THE THEORY OF p-LOCAL GROUPS: A SURVEY

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The motivation for this project comes from the study of the p-local homotopy theory of classifying spaces of finite groups, or more generally of compact Lie groups. By "p-local homotopy theory" of a space we mean the homotopy theory of its p-completion. It turns out that there is a close connection between the p-local homotopy theory of BG and the "p-local structure" of the group G, by which we mean the fusion (conjugacy relations) in a Sylow p-subgroup of G. This connection then suggested to us the construction of certain spaces (classifying spaces of "p-local finite groups" and "p-local compact groups") which have many of the same properties as have p-completed classifying spaces of finite and compact Lie groups.

A brief survey of the Bousfield-Kan p-completion functor will be given in Section 1. For the purpose of this introduction it suffices to say that this is a functor from spaces to spaces, which focuses on the properties of a space which are visible through its mod p homology.

The p-fusion data of a finite group G consists of a Sylow p-subgroup $S \leq G$, together with information on how subgroups of S are related to each other via conjugation in G. This data includes an abundance of information about BG. For example, the well known theorem of Cartan and Eilenberg [CE, Theorem XII.10.1], stating that $H^*(G; \mathbb{F}_p)$ is given by the subring of "stable elements" in $H^*(S; \mathbb{F}_p)$, can be interpreted as saying that mod p cohomology of finite groups is determined by p-fusion.

A much stronger version of the Cartan-Eilenberg theorem is given by the Martino-Priddy conjecture [MP]. The conjecture, recently proved by the third author, states roughly that the classifying spaces of two groups have the same p-local homotopy type if and only if the groups share the same p-fusion data. A more precise formulation of this result, as well as a discussion of some of the ideas which go into its proof, are presented in Section 3.

Unfortunately, we do not know how to describe the homotopy theoretic information associated to the classifying space of a finite group G from the p-fusion alone. For instance, the topological monoid of homotopy equivalences from $BG^\wedge_p$ to itself cannot easily be described using the fusion alone, and the work outlined below in Sections 2–3 was initially motivated by the need to obtain an algebraic description of this space. A partial answer to this problem, phrased in terms of p-fusion, appears in [BL]; but the complete description in [BLO1] requires the use of an additional structure, the "centric linking system" of the finite group.

The introduction of this new concept led to several new results, as described in detail in Section 2. For instance, it gave a different condition for determining when two finite groups have equivalent p-completed classifying spaces: a condition which is less useful than that given by the Martino-Priddy conjecture, but which is much easier to prove (Theorem 2.2). It also made it possible to give a complete algebraic description of the

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topological monoid of self homotopy equivalences of $BG^\wedge_p$ for G finite. In addition, the introduction of centric linking systems for groups paved the way to the more general concept of p-local finite groups.

The p-fusion in a finite group G with a Sylow p-subgroup S can be made into a category, whose objects are the subgroups of S, and whose morphisms are those monomorphisms induced by conjugation in G. This category is called the "fusion category of G at the prime p". Fusion systems can be defined more abstractly; they are categories of subgroups of a given finite p-group whose morphisms are certain monomorphisms between them, and which satisfy certain axioms. The fusion systems we will be interested in satisfy certain extra conditions first formulated by Puig [Pu3],[Pu4], and are known as "saturated fusion systems".

Our first goal is to construct a classifying space for each saturated fusion system. This is motivated in part by representation theory: we refer to Section 5.3 for a brief discussion of the fusion system of a block as defined by Alperin & Broué. But it also provides a new class of spaces, which have many of the same properties of p-completed classifying spaces of finite groups, and which can be characterized in homotopy theoretic terms. To construct these classifying spaces, we define what it means to be a centric linking system associated to a saturated fusion system. Just like the examples arising from finite groups, centric linking systems associated to abstract saturated fusion systems, if they exist, are categories which can be thought of as lifts of the respective fusion systems. The p-completed nerve of a centric linking system is the desired classifying space of the corresponding fusion system. This does, however, lead to the unsatisfactory situation that we do not yet know whether or not there exists a linking system associated to a given fusion system, and whether or not it is unique (see Section 4.8).

A "p-local finite group" is now defined to be a triple consisting of a finite p-group, a saturated fusion system over it and an associated centric linking system. The classifying space of a p-local finite group is defined to be the p-completed nerve of its linking system. When the triple consists of a Sylow p-subgroup of a finite group G, together with its fusion and linking systems, then its classifying space has the homotopy type of $BG^\wedge_p$.

