FIXED POINT SETS AND TANGENT BUNDLES OF ACTIONS ON DISKS AND EUCLIDEAN SPACES

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The main result of this paper is the determination, for any given finite group G not of prime power order, of exactly which smooth manifolds can be fixed point sets of smooth G-actions on disks or on euclidean spaces. General techniques for constructing smooth actions on disks with fixed point set of a given homotopy type were developed in [O1], and the procedure for constructing actions on euclidean spaces is similar (but simpler). What is new here is a way of constructing a G-vector bundle over a G-complex of given homotopy type which extends a given G-bundle over the fixed point set. Such a G-bundle can then be used to control the process of "thickening up" the G-complex to get a manifold with smooth G-action; and in particular to control the diffeomorphism type of the fixed point set. Here "G-complex" always means G-CW complex: a complex built up of orbits $G/H\times D^n$ of cells (where G acts trivially on the disk D^n).

The main technical result for constructing G-bundles, for a finite group G not of prime power order, is given in Theorem 2.4. Let $\mathcal{P}(G)$ denote the set of subgroups of G of prime power order. Very roughly, given a finite G-complex X, a G-vector bundle η over $X^{\mathcal{NP}} \overset{\text{def}}{=} \cup_{H\notin\mathcal{P}(G)} X^H$, and P-vector bundles $\xi_P{\downarrow}X$ for all $P\in\mathcal{P}(G)$, Theorem 2.4 gives conditions for being able to combine η and the ξ_P (after stabilization) to get a G-bundle over a G-complex X' of the same (nonequivariant) homotopy type as X, and with $(X')^{\mathcal{NP}} = X^{\mathcal{NP}}$. This result can then be combined with the equivariant thickening theorem of Edmonds & Lee [EL] and Pawałowski [Pa2] (see Theorem A.12 below), to construct manifolds with smooth G-action having given homotopy type and given tangential structure on the fixed point sets. Note that this procedure does not (directly) apply to construct closed manifolds with G-action, but only open (noncompact) manifolds, or compact manifolds with boundary.

Instead of trying to formulate a general (and very messy) theorem about the construction of manifolds with smooth actions, we concentrate our applications here to the case of smooth actions on disks and euclidean spaces. The study of this problem goes back to P. A. Smith [Sm], who showed that the fixed point set of any continuous action of a p-group (for any prime p) on a finite dimensional $\mathbb{F}_p$-acyclic space is itself $\mathbb{F}_p$-acyclic. A converse to Smith's theorem was proven by Lowell Jones [Jo], who showed among other things that any compact smooth stably complex $\mathbb{F}_p$-acyclic manifold can be the fixed point set of a smooth action of the group of order p on a disk. Thus, if G is any nontrivial p-group, then a compact smooth manifold can be the fixed point set of a smooth G-action on a disk if and only if it is stably complex and $\mathbb{F}_p$-acyclic. A similar (but simpler) construction can be used to prove the corresponding result for smooth p-group actions on euclidean spaces.

Later examples by various authors (cf. [Br, §I.8]) showed that when G does not have prime power order, then the situation is much less rigid. For example, one can find a smooth action of any such G on some euclidean space with fixed point set having the homotopy type of any given countable finite dimensional complex. However, the situation for actions on disks is more complicated. The main result in [O1] says that there is an integer $n_G \geq 0$ with the property that a finite CW complex F is homotopy equivalent to the fixed point set of some smooth G-action on a disk if and only if $\chi(F) \equiv 1 \pmod{n_G}$. The results here now make it possible to determine exactly which manifolds can be fixed point sets of smooth actions on disks or on euclidean spaces; and also to describe (at least stably) the possibilities for the normal bundle of the fixed point set. These results are summarized in the following theorem:

Theorem 0.1. *Let G be any finite group not of prime power order. Fix a smooth manifold F and a G-vector bundle $\eta\downarrow F$ satisfying the following three conditions:*

(1) η is nonequivariantly a product bundle;

(2) for each prime $p\big||G|$ and each p-subgroup $P \subseteq G$, $[\eta|P]$ is infinitely p-divisible in $\widetilde{KO}_P(F)_{(p)}$ (where $\widetilde{KO}_P(F) = KO_P(F)/KO_P(\mathrm{pt})$); and

(3) $\eta^G \cong \tau(F)$ (the tangent bundle of F).

Then there is a smooth action of G on a contractible manifold M such that $M^G = F$, and such that $\tau(M)|F \cong \eta \oplus (V{\times}F)$ for some G-representation V with $V^G = 0$. If $\partial F = \emptyset$, then M can be chosen to be a euclidean space. If F is compact and $\chi(F) \equiv 1 \pmod{n_G}$, then M can be chosen to be a disk.

Conditions (1)–(3) in Theorem 0.1 are also necessary: if G acts smoothly on any contractible manifold M, then they hold for the pair $(F, \eta) = (M^G, \tau(M)|M^G)$. Note in particular point (2): $[\eta|P]$ is infinitely p-divisible in $\widetilde{KO}_P(F)$ since $\eta|P$ is the restriction of a P-bundle over the $\mathbb{F}_p$-acyclic manifold M^P (hence the group $\widetilde{KO}_P(M^P)$ is uniquely p-divisible).

Theorem 0.1 still leaves it rather unclear exactly which manifolds can be the fixed point set of a G-action on a disk or euclidean space. In order to make this more precise, we first need some definitions. Let $\mathcal{M}_{\mathbb{C}} \supseteq \mathcal{M}_{\mathbb{C}+} \supseteq \mathcal{M}_{\mathbb{R}}$ be the classes of finite groups for which there exist G-representations V and W which are complex, self-conjugate, or real, respectively, such that $V|P \cong W|P$ for any $P \subseteq G$ of prime power order, and such that $\dim(V^G) = 1$ and $\dim(W^G) = 0$. By Lemma 3.1 below, $G \in \mathcal{M}_{\mathbb{C}}$ if and only if it contains an element not of prime power order, $G \in \mathcal{M}_{\mathbb{C}+}$ if and only if it contains an element not of prime power order which is conjugate to its inverse, and $G \in \mathcal{M}_{\mathbb{R}}$ if and only if it contains a subquotient which is dihedral of order $2n$ for some n not a prime power. (This condition for a group to be in $\mathcal{M}_{\mathbb{R}}$ was pointed out to me by Erkki Laitinen.)

In the following theorem, for any abelian group A, we let $\mathrm{qdiv}(A)$ (the subgroup of "quasidivisible" elements) denote the intersection of all kernels of homomorphisms from A into free abelian groups. When A is finitely generated, this is just the torsion subgroup of A. Also, the standard induction and forgetful maps between the groups of real, complex, and quaternion vector bundles over a space X are denoted as follows:

$$\widetilde{KO}(X) \underset{r}{\overset{c}{\rightleftarrows}} \widetilde{K}(X) \underset{q}{\overset{c'}{\rightleftarrows}} \widetilde{KSp}(X).$$

Theorem 0.2. *Let G be a finite group not of prime power order. Let $\mathsf{Fix}(G)$ be the class of smooth manifolds F (without action) for which there is a G-bundle $\eta{\downarrow}F$ satisfying conditions (1)–(3) in Theorem 0.1. Then a smooth manifold F is the fixed point set of a smooth action of G on a euclidean space if and only if $F \in \mathsf{Fix}(G)$ and $\partial F = \emptyset$; while F is the fixed point set of a smooth action of G on a disk if and only if F is compact, $F \in \mathsf{Fix}(G)$, and $\chi(F) \equiv 1 \pmod{n_G}$. Furthermore, $\mathsf{Fix}(G)$ is described as follows (where $\mathrm{Syl}_2(G)$ is a Sylow 2-subgroup of G):*

$F \in \mathsf{Fix}(G) \iff$	$\mathrm{Syl}_2(G) \ntriangleleft G$	$\mathrm{Syl}_2(G) \triangleleft G$
$G \in \mathcal{M}_{\mathbb{R}}$	(A) (no restriction)	———
$G \in \mathcal{M}_{\mathbb{C}+} \smallsetminus \mathcal{M}_{\mathbb{R}}$	(B) $c([\tau(F)]) \in c'(\widetilde{KSp}(F)) + qdiv(\widetilde{K}(F))$	———
$G \in \mathcal{M}_{\mathbb{C}} \smallsetminus \mathcal{M}_{\mathbb{C}+}$	(C) $[\tau(F)] \in r(\widetilde{K}(F)) + qdiv(\widetilde{KO}(F))$	(D) $[\tau(F)] \in r(\widetilde{K}(F))$ (F is stably complex)
$G \notin \mathcal{M}_{\mathbb{C}}$	(E) $[\tau(F)] \in qdiv(\widetilde{KO}(F))$	(F) $[\tau(F)] \in r\big(qdiv(\widetilde{K}(F))\big)$

This theorem extends results of Edmonds & Lee [EL, Theorem A] and Pawałowski [Pa2, Theorems 5.6 & 5.9]. However, all of their constructions give fixed point sets with stably complex tangent bundle, and the possibility of having other fixed point sets (when $\mathrm{Syl}_2(G) \ntriangleleft G$) is new. Note that if $G \notin \mathcal{M}_{\mathbb{C}}$, then all connected components of the fixed point set of a G-action on a disk or euclidean space must have the same dimension; while if $G \in \mathcal{M}_{\mathbb{C}} \smallsetminus \mathcal{M}_{\mathbb{R}}$ then the dimensions of the components can be different but must have the same parity (see [Pa1, Theorem A]). In contrast, if $G \in \mathcal{M}_{\mathbb{R}}$, then the components of the fixed point set can have arbitrary dimensions.

Examples of groups in the above classes include: (A) $D(2n)$, (B) $Q(4p^a)$, (C) $D(2p^a) \times C_{q^b}$, (D) C_n, (E) $D(2p^a)$, (F) $\mathbb{F}_{2^i} \rtimes C_{p^a}$ where $p^a | 2^i - 1$. Here, in all cases, n denotes any integer not a prime power, p and q denote distinct odd primes; and C_m, $D(m)$, and $Q(m)$ denote cyclic, dihedral, and quaternion groups of order m.

The conditions on F in cases (B) and (C) above are very similar, and it is not immediately clear that they give distinct classes $\mathsf{Fix}(G)$. To see that they do, set $X = S^5 \cup_{\eta^2} e^8$: the complex obtained by attaching an 8-cell to S^5 via the nontrivial element $\eta^2 \in \pi_7(S^5)$. We leave it as an exercise to check that $\widetilde{KO}(X) \cong \mathbb{Z}$, and that the maps

$$\widetilde{KO}(X) \xrightarrow[\cong]{c} \widetilde{K}(X) \xleftarrow[\cong]{c'} \widetilde{KSp}(X)$$

are isomorphisms. Thus, if F is a compact manifold with the homotopy type of X such that $[\tau(F)]$ generates $\widetilde{KO}(F)$, then $F \in \mathsf{Fix}(G)$ for G of type (B), but not for G of type (C).

So far, we have only discussed the case of actions of finite groups. If G is a compact Lie group with identity component G_0 which acts smoothly on a contractible manifold M with fixed point set F, then one can show (using [JO, Proposition 4.6] and the definition of $\mathsf{Fix}(-)$) that $F \in \mathsf{Fix}(G/G_0)$. More precisely, $(\tau(M)|F)^{G_0}$ is a G/G_0-bundle which satisfies conditions (1)–(3) in Theorem 0.1. In particular, if G is connected and nonabelian, then F can be the fixed point set of a smooth action of G on a disk if and only if it is stably parallelizable (see [O2, Theorems 3 & 5]). This is, however, still far from answering the question of which manifolds can be fixed point sets, since in general more homotopy types can occur as fixed point sets of G-actions on disks than of G/G_0-actions on disks.

Since the numbers n_G play such a key role in the above theorem, we summarize here their computation in [O1, Corollary to Theorem 5] and [O3, Theorem 7]. Let $\mathcal{G}^1$ be the class of all finite groups G which contain a normal subgroup $P \lhd G$ of prime power order such that G/P is cyclic. For each prime p, let $\mathcal{G}^p$ denote the class of all finite groups G which contain a normal subgroup in $\mathcal{G}^1$ of p-power index.

Theorem 0.3. *Fix a finite group G not of prime power order. For any prime p, $p|n_G$ if and only if $G \in \mathcal{G}^p$. Thus, $n_G = 0$ if and only if $G \in \mathcal{G}^1$, and $n_G = 1$ if and only if $G \notin \cup_p \mathcal{G}^p$. In general, n_G is equal to 0, 1, a product of distinct primes, or 4; and $n_G = 4$ if and only if*

(1) G lies in an extension $1 \to C_m \to G \to C_{2^k} \to 1$, where C_m is cyclic of odd order m and C_{2^k} is cyclic of order 2^k,

(2) $G \notin \mathcal{G}^1$, but its subgroup of index 2 does lie in $\mathcal{G}^1$, and

(3) there is no unit $u \in (\mathbb{Z}\zeta_m)^*$ such that $\alpha(u) = -u$, where ζ_m is a primitive m-th root of unity, and $\alpha \in \mathrm{Gal}(\mathbb{Q}\zeta_m/\mathbb{Q})$ is induced by the conjugation action of a generator of $G/C_m \cong C_{2^k}$.

As another special case of Theorem 0.1, we note the the following theorem about tangential representations at isolated fixed points. This generalizes results of Edmonds & Lee [EL, Theorem B] and Pawałowski [Pa3, Theorem 1]:

Theorem 0.4. *Let G be any group not of prime power order. Let $V_0, V_1, \ldots, V_m$ be (real) G-representations such that $V_0|P \cong V_1|P \cong \cdots \cong V_m|P$ for any $P \subseteq G$ of prime power order, and such that $V_i^G = 0$ for all i. Then there exists a G-representation W with $W^G = 0$, and a smooth action of G on a euclidean space (or a disk if $n_G|m$) with exactly $m+1$ fixed points $x_0, \ldots, x_m$, such that the tangential representation at x_i is $V_i \oplus W$.*

The paper is organized as follows. In Section 1, a space B_G^*O is constructed which has the following property (Proposition 1.3): for any finite G-complex X, $[X, B_G^*O]^G$ is (roughly) the inverse limit of the groups $KO_P(X)_{(p)}$, taken over all p-subgroups $P \subseteq G$ and all primes $p\big||G|$. The problem of lifting maps $X \to B_G^*O$ to B_GO is then handled in Section 2, and this leads to a general criterion (Theorem 2.4) for constructing G-bundles over G-complexes of given (nonequivariant) homotopy type and with given fixed point data. The proofs of Theorems 0.1 and 0.2, and some examples, are then given in Section 3. Finally, in an appendix, some technical results are listed, most of which are well known but seem hard to find in the literature. The equivariant thickening theorem in the version of Pawałowski is also stated there (Theorem A.12).

The proof of these results — more precisely the constructions in Section 1 — were to a great extent motivated by my joint work with Stefan Jackowski on vector bundles over classifying spaces of compact Lie groups [JO]. The connections with [JO] have largely disappeared while this work has evolved, but it probably would not have been possible without the discussions we had while writing that paper.

1. An approximation to the classifying spaces for G-bundles

Throughout this section, G will be a fixed finite group. Let $\mathcal{O}(G)$ denote the orbit category of G: the category whose objects are the orbits G/H for all subgroups $H \subseteq G$, and where $\mathrm{Mor}_{\mathcal{O}(G)}(G/H, G/K)$ is the set of all G-maps $G/H \to G/K$. For each prime p, $\mathcal{O}_p(G) \subseteq \mathcal{O}(G)$ will denote the full subcategory whose objects are the orbits G/P for p-subgroups $P \subseteq G$. Also, $\mathcal{O}_1(G)$ denotes the full subcategory with one object $G/1$.

For any full subcategory $\mathcal{C} \subseteq \mathcal{O}(G)$, we define

$$E\mathcal{C} = \underset{G/H \in \mathcal{C}}{\underrightarrow{\mathrm{hocolim}}}\big(G/H\big).$$

This can be regarded as the nerve of the category whose objects are the cosets aH for $G/H \in \mathcal{C}$, and where there is one morphism from aH to bK for each $\mathcal{C}$-morphism $G/H \to G/K$ which sends aH to bK. (In particular, there is at most one morphism between any pair of objects.) From this definition, $E\mathcal{C}$ is seen to be a G-complex all of whose orbit types lie in $\mathcal{C}$. Also, $E\mathcal{C}^H$ is contractible for any G/H in $\mathcal{C}$, since it is the nerve of a category with initial object the coset eH. In particular, $E\mathcal{O}_1(G) \cong EG$. More generally, by equivariant obstruction theory, $E\mathcal{C}$ is "universal" among G-complexes with orbits in $\mathcal{C}$: for any such X there is a G-map $X \to E\mathcal{C}$ which is unique up to G-homotopy.

In the appendix, B_GO is defined to be the infinite mapping cylinder of maps $B_GO(0) \to B_GO(d) \to B_GO(2d) \to \ldots$, where $d = |G|$, where $B_GO(n)$ is the base space of the universal n-dimensional G-bundle, and where the maps are stabilization by the regular representation $\mathbb{R}G$. Bundle direct sum defines product maps $B_GO(n) \times B_GO(m) \to B_GO(n+m)$, and these combine to define a G-map $B_GO \times B_GO \to B_GO$ which makes B_GO into a G-equivariant H-space. Alternatively, one can define B_GO as the identity component of the loop space $\Omega B\Big(\coprod_{n=0}^{\infty} B_GO(n)\Big)$ (once the $B_GO(n)$ have been defined precisely enough to make their disjoint union into a topological monoid); and then the H-space structure on B_GO is automatic.

We will also have need for the p-localization $B_GO_{(p)}$ of B_GO, for any prime $p\big||G|$. One elementary way to define this is as the infinite mapping cylinder of the maps

$$B_GO \xrightarrow{\cdot n_1} B_GO \xrightarrow{\cdot n_2} B_GO \xrightarrow{\cdot n_3} B_GO \xrightarrow{\cdot n_4} \ldots,$$

where $B_GO \xrightarrow{\cdot n} B_GO$ is multiplication by n (using the H-space structure), and where $n_1, n_2, \ldots$ is any sequence of positive integers prime to p such that each prime different from p divides infinitely many of the n_i. Thus, for any finite G-complex X, $[X, \mathbb{Z}_{(p)} \times B_GO_{(p)}]^G \cong KO_G(X)_{(p)}$. We regard B_GO as a subcomplex of $B_GO_{(p)}$ via inclusion into the first term of the cylinder. By construction, for each subgroup $H \subseteq G$, $(B_GO_{(p)})^H$ is the p-localization of $(B_GO)^H$ (where the group of components has also been p-localized). Hence by equivariant obstruction theory, it is immediate that the equivariant H-space structure on B_GO extends to an equivariant H-space structure on $B_GO_{(p)}$.

