Delta(g)$

∞-topoi

Definition

An ∞-topos is a left exact localization of a simplicial presheaf category (on a small category).

Example

– simplicial sets $\mathcal{S}$ = "spaces"

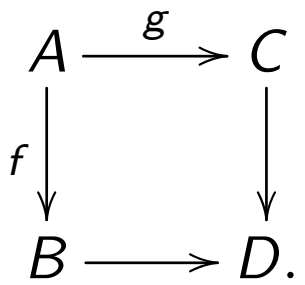
– functors to simplicial sets (from a small category)

– n-excisive functors to simplicial sets

– spectra parametrized by spaces

Gap and cogaps

Given a commutative diagram



we call the canonical map

$$(f, g)\colon A \to B \times_D C$$

the *(cartesian) gap (map)*. The canonical map

$$B \sqcup_A C \to D$$

will be the *cocartesian gap map/cogap*.

The Diagonal of a Map

Definition

Given a map

$$f\colon A \to B$$

the *diagonal* is the canonical map

$$\Delta(f)\colon A \to A \times_B A.$$

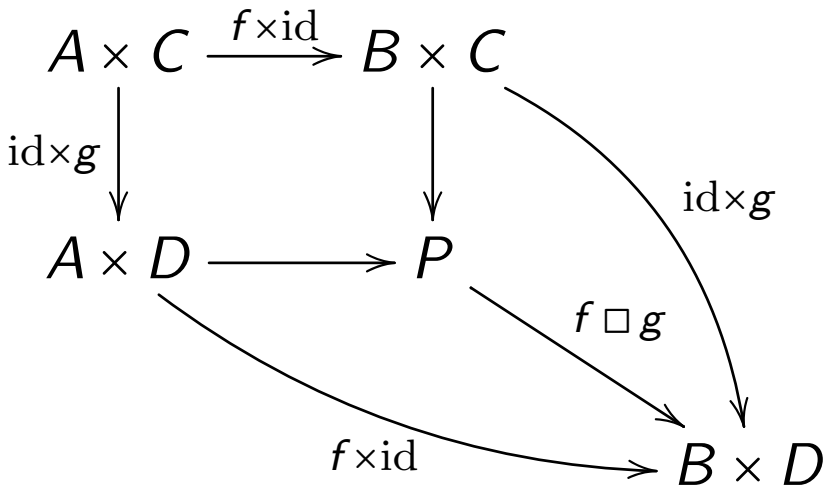
Example

– $\Delta(* \to X) = * \to \Omega X$

– $\Delta(X \to *) = X \to X \times X$, the diagonal.

The Pushout Product

Consider maps $f\colon A \to B$ and $g\colon C \to D$.



One has:

$$(f \square g) \square h = f \square (g \square h)$$

Two examples

Given two objects A and B.

Example

We have

$$(A \to *) \,\square\, (B \to *) = (A \star B \to *),$$

where $A \star B$ denotes the join of A and B.