A closely related topic, which provides extra motivation, is the study of p-compact groups. These objects were introduced in the 1980's by Dwyer and Wilkerson [DW3] and extensively studied by them and several other authors. A p-compact group is a loop space X (i.e., $X \simeq \Omega(BX)$ for some pointed "classifying space" BX), such that $H^*(X;\mathbb{F}_p)$ is finite and BX is p-complete. The concept of a p-compact group was designed to be a homotopy theoretic analogue of a classifying space of a compact Lie group. For instance if G is a compact Lie group whose group of components is a p-group, then $BG^\wedge_p$ is the classifying space of a p-compact group. On the other hand if $\pi_0(G)$ is not a p-group then the loop space $\Omega(BG^\wedge_p)$ is not generally $\mathbb{F}_p$-finite. In spite of the fact that there are many common aspects to the homotopy theory of classifying spaces of finite, compact Lie and p-compact groups, the techniques used in the study of p-compact groups fail or become hard to use if the $\mathbb{F}_p$-finiteness condition is dropped.

Our approach suggests a way to fix this problem. The report on "p-local compact groups" given here is incomplete, as the research is still in progress at the time of writing this survey. However, the definition of p-local compact groups is a natural extension of that of p-local finite groups, the main change being the replacement of the finite p-group S in the definition of a p-local finite group by what we call a "discrete p-toral group" (Section 6.1). Once we have defined saturated fusion systems and associated centric linking systems over such groups, the definition of a p-local compact group becomes the obvious generalization of that of p-local finite groups. We are then able to show that the family of spaces obtained as classifying spaces of p-local compact groups contains all p-completed classifying spaces of compact Lie groups and p-compact groups.

The theory presented here suggests a vast ground for further exploration. As one example, we outline some results concerning extensions of p-local finite groups (Section 5.4). Some of the other sections also discuss open problems and topics for further investigation.

This survey is written in order to make it easier for interested readers to learn about the subject. As the theory of p-local groups is still relatively new, the main references are contained in fairly large and quite technical papers. It is our hope that this survey will be a valuable reference for readers to get the general picture before they plunge into more comprehensive articles. The paper is designed respectively, namely, we attempted to give good motivation to all statements, but most of them appear either with a very brief sketch of proof or with no proof at all. For all statements however, a precise reference for where a proof can be found is listed.

Contents

1. The p-completion functor

We start with a brief description of the p-completion functor of Bousfield and Kan [BK], which we denote by $(-)^\wedge_p$. It is a functor from the category of spaces to itself, and comes equipped with a natural transformation $\lambda\colon \mathrm{Id} \longrightarrow (-)^\wedge_p$. A space X is *p-complete* if $\lambda_X\colon X \longrightarrow X^\wedge_p$ is a homotopy equivalence. One property of p-completion (one of the few) which applies to all spaces is that a map $f\colon X \longrightarrow Y$ induces a homotopy equivalence $f^\wedge_p\colon X^\wedge_p \longrightarrow Y^\wedge_p$ if and only if f induces an isomorphism $H^*(X;\mathbb{F}_p) \cong H^*(Y;\mathbb{F}_p)$.

A space X is called *p-good* if $\lambda_X\colon X \longrightarrow X^\wedge_p$ induces an isomorphism $H^*(X^\wedge_p;\mathbb{F}_p) \cong H^*(X;\mathbb{F}_p)$, or equivalently if $X^\wedge_p$ is p-complete. Spaces which are p-bad (i.e., not p-good) remain so permanently: repeated application of the completion functor will never produce a p-complete space. Simply connected spaces, or more generally nilpotent spaces, are p-good. Spaces whose fundamental group is finite are p-good. More generally, any space X for which $\pi_1(X)$ contains a p-perfect subgroup of finite index (i.e., a subgroup of finite index generated by its commutators and p-th powers) is also p-good. All spaces discussed in this article (at least, all spaces for which we want to take the p-completion) are p-good.

For any p-good space X, the map $\lambda_X\colon X \longrightarrow X^\wedge_p$ is a final object among homotopy classes of maps out of X which induce a mod-p homology isomorphism. If X and Y are any two p-good spaces, then their p-completions are homotopy equivalent if and only if there exists some space Z, and maps $X \xrightarrow{f} Z \xleftarrow{g} Y$, such that f and g are both mod-p homology equivalences. We say that X and Y have the same p-local homotopy type, or that they are mod p equivalent, if $X^\wedge_p \simeq Y^\wedge_p$.

Among our main objects of study here are the p-completed classifying spaces of compact Lie groups. For any compact Lie group G, BG is p-good since its fundamental group is finite. Also, $\pi_1(BG^\wedge_p) \cong \pi_0(G)/O^p(\pi_0(G))$, where $O^p(\Gamma)$ is the maximal p-perfect subgroup of a finite group Γ (equivalently, the smallest normal subgroup of p-power index).

While p-completion does not change the mod-p homology of BG when G (or $\pi_0(G)$) is finite, the space $BG^\wedge_p$ is not generally aspherical. In fact, for finite G, $BG^\wedge_p$ is aspherical only if G contains a normal subgroup of p-power index and order prime to p; in all other cases $BG^\wedge_p$ has infinitely many non-trivial homotopy groups. The homotopy theory of spaces of this form has some fascinating aspects. For a survey on some of the classical homotopy theory associated with $BG^\wedge_p$, the reader is referred to [CL].