Definition 1.1. *Define the G-space B_G^*O to be the pullback in the diagram:*

$$\begin{array}{ccc}
B_G^*O & \longrightarrow & \prod_{p||G|} \operatorname{map}(E\mathcal{O}_p(G), B_GO_{(p)}) \\
\big\downarrow & & \big\downarrow \\
\operatorname{map}(E\mathcal{O}_1(G), B_GO) & \xrightarrow{\text{diag}} & \prod_{p||G|} \operatorname{map}(E\mathcal{O}_1(G), B_GO_{(p)}),
\end{array}$$

where the right hand vertical map is induced by restriction to $E\mathcal{O}_1(G)$ (regarded as a subspace of $E\mathcal{O}_p(G)$). Let

$$L_G : B_GO \hookrightarrow B_G^*O$$

be the G-equivariant map induced by inclusions into constant maps in the above square.

By the homotopy extension property for inclusions of simplicial complexes, the right hand vertical map in the above square satisfies the equivariant homotopy lifting property. Thus, B_G^*O is also a homotopy pullback of that square.

We want to study maps from finite G-complexes to B_G^*O. The following lemma, a special case of a theorem of [JM], will be needed to handle the higher inverse limits which arise as obstructions.

Lemma 1.2. *For any finite G-complex X and any prime $p\big||G|$,*

$$\varprojlim_{G/P\in\mathcal{O}_p(G)}{}^{j} \Big(KO_G(G/P\times X)_{(p)}\Big) = 0 \qquad \textit{for all } j > 0.$$

Proof. A contravariant functor $F : \mathcal{O}_p(G) \to \mathsf{Ab}$ is called a Mackey functor if there is a covariant functor $F_* : \mathcal{O}_p(G) \to \mathsf{Ab}$ which takes the same values on objects, and such that any pullback square

$$\begin{array}{ccc}
\coprod_{i=1}^{k} G/K_i & \xrightarrow{\alpha_1} & G/H_1 \\
\big\downarrow{\scriptstyle \alpha_2} & & \big\downarrow{\scriptstyle \beta_1} \\
G/H_2 & \xrightarrow{\beta_2} & G/H
\end{array}$$

induces a commutative square

$$\begin{array}{ccc}
\bigoplus_{i=1}^{k} F(G/K_i) & \xrightarrow{F_*(\alpha_1)} & F(G/H_1) \\
\big\uparrow{\scriptstyle F(\alpha_2)} & & \big\uparrow{\scriptstyle F(\beta_1)} \\
F(G/H_2) & \xrightarrow{F_*(\beta_2)} & F(G/H).
\end{array}$$

By a theorem of Jackowski and McClure [JM, Proposition 5.14], for any Mackey functor $F : \mathcal{O}_p(G) \to \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$, $\varprojlim^j(F) = 0$ for all $j > 0$.

Now let F be the contravariant functor $F(G/P) = KO_G(G/P \times X)_{(p)}$. Then any map $f : G/P \to G/P'$ in $\mathcal{O}(G)$ induces a homomorphism $F_* : F(G/P) \to F(G/P')$: defined by sending a G-bundle $\xi{\downarrow}(G/P \times X)$ to the G-bundle $\xi'{\downarrow}(G/P' \times X)$ such that the fiber over any $(a', x) \in G/P' \times X$ is $\bigoplus_{a \in f^{-1}a'} \xi_{(a,x)}$. This makes F into a Mackey functor; and hence its higher limits over $\mathcal{O}_p(G)$ vanish by [JM]. $\square$

It will be convenient, when X is a (finite) G-complex, to write $\overline{KO}_G(X) = [X, B_G O]^G$; i.e., the group of virtual G-bundles which have virtual dimension zero over all connected components of X. We are now ready to prove the following proposition, which describes how to construct maps from a finite G-complex X to $B_G^* O$.

Proposition 1.3. *For any finite G-complex X, the square*

$$\begin{array}{ccc}
[X, B_G^* O]^G & \longrightarrow & \prod_{p \mid |G|} \Bigl(\varprojlim_{G/P \in \mathcal{O}_p(G)} \overline{KO}_P(X)_{(p)} \Bigr) \\
\Big\downarrow & & \Big\downarrow \\
\overline{KO}(X)^G & \longrightarrow & \prod_{p \mid |G|} \bigl(\overline{KO}(X)_{(p)} \bigr)^G
\end{array} \tag{1}$$

is a pullback square. Here, functoriality on the right is induced by the identification $\overline{KO}_P(X) \cong \overline{KO}_G(G/P \times X)$; and the right hand vertical arrow is induced by restricting the limit to the subcategory $\mathcal{O}_1(G) \subseteq \mathcal{O}_p(G)$ and identifying $\varprojlim_{\mathcal{O}_1(G)}(-)$ with $(-)^G$.

Proof. The basic idea of the proof is to regard $\mathrm{map}_G(X, B_G^* O)$ as the homotopy inverse limit, over an appropriate category, of the spaces $\mathrm{map}_P(X, B_G O_{(p)})$ (for p-subgroups $P \subseteq G$) and $\mathrm{map}(X, B_G O)$. We want to show that $[X, B_G^* O]^G$ is the inverse limit of the corresponding sets of components; and this follows upon showing that certain higher inverse limits vanish. The argument given here is a more direct version of this idea; and is similar to the approach used by Wojtkowiak [Wo] to describe maps from a homotopy direct limit to a space.

Square (1) above is equivalent to the diagram

$$\begin{array}{ccccc}
[X, B_G^* O]^G & \xrightarrow{\ \Phi_X\ } & S(X) & \longrightarrow & \prod_{p \mid |G|} \varprojlim_{\mathcal{O}_p(G)} [-\times X, B_G O_{(p)}]^G \\
& & \Big\downarrow & & \Big\downarrow \\
& & \varprojlim_{\mathcal{O}_1(G)} [G \times X, B_G O]^G & \longrightarrow & \prod_{p \mid |G|} \varprojlim_{\mathcal{O}_1(G)} [G \times X, B_G O_{(p)}]^G ,
\end{array} \tag{2}$$

where $S(X)$ is defined to be the pullback. We must show that Φ_X is a bijection. In Step 1, certain cochain complexes $D^*(X, n)$ are defined, and their homology groups are shown to vanish. And in Step 2, the obstructions to constructing maps $X \to B^*_G O$ (or to constructing a homotopy between two such maps) are shown to be homology groups of the $D^*(X,n)$.

Step 1: For any category $\mathcal{C}$ and any *contravariant* functor $F : \mathcal{C} \to \mathsf{Ab}$, let $C^*(\mathcal{C};F)$ denote the cochain complex

$$C^*(\mathcal{C};F) = \Big(0 \to \prod_{c} F(c) \to \prod_{c_0 \to c_1} F(c_0) \to \prod_{c_0 \to c_1 \to c_2} F(c_0) \to \dots\Big),$$

where the differentials are alternating sums of face maps. The homology groups of $C^*(\mathcal{C};F)$ are the higher limits $\varprojlim^*(F)$ (cf. [O4, Lemma 2]).

For each $n \geq 1$, $F^X_n : \mathcal{O}(G) \to \mathsf{Ab}$ will denote the functor

$$\begin{aligned} F^X_n(G/H) &= \operatorname{Coker}\Big[KO_G(G/H \times D^{n+1} \times X) \xrightarrow{\text{restr.}} KO_G(G/H \times S^n \times X)\Big] \\ &\cong \widetilde{KO}_H(\Sigma^n(X_+)). \end{aligned}$$

Let $D^*(X,n)$ be the cochain complex defined via the short exact sequence

$$\begin{aligned} 0 \to D^*(X,n) \longrightarrow C^*(\mathcal{O}_1(G); F^X_n) \oplus \prod_{p \mid |G|} C^*\big(\mathcal{O}_p(G); F^X_n(-)_{(p)}\big) \\ \longrightarrow \prod_{p \mid |G|} C^*\big(\mathcal{O}_1(G); F^X_n(-)_{(p)}\big) \to 0. \end{aligned} \tag{3}$$

For all $j > 0$, $\varprojlim^j_{\mathcal{O}_p(G)} F^X_n(-)_{(p)} = 0$ by Lemma 1.2 (applied to the G-complexes $S^n \times X$ and X). Also, a functor $M : \mathcal{O}_1(G) \to \mathsf{Ab}$ is the same as a $\mathbb{Z}[G]$-module, and $\varprojlim^*_{\mathcal{O}_1(G)} M \cong H^*(G;M)$. Since $H^j(G;M) \cong \prod_{p \mid |G|} H^j(G;M_{(p)})$ for any $\mathbb{Z}[G]$-module M, the long exact cohomology sequence for (3) reduces to an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0\big(D(X,n)\big) \longrightarrow \widetilde{KO}(\Sigma^n(X_+))^G \oplus \prod_{p \mid |G|} \varprojlim^0_{\mathcal{O}_p(G)} \widetilde{KO}_P(\Sigma^n(X_+))_{(p)} \\ \xrightarrow{\ \varphi\ } \prod_{p \mid |G|} \big(\widetilde{KO}(\Sigma^n(X_+))_{(p)}\big)^G \longrightarrow H^1\big(D(X,n)\big) \longrightarrow 0; \end{aligned} \tag{4}$$

and $H^j\big(D(X,n)\big) = 0$ for all $j \geq 2$.

For any $P \subseteq G$, the composite

$$\widetilde{KO}(\Sigma^n(X_+)) \xrightarrow{\text{transfer}} \widetilde{KO}_P(\Sigma^n(X_+)) \xrightarrow{\text{restr.}} \widetilde{KO}(\Sigma^n(X_+))$$

is the norm homomorphism for the action of P, and in particular sends any $x \in \widetilde{KO}(\Sigma^n(X_+))^G$ to $|P| {\cdot} x$. Here, the transfer map sends a vector bundle over $\Sigma^n(X_+)$ to the direct sum of its translates under the action of P (considered as a P-bundle). Thus, if p^m is the highest power of p dividing $|G|$, then

$$\operatorname{Im}\Big[\varprojlim^0_{\mathcal{O}_p(G)} \widetilde{KO}_P(\Sigma^n(X_+)) \longrightarrow \widetilde{KO}(\Sigma^n(X_+))^G\Big] \supseteq p^m {\cdot} \widetilde{KO}(\Sigma^n(X_+))^G.$$

Since $\widetilde{KO}(\Sigma^n(X_+))^G$ maps onto the sum of the $\mathbb{Z}/p^m \otimes \widetilde{KO}(\Sigma^n(X_+))^G$, this shows that the map φ in (4) is surjective, and hence that $H^1\big(D(X,n)\big) = 0$.

Step 2: We now consider maps from X to $B^*_G O$. For each $0 \le n \le \infty$, let $U_n(X)$ be the space defined by the pullback square

$$\begin{array}{ccc} U_n(X) & \longrightarrow & \prod_{p \mid |G|} \mathrm{map}(E\mathcal{O}_p(G)^{(n)} \times X, B_G O_{(p)}) \\ \downarrow & & \downarrow \\ \mathrm{map}(E\mathcal{O}_1(G)^{(n)} \times X, B_G O) & \xrightarrow{\text{diag}} & \prod_{p \mid |G|} \mathrm{map}(E\mathcal{O}_1(G)^{(n)} \times X, B_G O_{(p)}). \end{array} \tag{5}$$

Here, $E\mathcal{C}^{(n)}$ (for $\mathcal{C} = \mathcal{O}_p(G)$ or $\mathcal{O}_1(G)$) denotes the n-skeleton of the complex

$$E\mathcal{C} = \Big(\coprod_{n\ge 0}\Big(\coprod_{G/H_0 \to \dots \to G/H_n} (G/H_0 \times \Delta^n)\Big)\Big)\Big/\sim, \tag{6}$$

with the usual identifications induced by face and degeneracy maps. By Definition 1.1, $U_\infty(X) \simeq \mathrm{map}(X, B^*_G O)$.

By the pullback square in (2), an element of $S(X)$ corresponds to a choice of G-maps $G \times X \to B_G O$, and $G/P \times X \to B_G O_{(p)}$ for all p and all p-subgroups $P \subseteq G$: maps which agree up to homotopy with respect to morphisms in $\mathcal{O}_p(G)$. In other words, by (6), we can identify $S(X)$ with $\mathrm{Im}[\pi_0(U_1(X)) \to \pi_0(U_0(X))]$. To show that $\Phi_X : [X, B^*_G O]^G \to S(X)$ is onto, we must thus show that any element of $U_1(X)$ can be lifted to an element of $U_\infty(X)$ which has the same image in $\pi_0(U_0(X))$.

Fix an element $f_1 \in U_1(X)$, and consider the obstructions to lifting it to $U_2(X)$. By (6) again, a 2-simplex in $E\mathcal{O}_p(G)$ corresponds to a sequence of maps $G/P_0 \to G/P_1 \to G/P_2$ in $\mathcal{O}_p(G)$, and the obstruction to extending f_1 to that 2-simplex lies in the group

$$\mathrm{Coker}\Big[\overline{KO}_G(G/P_0 \times D^2 \times X)_{(p)} \longrightarrow \overline{KO}_G(G/P_0 \times S^1 \times X)_{(p)}\Big] = F^X_1(G/P_0)_{(p)}.$$

Similarly, the obstruction to extending f_1 to any 2-simplex in $E\mathcal{O}_1(G)$ lies in $F^X_1(G/1)$. These individual obstructions combine to give an element $\alpha_2 \in D^2(X,1)$ as the total obstruction to lifting f_1 to some $f_2 \in U_2(X)$. This element is easily seen to be a cocycle, and hence is a coboundary by Step 1. And if $\alpha_1 = \delta(\beta_1)$ for $\beta_1 \in D^1(X,1)$, then f_1 can be changed on 1-simplices (in a way specified by β_1) to remove the obstruction; after which the "modified" map can be lifted to an element $f_2 \in U_2(X)$. (Note that the co-H-space structure on the suspensions induces the usual addition on the groups $\widetilde{KO}_P(\Sigma(X_+))$.) Upon continuing this process, we see that at each stage the obstruction to lifting $f_n \in U_n(X)$ to $f_{n+1} \in U_{n+1}(X)$ (while allowing f_n to be changed on n-simplices) lies in $H^{n+1}(D^*(X,n))$, which again vanishes by Step 1.

This shows that $\Phi_X : [X, B^*_G O]^G \to S(X)$ is onto. To show that it is injective, we start with two elements $f, f' \in \mathrm{map}_G(X, B^*_G O) \cong U_\infty(X)$, together with a homotopy

$F_0 \in U_0(X{\times}I)$, and then lift the homotopy one step at a time. For each $n \geq 0$, the obstruction to lifting a homotopy $F_n \in U_n(X{\times}I)$ to $U_{n+1}(X{\times}I)$ (while taking a given value on $X{\times}\{0,1\}$) lies in $H^{n+1}(D^*(X,n{+}1))$. And this again vanishes by Step 1. □

Proposition 1.3 does not in general hold for infinite G-complexes. But it does hold for countable complexes with fixed G-action.

The following corollary to Proposition 1.3 will be needed in Section 3.

Corollary 1.4. *$B^*_G O$ can be given the structure of a G-equivariant H-space in a way such that $L_G : B_G O \to B^*_G O$ is an H-space homomorphism. Also, for any finite G-complex X, $[X, B^*_G O]^G$ is an abelian group with the property that $(\mathbb{Z}/n) \otimes [X, B^*_G O]^G$ is finite for all $n > 0$.*

Proof. The H-space structure on $B^*_G O$, and L_G being an H-space homomorphism, follow immediately from the pullback square in Definition 1.1, together with the H-space structures on $B_G O$ and its localizations (see the discussion before Definition 1.1). By Proposition 1.3, for any finite G-complex X, there is an exact sequence

$$0 \to [X, B^*_G O]^G \to \overline{KO}(X)^G \oplus \prod_{p \mid |G|} \Big(\varprojlim_{\mathcal{O}_p(G)} \overline{KO}_P(X)_{(p)} \Big) \to \prod_{p \mid |G|} \big(\overline{KO}(X)_{(p)} \big)^G. \quad (1)$$

So $[X, B^*_G O]^G$ is abelian, and $(\mathbb{Z}/n) \otimes [X, B^*_G O]^G$ is finite for any $n > 0$ since $(\mathbb{Z}/n) \otimes -$ and $\mathrm{Tor}(\mathbb{Z}/n, -)$ are finite for the other two terms in (1). □

2. Construction of G-bundles

Proposition 1.3 describes a procedure for constructing G-maps from a finite G-complex X to $B^*_G O$. What we really are interested in is the construction of G-maps from X to $B_G O$. In general, of course, G-maps from $B^*_G O$ cannot be lifted to $B_G O$ (L_G is not a G-homotopy equivalence). To get around this, we prove a rather complicated lifting result (Proposition 2.3); and then apply it in Theorem 2.4 to construct G-vector bundles by "pasting together" bundles over certain subcomplexes and for certain subgroups.

The first step is to compare the homotopy groups of fixed point sets in $B^*_G O$ with those in $B_G O$. Let $\mathrm{RO}(G) \cong KO_G(\mathrm{pt})$ denote the orthogonal representation ring of G, and let $\mathrm{IRO}(G) = \mathrm{Ker}[\mathrm{RO}(G) \xrightarrow{\dim} \mathbb{Z}] \cong \overline{KO}_G(\mathrm{pt})$ be its augmentation ideal.

Lemma 2.1. *Let G be any finite group, and let $L_G : B_G O \to B^*_G O$ be the map of Definition 1.1. Then the following hold.*

(a) L_G is (nonequivariantly) a homotopy equivalence.

(b) Fix a prime $p \mid |G|$ and a p-subgroup $1 \neq P \subseteq G$. Then $\pi_0((B^*_G O)^P) \cong \mathrm{IRO}(P)_{(p)}$, and $\pi_0(L^P_G)$ is isomorphic to the inclusion of $\mathrm{IRO}(P)$ into $\mathrm{IRO}(P)_{(p)}$. For each $x \in (B_G O)^P$,

$$\pi_i(L^P_G, x) : \pi_i\big((B_G O)^P, x\big) \longrightarrow \pi_i\big((B^*_G O)^P, L_G(x)\big)$$

is an isomorphism for $i = 1$; and for $i > 1$ its kernel and cokernel are torsion prime to p with the additional property that their m-torsion subgroups are finite for all m.