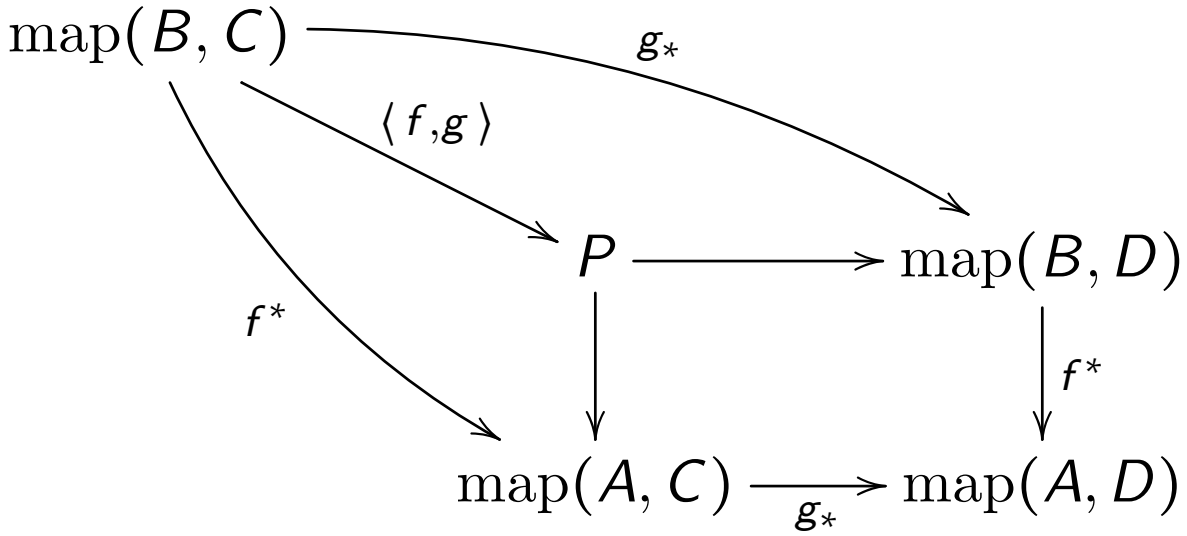
Example

We have

$$(* \to A) \,\square\, (* \to B) = (A \vee B \to A \times B).$$

The Pullback Bracket

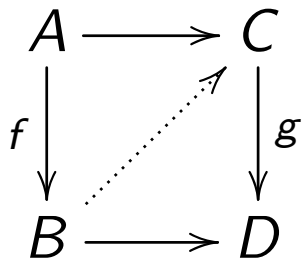
Adjointly:



One has:

$$\langle f, \langle g, h \rangle \rangle = \langle f \ \square \ g, h \rangle$$

Lifting properties



Definition

A map f is *left orthogonal to* g, if $\langle f, g \rangle$ is a weak equivalence. We write:

$$f \perp g$$

Given a class $\mathcal{R}$ of maps, we write ${}^{\perp}\mathcal{R}$ for the class of maps that are left orthogonal to all maps in $\mathcal{R}$. Similarly, $\mathcal{L}^{\perp}$, $({}^{\perp}\mathcal{R})^{\perp}$, ...

Factorization systems

Let $\mathcal{L}$ and $\mathcal{R}$ be two classes of maps. The pair $(\mathcal{L}, \mathcal{R})$ forms a *factorization system* if

1. each map f can be functorially factored (uniquely up to homotopy) into $f = r\ell$ where $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$

and

2. $\mathcal{L}^{\perp} = \mathcal{R}$ and $\mathcal{L} = {}^{\perp}\mathcal{R}$.

Example

For $n \geq -1$ the pair (n-connected, $(n-1)$-truncated) form a factorization system on spaces.

Modalities

Definition

A *modality* is a factorization system $(\mathcal{L}, \mathcal{R})$ such that the left class $\mathcal{L}$ is closed under base change.

Example

Given a left exact localization F of an ∞-topos. Then

$$(F\text{-equivalences}, F\text{-local maps})$$

form a modality.

Modality generated by S

Proposition (ABFJ)

Given a set S of maps in an ∞-topos, there exists a smallest modality such that $S \subset \mathcal{L}$.

Definition

This smallest modality is called *the modality generated by S*.

We then write $\ell > S$ for any $\ell \in \mathcal{L}$.

In particular, if $S = \{f\}$, then $\ell > f$.

If $\ell\colon A \to *$ and $f\colon B \to *$, then this is equivalent to

$$A > B$$

in the sense of Dror Farjoun ("A is killed by B", $P_B(A) = *$).

Fiberwise Orthogonality

Definition

Fix a map ℓ. We say that a map f is *fiberwise right orthogonal* and write

$$\ell \perp\!\!\!\perp f$$

if f is right orthogonal to any base change of ℓ.