2. Fusion and linking systems of finite groups

The main idea in this section is to describe how the homotopy theory of $BG^\wedge_p$ is related to certain categories, which we call the fusion and *centric* linking systems of G at the prime p (we emphasize "centric" since there is a notion, presently less useful, of a more general linking system, which will not be mentioned here). Throughout the section, we fix a finite group G and a prime p.

2.1. **Fusion systems of groups.** The *fusion system* $\mathcal{F}_p(G)$ of G is the category whose objects are the p-subgroups of G, and where for any pair of p-subgroups $P, Q \le G$,

$$\mathrm{Mor}_{\mathcal{F}_p(G)}(P,Q) = \mathrm{Hom}_G(P,Q) \overset{\mathrm{def}}{=} \{\alpha \in \mathrm{Hom}(P,Q) \mid \alpha = c_x, \text{ some } x \in G\}.$$

Thus, if we define

$$N_G(P,Q) = \{x \in G \mid xPx^{-1} \le Q\}$$

(the *transporter*), then

$$\mathrm{Mor}_{\mathcal{F}_p(G)}(P,Q) \cong N_G(P,Q)/C_G(P).$$

If S is any Sylow p-subgroup of G, then $\mathcal{F}_S(G) \subseteq \mathcal{F}_p(G)$ will denote the full subcategory whose objects are the subgroups of S. Since every p-subgroup of G is conjugate to a subgroup of any given $S \in \mathrm{Syl}_p(G)$, the inclusion of $\mathcal{F}_S(G)$ in $\mathcal{F}_p(G)$ is an equivalence of categories.

2.2. **Centric linking systems of groups.** For any finite group H, let $O^p(H)$ denote the minimal normal subgroup of p power index, or equivalently the maximal normal p-perfect subgroup of H. A p-subgroup $P \le G$ is *p-centric* if $Z(P) \in \mathrm{Syl}_p(C_G(P))$, or equivalently if $C_G(P) = Z(P) \times O^p(C_G(P))$ and $O^p(C_G(P))$ has order prime to p. For such subgroups we write $O^p(C_G(P)) = C'_G(P)$ for short.

The *centric linking system* $\mathcal{L}^c_p(G)$ of G is the category whose objects are the p-centric subgroups of G, and where

$$\mathrm{Mor}_{\mathcal{L}^c_p(G)}(P,Q) = N_G(P,Q)/C'_G(P)$$

for any pair of objects.

The category $\mathcal{L}_p^c(G)$ is the same as what Puig [Pu1, §VI.1] called the "O-localité" of G (when restricted to p-centric subgroups).

If S is any Sylow p-subgroup of G, then $\mathcal{L}_S^c(G) \subseteq \mathcal{L}_p^c(G)$ will denote the full subcategory whose objects are the subgroups of S which are p-centric in G. As was the case for the corresponding inclusion of fusion systems, this inclusion is always an equivalence of categories.

The following theorem helps to explain the usefulness of centric linking systems when studying the homotopy theory of $BG^\wedge_p$.

Theorem 2.1. *For any finite group G and any prime p, $|\mathcal{L}_p^c(G)|^\wedge_p \simeq BG^\wedge_p$.*

Proof. See [BLO1, Proposition 1.1]. The idea of the proof is to construct a larger category $\widetilde{\mathcal{L}}_p^c(G)$, with the same objects as $\mathcal{L}_p^c(G)$, but where $\mathrm{Mor}_{\widetilde{\mathcal{L}}_p^c(G)}(P,Q) = N_G(P,Q)$ for all P,Q. Let $\mathcal{B}(G)$ be the category with one object o_G, and where $\mathrm{End}_{\mathcal{B}(G)}(o_G) = G$. One then shows that the maps

$$|\mathcal{L}_p^c(G)| \xleftarrow{\ |\pi|\ } |\widetilde{\mathcal{L}}_p^c(G)| \xrightarrow{\ \widetilde{\alpha}_G\ } |\mathcal{B}(G)| \cong BG,$$

are $\mathbb{F}_p$-homology equivalences, where π is the obvious surjective functor, and where $\widetilde{\alpha}_G$ is induced by the functor which is the inclusion on all morphism sets $N_G(P,Q) \subseteq G$. □

2.3. **Equivalences of p-completed classifying spaces.** The Martino-Priddy conjecture states roughly that the homotopy type of $BG^\wedge_p$ depends only on the fusion of G in $S \in \mathrm{Syl}_p(G)$, or equivalently on the fusion system $\mathcal{F}_p(G)$. This will be discussed in more detail in Section 3, and stated precisely in Theorem 3.1.