Proof. Fix a prime $p\big||G|$ and a p-subgroup $P \subseteq G$. By Proposition 1.3 (applied with $X = G/P$) there is a pullback square

$$\begin{array}{ccc} [G/P, B^*_G O]^G \cong \pi_0((B^*_G O)^P) & \longrightarrow & \prod_{q||G|} \Big(\varprojlim_{G/Q \in \mathcal{O}_q(G)} \overline{KO}_Q(G/P)_{(q)} \Big) \\ \Big\downarrow & & \Big\downarrow \\ \overline{KO}(G/P)^G & \longrightarrow & \prod_{q||G|} \big(\overline{KO}(G/P)_{(q)} \big)^G . \end{array} \tag{1}$$

Similarly, for any $i > 0$, Proposition 1.3 yields a pullback square

$$\begin{array}{ccc} \dfrac{[\Sigma^i(G/P_+), B^*_G O]^G}{[\mathrm{pt}, B^*_G O]^G} \cong \pi_i((B^*_G O)^P) & \longrightarrow & \prod_{q||G|} \Big(\varprojlim_{G/Q \in \mathcal{O}_q(G)} KO^{-i}_Q(G/P)_{(q)} \Big) \\ \Big\downarrow & & \Big\downarrow \\ KO^{-i}(G/P)^G & \longrightarrow & \prod_{q||G|} \big(KO^{-i}(G/P)_{(q)} \big)^G . \end{array} \tag{2}$$

For any $i \geq 0$ and any $Q \subseteq G$,

$$KO^{-i}_Q(G/P) \cong KO^{-i}_G(G/P \times G/Q) \cong KO^{-i}_P(G/Q).$$

In general, if F_P is any contravariant functor from P-complexes to abelian groups such that $F_P(X \amalg Y) = F_P(X) \oplus F_P(Y)$, then

$$\varprojlim_{G/Q \in \mathcal{O}_q(G)} F_P(G/Q) \cong \varprojlim_{\mathcal{O}_q(P)} (F_P) \cong \begin{cases} F_P(\mathrm{pt}) & \text{if } q = p \\ \big(F_P(P)\big)^P & \text{if } q \neq p. \end{cases} \tag{3}$$

The first isomorphism holds since for each P-orbit $P{\cdot}gQ$ in any G/Q, there is an $\mathcal{O}_q(G)$-morphism $G/Q' \to G/Q$ (where $Q' = P \cap gQg^{-1}$) which sends the P-orbit P/Q' in $\mathcal{O}_q(P)$ isomorphically to $P{\cdot}gQ$. Also, $\mathcal{O}_q(P)$ has a final object if $p = q$, and contains only the free orbit if $p \neq q$.

If we now apply (3) with $F_P = \overline{KO}_P(-)_{(q)}$, then square (1) is reduced to an isomorphism

$$\pi_0\big((B^*_G O)^P\big) \xrightarrow{\ \cong\ } \overline{KO}_P(\mathrm{pt})_{(p)} \cong \mathrm{IRO}(P)_{(p)}$$

(note that $\overline{KO}(G/P) = [G/P, BO] = 0$). Thus, $\pi_0(L^P_G)$ is isomorphic to the inclusion of $\mathrm{IRO}(P)$ into $\mathrm{IRO}(P)_{(p)}$. And if (3) is applied with $F_P = KO^{-i}_P(-)_{(q)}$, then square (2) reduces to a pullback square

$$\begin{array}{ccc} \pi_i((B^*_G O)^P) & \longrightarrow & KO^{-i}_P(\mathrm{pt})_{(p)} \\ \Big\downarrow & & \Big\downarrow \\ KO^{-i}(\mathrm{pt}) & \longrightarrow & KO^{-i}(\mathrm{pt})_{(p)} . \end{array} \tag{4}$$

When $P=1$, this shows that $\pi_i(B^*_G O)\cong KO^{-i}(\mathrm{pt})\cong \pi_i(B_G O)$ for all $i>0$, and hence that L_G is nonequivariantly a homotopy equivalence.

Now set $M_i = \mathrm{Ker}\big[KO_P^{-i}(\mathrm{pt}) \xrightarrow{\text{forget}} KO^{-i}(\mathrm{pt})\big]$. The forgetful map is a surjection: split by regarding a bundle over S^i as a P-bundle with trivial action. So by (4), the kernel and cokernel of the homomorphism

$$KO_P^{-i}(\mathrm{pt}) \cong \pi_i\big((B_G O)^P\big) \xrightarrow{\pi_i(L_G^P)} \pi_i\big((B^*_G O)^P\big)$$

are given by

$$\mathrm{Ker}(\pi_i(L_G^P)) \cong \mathrm{Ker}\big[M_i \to M_{i(p)}\big] \quad \text{and} \quad \mathrm{Coker}(\pi_i(L_G^P)) \cong \mathrm{Coker}\big[M_i \to M_{i(p)}\big]. \tag{5}$$

In particular, since M_i is finitely generated, $\mathrm{Ker}\big(\pi_i(L_G^P)\big)$ and $\mathrm{Coker}\big(\pi_i(L_G^P)\big)$ are torsion of order prime to p, and have finite m-torsion for any $m>0$.

It remains to show that $\pi_1(L_G^P)$ is an isomorphism. By Proposition A.2(b), $KO_P^{-1}(\mathrm{pt}) \cong \pi_1((B_G O)^P)$ is a sum of one copy of $\mathbb{Z}/2$ for each irreducible P-representation of real type. So if $p=2$, then $\pi_1(L_G^P)$ is an isomorphism by (5), since M_1 is a finite 2-group. If p is odd, then the only irreducible P-representation of real type is the trivial one (see Proposition A.1(c)); so $KO_P^{-i}(\mathrm{pt})\cong KO^{-i}(\mathrm{pt})\cong\mathbb{Z}/2$, $M_1=0$, and $\pi_1(L_G^P)$ is again an isomorphism. □

When $f: X\to Y$ is a given map between spaces, we will frequently write $\pi_i(Y,X,x)$ (for $x\in X$) to denote the relative homotopy group $\pi_i(Z_f,X,x)$. Here, Z_f is the mapping cylinder of f. Also, we write $\pi_i(Y,X)$ when X is connected and the basepoint need not be specified.

The next lemma will provide the induction step in our construction of G-bundles.

Lemma 2.2. *Fix a finite group G and a prime p. Let $X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y$ be G-maps, where*

(1) X and Y are countable finite dimensional (nonempty) G-complexes;

(2) Z and Y are connected, $\pi_1(Y,Z)=1$, $\pi_2(Y,Z)$ is abelian, and $\pi_i(Y,Z)\otimes\mathbb{Z}_{(p)}=0$ for all $i\geq 2$; and

(3) for any nontrivial p-subgroup $1\neq P\subseteq G$, $(\beta\alpha)^P: X^P\to Y^P$ is an $\mathbb{F}_p$-homology equivalence.

Then there exist a countable finite dimensional G-complex $\overline{X}\supseteq X$ and an extension $\overline{\alpha}:\overline{X}\to Z$ of α, such that G acts freely on $\overline{X}\smallsetminus X$ and such that $\beta\overline{\alpha}:\overline{X}\to Y$ is an $\mathbb{F}_p$-homology equivalence. If in addition,

(4) X and Y are finite G-complexes,

(5) $\mathrm{Ker}(\pi_1(\beta))$ has finite n-torsion for all n, and

(6) $\chi(X^H)=\chi(Y^H)$ for all cyclic subgroups $H\subseteq G$ of order prime to p,

then $\overline{X}$ can be chosen to be a finite G-complex.

Proof. **Finite case:** Assume that all points (1)–(6) hold. By attaching some finite number of free orbits of 1-cells to X if necessary (and extending α accordingly), we can assume that X is connected and that $\pi_1(\beta\alpha) : \pi_1(X) \twoheadrightarrow \pi_1(Y)$ is onto. Set

$$K_X = \operatorname{Ker}\bigl[\pi_1(X) \xrightarrow{\pi_1(\beta\alpha)} \pi_1(Y)\bigr] \qquad \text{and} \qquad K_Z = \operatorname{Ker}\bigl[\pi_1(Z) \xrightarrow{\pi_1(\beta)} \pi_1(Y)\bigr].$$

Since $\pi_1(X)$ and $\pi_1(Y)$ are both finitely presented, K_X is finitely generated as a normal subgroup of $\pi_1(X)$ (cf. [Ro, Lemma 1.43(i)]). By (2), K_Z is abelian and torsion prime to p, and by (5) its n-torsion subgroup is finite for any n. Then $\operatorname{Im}[K_X \xrightarrow{\alpha_*} K_Z]$ is finite (it has bounded torsion since K_X is finitely generated); and hence $\operatorname{Im}[\pi_1(X) \xrightarrow{\pi_1\alpha} \pi_1(Z)]$ is finitely presented (an extension of one finitely presented group by another). So by [Ro] again, $\operatorname{Ker}(\pi_1(\alpha))$ is finitely generated as a normal subgroup of $\pi_1(X)$. We can thus attach finitely many free orbits of 2-cells to X, to obtain a finite G-complex $X_1 \supseteq X$ and a map $\alpha_1 : X_1 \to Z$ extending α, such that $\pi_1(\alpha_1)$ is injective, and $\operatorname{Ker}(\pi_1(\beta\alpha_1))$ is finite abelian of order prime to p.

Set $d = \max\{\dim(X), \dim(Y), 2\}$. We next construct a sequence of finite complexes

$$X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_d,$$

together with G-maps $\alpha_m : X_m \to Z$ extending α_1, such that each X_m (for $2 \le m \le d$) is constructed from X_{m-1} by attaching free orbits of m-cells, and such that $\pi_i(\widetilde{Z}, \widetilde{X}_m)\otimes\mathbb{Z}_{(p)} = 0$ for each $i \le m$. Assume that X_{m-1} has been constructed. Then by Lemma A.7,

$$\pi_m(\widetilde{Z}, \widetilde{X}_{m-1}) \otimes \mathbb{Z}_{(p)} \cong \pi_m(\widetilde{Y}, \widetilde{X}_{m-1}) \otimes \mathbb{Z}_{(p)}$$

is finitely generated as a $\mathbb{Z}_{(p)}[\pi_1(X_{m-1})]$-module. We can thus attach some finite number of free orbits of m-cells to X_{m-1} (while extending α_{m-1}) to get a new complex X_m such that $\pi_m(\widetilde{Z}, \widetilde{X}_m)\otimes\mathbb{Z}_{(p)} \cong \pi_m(\widetilde{Y}, \widetilde{X}_m)\otimes\mathbb{Z}_{(p)} = 0$.

Now consider the sequence of maps $X_d \xrightarrow{\alpha_d} Z \xrightarrow{\beta} Y$. By assumption, $\pi_i(Y, X_d)\otimes\mathbb{Z}_{(p)} = 0$ for all $i \le d = \dim(X_d) \ge \dim(Y)$. Also, by Lemma A.7, applied to the pairs (Y, X_d) and (Y, Z), $H_i(Y, X_d; \mathbb{Z}_{(p)}) \cong H_i(Z, X_d; \mathbb{Z}_{(p)}) = 0$ for all $i \le d$, and the Hurewicz homomorphism

$$\pi_{d+1}(Z, X_d)\otimes\mathbb{Z}_{(p)} \cong \pi_{d+1}(Y, X_d)\otimes\mathbb{Z}_{(p)} \twoheadrightarrow H_{d+1}(Y, X_d; \mathbb{Z}_{(p)})$$

is surjective. By Lemma A.10, together with conditions (3) and (6), $H_{d+1}(Y, X_d; \mathbb{F}_p)$ is free as an $\mathbb{F}_p[G]$-module. We can thus choose elements of $\pi_{d+1}(Z, X_d)$ which represent a basis of $H_{d+1}(Y, X_d; \mathbb{F}_p)$, and attach accordingly free orbits of $(d+1)$-cells to X_d, extending α_d, to get a finite G-complex $\overline{X} \supseteq X_d$ and a map $\bar{\alpha} : \overline{X} \to Z$ which is an $\mathbb{F}_p$-homology equivalence.

Countable case: The proof is similar in the countable case (i.e., when we only assume conditions (1)–(3)), but much simpler. All homotopy groups of Y and the X_i are countable by Lemma A.6. So at each stage, the relevant homotopy elements can be eliminated by attaching only countably many cells. And in the last step, $H_{d+1}(Y, X_d; \mathbb{F}_p)$ is stably free as a countably generated projective $\mathbb{F}_p[G]$-module by Lemma A.10, and so free orbits of d- and $(d+1)$-dimensional cells can be attached to X_d to construct the $\mathbb{F}_p$-homology equivalence $\overline{X} \to Z$. □

We are now ready lift maps from $B^*_G O$ to $B_G O$.

Proposition 2.3. *Assume that G is a finite group not of prime power order. Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & B_GO \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle L_G} \\ Y & \xrightarrow{f_Y} & B_G^*O \end{array} \tag{1}$$

be a homotopy commutative square of G-maps, where X and Y are countable finite dimensional G-complexes. Assume, for each prime $p\big||G|$ and each p-subgroup $1\ne P\subseteq G$, that

$$\mathrm{Im}\big[\pi_0(Y^P)\xrightarrow{(f_Y^P)_*}\pi_0((B_G^*O)^P)\big]\subseteq\mathrm{Im}\big[\pi_0((B_GO)^P)\xrightarrow{(L_G^P)_*}\pi_0((B_G^*O)^P)\big]. \tag{2}$$

Then there is a countable finite dimensional G-complex $\overline{X}\supseteq X$ such that all isotropy subgroups of $\overline{X}\smallsetminus X$ have prime power order, and extensions $\overline{\varphi}:\overline{X}\to Y$ of φ and $\overline{f}:\overline{X}\to B_GO$ of f such that $\overline{\varphi}$ is (nonequivariantly) a homotopy equivalence and $f_Y\circ\overline{\varphi}\simeq L_G\circ\overline{f}$. If, furthermore,

(3) X and Y are finite G-complexes with $X^G\ne\emptyset$ and Y connected,

(4) $\chi(X^H)=\chi(Y^H)$ for all $H\subseteq G$ not of prime power order, and

(5) $\chi\big((\varphi^{-1}Y_i^P)^H\big)=\chi\big((Y_i^P)^H\big)$ for each prime $p\big||G|$, each p-subgroup $1\ne P\subseteq G$, each connected component Y_i^P of Y^P, and each cyclic subgroup $1\ne H/P\subseteq N(P)/P$ of order prime to p,

then $\overline{X}$ can be chosen to be a finite G-complex.

Proof. We concentrate on the proof in the case where X and Y are finite complexes and conditions (3)–(5) hold. The proof in the finite dimensional case is simpler, and some remarks will be made afterwards as to how it differs from that for finite complexes.

Define Z to be the (G-equivariant) homotopy pullback in the following square:

$$\begin{array}{ccc} Z & \xrightarrow{\gamma} & B_GO \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle L_G} \\ Y & \xrightarrow{f_Y} & B_G^*O. \end{array} \tag{6}$$

Lemma 2.1(b) and Condition (2) imply that for any $P\subseteq G$ of prime power order,

$$\pi_0(Z^P)\xrightarrow[\cong]{\pi_0(\beta^P)}\pi_0(Y^P)\quad\text{and}\quad\pi_1(Z^P,x)\xrightarrow[\text{onto}]{\pi_1(\beta^P)}\pi_1(Y^P,\beta(x))\ \forall x\in Z^P. \tag{7}$$

By the homotopy commutativity of (1), there is a G-map $\alpha:X\to Z$ such that $\beta\circ\alpha\simeq\varphi$ and $\gamma\circ\alpha\simeq f$.

Finite case: Step 1: Let $P_1,\dots,P_k$ be conjugacy class representatives for all subgroups $1\ne P\subseteq G$ of prime power order, ordered from largest to smallest (i.e., $i\le j$ if P_i contains a subgroup conjugate to P_j). We first construct finite G-complexes

$X = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k$, together with maps $\alpha_i : X_i \to Z$ (where $\alpha_0 = \alpha$), satisfying the following conditions for all $1 \le i \le k$:

(a) $X_i \smallsetminus X_{i-1}$ contains only orbits of type G/P_i,

(b) $\alpha_i | X_{i-1} = \alpha_{i-1}$, and

(c) $(\beta \circ \alpha_i)^{P_i} : (X_i)^{P_i} \to Y^{P_i}$ is an $\mathbb{F}_{p_i}$-homology equivalence, where p_i is the prime such that P_i is a p_i-group.

Fix some $1 \le i \le k$, and assume that X_{i-1} and α_{i-1} have been constructed. Consider the maps

$$(X_{i-1})^{P_i} \xrightarrow{(\alpha_{i-1})^{P_i}} Z^{P_i} \xrightarrow{\beta^{P_i}} Y^{P_i}.$$

After restricting to any connected component of Y^{P_i} (and to those connected components of Z^{P_i} and $(X_{i-1})^{P_i}$ which map into it), these maps satisfy hypotheses (2)–(6) of Lemma 2.2, applied with the action of the group $N(P_i)/P_i$. Note in particular that conditions (2) and (5) in Lemma 2.2 follow from (7) and Lemma 2.1 (and the pullback square (6)), that condition (3) follows from the assumptions on X_{i-1}, and that condition (6) follows from condition (5) here. So by Lemma 2.2, there is a finite $N(P_i)/P_i$-complex $W \supseteq (X_{i-1})^{P_i}$, and an equivariant map $\widehat{\alpha} : W \to Z^{P_i}$ which extends $(\alpha_{i-1})^{P_i}$, such that $\beta^{P_i} \circ \widehat{\alpha}$ is an $\mathbb{F}_{p_i}$-homology equivalence. And if we set $X_i = X_{i-1} \cup_{G \times (\text{incl})} (G \times_{N(P_i)} W)$ and $\alpha_i = \alpha_{i-1} \cup (G \times \widehat{\alpha})$, then the pair (X_i, α_i) satisfy conditions (a), (b), and (c) above.

Step 2: It remains to deal with the free orbits. Note that $\beta : Z \to Y$ is (nonequivariantly) a homotopy equivalence since L_G is (by Lemma 2.1). Since X_k and Y are both finite, we can attach free orbits of cells to X_k, eliminating all relative homotopy groups $\pi_i(Y, X_k) \cong \pi_i(Z, X_k)$ for small i, until we get a new finite G-complex $X' \supseteq X_k$ and a map $\alpha' : X' \to Z$ extending α_k, such that $H_i(Y, X') = 0$ for all $i \le \dim(X') \ge \dim(Y)$. Set $d = \dim(X') + 1$.

By Lemma A.11 below, $H_d(Y, X') \cong H_d(Z, X')$ is a projective $\mathbb{Z}[G]$-module; and there exists a finite G-complex T such that $T^H = \text{pt}$ for all $H \subseteq G$ not of prime power order, and such that for some m, T is $(m-1)$-connected and $\widetilde{H}_*(T) = H_m(T) \cong H_d(Y, X')$ as $\mathbb{Z}[G]$-modules. Upon replacing T by an appropriate suspension, we can assume that $m \ge d$, and that $m - d$ is even. Set $X'' = X' \vee T$ (recall $(X')^G = X^G \ne \emptyset$), and extend α' to $\alpha'' : X'' \to Z$ by sending T to a point. Then $H_*(Y, X'') \cong H_*(Z, X'')$ vanishes except in dimensions d and $m+1$, and $H_d(Y, X'') \cong H_{m+1}(Y, X'')$ as $\mathbb{Z}[G]$-modules. Since Y and Z are connected, we can now attach (finitely many) free orbits of cells $G \times D^i$ to X'', for $d + 1 \le i \le m + 1$, to obtain a G-map $\overline{\alpha} : \overline{X} \to Z$ which is a (nonequivariant) homotopy equivalence. By construction, all isotropy subgroups of $\overline{X} \smallsetminus X$ have prime power order.