Observe: A modality $(\mathcal{L}, \mathcal{R})$ is just an ordinary factorization system where each map in $\mathcal{R}$ is fiberwise right orthogonal to any map in $\mathcal{L}$.

Blakers-Massey Theorem

Theorem (ABFJ)

Let $(\mathcal{L}, \mathcal{R})$ be a modality in an ∞-topos. Consider a pushout

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & D. \end{array}$$

If

$$\Delta(f) \square \Delta(g) \in \mathcal{L}$$

then the cartesian gap map

$$(f, g)\colon A \to B \times_D C$$

is in $\mathcal{L}$. In other words:

$$(f, g) > \Delta(f) \square \Delta(g)$$

Chacholski-Scherer-Werndli Theorem

Chacholski-Scherer-Werndli ("Homotopy Excision and Cellularity") prove for spaces:

$$\mathrm{fib}\,(f,g) > (\Omega\,\mathrm{fib}\ f) * (\Omega\,\mathrm{fib}\ g).$$

Note:

$$\mathrm{fib}\big(\Delta(f) \square \Delta(g)\big) = (\Omega\,\mathrm{fib}\ f) * (\Omega\,\mathrm{fib}\ g)$$

The classical BMT

Observe that the following are equivalent:

– $h > (S^n \to *)$

– h is n-connected

– $\Delta(h)$ is $(n-1)$-connected

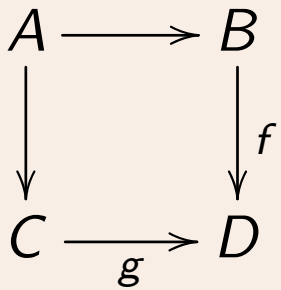
Proof:

$$(f,g) > \Delta(f) \square \Delta(g) > (S^{m-1} \to *) \square (S^{n-1} \to *)$$
$$= (S^{m-1} \star S^{n-1} \to *) = (S^{m+n-1} \to *)$$

"Dual" BMT

Theorem (ABFJ)

In an ∞-topo consider a pullback:



Then for the cogap map

$$B \cup_A C \to D > f \,\square\, g.$$

Goodwillie calculus

Let $\mathcal{F} = \mathrm{Fun}_{\mathcal{S}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{S})$ or $\mathcal{F} = \mathrm{Fun}_{\mathcal{S}}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$; these are ∞-topoi.

A *homotopy functor* is a functor that preserves the class of weak equivalences.

Goodwillie: Treat homotopy functors as analogues of C^∞-functions and study a Taylor expansion.

The Goodwillie tower

Theorem (Goodwillie)

For each homotopy functor F exist a tower of functors

$$F \to \cdots \to P_n F \to \cdots \to P_1 F \to P_0 F,$$

such that $F \to P_n F$ is initial among all maps to n-excisive functors.

Higher excision

Definition

A homotopy functor is *n-excisive* if it sends all strongly cocartesian $(n+1)$-cubes to cartesian ones.

Examples

1. F 0-excisive $\iff$ F constant up to homotopy
2. F 1-excisive $\iff$ $\pi_* F$ gen. homology theory

Here, $P_n F = \operatorname{hocolim}_k T_n^k F$ with

$$F(X) \to T_n F(X) = \operatorname{holim}_{U \neq \emptyset} F(X \star U),$$

where $U \subset \{1, \cdots, n+1\}$.

Linearization

Let F be reduced, ie. $F(*) = *$. Then

$$F(X) \to T_1F(X) = \Omega F(\Sigma X),$$

and

$$P_1F(X) = \operatorname{hocolim}_k \Omega^k F(\Sigma^k X) = \Omega^\infty F(\Sigma^\infty X)$$

For $F = \mathrm{id}$

$$P_1\mathrm{id} = \Omega^\infty\Sigma^\infty.$$

Stable homotopy is the closest homology theory to the identity of pointed spaces.