In this section, we state a weaker result, which says that the homotopy type of $BG^\wedge_p$ depends only on its linking system $\mathcal{L}_p^c(G)$. This does provide a (finite) combinatorial condition for two p-completed classifying spaces to be homotopy equivalent. The condition is, however, more complicated to check than the one stated in the Martino-Priddy conjecture, and hence less satisfactory. The proof of this statement is however much easier than the proof of the Martino-Priddy conjecture.

Theorem 2.2. *For any pair G, G' of finite groups and any prime p, $BG^\wedge_p \simeq BG'^\wedge_p$ if and only if $\mathcal{L}_p^c(G) \simeq \mathcal{L}_p^c(G')$.*

Proof. See [BLO1, Theorem A]. The "if" part of this theorem follows immediately from Theorem 2.1. For the "only if part" a construction of a centric linking system for a space is required. See Section 2.5 for more details. □

2.4. **Maps from classifying spaces of finite p-groups.** One of our goals is to describe in many cases maps between p-completed classifying spaces of finite groups. The simplest case to consider is that where the source is the classifying space of a finite p-group.

If H and K are any two discrete groups then the space of (unpointed) maps from BH to BK is very simple to describe, and the result is classical. Set

$$\mathrm{Rep}(H,K) = \mathrm{Hom}(H,K)/\,\mathrm{Inn}(K),$$

the set of conjugacy classes of homomorphisms ("representations") from H to K. Then the natural map

$$B\colon\ \mathrm{Rep}(H,K) \longrightarrow [BH,BK]$$

is a bijection. Also, for each $\rho\colon H \longrightarrow K$, the homomorphism

$$C_K(\rho) \times H \xrightarrow{\ \text{incl}\cdot\rho\ } K,$$

where $C_K(\rho) \overset{\text{def}}{=} C_K(\text{Im}(\rho))$, induces a map of spaces $BC_K(\rho) \times BH \longrightarrow BK$, whose adjoint

$$BC_K(\rho) \xrightarrow{\ \cong\ } \text{Map}(BH, BK)_{B\rho}$$

is a homotopy equivalence. The simplest way to see this result is to first show that $[BH, BK]_* \cong \text{Hom}(H, K)$ (i.e., *pointed* homotopy classes of pointed maps) and that each component of the pointed mapping space $\text{Map}_*(BH, BK)$ is contractible; and then examine the fibration

$$\text{Map}_*(BH, BK) \longrightarrow \text{Map}(BH, BK) \longrightarrow BK.$$

The following "folk theorem" describes one situation in which this result can be generalized from classifying spaces to p-completed classifying spaces.

Theorem 2.3. *For any finite p-group P, and any finite group G, the p-completion map $BG \longrightarrow BG^\wedge_p$ induces a (weak) homotopy equivalence*

$$\text{Map}(BP, BG)^\wedge_p \xrightarrow{\ \simeq\ } \text{Map}(BP, BG^\wedge_p).$$

In particular, the map $(\rho \mapsto B\rho^\wedge_p)$ defines a bijection

$$\text{Rep}(P, G) \ \xrightarrow[\cong]{\ B\ }\ [BP, BG^\wedge_p].$$

For each $\rho\colon P \longrightarrow G$, the induced product map $C_G(\rho) \times P \longrightarrow G$ induces (after taking adjoints) a homotopy equivalence

$$BC_G(\rho)^\wedge_p \longrightarrow \text{Map}(BP, BG^\wedge_p)_{B\rho}\,.$$

In particular, the mapping space $\text{Map}(BP, BG^\wedge_p)$ *is p-complete.*

Proof. See [BL, Proposition 2.1], but this theorem was known to the experts well before we wrote down a proof. □

This result will be generalized in Section 4 (Theorem 4.2).

2.5. **Fusion and linking systems for spaces.** The "only if" part of Theorem 2.2 follows from another construction: a functor from spaces to categories which in particular sends $BG^\wedge_p$ to $\mathcal{L}^c_S(G)$. This is described as follows.

Fix a space X, a finite p-group S, and a map $f\colon BS \longrightarrow X$. We define $\mathcal{F}_{S,f}(X)$ to be the category whose objects are the subgroups of S, and where for $P, Q \le S$,

$$\text{Mor}_{\mathcal{F}_{S,f}(X)}(P, Q) = \bigl\{\alpha \in \text{Hom}(P, Q) \bigm| f|_{BP} \simeq f|_{BQ} \circ B\alpha\bigr\}.$$

This is clearly a fusion system over S (though not necessarily saturated!), and can be thought of as the fusion system of the space X with respect to the pair (S, f).