Countable case: The main difference in the proof when X and Y are countable and finite dimensional is that since we are working with countably generated modules, the group $H_d(Y, X')$ is stably free by Lemma A.11. So the last part of Step 2, and in particular the replacement of X' by a wedge product, are not needed. □

Note that the condition $X^G \ne \emptyset$ in Proposition 2.3 is needed only in the last step of the construction, when removing the projective obstruction to making $X' \to Y$ a homotopy equivalence. If X^G is empty, the calculations of projective obstructions in [OP, §4] provide other conditions under which the lemma still holds.

The main technical theorem on the construction of G-vector bundles can now be shown.

Theorem 2.4. *Assume that G is a finite group not of prime power order. Let $\varphi : X \to Y$ be a G-map, where X is a finite G-complex and Y is countable and finite dimensional. Fix a G-bundle $\eta{\downarrow}X$, and a P-vector bundle $\xi_P{\downarrow}Y$ for each subgroup $P \subseteq G$ of prime power order, such that the following conditions hold:*

(1) For each prime p, the ξ_P (for p-subgroups $P \subseteq G$) are p-locally "consistent up to isomorphism" with respect to morphisms in $\mathcal{O}_p(G)$; i.e., they define an element in the inverse limit

$$\varprojlim_{G/P \in \mathcal{O}_p(G)} KO_P(Y)_{(p)} = \varprojlim_{G/P \in \mathcal{O}_p(G)} KO_G(G/P \times Y)_{(p)}.$$

(2) If $1 \neq P \subseteq G$ is a p-subgroup, then $[\varphi^*(\xi_P)] = [\eta|P]$ in $KO_P(X)_{(p)}$.

(3) $[\xi_1] \in KO(Y)^G$, and $\varphi^*(\xi_1) \cong \eta$ (as nonequivariant vector bundles over X).

Then there is a countable finite dimensional G-complex $\overline{X} \supseteq X$ such that all isotropy subgroups of $\overline{X} \smallsetminus X$ have prime power order, a G-map $\overline{\varphi} : \overline{X} \to Y$ which extends φ and is a homotopy equivalence, a G-vector bundle $\overline{\eta}{\downarrow}\overline{X}$, and a (real) G-representation V, so that $\overline{\eta}|X \cong \eta \oplus (V \times X)$ as G-vector bundles; and such that $[\overline{\eta}] = [\overline{\varphi}^*(\xi_P) \oplus (V \times \overline{X})]$ in $KO(\overline{X})$ (if $P = 1$) or in $KO_P(\overline{X})_{(p)}$ (if $P \subseteq G$ is a p-subgroup). If, furthermore,

(4) X and Y are finite G-complexes with $X^G \neq \emptyset$ and Y connected,

(5) $\chi(X^H) = \chi(Y^H)$ for all $H \subseteq G$ not of prime power order, and

(6) $\chi\big((\varphi^{-1}Y_i^P)^H\big) = \chi\big((Y_i^P)^H\big)$ for each prime $p \big| |G|$, each p-subgroup $1 \neq P \subseteq G$, each connected component Y_i^P of Y^P, and each cyclic subgroup $1 \neq H/P \subseteq N(P)/P$ of order prime to p,

then $\overline{X}$ can be chosen to be a finite G-complex.

Proof. Let $f : X \to B_G O$ be the classifying map for η (Lemma A.3). Write $Y = \cup_{i=1}^{\infty} Y_i$, where $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y$ are finite G-subcomplexes. By Proposition 1.3 (and Lemma A.3 again), for each i, the ξ_P combine to define a G-map $f'_i : Y_i \to B_G^* O$ which is unique up to G-homotopy. In particular, $f'_i|Y_{i-1} \simeq f'_{i-1}$ for all i, and hence the f'_i combine to give a map $f_Y : Y \to B_G^* O$. Since X is a finite G-complex, conditions (2) and (3) (and Proposition 1.3 again) show that $f_Y \circ \varphi \simeq L_G \circ f$. These maps satisfy all of the appropriate hypotheses of Proposition 2.3. Note in particular that condition (2) in Proposition 2.3 is satisfied since the ξ_P are actual bundles (see Lemma 2.1(b)).

Proposition 2.3 now applies to give a countable finite dimensional (or finite) G-complex $\overline{X} \supseteq X$, and G-maps $\overline{\varphi} : \overline{X} \to Y$ and $\overline{f} : \overline{X} \to B_G O$ extending φ and f, such that $\overline{\varphi}$ is a homotopy equivalence, such that $L_G \circ \overline{f} \simeq f_Y \circ \overline{\varphi}$, and such that all isotropy subgroups of $\overline{X} \smallsetminus X$ have prime power order. It remains to check that $\overline{f}$ is induced by a G-bundle. For all $H \subseteq G$ not of prime power order, $\mathrm{Im}(\pi_0(\overline{f}^H)) = \mathrm{Im}(\pi_0(f^H))$ is finite since f is induced by an actual bundle. If $P \subseteq G$ has prime power order, then $\mathrm{Im}(\pi_0(\overline{f}^P))$ is finite since the composite $\overline{X} \xrightarrow{\overline{f}} B_G O \to B_P O_{(p)}$ is P-homotopic to the classifying map for the P-bundle $(\overline{\varphi})^*\xi_P$ (and $\pi_0((B_GO)^P) \cong \mathrm{IRO}(P)$ injects into $\pi_0((B_PO_{(p)})^P)$ by Proposition A.2). Thus, by Lemma A.3, $\overline{f}$ factors through $B_GO(n)$ for some n, and hence is the classifying map of some G-bundle $\overline{\eta}{\downarrow}\overline{X}$, whose restriction to X is stably isomorphic to η, and which is P-equivariantly stably isomorphic to $\overline{\varphi}^*(\xi_P)$ for each prime $p\big||G|$ and each p-subgroup $P \subseteq G$. $\square$

Theorem 2.4 can easily be combined with Theorem A.12 below, to allow the construction of smooth G-manifolds with various properties. But since it seems quite difficult to formulate such a theorem in the greatest possible generality, we limit the applications to the case of actions on disks and euclidean spaces described in the next section.

3. Smooth actions on disks and euclidean spaces

We are now ready to describe the fixed point sets, and the tangent bundles over fixed point sets, for actions of a finite group not of prime power order on a disk or euclidean space. We first recall the definition of the number n_G which determines which homotopy types can occur among fixed point sets of actions of G on disks.

Consider the set $\{\chi(X^G){-}1 \,|\, X \text{ a finite contractible } G\text{-complex}\} \subseteq \mathbb{Z}$. This is a subgroup of $\mathbb{Z}$ (as seen by taking wedge products and suspensions of G-complexes), and hence has the form $n_G{\cdot}\mathbb{Z}$ for some unique $n_G \geq 0$. Thus, by definition, for any $k \in \mathbb{Z}$, there is a finite contractible G-complex X such that $\chi(X^G) = k$ if and only if $k \equiv 1 \pmod{n_G}$; and the main theorem in [O1] says that any finite complex F with an "allowable" Euler characteristic can be realized as a fixed point set in this way. This is also a special case of Theorem 2.4 above: if Y is a finite contractible G-complex, and if $\chi(F) = \chi(Y^G)$, then that theorem describes how to construct a finite contractible G-complex $\overline{X}$ with $\overline{X}^G = F$ (while taking all maps to B_GO and B_G^*O to be trivial).

Proof of Theorem 0.1. By assumption, F is a smooth manifold, and $\eta{\downarrow}F$ is a G-bundle such that (1) η is nonequivariantly a product bundle, (2) $[\eta|P] \in \widetilde{KO}_P(F)$ is infinitely p-divisible for all primes p and all p-subgroups $P \subseteq G$, and (3) $\eta^G \cong \tau(F)$. Let V be the fiber over any point of F (regarded as a G-representation).

Finite case: Assume that F is compact and $\chi(F) \equiv 1 \pmod{n_G}$. If $F = \emptyset$, then $n_G = 1$, and G has a fixed point free action on a disk by [O1, corollary to Theorem 3]. So we can assume $F \neq \emptyset$. By the above definition of n_G, there is a finite contractible G-complex Y with $\chi(Y^G) = \chi(F)$ (and $Y^G \neq \emptyset$). Set $X = F \vee (Y/Y^G)$, let $\varphi : X \to Y$ be any G-map, extend η to a G-bundle $\eta{\downarrow}X$ by letting it be trivial over Y/Y^G, and set $\xi_P = (V|P){\times}Y$ for each P. Then $\chi(X^H) = \chi(Y^H)$ for all $H \subseteq G$, and Y^P is acyclic and hence connected for each $P \subseteq G$ of prime power order. By Theorem 2.4, there is a finite contractible complex $\overline{X} \supseteq X$ and a G-bundle $\overline{\eta}{\downarrow}\overline{X}$ such that $\overline{\eta}|X$ is stably isomorphic to η. In particular, by condition (1) above, $(\overline{\eta}|F)^G$ is stably isomorphic to $\tau(F)$; and so by Theorem A.12 there is a smooth action of G on a compact contractible manifold M with fixed point set F and with $\tau(M)|F$ stably isomorphic to η. By the h-cobordism theorem (cf. [Mi]), M is a disk if its boundary is simply connected. So if M is not itself a disk, then we can replace it by $M{\times}D(V)$ for any G-representation $V \neq 0$ with $V^G = 0$.

Countable case: Let $f : F \to B_GO$ be the classifying map for $[\eta] - [F{\times}V]$. By

Proposition 1.3 (and the assumptions on η), $(L_G\circ f)|F' \simeq *$ for any finite subcomplex $F' \subseteq F$. In particular, the image of $L_G\circ f$ is contained in the identity connected component of $(B^*_GO)^G$. By Corollary 1.4, $(B^*_GO)^G$ is an H-space, and $(\mathbb{Z}/n)\otimes[\Sigma X,(B^*_GO)^G]$ is finite for any finite complex X and any $n > 0$. Hence Lemma A.9 applies to show that there is a countable finite dimensional $\mathbb{Z}/|G|$-acyclic complex $Y \supseteq F$ and a map $f_Y : Y \to (B^*_GO)^G$ which extends $L_G\circ f$.

Recall that $L_G : B_GO \to B^*_GO$ and the forgetful map $B_GO \to BO$ are both nonequivariantly homotopy equivalences: the first by Lemma 2.1(a) and the second by Proposition A.2(b). Let $\xi\downarrow Y$ be any bundle which is classified by $f_Y : Y \to B^*_GO \simeq BO$. Then $\xi|F$ is a (stably) product bundle, since $\eta\downarrow F$ is by assumption a product bundle. Let $\xi'\downarrow(Y/F)$ be an inverse bundle to ξ (Y is finite dimensional), consider it as a G-bundle over Y with trivial G-action, and let $\psi : Y \to Y/F \to (B_GO)^G$ be its classifying map. We can now replace $[f_Y]$ by $[f_Y]+[L_G\circ\psi]$ (L_G is a homomorphism of H-spaces by Corollary 1.4), and thus arrange that $f_Y : Y \to B^*_GO$ be nonequivariantly nullhomotopic.

By Proposition 2.3, applied with $X = F$ and $\varphi : X \to Y$ the inclusion, there is a countable finite dimensional G-complex $\overline{X} \supseteq F$, together with extensions $\overline{\varphi} : \overline{X} \to Y$ and $\overline{f} : \overline{X} \to B_GO$ of φ and f, such that $\overline{X}^G = F$ and $\overline{\varphi}$ is nonequivariantly a homotopy equivalence. In particular, $\overline{X}$ is $\mathbb{Z}/|G|$-acyclic; and so by Smith theory (cf. [Br, Theorem III.7.11]), $\overline{X}^P$ is $\mathbb{F}_p$-acyclic for each prime $p\big||G|$ and each p-subgroup $P \subseteq G$. Set $d = \dim(\overline{X})$, and consider the d-skeleton of the join $\overline{X}*EG$. By Lemma A.11, $H_d\big((\overline{X}*EG)^{(d)}\big)$ is $\mathbb{Z}[G]$-stably free, and so free orbits of d- and $(d+1)$-cells can be attached to produce a countable finite dimensional contractible G-complex $Z \supseteq \overline{X}$. By construction, all orbits in $Z\smallsetminus\overline{X}$ are free. Since $\overline{X}*EG$ is contractible, the inclusion of the d-skeleton extends to a map $\psi : Z \to \overline{X}*EG^{(d+1)}$.

We next show, inductively on n, that there are G-maps $f_n : \overline{X}*EG^{(n)} \to B_GO$ for all $n \geq 0$ which extend $\overline{f} : \overline{X} \to B_GO$. Since f_Y is (nonequivariantly) nullhomotopic and $L_G : B_GO \to B^*_GO$ is a homotopy equivalence, $\overline{f}$ is also nullhomotopic. So we can construct $f_0 : \overline{X}*EG^{(0)} \to B_GO$.

Now assume, for some $n \geq 1$, that $f_{n-1} : \overline{X}*EG^{(n-1)} \to B_GO$ has been constructed. The obstruction to extending f_{n-1} to $\overline{X}*EG^{(n)}$ is an element

$$\epsilon_n \in C^n\big(EG;\widetilde{KO}(\Sigma^n\overline{X})\big) \stackrel{\text{def}}{=} \operatorname{Hom}_{\mathbb{Z}G}\big(C_n(EG),\widetilde{KO}(\Sigma^n\overline{X})\big).$$

We can regard $\widetilde{KO}(\Sigma^n\overline{X})$ as the set of homotopy classes of maps of pairs

$$(D^n,S^{n-1}) \longrightarrow \Big(\operatorname{map}(\overline{X},B_GO)\,,\,(\text{constant maps})\Big);$$

and under this identification addition is given by juxtaposition of disks. This viewpoint makes it clear that ϵ_n is a cocycle, and hence is a coboundary since $\widetilde{KO}^*(\overline{X})$ is uniquely $|G|$-divisible ($\overline{X}$ is $\mathbb{Z}/|G|$-acyclic). And if ϵ_n is the coboundary of α_{n-1} in

$C^{n-1}\big(EG;\widetilde{KO}(\Sigma^n\overline{X})\big)$, then α_{n-1} provides the "recipe" for changing f_{n-1} on $(n-1)$-simplices in EG to obtain a map which can be extended to $\overline{X}*EG^{(n)}$.

By Lemma A.3, $f_{d+1}\circ\psi : Z \to B_GO$ induces a G-bundle $\bar\eta\downarrow Z$, and $\bar\eta|(F{=}Z^G)$ is stably isomorphic to η. Theorem A.12 can now be applied to construct a smooth G-action on a contractible manifold M with fixed point set F, such that $\tau(M)|F$ is stably isomorphic to η. If $\partial F = \emptyset$, then we may assume that $\partial M = \emptyset$ (otherwise just replace M by its interior). If $\dim(M) \ge 5$, then by [St, Theorem 5.1], M is a euclidean space if it is simply connected at infinity; in particular, if there is a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subspaces whose union is M and such that each $M\smallsetminus K_i$ is simply connected. And if M does not satisfy this property, then $M\times\mathbb{R}$ does; and so we can get a euclidean space by replacing M by $M\times V$ for any G-representation $V \ne 0$ with $V^G = 0$. $\square$

It remains to prove Theorem 0.2; and in particular to characterize which smooth manifolds have G-vector bundles which satisfy the hypotheses of Theorem 0.1.

In the proofs of the next three lemmas, a pair of (real or complex) G-representations (V, W) will be called a "$\mathcal{P}$-matched pair" of type n if $V|P \cong W|P$ for all $P \subseteq G$ of prime power order, and $\dim(V^G) - \dim(W^G) = n$. In these terms, $\mathcal{M}_\mathbb{C} \supseteq \mathcal{M}_{\mathbb{C}+} \supseteq \mathcal{M}_\mathbb{R}$ are the classes of finite groups for which there exist a $\mathcal{P}$-matched pair (V, W) of complex, self-conjugate, or real G-representations, respectively, of type 1. If we only assume that $V|P \cong W|P$ for p-subgroups $P \subseteq G$ (for some given prime p), then (V, W) will be called a p-matched pair.

Lemma 3.1. *The following hold for any finite group G not of prime power order.*

(a) $G \in \mathcal{M}_\mathbb{C}$ if and only if G contains an element not of prime power order; and $G \in \mathcal{M}_{\mathbb{C}+}$ if and only if G contains an element not of prime power order which is conjugate to its inverse.

(b) $G \in \mathcal{M}_\mathbb{R}$ if and only if there are subgroups $K \lhd H \subseteq G$ such that H/K is dihedral of order $2n$ for some n not a prime power.

Proof. **(a)** Assume $g \in G$ has order $n = km$, where $k, m > 1$ and $(k, m) = 1$. For any n-th root of unity ψ, we let $\mathbb{C}_\psi$ be the 1-dimensional $\langle g\rangle$-representation where g acts via multiplication by ψ. Set $\alpha = \exp(2\pi i/k)$ and $\beta = \exp(2\pi i/m)$, and define

$$V = \mathrm{Ind}_{\langle g\rangle}^G\big(\mathbb{C}_1 \oplus \mathbb{C}_{\alpha\beta}\big) \text{ and } W = \mathrm{Ind}_{\langle g\rangle}^G\big(\mathbb{C}_\alpha \oplus \mathbb{C}_\beta\big).$$

Then (V, W) is a $\mathcal{P}$-matched pair of type 1, and so $G \in \mathcal{M}_\mathbb{C}$. If g is conjugate to g^{-1}, then V and W are self-conjugate — the character of any G-representation induced from $\langle g\rangle$ is real valued by the formula for the character of an induced representation (cf. [Se, §7.2, Proposition 20]) — and so $G \in \mathcal{M}_{\mathbb{C}+}$.

If $G \in \mathcal{M}_\mathbb{C}$, let (V, W) be a $\mathcal{P}$-matched pair of complex representations of type 1. Then $V \ncong W$, but the characters of V and W agree on elements of prime power order. So G must contain an element not of prime power order.