Modality associated to P_n

Definition

A map $f\colon F \to G$ is *n-excisive* (or P_n-local) if

$$\begin{array}{ccc} F & \longrightarrow & P_nF \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P_nf} \\ G & \longrightarrow & P_nG \end{array}$$

is a homotopy pullback.

A map f is a *P_n-equivalence* if the induced map P_nf is an equivalence.

Recall: The pair (P_n-equiv., n-exc. maps) is a modality.
(P_n is left exact!)

Theorem (Anel/B/Finster/Joyal)

In a homotopy pushout square in $\mathcal{F} = \mathrm{Fun}(\mathcal{C}, \mathcal{S})$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

let f *be a* P_m*-equivalence and* g *a* P_n*-equivalence. Then*

$$(f, g)\colon A \to B \times_D C$$

is a P_{m+n+1}*-equivalence.*

Holds also for functors to $\mathcal{S}_*$*.*

Corollaries

Corollary

Let F be n-reduced. Then $P_{2n-1}F \simeq \Omega P_{2n-1}\Sigma F$.

Corollary (Arone-Dwyer-Lesh)

If a functor has derivatives only in the range between n and $2n-1$ then it is infinitly deloopable.

Corollary (Goodwillie)

If a functor is n-homogeneous it is infinitly deloopable.

In fact, $BF = P_n\Sigma F$ for F n-homogeneous.

Representable functors

We write for $K \in \mathcal{S}_*^{\mathrm{fin}}$

$$R^K = \mathrm{map}_{\mathcal{S}_*}(K, -).$$

We write

$$w_K \colon * \to R^K$$

for the canonical map picking out the constant map.

Idea: In the same way as $S^0 \to *$ generates the connected-truncated modalities, the maps w_K generate the Goodwillie tower.

Consider the map

$$w_{K_0} \square \cdots \square w_{K_n} \colon W_{n+1}(R^{K_0}, \cdots, R^{K_n}) \to R^{K_0} \times \cdots \times R^{K_n}.$$

For $K_0 = \cdots = K_n = K$ there is a pullback

$$\begin{array}{ccc} \Gamma_n(R^K) & \longrightarrow & W_{n+1}^{R^K} \\ \downarrow{\scriptstyle \gamma_{n+1}^K} & & \downarrow{\scriptstyle w_K^{\square n+1}} \\ R^K & \xrightarrow[\Delta]{} & (R^K)^{n+1} \end{array}$$

and γ_K is called the *n-th Ganea fibration* of R^K.

Observations on this pullback

Proposition (ABFJ)

Using Yoneda γ_n^K *induces the map*

$$t_nF\colon F \to T_nF$$

used by Goodwillie to define T_nF *and then* P_nF*.*

Proposition (ABFJ)

The maps γ_n^K *and* w_{n+1}^K *possess the same fiberwise right orthogonal class of maps.*

Corollary (ABFJ)

P_m-$equiv \square P_n$-$equiv \subset P_{m+n+1}$-$equiv$

Monos and Epis in any ∞-topos

Proposition

In an ∞-topos one has:

1. *A map f is mono iff $\pi_0 f$ is mono and the square*

$$\begin{array}{ccc} X & \longrightarrow & \pi_0 X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_0 f} \\ Y & \longrightarrow & \pi_0 Y \end{array}$$

   is a homotopy pullback.

2. *A map f is epi iff $\pi_0 f$ is epi.*

Monos and Epis in $P_n\mathcal{F}$

Proposition (ABFJ)

In the ∞-topos $P_n\mathcal{F}$ we have:

1. *A map f is mono iff $P_0 f$ is mono and the square*

$$\begin{array}{ccc} X & \longrightarrow & P_0 X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P_0 f} \\ Y & \longrightarrow & P_0 Y \end{array}$$

   is a homotopy pullback.

2. *A map f is epi iff $P_0 f$ is epi.*