We next define the analogous linking system for X with respect to (S, f). Let $\mathcal{L}_{S,f}(X)$ be the category whose objects are the subgroups of S, and where for $P, Q \le S$,

$$\text{Mor}_{\mathcal{L}_{S,f}(X)}(P, Q) = \bigl\{(\alpha, [\phi]) \bigm| \alpha \in \text{Hom}(P, Q),\ [\phi] \text{ a homotopy class of paths in } \text{Map}(BP, X) \text{ from } f|_{BP} \text{ to } f|_{BQ} \circ B\alpha\bigr\}.$$

A morphism of $\mathcal{L}_{S,f}(X)$ from P to Q thus consists of a morphism $\alpha \in \text{Hom}_{\mathcal{F}_{S,f}(X)}(P, Q)$, together with a homotopy class of homotopies from $f|_{BP}$ to $f|_{BQ} \circ B\alpha$.

There is an obvious "forgetful" functor

$$\pi_{S,f}(X)\colon \mathcal{L}_{S,f}(X) \longrightarrow \mathcal{F}_{S,f}(X)$$

which is the identity on objects and sends a morphism $(\alpha,[\phi])$ to α. Note that the set of morphisms in $\mathcal{L}_{S,f}(X)$ sitting over any given $\alpha \in \mathrm{Hom}_{\mathcal{F}_{S,f}(X)}(P,Q)$ is in bijective correspondence with $\pi_1(\mathrm{Map}(BP,X)_{f|_{BP}})$.

We also want a centric linking system in this situation. Let $\mathcal{L}^c_{S,f}(X)$ be the full subcategory of $\mathcal{L}_{S,f}(X)$ whose objects are the subgroups $P \le S$ such that $C_S(P') = Z(P')$ for all P' isomorphic to P in $\mathcal{F}_{S,f}(X)$.

The "only if" part of Theorem 2.2 now follows easily from the following:

Theorem 2.4. *For any finite group G and any prime p, fix $S \in \mathrm{Syl}_p(G)$, and let $\theta\colon BS \longrightarrow BG^\wedge_p$ be the inclusion. Then there are equivalences of categories*

$$\mathcal{F}_{S,\theta}(BG^\wedge_p) \cong \mathcal{F}_S(G) \qquad \text{and} \qquad \mathcal{L}^c_{S,\theta}(BG^\wedge_p) \cong \mathcal{L}^c_S(G).$$

Proof. See [BLO1, Proposition 2.7]. The first isomorphism says that for any $P,Q \le S$ and any $\alpha \in \mathrm{Hom}(P,Q)$, $\alpha \in \mathrm{Hom}_G(P,Q)$ (i.e., α is induced by conjugation in G) if and only if the composite $BP \xrightarrow{B\alpha} BQ \xrightarrow{\text{incl}} BG$ is homotopic to the inclusion $BP \subseteq BG$, and this follows from Theorem 2.3.

The isomorphism between the two linking categories is slightly more delicate. Note first that by definition (and the isomorphism of fusion systems), both categories have as objects the subgroups of S which are p-centric in G. For any $\alpha \in \mathrm{Hom}_G(P,Q)$ (any $P,Q \le S$), the number of morphisms in $\mathcal{F}_S(G)$ which cover α is equal to $|Z(P)|$, and the number of morphisms in $\mathcal{F}_{S,\theta}(BG^\wedge_p)$ which cover α is equal to the order of $\pi_1(\mathrm{Map}(BP,BG^\wedge_p)_{\text{incl}}) \cong Z(P)$ (see Theorem 2.3). This suggests that there should be a natural bijection between these morphism sets, and such a bijection is constructed by mapping the transporter $N_G(P,Q)$ surjectively to both sets and showing that one has the same identifications in the two cases. □

Theorem 2.4 suggests a way to extend the centric linking system of a finite group G to all p-subgroups. Let $\mathcal{L}_p(G)$ be the category whose objects are the p-subgroups of G, and where $\mathrm{Mor}_{\mathcal{L}_p(G)}(P,Q) = N_G(P,Q)/O^p(C_G(P))$. This is easily seen to be well defined on morphisms. As usual, for $S \in \mathrm{Syl}_p(G)$, $\mathcal{L}_S(G) \subseteq \mathcal{L}_p(G)$ denotes the full subcategory whose objects are the subgroups of S. Then the argument just sketched also shows that $\mathcal{L}_S(G) \cong \mathcal{L}_{S,\theta}(BG^\wedge_p)$.

2.6. **Isotypical equivalences of fusion and linking categories.** Let $\mathcal{C}$ be any of the categories $\mathcal{F}_p(G)$, $\mathcal{L}^c_p(G)$, $\mathcal{F}_{S,f}(X)$, or $\mathcal{L}_{S,f}(X)$, or any subcategory of one of these categories. Let $\mathcal{C} \xrightarrow{\epsilon} \mathsf{Gr}$ be the forgetful functor to the category of groups. A self equivalence $\varphi\colon \mathcal{C} \longrightarrow \mathcal{C}$ is called *isotypical* if there is a natural isomorphism of functors from $\epsilon \circ \varphi$ to ϵ.