Now assume that $G \in \mathcal{M}_{\mathbb{C}+}$. Recall that a subgroup $H \subseteq G$ is p-elementary (for a prime p) if it is a product of a p-group with a cyclic group; and is 2-$\mathbb{R}$-elementary if it is a semidirect product $H = C_n \rtimes P$, where C_n is cyclic of order n, P is a 2-group, and each element of P centralizes C_n or acts on it via $(g \mapsto g^{-1})$. For each

$H \subseteq G$, let $I_{\mathcal{P}}(H)^+ \subseteq \mathrm{R}(H)$ be the subgroup of self-conjugate elements which vanish upon restriction to prime power order subgroups of H; or equivalently the subgroup of elements whose characters are real valued and vanish on elements of prime power order. Then $I_{\mathcal{P}}(H)^+$ is a module over $\mathrm{RO}(G)$, where multiplication by $[V]$ is induced by tensor product with $\mathbb{C}\otimes_{\mathbb{R}}V$ (or by multiplication with its character). Also, the $I_{\mathcal{P}}(H)^+$ are preserved under induction and restriction maps (by the formula for the character of an induced representation again). Since $\mathrm{RO}(G)$ is generated by induction from subgroups which are p-elementary or 2-$\mathbb{R}$-elementary [Se, §12.6, Theorem 27], it now follows by Frobenius reciprocity that $I_{\mathcal{P}}(G)^+$ is also generated by induction from such subgroups (cf. [Lam, Theorem 3.4(III)]). Hence, since induction leaves unchanged the dimensions of fixed point sets, there is some p-elementary or 2-$\mathbb{R}$-elementary subgroup $G' \subseteq G$ and a $\mathcal{P}$-matched pair (V, W) of self-conjugate G'-representations of type a with $2\nmid a$. In particular, G' contains elements not of prime power order, and so $G' \in \mathcal{M}_{\mathbb{C}}$. Thus, there exist $\mathcal{P}$-matched pairs of self-conjugate G'-representations of type 2; hence of type a for any $a \in \mathbb{Z}$; and so $G' \in \mathcal{M}_{\mathbb{C}+}$.

Now note that $\mathrm{Syl}_2(G') \ntriangleleft G'$: since otherwise

$$\dim(V^{G'}) \equiv \dim(V^{\mathrm{Syl}_2(G')}) = \dim(W^{\mathrm{Syl}_2(G')}) \equiv \dim(W^{G'}) \pmod 2.$$

(The representations $V^{\mathrm{Syl}_2(G')}/V^{G'}$ and $W^{\mathrm{Syl}_2(G')}/W^{G'}$ of the odd order group $G'/\mathrm{Syl}_2(G')$ both split as sums $U \oplus \overline{U}$, and hence are even dimensional, by Proposition A.1(c)). Hence G' is 2-$\mathbb{R}$-elementary but not 2-elementary. Write $G' = C_n \rtimes P$, where $C_n = \langle g \rangle$ is cyclic of odd order n and P is a 2-group. We must show that G' contains an element not of prime power order conjugate to its inverse, and g is such an element if n is not a prime power. If $n > 1$ is an odd prime power, and if $|P| \geq 4$, then write $g' = gx$ for any element $x \in Z(P)$ of order 2 which centralizes g; and g' has order $2n$ and is conjugate to its inverse. And the remaining possibilities — $|P| = 2$ and n a prime power, or $n = 1$ — both contradict the above observation that G' contains elements not of prime power order.

(b) Assume first that G is dihedral of order $2n$, where n is not a prime power. Then $G \in \mathcal{M}_{\mathbb{C}}$ by (a). Also, every $\mathbb{C}G$-representation has the form $\mathbb{C} \otimes_{\mathbb{R}} V$ for some $\mathbb{R}G$-representation V (this holds for any dihedral group). Thus, there is a $\mathcal{P}$-matched pair $(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C} \otimes_{\mathbb{R}} W)$ of complex G-representations of type 1, so (V, W) is a $\mathcal{P}$-matched pair of real G-representations of type 1, and $G \in \mathcal{M}_{\mathbb{R}}$.

Clearly, $G \in \mathcal{M}_{\mathbb{R}}$ if any quotient group of G lies in $\mathcal{M}_{\mathbb{R}}$. And if $H \subseteq G$ and (V, W) is a $\mathcal{P}$-matched pair of real H-representations of type 1, then $(\mathrm{Ind}_H^G(V), \mathrm{Ind}_H^G(W))$ is a $\mathcal{P}$-matched pair of real G-representations of type 1.

Conversely, assume that $G \in \mathcal{M}_{\mathbb{R}}$. We must show that G contains a dihedral subquotient of order $2n$ for some n not a prime power. The same argument as that used in (a) shows that G contains a 2-$\mathbb{R}$-elementary subgroup which is not 2-elementary and which lies in $\mathcal{M}_{\mathbb{R}}$. Upon replacing G by this subgroup, we may assume that $G \cong C_n \rtimes P$, where n is odd and P is a 2-group, where $P_0 = P \cap C_G(C_n)$ has index 2 in P, and where elements in $P \smallsetminus P_0$ act on C_n via $(a \mapsto a^{-1})$. Then G/P_0 is dihedral of order $2n$, and we are done if n is not a prime power. If P is not cyclic, then let $\mathrm{Fr}(P)$ denote its Frattini subgroup (generated by squares and commutators in P); $P/\mathrm{Fr}(P)$ is elementary abelian of order at least 4 (cf. [Go, Corollary 5.1.2 & Theorem 5.1.3]), and so $G/\mathrm{Fr}(P)$ contains a subgroup which is dihedral of order $4n$.

It remains to consider the case where P is cyclic; i.e., where

$$G = \langle a, b \,|\, a^{p^k} = 1 = b^{2^m},\, bab^{-1} = a^{-1} \rangle.$$

We must show that $G \notin \mathcal{M}_{\mathbb{R}}$. Assume otherwise: let (V, W) be a $\mathcal{P}$-matched pair of $\mathbb{R}G$-representations of type 1. Decompose V and W as sums

$$V = V_{11} \oplus V_{1x} \oplus V_{x1} \oplus V_{xx} \quad \text{and} \quad W = W_{11} \oplus W_{1x} \oplus W_{x1} \oplus W_{xx},$$

where $V_{11} \oplus V_{1x} = V^{\langle a \rangle}$ and $V_{11} \oplus V_{x1} = V^{\langle b^2 \rangle}$ (and similarly for W). Thus, for example, V_{xx} and W_{xx} are the sums of those irreducible components where neither a nor b^2 acts trivially. Then $4 | \dim(V_{xx})$ and $4 | \dim(W_{xx})$: each irreducible real G-representation on which neither a nor b^2 acts trivially is 4-dimensional (and of complex or quaternion type). Since $\dim(V) = \dim(W)$ and $\dim(V^{\langle a \rangle}) = \dim(W^{\langle a \rangle})$ by assumption, this shows that $\dim(V_{x1}) \equiv \dim(W_{x1}) \pmod 4$. And since $\dim(V_{x1}^{\langle b \rangle}) = \frac{1}{2}\dim(V_{x1})$ (b acts on each irreducible representation in V_{x1} with equally many eigenvalues $+1$ and -1), we now see that

$$\dim(V^G) = \dim(V^{\langle b \rangle}) - \tfrac{1}{2}\dim(V_{x1}) \equiv \dim(W^{\langle b \rangle}) - \tfrac{1}{2}\dim(W_{x1}) = \dim(W^G) \pmod 2.$$

And this contradicts the assumption that $\dim(V^G) - \dim(W^G) = 1$. $\square$

The condition in Lemma 3.1(b) for G to lie in $\mathcal{M}_{\mathbb{R}}$ was pointed out to me by Erkki Laitinen.

Recall that $\mathsf{Fix}(G)$ denotes the class of smooth manifolds F for which there is a G-vector bundle η such that η is nonequivariantly a product, $[\eta|P] \in \widetilde{KO}_P(F)$ is infinitely p-divisible for all primes $p \big| |G|$ and all p-subgroups $P \subseteq G$, and $\eta^G \cong \tau(F)$. We are now ready to start proving necessary and sufficient conditions for a manifold F to lie in $\mathsf{Fix}(G)$, for a given group G.

The standard induction and forgetful maps between the groups of real, complex, and quaternion vector bundles over F are denoted here as follows:

$$\widetilde{KO}(F) \underset{r}{\overset{c}{\rightleftarrows}} \widetilde{K}(F) \underset{q}{\overset{c'}{\leftrightarrows}} \widetilde{KSp}(F).$$

As usual, F is called stably complex if $[\tau(F)] \in r(\widetilde{K}(F))$; or equivalently if $\tau(F) \oplus \mathbb{R}^k$ has a complex structure for some k. Note that this requires that the dimensions of the connected components of F all have the same parity.

Recall that for any abelian group A, $\operatorname{qdiv}(A)$ denotes the intersection of the kernels of all homomorphisms from A to free abelian groups. In particular, $\operatorname{qdiv}(A) = \operatorname{tors}(A)$ if A is finitely generated.

Lemma 3.2. *Fix a finite group G not of prime power order, and a smooth manifold F. Then the following hold.*

(a) $F \in \mathsf{Fix}(G)$ if $G \in \mathcal{M}_{\mathbb{R}}$, or if $G \in \mathcal{M}_{\mathbb{C}}$ and F is stably complex.

(b) $F \in \mathsf{Fix}(G)$ if $G \in \mathcal{M}_{\mathbb{C}+}$ and $c([\tau(F)]) \in c'(\widetilde{KSp}(F)) + qdiv(\widetilde{K}(F))$.

(c) $F \in \mathsf{Fix}(G)$ *if* $[\tau(F)] \in r\bigl(qdiv(\widetilde{K}(F))\bigr)$, *or if* $\mathrm{Syl}_2(G) \not\trianglelefteq G$ *and* $\tau(F) \in qdiv(\widetilde{KO}(F))$.

Proof. For any $F \in \mathsf{Fix}(G)$, we let $\mathsf{Tg}_G(F)$ denote the class of G-bundles over F satisfying conditions (1)–(3) in Theorem 0.1.

(a) Assume first that $G \in \mathcal{M}_{\mathbb{R}}$, and let (V,W) be a $\mathcal{P}$-matched pair of real G-representations such that $\dim(V^G) = 1$ and $W^G = 0$. If F is any compact manifold, let $\tau(F)$ and $\nu(F)$ denote the tangent and normal bundles, and set

$$\eta = \bigl(\tau(F)\otimes V\bigr) \oplus \bigl(\nu(F)\otimes W\bigr) \tag{1}$$

(as a real G-bundle over F). Then $\eta|P$ is a product P-vector bundle for any $P \subseteq G$ of prime power order, and $\eta^G \cong \tau(F)$. Thus $\eta \in \mathsf{Tg}_G(F)$, and so $F \in \mathsf{Fix}(G)$. If $G \in \mathcal{M}_{\mathbb{C}}$ and F is stably complex, then the same construction as in (1), but with complex bundles and representations, again produces a bundle $\eta \in \mathsf{Tg}_G(F)$.

(b) Assume $G \in \mathcal{M}_{\mathbb{C}+}$ and $c([\tau(F)]) \in c'(\widetilde{KSp}(F)) + \mathrm{qdiv}(\widetilde{K}(F))$. By (a), there are elements $g, x \in G$ such that $|g|$ is not a prime power and $xgx^{-1} = g^{-1}$. Set $G' = \langle g, x\rangle$. We can choose a subgroup $K \lhd G'$ such that G'/K is either dihedral of order $2n$ where n is not a prime power, or quaternion of order $4p$ for some odd prime p. In the first case, $G \in \mathcal{M}_{\mathbb{R}}$ by Lemma 3.1(b), and so $F \in \mathsf{Fix}(G)$ by (a). So we are reduced to the case where G'/K is quaternion of order $4p$.

Fix $a \in G'$ which generates the cyclic subgroup of order $2p$ in G'/K, and set $H = \langle a, K\rangle \lhd G'$. Set $\zeta = \exp(2\pi i/p)$. Then there are $\mathbb{R}G$-representations V', W' and $\mathbb{H}G$-representations V'', W'' such that

$$\mathbb{C}\otimes_{\mathbb{R}}V' \cong \mathrm{Ind}_H^G(\mathbb{C}_1), \quad \mathbb{C}\otimes_{\mathbb{R}}W' \cong \mathrm{Ind}_H^G(\mathbb{C}_\zeta), \quad V''|\mathbb{C} \cong \mathrm{Ind}_H^G(\mathbb{C}_{-\zeta}),$$
$$W''|\mathbb{C} \cong \mathrm{Ind}_H^G(\mathbb{C}_{-1}).$$

Here $\mathbb{C}_\psi$ denotes the 1-dimensional H/K-representation where a acts via multiplication by ψ. Set

$$V = (\mathbb{C}\otimes_{\mathbb{R}}V')\oplus(V''|\mathbb{C}) \qquad \text{and} \qquad W = (\mathbb{C}\otimes_{\mathbb{R}}W')\oplus(W''|\mathbb{C}).$$

Then (V,W) is a self-conjugate $\mathcal{P}$-matched pair of G-representations of type 1; and (V',W') and (V'',W'') are 2-matched pairs of G-representations of types 1 and 0, respectively.

Set $\tau = \tau(F)$, and let ν be a normal bundle for F. By assumption on F, there are $\mathbb{H}$-bundles τ'' and ν'' such that $\tau''\oplus\nu''$ is a product $\mathbb{H}$-bundle, and such that $[\tau''|\mathbb{C}] = [\mathbb{C}\otimes_{\mathbb{R}}\tau] \in \widetilde{K}(F)/(\mathrm{qdiv})$. Since all infinitely 2-divisible elements in $\widetilde{K}(F)^+$ (the elements invariant under complex conjugation) are in the image of infinitely 2-divisible elements in $\widetilde{KSp}(F)$ (see Lemma A.5(a)), we can assume that the difference $[\tau''|\mathbb{C}] - [\mathbb{C}\otimes_{\mathbb{R}}\tau]$ is infinitely p-divisible for all odd primes $p\bigm||G|$ (Lemma A.5(b)). Set

$$\eta = (\tau\otimes_{\mathbb{R}}V')\oplus(\tau''\otimes_{\mathbb{H}}V'')\oplus(\nu\otimes_{\mathbb{R}}W')\oplus(\nu''\otimes_{\mathbb{H}}W'').$$

By construction, $[\eta|P] = 0 \in \widetilde{KO}_P(F)$ for any 2-subgroup $P \subseteq G$. Also,

$$\begin{aligned}\mathbb{C}\otimes_{\mathbb{R}}\eta &= ((\mathbb{C}\otimes_{\mathbb{R}}\tau)\otimes_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}}V'))\oplus(\tau''\otimes_{\mathbb{C}}V'')\oplus((\mathbb{C}\otimes_{\mathbb{R}}\nu)\otimes_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}}W'))\oplus(\nu''\otimes_{\mathbb{C}}W'')\\ &\equiv (\mathbb{C}\otimes_{\mathbb{R}}\tau)\otimes_{\mathbb{C}}V \oplus (\mathbb{C}\otimes_{\mathbb{R}}\nu)\otimes_{\mathbb{C}}W\end{aligned}$$

modulo elements which are infinitely p-divisible for all odd primes $p\big||G|$. So for each such p and each p-subgroup $P \subseteq G$, $2{\cdot}[\eta|P] = r([\mathbb{C}\otimes_{\mathbb{R}}\eta|P])$ is infinitely p-divisible, and hence $[\eta|P]$ is infinitely p-divisible by Lemma A.5(a). Thus, $\eta \in \mathsf{Tg}_G(F)$, and so $F \in \mathsf{Fix}(G)$.

(c) Assume first that G has the property that for each prime p, there is a p-matched pair (V_p, W_p) of real G-representations of type 1. We can assume that $\dim((V_p)^G) = 1$ and $(W_p)^G = 0$. Let F be such that $[\tau(F)] \in \operatorname{qdiv}(\widetilde{KO}(F))$. By Lemma A.5(b), we can write $[\tau(F)] = \sum_{p||G|}[\tau_p]$, where each $[\tau_p]$ is infinitely q-divisible for all primes $q\big||G|$ different from p. Choose ν_p such that each $\tau_p\oplus\nu_p$ is a product bundle. Set

$$\eta = \bigoplus_{p||G|} \Big((\tau_p\otimes V_p) \oplus (\nu_p\otimes W_p)\Big).$$

Then $\eta \in \mathsf{Tg}_G(F)$, and so $F \in \mathsf{Fix}(G)$.

If $\tau(F) \in r\big(\operatorname{qdiv}(\widetilde{K}(F))\big)$, and if there is for each prime p a p-matched pair of complex representations of type 1, then the same construction (taken with complex bundles) gives a G-bundle $\eta \in \mathsf{Tg}_G(F)$.

It remains to show that for each prime p, G has a p-matched pair (V_p, W_p) of complex representations of type 1, and a p-matched pair of real representations of type 1 if $\mathrm{Syl}_2(G) \ntriangleleft G$. The complex representations are easily constructed: let $g \in G\smallsetminus 1$ be any element of order m prime to p, let $\mathbb{C}_1$ and $\mathbb{C}_\zeta$ be the 1-dimensional $\langle g\rangle$-representations where g acts via multiplication by 1 or $\zeta = \exp(2\pi i/m)$, and set

$$V_p = \operatorname{Ind}_{\langle g\rangle}^G(\mathbb{C}_1) \qquad \text{and} \qquad W_p = \operatorname{Ind}_{\langle g\rangle}^G(\mathbb{C}_\zeta). \tag{2}$$

Also, if $2\big||G|$ and g is any element of order 2, then $V_p = \operatorname{Ind}_{\langle g\rangle}^G(\mathbb{R}^+)$ and $W_p = \operatorname{Ind}_{\langle g\rangle}^G(\mathbb{R}^-)$ form for any odd prime p a p-matched pair of real representations of type 1.

If G is dihedral of order $2m$, where $m > 1$ is odd, and if g generates the subgroup of index 2, then the representations in (2) are induced from a 2-matched pair of real G-representations. So to finish the proof, we need only show that any group G such that $\mathrm{Syl}_2(G) \ntriangleleft G$ contains a subquotient of that form. Upon dividing out by the intersection $O_2(G)$ of the Sylow 2-subgroups, we can assume that this intersection is trivial. Let S be any conjugacy class of elements of order 2 in G. By [Go, Theorem 3.8.2], either S generates a normal 2-subgroup of G (which is clearly not the case), or some pair x, y of elements in S generates a subgroup not of 2-power order. And then $\langle x, y\rangle$ is dihedral, and contains a dihedral subgroup of order 2m for some odd $m > 1$. □

Lemma 3.2 gave sufficient conditions for a manifold to be contained in $\mathsf{Fix}(G)$. It remains to show that these conditions are also necessary.

Lemma 3.3. *Fix a finite group G not of prime power order, and assume that $F \in \mathsf{Fix}(G)$.*

(a) If $\mathrm{Syl}_2(G) \lhd G$, then F is stably complex.

(b) If $G \notin \mathcal{M}_{\mathbb{C}}$, then $\tau(F) \in qdiv(\widetilde{KO}(F))$, and $\tau(F) \in r\big(qdiv(\widetilde{K}(F))\big)$ if $\mathrm{Syl}_2(G) \lhd G$.

(c) If $G \notin \mathcal{M}_{\mathbb{R}}$, then $c([\tau(F)]) \in c'(\widetilde{KSp}(F)) + qdiv(\widetilde{K}(F))$. If $G \notin \mathcal{M}_{\mathbb{C}+}$, then $[\tau(F)] \in r(\widetilde{K}(F)) + qdiv(\widetilde{KO}(F))$.

Proof. Fix some $F \in \mathsf{Fix}(G)$. Let $\eta{\downarrow}F$ be a G-vector bundle which satisfies conditions (1)–(3) in Theorem 0.1: $\eta^G \cong \tau(F)$, $[\eta] = 0$ in $\widetilde{KO}(F)$, and $\eta|P$ is infinitely p-divisible in $\widetilde{KO}_P(F)$ for each prime $p\big||G|$ and each p-subgroup $P \subseteq G$.

(a,b) If all elements in G have prime power order, then $\mathrm{R}(G)$ is detected by restriction to the Sylow subgroups, and so $c([\eta]) = [\mathbb{C}\otimes_{\mathbb{R}}\eta]$ lies in $\mathrm{qdiv}(\widetilde{K}_G(F)) \cong \mathrm{qdiv}(\widetilde{K}(F))\otimes\mathrm{R}(G)$. Since $r{\circ}c([\eta]) = 2{\cdot}[\eta]$, it now follows that $[\eta] \in \mathrm{qdiv}(\widetilde{KO}_G(F))$, and in particular that $[\eta^G] = [\tau(F)]$ lies in $\mathrm{qdiv}(\widetilde{KO}(F))$.

Now assume that $\mathrm{Syl}_2(G) \lhd G$, and write $G_2 = \mathrm{Syl}_2(G)$ for short. Then $[\eta^{G_2}]$ is infinitely 2-divisible in $\widetilde{KO}(F)$, and hence is the image of the infinitely 2-divisible element $c\big(\frac{1}{2}{\cdot}[\eta^{G_2}]\big) \in \widetilde{K}(F)$. In particular, $[\eta^{G_2}] \in r\big(\mathrm{qdiv}(\widetilde{K}(F))\big)$. Let $V_0 = \mathbb{R}, V_1, \ldots V_k$ be the distinct irreducible real representations of G/G_2. Write

$$\eta^{G_2} = \eta_0 \oplus \eta_1 \oplus \cdots \oplus \eta_k,$$

where each fiber in η_i is a sum of copies of V_i (in particular, $\eta_0 = \eta^G$). Since $|G/G_2|$ is odd, each representation $V_1, \ldots, V_k$ can be given a complex structure by Proposition A.1(c), and hence (by Proposition A.1(a)) each η_i has the form $\eta_i \cong \xi_i\otimes_{\mathbb{C}}V_i$ for some complex bundle $\xi_i{\downarrow}F$. Thus $[\eta_i]$ is a complex bundle for each $i \geq 1$, and so $\tau(F) \cong \eta_0$ is a stably complex bundle. If, in addition, all elements of G have prime power order, then we have seen that $[\eta] \in \mathrm{qdiv}(\widetilde{KO}_G(F))$, so $[\xi_i] \in \mathrm{qdiv}(\widetilde{K}(F))$ for all $1 \leq i \leq k$, and $[\eta_i] \in r\big(\mathrm{qdiv}(\widetilde{K}(F))\big)$ for each $i \geq 1$. Also, $[\eta^{G_2}] \in r\big(\mathrm{qdiv}(\widetilde{K}(F))\big)$ as seen above; and hence $[\tau(F)] = [\eta_0]$ lies in $r\big(\mathrm{qdiv}(\widetilde{K}(F))\big)$.

(c) Note first that $G \in \mathcal{M}_{\mathbb{R}}$ ($G \in \mathcal{M}_{\mathbb{C}+}$) if there is a $\mathcal{P}$-matched pair (V, W) of real (self conjugate complex) representations of type a for any odd a. To see this, note first that if there is such a $\mathcal{P}$-matched pair, then G has an element not of prime power order, and hence $G \in \mathcal{M}_{\mathbb{C}}$ by Lemma 3.1(a). This implies that there is a $\mathcal{P}$-matched pair of real G-representations of type 2; and hence a $\mathcal{P}$-matched pair of real (or self conjugate) G-representations of type 1.

Let $T : \widetilde{K}(F) \to \mathbb{Z}$ be any conjugation invariant homomorphism (i.e., $T(\xi) = T(\bar{\xi})$); and let T_G denote the induced homomorphism

$$T_G = T\otimes\mathrm{Id} : \widetilde{K}_G(F) \cong \widetilde{K}(F)\otimes\mathrm{R}(G) \longrightarrow \mathrm{R}(G).$$

We first analyze $T_G(\mathbb{C}\otimes_{\mathbb{R}}\eta)$. Let $V_0, V_1, \ldots, V_n$ be the distinct irreducible $\mathbb{R}G$-representations, where $V_0 \cong \mathbb{R}$ is the trivial representation, and where V_i has real type for $0 \leq i \leq k$, has complex type (with given complex structure) for $k+1 \leq i \leq m$, and has quaternion type (with given structure) for $m+1 \leq i \leq n$. Then

$$\eta \cong \Big(\bigoplus_{i=0}^{k} \xi_i\otimes_{\mathbb{R}}V_i\Big) \oplus \Big(\bigoplus_{i=k+1}^{m} \xi_i\otimes_{\mathbb{C}}V_i\Big) \oplus \Big(\bigoplus_{i=m+1}^{n} \xi_i\otimes_{\mathbb{H}}V_i\Big),$$

(Proposition A.1(a)), where the ξ_i are real, complex, or quaternion vector bundles, respectively, and where $\xi_0 \cong \tau(F)$. For convenience, we write $T(\xi_i) = T([\mathbb{C}\otimes_{\mathbb{R}}\xi_i])$ if $i \le k$, $T(\xi_i) = T([\xi_i])$ if $k{+}1 \le i \le m$, and $T(\xi_i) = T([\xi_i|\mathbb{C}])$ if $m+1 \le i \le n$. Then

$$T_G([\mathbb{C}\otimes_{\mathbb{R}}\eta]) = \sum_{i=0}^{k} T(\xi_i)\cdot c([V_i]) + \sum_{i=k+1}^{m} T(\xi_i)\cdot\big([V_i] + [\overline{V}_i]\big) + \sum_{i=m+1}^{n} T(\xi_i)\cdot c'([V_i]) \in \mathrm{R}(G). \tag{1}$$

Since (for given T) the T_G commute with restriction of subgroups, we see that $T_G([\mathbb{C}\otimes_{\mathbb{R}}\eta])|P = 0$ for any $P \subseteq G$ of prime power order. Thus $T_G([\mathbb{C}\otimes_{\mathbb{R}}\eta]) = [V] - [W]$, where (V, W) is a $\mathcal{P}$-matched pair of self-conjugate G-representations of type $T(\xi_0)$.

Assume that $[\tau(F)] = [\xi_0] \notin r(\widetilde{K}(F)) + \mathrm{qdiv}(\widetilde{KO}(F))$. Then $c([\xi_0])$ is not a multiple of 2 in $\widetilde{K}(F)/\langle[\xi] \,|\, [\xi\oplus\bar{\xi}] \in (\mathrm{qdiv})\rangle$. So by Lemma A.5(c), we can choose T such that $T(\xi_0)$ is odd. Then $T_G([\mathbb{C}\otimes_{\mathbb{R}}\eta]) = [V] - [W]$ where (V, W) is a $\mathcal{P}$-matched pair of self-conjugate G-representations of type $T(\xi_0)$, and $G \in \mathcal{M}_{\mathbb{C}+}$ by the above remarks.

If $c([\tau(F)]) = c([\xi_0]) \notin c'(\widetilde{KSp}(F)) + \mathrm{qdiv}(\widetilde{K}(F))$, then $q{\circ}c([\xi_0])$ is not a multiple of 2 in $\widetilde{KSp}(F)/(\mathrm{qdiv})$. By Lemma A.5(c) again, we can choose T to be a composite of the form $\widetilde{K}(F) \xrightarrow{q} \widetilde{KSp}(F) \to \mathbb{Z}$, and such that $T(\xi_0)$ is odd. In particular, $T(\xi_i) \in 2\mathbb{Z}$ for $m+1 \le i \le n$ (the quaternion case). By (1), $T_G([\mathbb{C}\otimes_{\mathbb{R}}\eta]) = c([V] - [W])$, where

$$[V] - [W] = \sum_{i=0}^{k} T(\xi_i)\cdot[V_i] + \sum_{i=k+1}^{m} T(\xi_i)\cdot r([V_i]) + \sum_{i=m+1}^{n} \frac{T(\xi_i)}{2}\cdot r{\circ}c'([V_i]);$$

and (V, W) is a $\mathcal{P}$-matched pair of real G-representations of type $T(\xi_0)$. It follows that $G \in \mathcal{M}_{\mathbb{R}}$. $\square$

We now get immediately:

Proof of Theorem 0.2. If $F = M^G$ for any contractible manifold M with smooth G-action, then the G-vector bundle $\eta = \tau(M)|F$ satisfies conditions (1)–(3) in Theorem 0.1, and so $F \in \mathsf{Fix}(G)$. Also, $\chi(F) \equiv 1 \pmod{n_G}$ if M is a disk. Conversely, by Theorem 0.1, if $F \in \mathsf{Fix}(G)$, then F is the fixed point set of a smooth G-action on a euclidean space if $\partial F = \emptyset$, and F is the fixed point set of a smooth G-action on a disk if F is compact and $\chi(F) \equiv 1 \pmod{n_G}$.

The necessary and sufficient conditions for F to be in $\mathsf{Fix}(G)$ were shown in Lemmas 3.2 and 3.3. Note in particular case (C) in Theorem 0.2. If $G \in \mathcal{M}_{\mathbb{C}} \smallsetminus \mathcal{M}_{\mathbb{C}+}$ and $\mathrm{Syl}_2(G) \ntrianglelefteq G$, then $F \in \mathsf{Fix}(G)$ if $[\tau(F)] \in r(\widetilde{K}(F))$ (Lemma 3.2(a)) or if $[\tau(F)] \in \mathrm{qdiv}(\widetilde{KO}(F))$ (Lemma 3.2(c)). So from the definition of $\mathsf{Fix}(G)$, it follows that $F \in \mathsf{Fix}(G)$ if $[\tau(F)] \in r(\widetilde{K}(F)) + \mathrm{qdiv}(\widetilde{KO}(F))$. $\square$

The following example shows how Theorem 0.2 applies in the case of a dihedral or quaternion group acting on a disk.

Example 3.4. *If G is dihedral of order $2n$ or quaternion of order $4n$, where n is not a prime power, then a compact manifold F is the fixed point set of a G-action on some disk if and only if $\chi(F)$ is odd. If G is dihedral of order $2p^a$ for some odd prime p, then a compact manifold is the fixed point set of a G action on a disk if and only if $\chi(F) = 1$ and $[\tau(F)]$ is torsion in $\widetilde{KO}(F)$. If G is quaternion of order $4p^a$ for some odd prime p, then a compact manifold is the fixed point set of a G action on a disk if and only if $\chi(F) = 1$, and there is an $\mathbb{H}$-vector bundle $\xi{\downarrow}F$ such that $c([\tau(F)]) \equiv c'([\xi])$ modulo torsion in $\widetilde{K}(F)$.*

Proof. If G is dihedral of order $2n$ or quaternion of order $4n$, then by Theorem 0.3, $n_G = 2$ if n is not a prime power, and $n_G = 0$ if n is a power of an odd prime. The rest of the corollary follows from Lemma 3.1 and Theorem 0.2. □

As another example, note that for $G = A_4 \times \Sigma_3$, any compact smooth manifold F can be the fixed point set of a smooth G-action on a disk. This is in fact the smallest group with that property (see [O1, Theorem 8]).

Appendix

We collect here some results which are well known, but which either are hard to find in the literature, or which have been used often enough to state here explicitly.

Real G-vector bundles and their classifying spaces

We start with the following proposition, which describes some of the basic structure of real G-vector bundles and real G-representations.

Proposition A.1. *Fix a finite group G. Let $V_0, V_1, \ldots, V_k$ be the distinct irreducible $\mathbb{R}G$-representations, where $V_0 \cong \mathbb{R}$ with the trivial G-action. For each i, set $D_i = \mathrm{End}_{\mathbb{R}G}(V_i)$ ($\cong \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$).*

(a) Let X be space with trivial G-action, and let $\xi{\downarrow}X$ be a real G-vector bundle. Then $\xi \cong \bigoplus_{i=0}^{k}(V_i \otimes_{D_i} \xi_i)$, where each ξ_i is a (nonequivariant) D_i-vector bundle over X.

(b) Let V be any orthogonal G-representation, and let $O_G(V)$ be the group of G-equivariant orthogonal self maps of V. Then

$$V \cong \bigoplus_{i=0}^{k} (V_i)^{n_i} \qquad \text{implies} \qquad O_G(V) \cong \prod_{i=0}^{k} O(n_i, D_i),$$

where we write $O(n, \mathbb{R}) = O(n)$, $O(n, \mathbb{C}) = U(n)$, and $O(n, \mathbb{H}) = Sp(n)$.

(c) If $|G|$ is odd, then $D_i \cong \mathbb{C}$ for all $i \neq 0$.

Proof. For each i, $\mathrm{End}_{\mathbb{R}G}(V_i)$ is a division algebra over $\mathbb{R}$ by Schur's lemma (cf. [Ad, Lemma 3.22]), and hence is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Part (b) also follows from Schur's lemma, and part (c) from [Se, Exercise 13.12].

To see part (a), set $\xi_i = \mathrm{Hom}_{\mathbb{R}G}(V_i, \xi)$ (defined fiberwise) for each i. Then ξ_i is a D_i-bundle, and the evaluation maps define an isomorphism

$$\bigoplus_{i=0}^{k} V_i \otimes_{D_i} \mathrm{Hom}_{\mathbb{R}G}(V_i, \xi) \xrightarrow{\cong} \xi. \qquad \square$$

An irreducible $\mathbb{R}G$-representation V will be said to have real, complex, or quaternion type, depending on whether $\mathrm{End}_{\mathbb{R}G}(V)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

For each $n \ge 0$, $B_GO(n)$ will denote the classifying space for n-dimensional G-vector bundles: constructed using either infinite joins (cf. [tD, §I.8]), or Grassmannians of n-dimensional subspaces in an appropriate infinite dimensional G-representation. It has a universal G-vector bundle $E^n_G {\downarrow} B_GO(n)$ with respect to which pullback defines a bijection between $[X, B_GO(n)]^G$ and the set of locally trivial n-dimensional orthogonal G-bundles over X, for any countable G-complex X (cf. [tD, Theorem I.8.12], where the classifying space is denoted $B(G, O(n))$). Note that $B_GO(n)$ is connected for all n, since $\pi_0(B_GO(n))$ contains just one element: the class of the product bundle $G \times \mathbb{R}^n {\downarrow} G$.

For each orthogonal G-representation V, and each $m \ge 0$, direct sum with V defines a G-map

$$\oplus_V : B_GO(m) \longrightarrow B_GO(m + \dim(V)),$$

which is well defined up to G-homotopy. We define B_GO to be the homotopy direct limit (i.e., infinite mapping cylinder, or mapping telescope)

$$B_GO = \underrightarrow{\mathrm{hocolim}}\Big(B_GO(0) \xrightarrow{\oplus_{\mathbb{R}G}} B_GO(d) \xrightarrow{\oplus_{\mathbb{R}G}} B_GO(2d) \xrightarrow{\oplus_{\mathbb{R}G}} \cdots\Big),$$

where $\mathbb{R}G$ denotes the regular representation and $d = \dim(\mathbb{R}G) = |G|$. For each n, we let

$$\iota_n : B_GO(nd) \to B_GO$$

denote the inclusion of the n-th stage into this telescope.

If X is any finite G-complex, then any map $X \to B_GO$ factors through some finite stage $B_GO(nd)$ in the mapping telescope, and similarly for homotopies between maps. Hence

$$\begin{aligned}
[X, B_GO] &\cong \underrightarrow{\lim}\Big([X, B_GO(0)] \xrightarrow{\oplus_{\mathbb{R}G}} [X, B_GO(d)] \xrightarrow{\oplus_{\mathbb{R}G}} [X, B_GO(2d)] \xrightarrow{\oplus_{\mathbb{R}G}} \cdots\Big) \\
&\cong \underrightarrow{\lim}\Big(\mathrm{Vect}^{\mathbb{R},G}_0(X) \xrightarrow{\oplus_{\mathbb{R}G}} \mathrm{Vect}^{\mathbb{R},G}_d(X) \xrightarrow{\oplus_{\mathbb{R}G}} \mathrm{Vect}^{\mathbb{R},G}_{2d}(X) \xrightarrow{\oplus_{\mathbb{R}G}} \cdots\Big) \qquad (d = |G|) \\
&\cong \mathrm{Ker}\big[KO_G(X) \xrightarrow{\dim} \mathbb{Z}\big];
\end{aligned}$$

Here, $\mathrm{Vect}^{\mathbb{R},G}_m(X)$ denotes the set of isomorphism classes of m-dimensional orthogonal G-vector bundles over X; and the last step holds since any G-vector bundle over X is a summand of a product bundle $\mathbb{R}G^k \times X$ for some k (since any G-representation is contained in some multiple of the regular representation $\mathbb{R}G$). In particular, this shows that $\mathbb{Z} \times B_GO$ is the classifying space for the equivariant K-theory functor $KO_G(-)$.

We next look more closely at the fixed point sets $(B_GO)^H$ and $(B_GO(n))^H$.

Proposition A.2. *Fix a finite group G and a subgroup $H \subseteq G$.*

(a) For each $n \geq 0$,

$$(B_G O(n))^H \simeq \coprod_{[V] \in \mathrm{Rep}_n^{\mathbb{R}}(H)} BO_H(V)$$

where $\mathrm{Rep}_n^{\mathbb{R}}(H)$ is the set of isomorphism classes of n-dimensional orthogonal H-representations.

(b) Let $V_0, V_1, \ldots, V_k$ be the distinct irreducible orthogonal H-representations. Then

$$(B_G O)^H \simeq \mathrm{IRO}(H) \times \prod_{i=0}^{k} B_i,$$

where $\mathrm{IRO}(H) = \mathrm{Ker}[\mathrm{RO}(H) \xrightarrow{\dim} \mathbb{Z}]$ is the augmentation ideal, and where $B_i \cong BO$, BU, or BSp depending on whether $\mathrm{End}_{\mathbb{R}G}(V_i) \cong \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

(c) For any $n > 0$, $\iota_n^H : \big(B_G O(nd)\big)^H \to (B_G O)^H$ ($d = |G|$) sends the component $\big(B_G O(nd)\big)_V^H$ corresponding to the representation V to the component of $(B_G O)^H$ corresponding to $[V] - [\mathbb{R}G^n] \in \mathrm{IRO}(H)$; and the map between this pair of components is m-connected if each irreducible H-representation occurs in V with multiplicity at least m.