When $\mathcal{C}$ is one of the above categories, we let $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{Aut}_{\mathrm{typ}}(\mathcal{C})$ denote the monoids of self equivalences and isotypical self equivalences, respectively of $\mathcal{C}$. More generally, we let $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{C})$ be the strict monoidal category whose objects are the isotypical self equivalences of $\mathcal{C}$, and whose morphisms are the natural isomorphisms of functors. We can then define $\mathrm{Out}_{\mathrm{typ}}(\mathcal{C})$ to be the group of automorphisms modulo natural isomorphisms; i.e., the group of connected components of the nerve $|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{C})|$.

2.7. **Self homotopy equivalences of $BG^\wedge_p$.** Using Theorems 2.1 and 2.4, one can also describe the monoid of self homotopy equivalences of $BG^\wedge_p$ in terms of automorphisms of the category $\mathcal{L}^c_p(G)$. For any space X, we let $\mathrm{Aut}(X)$ denote the topological monoid of all self homotopy equivalences of X; and by analogy with the notation for self equivalences of categories, let $\mathrm{Out}(X) = \pi_0(\mathrm{Aut}(X))$ denote the group of homotopy classes of self equivalences.

For any finite group G, $O_{p'}(G) \lhd G$ denotes the largest normal subgroup of G of order prime to p.

Theorem 2.5. *For any finite group G and any prime p, the connected components of the space $\mathrm{Aut}(BG^\wedge_p)$ are all aspherical, and there are isomorphisms*

$$\mathrm{Out}(BG^\wedge_p) \cong \mathrm{Out_{typ}}(\mathcal{L}^c_p(G)) \qquad \text{and} \qquad \pi_1(\mathrm{Aut}(BG^\wedge_p)) \cong Z(G/O_{p'}(G))^\wedge_p.$$

Moreover, there is a homotopy equivalence

$$B\,\mathrm{Aut}(BG^\wedge_p) \cong B|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}^c_p(G))|.$$

Proof. See [BLO1, Theorems B & C] and [BL, Theorem 1.1]. □

3. The Martino-Priddy conjecture

Originally, the results in Section 2 were motivated partly as a means of studying $\mathrm{Aut}(BG^\wedge_p)$ (Theorem 2.5), but also partly as a means of finding algebraic or combinatorial conditions for two p-completed classifying spaces to be homotopy equivalent (Theorem 2.2). What we really want is to describe both of these in terms of the fusion systems, or equivalently in terms of fusion among subgroups of a Sylow subgroup.

3.1. **Fusion preserving isomorphisms.** Fix a pair of finite groups G and G', a prime p, and Sylow subgroups $S \in \mathrm{Syl}_p(G)$ and $S' \in \mathrm{Syl}_p(G')$. An isomorphism $S \xrightarrow{\varphi} S'$ is called *fusion preserving* if for all $P, Q \le S$ and all $P \xrightarrow[\cong]{\alpha} Q$, α is induced by conjugation in G if and only if $\varphi(P) \xrightarrow[\cong]{\varphi\alpha\varphi^{-1}} \varphi(Q)$ is induced by conjugation in G'.

Clearly, a fusion preserving isomorphism $S \longrightarrow S'$ in the above situation induces an isomorphism of categories $\mathcal{F}_S(G) \cong \mathcal{F}_{S'}(G')$, and hence an equivalence of the larger categories $\mathcal{F}_p(G) \simeq \mathcal{F}_p(G')$. Conversely, given an *isotypical* equivalence between the fusion systems of G and G', it is not hard to construct a fusion preserving isomorphism between their Sylow p-subgroups [BLO1, Lemma 5.1].

A classical result in group cohomology is a theorem by Cartan and Eilenberg (see [CE, Theorem XII.10.1]), which states roughly that for any finite group G, the cohomology ring $H^*(BG; \mathbb{F}_p)$ is determined by $S \in \mathrm{Syl}_p(G)$ and fusion in S (see Section 4.6). The key point of the Martino-Priddy conjecture is the much stronger statement that the homotopy type of the p-completed classifying space $BG^\wedge_p$ is determined by p-fusion.

Theorem 3.1 (Martino-Priddy conjecture)**.** *For any pair G, G' of finite groups and any prime p, the following three conditions are equivalent:*

(a) $BG^\wedge_p \simeq BG'^\wedge_p$.

(b) *There is a fusion preserving isomorphism* $S \xrightarrow[\cong]{\varphi} S'$ *between Sylow* p*-subgroups of* G *and* G'.

(c) *There is an isotypical equivalence* $\mathcal{F}_p(G) \simeq \mathcal{F}_p(G')$.

The implications (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c) were proved by Martino and Priddy [MP]. In particular, (b) and (c) are equivalent by the above remarks. The remaining implication was recently proven by the third author ([O1] and [O2]), using the classification theorem for finite simple groups.

3.2. **Obstruction theory.** The first step in analyzing the Martino-Priddy conjecture is to reduce it to a problem of higher derived functors of inverse limits over the orbit category of G. For any finite group G and any prime p, the *orbit category* $\mathcal{O}_p(G)$ of G is the category whose objects are the p-subgroups of G, and where for all p-subgroups $P, Q \le G$,

$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q) = \mathrm{Map}_G(G/P, G/Q) \cong Q\backslash N_G(P,Q).$$

Let $\mathcal{Z}_G\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ be the functor

$$\mathcal{Z}_G(P) = \begin{cases} Z(P) & \text{if } P \text{ is } p\text{-centric} \\ 0 & \text{otherwise.} \end{cases}$$

By [BLO1, Proposition 6.1], the obstruction (for $S \in \mathrm{Syl}_p(G)$ and $S' \in \mathrm{Syl}_p(G')$) to lifting a fusion preserving isomorphism

$$S \xrightarrow[\cong]{\varphi} S'$$

to an equivalence

$$\mathcal{L}_p^c(G) \xrightarrow{\simeq} \mathcal{L}_p^c(G')$$

lies in the group

$$\varprojlim_{\mathcal{O}_p(G)}{}^2(\mathcal{Z}_G).$$

This can be seen directly by choosing maps

$$\mathrm{Mor}_{\mathcal{L}_p^c(G)}(P,Q) \longrightarrow \mathrm{Mor}_{\mathcal{L}_p^c(G')}(\varphi(P),\varphi(Q))$$

(for all $P, Q \le S$), and then examining the 2-cocycle in

$$C^2(\mathcal{O}_p(G);\mathcal{Z}_G) \overset{\text{def}}{=} \prod_{P\to Q\to R} Z(P)$$

which describes the failure of the images of commutative triangles in $\mathcal{L}_p^c(G)$ to commute in $\mathcal{L}_p^c(G')$.

The Martino-Priddy conjecture was proven by showing:

Theorem 3.2. *For any finite group* G *and any prime* p, $\varprojlim_{\mathcal{O}_p(G)}{}^i(\mathcal{Z}_G) = 0$ *if* $i \ge 2$, *or if* p *is odd and* $i \ge 1$.

It is the proof of this result which depends on the classification of finite simple groups.

3.3. **Reduction to simple groups.** To reduce the proof of Theorem 3.2 to a problem about simple groups, consider a maximal normal series of G:

$$1 = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_n = G.$$

Then for each i, H_i/H_{i-1} is a product of copies of some fixed simple group L_i (see [Go, Theorem 2.1.5]). Define subfunctors $\mathcal{Z}_G^i \subseteq \mathcal{Z}_G$ by setting $\mathcal{Z}_G^i(P) = Z(P) \cap H_i$ if P is p-centric, and $\mathcal{Z}_G^i(P) = 0$ otherwise. From the long exact sequences

$$\cdots \longrightarrow \varprojlim_{\mathcal{O}_p(G)}{}^n(\mathcal{Z}_G^{i-1}) \longrightarrow \varprojlim_{\mathcal{O}_p(G)}{}^n(\mathcal{Z}_G^{i}) \longrightarrow \varprojlim_{\mathcal{O}_p(G)}{}^n(\mathcal{Z}_G^{i}/\mathcal{Z}_G^{i-1}) \longrightarrow \varprojlim_{\mathcal{O}_p(G)}{}^{n-1}(\mathcal{Z}_G^{i-1}) \longrightarrow \cdots,$$

we see that $\varprojlim^n(\mathcal{Z}_G) = 0$ (for any given n) if $\varprojlim_{\mathcal{O}_p(G)}{}^n(\mathcal{Z}_G^{i}/\mathcal{Z}_G^{i-1}) = 0$ for all i.

Higher limits of the functors $\mathcal{Z}_G^{i}/\mathcal{Z}_G^{i-1}$ are described in terms of higher limits of certain other functors $\mathcal{Y}_L^\Gamma$, defined when L is quasisimple (i.e., L is perfect and $L/Z(L)$ is simple) and $\mathrm{Inn}(L) \le \Gamma \le \mathrm{Aut}(L)$. For any such L and Γ, let $c\colon L \longrightarrow \Gamma$ denote the homomorphism which sends $g \in L$ to conjugation by g, and define

$$\mathcal{Y}_L^\Gamma \colon \mathcal{O}_p(\Gamma)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

by setting

$$\mathcal{Y}_\Gamma^L(P) = \begin{cases} (c^{-1}P)^P/Z(L)^P & \text{if } P \cap \mathrm{Inn}(L) \text{ is } p\text{-centric in } \mathrm{Inn}(L) \\ 0 & \text{otherwise}. \end{cases}$$