(d) For any finite H-complex X,

$$\pi_i\big(\mathrm{map}_H(X, B_G O)\big) \cong \begin{cases} KO_H^{-i}(X) & \text{if } i > 0 \\ \mathrm{Ker}\big[KO_H(X) \xrightarrow{\dim} \mathbb{Z}\big] & \text{if } i = 0. \end{cases}$$

Proof. Consider the H-equivariant maps $B_G O(n) \xrightarrow{f_1} B_H O(n) \xrightarrow{f_2} B_G O(n)$, where f_1 classifies the universal bundle $E_G^n {\downarrow} B_G O(n)$ regarded as an H-bundle, and where f_2 classifies the G-bundle $(G \times_H E_H^n) {\downarrow} (G \times_H B_H O(n))$. These are easily checked to be H-homotopy inverses; and show that $B_G O(n)$ is H-equivariantly homotopy equivalent to $B_H O(n)$. So $B_G O$ is H-homotopy equivalent to $B_H O$. In particular, it suffices to prove the proposition when $H = G$.

(a) For any n,

$$\pi_0((B_G O(n))^G) \cong \mathrm{Vect}_n^{\mathbb{R},G}(\mathrm{pt}) \cong \mathrm{Rep}_n^{\mathbb{R}}(G).$$

Let $\big(B_G O(n)\big)_V^G$ denote the component corresponding to the representation V. For any X (without group action),

$$[X, (B_G O(n))^G] \cong [X, B_G O(n)]^G \cong \mathrm{Vect}_n^{\mathbb{R},G}(X);$$

and so $[X, (B_G O(n))_V^G]$ is the set of isomorphism classes of G-vector bundles over X with fiber V. The structure group for such bundles is $O_G(V)$, and hence $(B_G O(n))_V^G \simeq BO_G(V)$.

(b,c) The descriptions of the components of $(B_GO)^G$, and of $\iota_n^G : (B_GO(nd))^G \to (B_GO)^G$, follow from part (a) and Proposition A.1(b), upon taking limits with respect to direct sum with the regular representation $\mathbb{R}G$. Note in particular that

$$\mathrm{IRO}(G) \cong \varinjlim\Big(\mathrm{Rep}_0^{\mathbb{R}}(G) \xrightarrow{\oplus\mathbb{R}G} \mathrm{Rep}_d^{\mathbb{R}}(G) \xrightarrow{\oplus\mathbb{R}G} \mathrm{Rep}_{2d}^{\mathbb{R}}(G) \xrightarrow{\oplus\mathbb{R}G} \dots\Big).$$

And the last statement in (c) follows since the inclusions $BO(m) \to BO$, $BU(m) \to BU$, and $BSp(m) \to BSp$ are m-connected for all m.

(d) We have already seen that $[X, B_GO]^G \cong \mathrm{Ker}[KO_G(X) \xrightarrow{\dim} \mathbb{Z}]$ when X is a finite G-complex. And for any $i > 0$,

$$\pi_i\big(\mathrm{map}_G(X, B_GO)\big) \cong [\Sigma^i(X_+), B_GO]_*^G \cong KO_G^{-i}(X). \qquad \square$$

As was noted above, any map from a finite G-complex X to B_GO factors through some $\iota_n : B_GO(nd) \to B_GO$, and hence induces (stably) a G-bundle over X. This does not hold in general for finite dimensional G-complexes, but the next lemma describes conditions under which maps $X \to B_GO$ do induce G-bundles.

Lemma A.3. *Fix a countable finite dimensional G-complex X. Then for each $n \geq 0$, pullback of the universal bundle $E_G^n{\downarrow}B_GO(n)$ defines a bijection between $[X, B_GO(n)]^G$ and the set of isomorphism classes of n-dimensional G-bundles over X. Also, a G-map $f : X \to B_GO$ factors through $\iota_m : B_GO(md) \to B_GO$ for some m (where $d = |G|$) if and only if $\mathrm{Im}(\pi_0(f^H)) \subseteq \pi_0((B_GO)^H)$ is finite for all $H \subseteq G$. And any two liftings $f_m, f_m' : X \to B_GO(md)$ of f are homotopic after some finite stabilization; i.e., the induced G-bundles over X are stably isomorphic.*

Proof. The bijection between n-dimensional G-bundles over X and $[X, B_GO(n)]^G$ is a special case of [tD, Theorem I.8.12].

If $f : X \to B_GO$ factors through some $B_GO(md)$, then $\mathrm{Im}(\pi_0(f^H))$ must be finite for all H since $(B_GO(md))^H$ has only finitely many connected components (corresponding to the finite set of m-dimensional H-representations). Conversely, set $n = \dim(X)$, and assume that $\mathrm{Im}(\pi_0(f^H))$ is finite for all $H \subseteq G$. For each H, we can choose some $m_H \geq 0$ large enough so that the image of f^H is contained in components of $(B_GO)^H$ corresponding to some family of virtual H-representations $v_i = [V_i] - [\mathbb{R}G^{m_H}] \in \mathrm{IRO}(H)$ $(1 \leq i \leq k)$, and such that each irreducible H-representation occurs in each V_i with multiplicity at least n. Thus, the image of any connected component of X^H is lies in one of the components $(B_GO)_{v_i}^H$, which is in the image of $(B_GO(m_H{\cdot}d))_{V_i}^H$; and the inclusion of those components is n-connected by Proposition A.2(c). Hence, if we set $m = \max\{m_H \mid H \subseteq G\}$, then $f : X \to B_GO$ factors through $B_GO(md)$. $\square$

Divisible and quasidivisible subgroups

The purpose of the following two lemmas is to set up some notation and results to work with cohomology and K-theory groups of countably infinite CW complexes. Hence, we concentrate on the class of what we call $\mathcal{PFG}$-groups ("pro-finitely generated"): abelian groups which are products of the form $\varprojlim(M_i) \times \varprojlim^1(M_i')$, where M_i and M_i' are two inverse systems of finitely generated abelian groups. We first note that the $\varprojlim^1$ factor is divisible (i.e., n-divisible for all $n > 0$).

Lemma A.4. *Fix a sequence* $\left(\cdots \to M_2 \to M_1 \to M_0\right)$ *of abelian groups. Then for any* $n > 0$, $\varprojlim^1(M_i)$ *is* n*-divisible if* M_i/nM_i *is finite for all* i*. In particular, if the* M_i *are all finitely generated, then* $\varprojlim^1(M_i)$ *is divisible, and hence injective.*

Proof. Since $\varprojlim^1$ is right exact, the sequence

$$\varprojlim_i{}^1(M_i) \xrightarrow{\ \cdot n\ } \varprojlim_i{}^1(M_i) \longrightarrow \varprojlim_i{}^1(M_i/nM_i) \to 0$$

is exact for all $n > 0$, and $\varprojlim^1(M_i/nM_i) = 0$ if the M_i/nM_i are finite. So $\varprojlim^1(M_i)$ is n-divisible in this case. □

For any abelian group A, we define $\mathrm{qdiv}(A)$ to be the smallest possible kernel of a homomorphism from A to a product of copies of $\mathbb{Z}$. Clearly, all elements in A which are infinitely p-divisible for any prime p are contained in $\mathrm{qdiv}(A)$. So by Lemma A.4, if $A = \varprojlim(M_i) \times \varprojlim^1(M'_i)$ where M_i and M'_i are inverse systems of finitely generated abelian groups, then $\mathrm{qdiv}(A) = \varprojlim(\mathrm{tors}(M_i)) \times \varprojlim^1(M'_i)$.

Lemma A.5. *If* X *is a countable CW complex, and if* h^* *is any (representable) cohomology theory such that* $h^i(pt)$ *is finitely generated for all* i*, then* $h^i(X)$ *is a* $\mathcal{PFG}$*-group for all* i*. In particular,* $\widetilde{K}(X)$, $\widetilde{KO}(X)$*, and* $\widetilde{KSp}(X)$ *are all* $\mathcal{PFG}$*-groups. Furthermore, the following hold for any* $\mathcal{PFG}$*-group* A*:*

(a) If $x \in A$ *is divisible, i.e., if* $x \in nA$ *for all* $n > 0$*, then* x *is "sequentially divisible" in that for any sequence* $n_1, n_2, \dots$ *of positive integers there is a sequence* $x = x_0, x_1, x_2, \dots$ *in* A *such that* $n_i x_i = x_{i-1}$ *for all* i*. Similarly, if* $x \in A$ *is infinitely* p*-divisible for any prime* p*, then there is a sequence* $x = x_0, x_1, x_2, \dots$ *such that* $px_i = x_{i-1}$ *for all* i*. And if* nx *is infinitely* p*-divisible for any prime* $p \nmid n$*, then* x *is also infinitely* p*-divisible.*

(b) For any n*, and any* $x \in \mathrm{qdiv}(A)$*, we can write* $x = \sum_{p|n} x_p$*, where each* x_p *is infinitely* q*-divisible for all primes* $q \neq p$ *dividing* n*.*

(c) If $x \in A$*, and* $x \notin 2A+\mathrm{qdiv}(A)$*, then there is a homomorphism* $\varphi : A \to \mathbb{Z}$ *such that* $\varphi(x)$ *is odd.*

Proof. Fix a countable CW complex X, and write $X = \cup_{i=1}^{\infty} X_i$, where $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ are finite subcomplexes. Then for any representable cohomology theory h^*, there is for each j a short exact sequence

$$0 \to \varprojlim_i{}^1\big(\widetilde{h}^{j-1}(X_i)\big) \longrightarrow h^j(X) \longrightarrow \varprojlim_i\big(h^j(X_i)\big) \to 0;$$

where the $h^j(X_i)$ and $\widetilde{h}^{j-1}(X_i)$ are all finitely generated. The extension splits, since the first term is injective by Lemma A.4, and so $h^j(X)$ is a $\mathcal{PFG}$-group.

Now assume A is a $\mathcal{PFG}$-group, and write $A = \varprojlim(M_i) \times \varprojlim^1(M'_i)$, where the M_i and M'_i are all finitely generated. Point (a) follows upon noting that the divisible elements in A are precisely those in $\varprojlim^1(M'_i)$, and that the p-divisible elements (for any prime p) are those in $\varprojlim\big(p'\text{-tors}(M_i)\big) \times \varprojlim^1(M'_i)$. Point (b) is immediate. Point (c) follows upon noting that if $x \notin 2A + \mathrm{qdiv}(A)$, then the image of x in some $M_i/(\mathrm{tors})$ is not a multiple of 2. And then there is a homomorphism $M_i/(\mathrm{tors}) \to \mathbb{Z}$ which sends the image of x to an odd integer. □

Homotopy and homology groups

We collect here some miscellaneous lemmas on homotopy and homology groups and the Hurewicz map.

Lemma A.6. *All homotopy groups of a countable CW complex are countable.*

Proof. The homotopy groups of a finite simply connected complex are finitely generated (cf. [Hu, Corollary X.8.3]). The fundamental group of a countable complex is countably generated and hence countable. So the homotopy groups of the universal cover of a countable complex are countable direct limits of finitely generated groups, and hence are countable. □

The following version of the relative Hurewicz theorem is needed in Section 2 when constructing spaces and maps. For convenience, when a map $f : X \to Y$ is understood, we write $\pi_*(Y, X)$ for $\pi_*(Z_f, X)$ (where Z_f denotes the mapping cylinder), and similarly for $H_*(Y, X)$.

Lemma A.7. *Fix a prime p and $n \geq 2$. Assume that $f : X \to Y$ is a map between connected complexes such that $\pi_1(f)$ is onto, such that $\operatorname{Ker}(\pi_1(f))$ is abelian and torsion prime to p, and such that $\pi_i(\widetilde{Y}, \widetilde{X})\otimes\mathbb{Z}_{(p)} = 0$ for all $i < n$. Here, $\widetilde{X}$ and $\widetilde{Y}$ denote the universal covers. Then $H_i(Y, X)\otimes\mathbb{Z}_{(p)} = 0$ for all $i < n$, and the Hurewicz map $\pi_n(\widetilde{Y}, \widetilde{X})\otimes\mathbb{Z}_{(p)} \twoheadrightarrow H_n(Y, X; \mathbb{Z}_{(p)})$ is onto. If, furthermore, X and Y are finite complexes and $\operatorname{Ker}(\pi_1(f))$ is finite, then $\pi_n(\widetilde{Y}, \widetilde{X})\otimes\mathbb{Z}_{(p)}$ is finitely generated as a $\mathbb{Z}_{(p)}[\pi_1(X)]$-module.*

(Note that $\pi_2(\widetilde{Y}, \widetilde{X}) \cong \operatorname{Im}[\pi_2(Y) \to \pi_2(Y, X)]$, and $\pi_i(\widetilde{Y}, \widetilde{X}) = \pi_i(Y, X)$ for $i > 2$. The lemma is formulated using $\pi_i(\widetilde{Y}, \widetilde{X})$ rather than $\pi_i(Y, X)$ to allow the possibility that $\pi_2(Y, X)$ is not abelian.)

Proof. Let F be the homotopy fiber of $f : X \to Y$, and let $\widetilde{F}$ be its universal cover (F is connected by assumption). Then

$$\pi_i(\widetilde{F})\otimes\mathbb{Z}_{(p)} \cong \pi_{i+1}(Y, X)\otimes\mathbb{Z}_{(p)} = 0 \qquad \text{for all } 2 \leq i < n-1.$$

So by the generalized Hurewicz theorem (cf. [Hu, Theorem X.8.1]), applied to the class of torsion abelian groups of order prime to p,

$$H_i(\widetilde{F}; \mathbb{Z}_{(p)}) = 0 \quad \text{for all } i < n-1 \quad \text{and} \quad H_{n-1}(\widetilde{F}; \mathbb{Z}_{(p)}) \cong \pi_{n-1}(\widetilde{F})\otimes\mathbb{Z}_{(p)}. \tag{1}$$

Set

$$\Gamma = \pi_2(\widetilde{Y}, \widetilde{X}) \cong \operatorname{Im}\Big[\pi_2(Y) \longrightarrow \pi_2(Y, X){\cong}\pi_1(F)\Big] = \operatorname{Ker}\big[\pi_1(F) \to \pi_1(X)\big],$$

and let $\overline{F} = \widetilde{F}/\Gamma$ be the covering of F with fundamental group Γ. By Lemma A.8 below, Γ acts trivially on $H_*(\widetilde{F})$. If $n \geq 3$, then Γ is abelian and torsion prime to p, by assumption, and the spectral sequence for the fibration $\widetilde{F} \to \overline{F} \to B\Gamma$ gives

$$H_*(\overline{F}; \mathbb{Z}_{(p)}) \cong H_0(\Gamma; H_*(\widetilde{F}; \mathbb{Z}_{(p)})) \cong H_*(\widetilde{F}; \mathbb{Z}_{(p)}).$$

Together with (1), this shows that

$$\pi_i(\overline{F})\otimes\mathbb{Z}_{(p)} \cong H_i(\overline{F};\mathbb{Z}_{(p)}) \qquad \text{for all } i \le n-1 \tag{2}$$

whenever $i \ge 2$, and this clearly also holds for $i = 1$.

Since $H_i(\overline{F};\mathbb{Z}_{(p)}) = 0 = H_i(\widetilde{F};\mathbb{Z}_{(p)})$ for $i < n-1$ by (1) and (2), the spectral sequence for the fibration $\overline{F} \to \widetilde{X} \to \widetilde{Y}$ shows that

$$H_i(\widetilde{Y},\widetilde{X};\mathbb{Z}_{(p)}) = 0 \quad \text{for } i < n \qquad \text{and} \qquad H_n(\widetilde{Y},\widetilde{X};\mathbb{Z}_{(p)}) \cong H_{n-1}(\overline{F};\mathbb{Z}_{(p)}). \tag{3}$$

And this together with (2) shows that the p-local Hurewicz homomomorphism for the pair $(\widetilde{Y},\widetilde{X})$ is a composite of isomorphisms:

$$\pi_n(\widetilde{Y},\widetilde{X})\otimes\mathbb{Z}_{(p)} \cong \pi_{n-1}(\overline{F})\otimes\mathbb{Z}_{(p)} \cong H_{n-1}(\overline{F};\mathbb{Z}_{(p)}) \cong H_n(\widetilde{Y},\widetilde{X};\mathbb{Z}_{(p)}). \tag{4}$$

Set $K = \mathrm{Ker}(\pi_1(f))$. By assumption, K is abelian and torsion prime to p. Hence $H_*(\widetilde{Y},\widetilde{X}/K;\mathbb{Z}_{(p)}) \cong H_0(K;H_*(\widetilde{Y},\widetilde{X};\mathbb{Z}_{(p)}))$. Since $H_i(\widetilde{Y},\widetilde{X};\mathbb{Z}_{(p)}) = 0$ for $i < n$ by (3), the spectral sequence for the fibration $(\widetilde{Y},\widetilde{X}/K) \to (Y,X) \to \pi_1(Y)$ shows that

$$H_i(Y,X;\mathbb{Z}_{(p)}) \cong H_0\big(\pi_1(Y);H_i(\widetilde{Y},\widetilde{X}/K;\mathbb{Z}_{(p)})\big) \cong H_0(\pi_1(X);H_i(\widetilde{Y},\widetilde{X};\mathbb{Z}_{(p)}))$$

for all $i \le n$. Together with (3) and (4), this shows that $H_i(Y,X;\mathbb{Z}_{(p)}) = 0$ for $i < n$, and that the Hurewicz homomorphism sends $\pi_n(\widetilde{Y},\widetilde{X})\otimes\mathbb{Z}_{(p)}$ onto $H_n(Y,X;\mathbb{Z}_{(p)})$.

Now assume that X and Y are finite complexes, and that $\mathrm{Ker}[\pi_1(X) \twoheadrightarrow \pi_1(Y)]$ is finite. The kernel has order prime to p, by assumption, and so any projective $\mathbb{Z}_{(p)}[\pi_1(Y)]$-module is also projective as a $\mathbb{Z}_{(p)}[\pi_1(X)]$-module. Each term in the relative cellular chain complex $C_* = C_*(\widetilde{Y},\widetilde{X};\mathbb{Z}_{(p)})$ is thus a finitely generated projective $\mathbb{Z}_{(p)}[\pi_1(X)]$-module. Since C_* has no homology below dimension n, $Z_n \stackrel{\text{def}}{=} \mathrm{Ker}[C_n \xrightarrow{\partial} C_{n-1}]$ is a direct summand of C_n and hence finitely generated. So $H_n(\widetilde{Y},\widetilde{X};\mathbb{Z}_{(p)}) = Z_n/\partial(C_{n+1})$ is also finitely generated over $\mathbb{Z}_{(p)}[\pi_1(X)]$; and is isomorphic to $\pi_n(\widetilde{Y},\widetilde{X})\otimes\mathbb{Z}_{(p)}$ by (4). □

It remains to prove the following lemma, which says in particular that the homotopy fiber of a map between connected and 1-connected spaces is simple.

Lemma A.8. *Let $F \to X \xrightarrow{f} Y$ be a fibration of path connected spaces such that F has a universal cover $\widetilde{F}$; and set $\Gamma = \mathrm{Ker}[\pi_1(F) \to \pi_1(X)]$. Then the translation action of any element of Γ on $\widetilde{F}$ is homotopic to the identity. In particular, Γ acts trivially on $\pi_*(\widetilde{F})$ and on $H_*(\widetilde{F})$.*

Proof. Fix a basepoint $x_0 \in F \subseteq X$, and set $y_0 = f(x_0) \in Y$. Let $\gamma : I \to F$ be any loop ($\gamma(0) = \gamma(1) = x_0$) which represents an element of Γ, and choose a homotopy $G : I\times I \to X$ such that $G(t,0) = \gamma(t)$, and $G(t,s) = x_0$ if $s = 1$ or $t \in \{0,1\}$. Then $[f \circ G] \in \pi_2(Y)$, and $\partial([f \circ G]) = [\gamma]$.

Define $\alpha = f \circ G \circ \mathrm{proj} : F \times I \times I \to B$. By the homotopy lifting property for the fibration, there exists a map $A : F \times I \times I \to X$ such that $A(x,t,s) = x$ if $s = 1$ or $t \in \{0,1\}$. Let $\beta : F \times I \to F$ be the map $\beta(x,t) = A(x,t,0)$. Then $\beta(-,0) = \beta(-,1) = \mathrm{Id}_F$, and so β can be lifted to a unique homotopy $\widetilde{\beta} : \widetilde{F} \times I \to \widetilde{F}$ such that $\widetilde{\beta}(-,0) = \mathrm{Id}$. Also, the loop $\beta(x_0,-)$ is homotopic to γ by construction, so $\widetilde{\beta}(-,1)$ is the covering transformation induced by γ, and is thus homotopic to the identity. □

The next lemma is much more technical. It is needed in the proof of Theorem 0.1, to handle fixed point sets not of finite homotopy type.

Lemma A.9. *Fix $n > 0$, and let B be a connected H-space with the property that $(\mathbb{Z}/n) \otimes [\Sigma K, B]$ is finite for any finite CW complex K. Let X be a countable finite dimensional complex, and let $f : X \to B$ be a map which is nullhomotopic on all finite subcomplexes of X. Then f factors as a composite $X \hookrightarrow \overline{X} \xrightarrow{\bar{f}} B$, for some countable finite dimensional $\mathbb{Z}/n$-acyclic complex $\overline{X} \supseteq X$.*

Proof. Write $X = \cup_{i=1}^{\infty} X_i$, where $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ are all finite subcomplexes. By assumption, $[f] \in [X, B]$ lies in the image of the first term in the short exact sequence

$$0 \to \varprojlim{}^1 [\Sigma(X_i), B] \longrightarrow [X, B] \longrightarrow \varprojlim [X_i, B] \to 0.$$

By Lemma A.4, the group $\varprojlim^1 [\Sigma(X_i), B]$ is n-divisible, and so there is a sequence of maps $f = f_0, f_1, f_2, \ldots : X \to B$ such that $n\cdot[f_i] = [f_{i-1}]$ for each i. Hence f factors as a composite

$$X \xrightarrow{\;\widehat{f}\;} \widehat{B} \longrightarrow B,$$

where $\widehat{B}$ is the homotopy inverse limit of the sequence $\left(\ldots \xrightarrow{\cdot n} B \xrightarrow{\cdot n} B \xrightarrow{\cdot n} B\right)$.

We next claim that $\widehat{B}$ is $\mathbb{Z}/n$-acyclic. To see this, note that each map $B \xrightarrow{\cdot n} B$ induces multiplication by n in homotopy groups, and so all homotopy groups of $\widehat{B}$ are uniquely n-divisible. Hence, via spectral sequences for the fibrations, it will suffice to show that $\widetilde{H}_*(K(M,i); \mathbb{Z}/n) = 0$ for all $i \geq 1$ and all $\mathbb{Z}[\frac{1}{n}]$-modules M. It suffices (by taking direct limits) to show this for finitely generated M; and hence for $M = \mathbb{Z}[\frac{1}{n}]$ and M finite of order prime to n. The latter case is clear. When $M = \mathbb{Z}[\frac{1}{n}]$, then $K(M,1)$ is a $\mathbb{Z}[\frac{1}{n}]$-Moore space (and hence $\mathbb{Z}/n$-acyclic); and its deloopings are seen to be $\mathbb{Z}/n$-acyclic using the usual spectral sequences.

It remains to show that $\widehat{f}$ extends to a countable finite dimensional $\mathbb{Z}/n$-acyclic complex $\overline{X} \supseteq X$. To see this, first replace $\widehat{B}$ by a CW complex (of the same weak homotopy type) which contains X as a subcomplex. Since homology is supported by finite complexes, there is a sequence $X = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ of countable subcomplexes of $\widehat{B}$ such that each inclusion $X_{i-1} \subseteq X_i$ is trivial in $\mathbb{Z}/n$-homology. Set $X_\infty = \cup_{i=1}^{\infty} X_i$. Then X_∞ is countable and $\mathbb{Z}/n$-acyclic, but need not be finite dimensional. Set $d = \dim(X)$, and consider the free abelian group $B_d(X_\infty) \overset{\text{def}}{=} \mathrm{Ker}[H_d((X_\infty)^{(d)}) \to H_d(X_\infty)]$. Every element in $B_d(X_\infty)$ is in the image of the Hurewicz homomorphism for $(X_\infty)^{(d)}$. So there is a $(d+1)$-dimensional complex $\overline{X}$ with the same d-skeleton as X_∞, such that the inclusion $\overline{X}^{(d)} \subseteq X_\infty$ extends to $\overline{X}$, and such that $H_i(\overline{X}) \cong H_i(X_\infty)$ for $i \le d$ and $H_i(\overline{X}) = 0$ for $i > d$. Then $\widetilde{H}_*(\overline{X})$ is uniquely n-divisible, so $\overline{X}$ is $\mathbb{Z}/n$-acyclic, and we are done. $\square$

Projective and stably free homology of G-complexes

The following lemma is basically taken from [O1], although not stated there explicitly.

Lemma A.10. *Let G be any finite group, and let $f : X \to Y$ be a map between countable finite dimensional G-complexes. Set $n = \max\{\dim(X), \dim(Y)\}$. Assume, for some prime p, that $\widetilde{H}_i(Z_f, X; \mathbb{F}_p) = 0$ for all $i \le n$, and that $f^P : X^P \to Y^P$ is an $\mathbb{F}_p$-homology equivalence for all p-subgroups $1 \ne P \subseteq G$. Then $H_{n+1}(Z_f, X; \mathbb{F}_p)$ is projective as an $\mathbb{F}_p[G]$-module, and hence is stably free as a countably generated $\mathbb{F}_p[G]$-module. If in addition, X and Y are finite complexes, and $\chi(X^H) = \chi(Y^H)$ for any cyclic subgroup $1 \ne H \subseteq G$ of order prime to p, then $H_{n+1}(Z_f, X; \mathbb{F}_p)$ is a free $\mathbb{F}_p[G]$-module.*

Proof. By replacing Y with the mapping cone of f, we can assume that X is a point (and $\dim(Y) \le n{+}1$). Throughout the proof, for any subcomplex $Y' \subseteq Y$, we let $C_*(Y, Y'; \mathbb{F}_p)$, denote the cellular chain complex of (Y, Y'): the complex whose n-th degree term is the free $\mathbb{F}_p$-module with one generator for each n-cell in Y not in Y'.

Fix a Sylow p-subgroup $S \subseteq G$, and let Y_s be the union of the fixed point sets Y^P taken over all nontrivial subgroups $1 \ne P \subseteq S$. Then Y_s is a union of $\mathbb{F}_p$-acyclic subcomplexes such that all intersections are also $\mathbb{F}_p$-acyclic. Hence Y_s is itself $\mathbb{F}_p$-acyclic (seen using Mayer-Vietoris sequences). We thus get an exact sequence

$$0 \to H_{n+1}(Y; \mathbb{F}_p) \to C_{n+1}(Y, Y_s; \mathbb{F}_p) \to C_n(Y, Y_s; \mathbb{F}_p) \to \ldots \to C_0(Y, Y_s; \mathbb{F}_p) \to 0,$$

and each term $C_i(Y, Y_s; \mathbb{F}_p)$ is free as an $\mathbb{F}_p[S]$-module since S acts freely on $Y \smallsetminus Y_s$. Thus, $H_{n+1}(Y; \mathbb{F}_p)$ is stably free as an $\mathbb{F}_p[S]$-module; and hence is projective as an $\mathbb{F}_p[G]$-module (cf. [Rim, Corollary 2.4 & Proposition 4.8]). And by the "Eilenberg swindle", any countably generated projective module M is stably free in the category of countably generated modules: we can write $M \oplus N \cong F$ for some countably generated free module F, and then

$$M \oplus F^\infty \cong M \oplus (N{\oplus}M) \oplus (N{\oplus}M) \oplus \cdots \cong (M{\oplus}N) \oplus (M{\oplus}N) \oplus \cdots \cong F^\infty.$$

Assume now that X and Y are finite complexes, and that $\chi(Y^H) = 1$ for any cyclic subgroup $1 \ne H \subseteq G$ of order prime to p. We want to show that $H_{n+1}(Y; \mathbb{F}_p)$ is $\mathbb{F}_p[G]$-free. In general, by [Se, §16.1], two projective $\mathbb{F}_p[G]$-modules M_1 and M_2 are isomorphic if and only if $[M_1] = [M_2] \in R_{\mathbb{F}_p}(G)$ (the representation ring of *all* finitely generated $\mathbb{F}_p[G]$-modules modulo short exact sequences). By [Se, §18.2], this is the case if and only if M_1 and M_2 have the same modular character, where the modular character of an $\mathbb{F}_p[G]$-module is a complex valued function defined on the set of elements of G of order prime to p. Thus, a projective $\mathbb{F}_p[G]$-module M is free if and only if $M \cong (\mathbb{F}_p[G])^s$ for $s = \mathrm{rk}_{\mathbb{F}_p}(M^G)$, if and only if $M^{|G|} \cong (\mathbb{F}_p[G])^r$ for $r = \mathrm{rk}_{\mathbb{F}_p}(M)$, if and only if M is free as a $\mathbb{F}_p[H]$-module for each cyclic subgroup $H \subseteq G$ of order prime to p. And for any such H, since $\chi(Y^K) = 1$ for all $1 \neq K \subseteq H$ by assumption, a count of the numbers of H-orbits of cells in Y shows that in $R_{\mathbb{F}_p}(H)$,

$$(-1)^n[H_{n+1}(Y;\mathbb{F}_p)] = \sum_{i=0}^{n}(-1)^i[C_i(Y,\mathrm{pt};\mathbb{F}_p)] \equiv 0 \pmod{\langle\text{free}\rangle}. \qquad \square$$

The next lemma provides an analogous result for integral homology.

Lemma A.11. *Let G be any finite group, and let $f : X \to Y$ be a map between countable finite dimensional G-complexes. Set $n = \max\{\dim(X), \dim(Y)\}$. Assume that $\widetilde{H}_i(Z_f, X;\mathbb{Z}) = 0$ for all $i \leq n$, and that $f^P : X^P \to Y^P$ is an $\mathbb{F}_p$-homology equivalence for all primes $p\big||G|$ and all p-subgroups $1 \neq P \subseteq G$. Then $H_{n+1}(Z_f, X;\mathbb{Z})$ is $\mathbb{Z}[G]$-projective, and hence is stably free as a countably generated $\mathbb{Z}[G]$-module. If in addition X and Y are finite complexes, and $\chi(X^H) = \chi(Y^H)$ for all $1 \neq H \subseteq G$, then there is a finite G-complex T such that $T^H = pt$ for all $H \subseteq G$ not of prime power order, and such that for some d, T is $(d{-}1)$-connected and $\widetilde{H}_*(T;\mathbb{Z}) = H_d(T;\mathbb{Z}) \cong H_{n+1}(Z_f, X;\mathbb{Z})$ as $\mathbb{Z}[G]$-modules.*

Proof. By Lemma A.10, $H_{n+1}(Z_f, X;\mathbb{F}_p) = \mathbb{Z}/p \otimes H_{n+1}(Z_f, X)$ is $\mathbb{F}_p[G]$-projective for all primes $p\big||G|$. Also, $H_{n+1}(Z_f, X;\mathbb{Z})$ is $\mathbb{Z}$-free, since $\dim(Z_f) = n{+}1$. In particular, $H_{n+1}(Z_f, X;\mathbb{Z})$ is G-cohomologically trivial; and hence is $\mathbb{Z}[G]$-projective by Rim's theorem [Rim, Theorem 4.11]. And by the "Eilenberg swindle" again, this implies that $H_{n+1}(Z_f, X;\mathbb{Z})$ is $\mathbb{Z}[G]$-stably free in the category of countably generated $\mathbb{Z}[G]$-modules.

Now assume that X and Y are finite complexes. Let C_f denote the mapping cone of $f : X \to Y$, so $H_{n+1}(C_f) \cong H_{n+1}(Z_f, X)$ is $\mathbb{Z}[G]$-projective. Let $C_f^{\mathcal{NP}}$ be the set of elements $x \in C_f$ whose isotropy subgroup G_x does not have prime power order. Then $\chi((C_f^{\mathcal{NP}})^H) = 1$ for all $H \subseteq G$ (since $\chi(X^H) = \chi(Y^H)$ when $H \neq 1$, and all free orbits have been removed). Hence, by [O2, Proposition 5], there is a finite contractible G-complex $Z \supseteq C_f^{\mathcal{NP}}$ such that all isotropy subgroups of $Z \smallsetminus C_f^{\mathcal{NP}}$ have prime power order. By [O2, Lemma 11], applied to the inclusion $Z \subseteq Z \cup_{C_f^{\mathcal{NP}}} C_f$, there exists a finite contractible G-complex $Z' \supseteq C_f$ such that all isotropy subgroups of $Z' \smallsetminus C_f$ have prime power order. If we now set $T = Z'/C_f$, then $T \simeq \Sigma(C_f)$ (nonequivariantly); and so T is $(n{+}1)$-connected and $\widetilde{H}_*(T) = H_{n+2}(T) \cong H_{n+1}(Z_f, X)$.

Since this last argument is rather indirect, we now outline a more direct argument to help explain what is really going on. The details are similar to those used in [O1, §3] to study fixed point sets. For any G-complex X, $X^{\mathcal{NP}} \subseteq X$ denotes the union of fixed point sets of subgroups not of prime power order. A finite G-complex X will be called *simple* if $\chi(X^H) = 1$ for all $H \subseteq G$. A finite G-complex X will be called a *G-resolution* if X is n-dimensional and $(n{-}1)$-connected for some n, and $H_n(X)$ is $\mathbb{Z}[G]$-projective. For any G-resolution X, set $\gamma_G(X) = (-1)^n[H_n(X)] \in \widetilde{K}_0(\mathbb{Z}[G])$ ($n = \dim(X)$). Let $\mathcal{B}_0(G) \subseteq \widetilde{K}_0(\mathbb{Z}[G])$ be the subset of all $\gamma_G(X)$ for G-resolutions X such that $X^{\mathcal{NP}}$ is a point. Using direct geometric constructions, one now shows that $\mathcal{B}_0(G)$ is a subgroup, and that there is a well defined function

$$\Gamma_G : \{\text{finite simple } G\text{-complexes}\} \longrightarrow \widetilde{K}_0(\mathbb{Z}[G])/\mathcal{B}_0(G)$$

which sends X to $\gamma_G(\overline{X})$ for any G-resolution $\overline{X}$ such that $\overline{X}^{\mathcal{NP}} \cong X^{\mathcal{NP}}$. Also, $\Gamma_G(X)$ depends only on the G-homotopy type of X, and Γ_G is additive in the sense that $\Gamma_G(Y/X) = \Gamma_G(Y) - \Gamma_G(X)$ for any pair $X \subseteq Y$. It is now straightforward to show that any such function from finite simple G-complexes to an abelian group is trivial. And when applied to the mapping cone C_f defined above (more precisely, to $C_f^{\mathcal{NP}}$), this shows that $[H_{n+1}(Z_f, X)] \in \mathcal{B}_0(G)$. □

An equivariant thickening theorem

The procedure for constructing manifolds with smooth G-action, starting with a G-vector bundle over a countable finite dimensional G-complex, is based on the following equivariant thickening theorem. As has been seen, it provides a good tool when constructing smooth G-actions on disks and euclidean spaces. In contrast, it cannot be used (at least not directly) to construct smooth actions on closed manifolds.

Theorem A.12. [Pawałowski] *Fix a finite group G, a countable finite dimensional G-complex X, and a G-vector bundle $\xi{\downarrow}X$. Assume that $F = X^G$ is given the structure of a smooth manifold, and that $(\xi|F)^G$ is stably isomorphic to the tangent bundle $\tau(F)$. Then there is a smooth manifold M with smooth G-action, containing X as a G-deformation retract, such that M^G is diffeomorphic to F, and such that $\tau(M)|X$ is stably G-isomorphic to ξ (i.e., $(\tau(M)|X)\oplus(V\times X) \cong \xi\oplus(W\times X)$ for some pair of G-representations V and W). If X is a finite G-complex, then M can be chosen to be compact.*

Proof. See [Pa2, Theorems 2.4 & 3.1] (where the result is stated more precisely). The idea of the proof is the following. After adding a product bundle to ξ, we can assume that $\xi^G{\downarrow}F \cong \tau(F)\oplus(\mathbb{R}^k\times F)$ for some k; and that

$$\dim((\xi_x)^H) > 2\cdot\dim(X^H) + k \quad \text{and} \quad \dim((\xi_x)^H) - \dim((\xi_x)^{>H}) > \dim(X^H)$$

for all $H \subsetneqq G$ and all $x \in X^H$. Here, $(\xi_x)^{>H}$ denotes the union of the fixed point sets of subgroups of G_x strictly containing H.

Choose G-invariant subcomplexes $F = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots X$, such that $X = \cup_{i=0}^N X_i$ (where $N \leq \infty$), and such that each X_i is obtained from X_{i-1} by attaching one orbit of cells $G/H \times D^j$ for some H and j. Manifolds $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ are now constructed such that for each i, $(M_i)^G = F$, $X_i \subseteq M_i$ and $(\partial M_i \smallsetminus \partial F) \subseteq (M_i \smallsetminus X_i)$ are G-deformation retracts, and $\tau(M_i)|X_i\oplus(\mathbb{R}^k\times X_i) \cong \xi|X_i$. To start the procedure, let M_0 be the disk bundle of $(\xi|F)/(\xi^G)$. The induction step is carried out using standard (nonequivariant) embedding theorems, and theorems about destabilizing vector bundles and isomorphisms between them. The manifold $M = \cup_{i=0}^N M_i$ now satisfies the conclusions of the theorem. □

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