For example, when L is simple and $\Gamma = \mathrm{Inn}(L)$, then $\mathcal{Y}_L^\Gamma = \mathcal{Z}_L$. In terms of these functors, we get

Proposition 3.3 ([O2, Theorem 3.3]). *Assume $H_{i-1} \lhd H_i \lhd G$ are normal subgroups, where H_i/H_{i-1} is a minimal normal subgroup of G/H_{i-1}, and define $\mathcal{Z}_G^{i-1} \subseteq \mathcal{Z}_G^i$ as above. If H_i/H_{i-1} is abelian, or if there is a p-subgroup $Q \le G$ such that $[Q, H_i] \le H_{i-1}$ and $C_{H_i}(Q) \le H_{i-1}$, then the quotient functor $\mathcal{Z}_G^{i}/\mathcal{Z}_G^{i-1}$ is acyclic (its higher limits vanish in degrees ≥ 1). Otherwise, there are a quasisimple group L such that H_i/H_{i-1} is a product of copies of the simple group $L/Z(L)$ and a subgroup $\Gamma \le \mathrm{Aut}(L)$ containing $\mathrm{Inn}(L)$, together with homomorphisms*

$$\varprojlim_{\mathcal{O}_p(\Gamma)}{}^n(\mathcal{Y}_L^\Gamma) \longrightarrow \varprojlim_{\mathcal{O}_p(G)}{}^n(\mathcal{Z}_G^{i}/\mathcal{Z}_G^{i-1})$$

which are onto when $n = 1$ and isomorphisms when $n \ge 2$.

Proposition 3.3 describes how the vanishing of higher limits of $\mathcal{Z}_G$ for arbitrary finite groups G is reduced to a question involving finite simple groups. More precisely, it involves the vanishing of higher limits of the functors $\mathcal{Y}_{\widetilde{L}}^\Gamma$, when $\widetilde{L}$ is a central extension of a simple group L and Γ is an extension of L by outer automorphisms. In fact, when p is odd, a more direct approach was found which does not involve these functors $\mathcal{Y}_{\widetilde{L}}^\Gamma$, but they do still have to be studied when $p = 2$.

3.4. **The odd primary case.** When p is odd, then for any finite p-group P, we define $\mathfrak{X}(P) \le P$ to be the largest subgroup for which there is a normal series

$$1 = Q_0 \lhd Q_1 \lhd \cdots \lhd Q_n = \mathfrak{X}(P)$$

such that $Q_i \lhd P$ for all i, and such that

$$[\Omega_1(C_P(Q_{i-1})), Q_i; p{-}1] = 1 \qquad \forall\, i = 1, \ldots, n.$$

Here, for any finite p-group P, $\Omega_1(P) = \langle g \in P \,|\, g^p = 1\rangle$. Also, for subgroups $H, K \le G$, $[H, K; n]$ denotes the n-fold commutator $[\cdots[[H, K], K]\cdots, K]$.

In the following proposition, when G is a group and $S \in \mathrm{Syl}_p(G)$, we say that a subgroup $P \le S$ is *weakly* $\mathrm{Aut}(G)$*-closed* in S if there is no other subgroup $P \ne P' \le S$ which lies in the $\mathrm{Aut}(G)$-orbit of P. Also, $J_e(P)$ denotes the Thompson subgroup: the subgroup generated by all (maximal) elementary abelian p-subgroups of P of maximal rank.

Proposition 3.4 ([O1, Propositions 4.1 & 3.7])**.** *For any odd prime p and any finite group G, $\mathcal{Z}_G$ is acyclic if for each nonabelian simple group L which occurs in the decomposition series of G, and any $S \in \mathrm{Syl}_p(L)$, there is a subgroup $Q \le \mathfrak{X}(S)$ which is centric and weakly* $\mathrm{Aut}(L)$*-closed in S. In particular, this always holds if $J_e(S) \le \mathfrak{X}(S)$.*

In fact, [O1, Proposition 4.1] is slightly stronger, in that $\mathfrak{X}(S)$ in the above statement is replaced by a larger subgroup (but depending also on L). When p is odd, L is simple, and $S \in \mathrm{Syl}_p(L)$, then in almost all cases, either $\mathfrak{X}(S) = S$, or S contains a unique elementary abelian subgroup of maximal rank (and clearly $J_e(S) \le \mathfrak{X}(S)$ when either of these happens). The only exceptions to this (i.e., the only cases where $\mathfrak{X}(S) \lneqq S$ and there is more than one elementary abelian subgroup of maximal rank) occur when $p = 3$ and $L \cong PSU_n(3^k)$; in which case the more restrictive hypothesis of Proposition 3.4 holds. In fact, no examples are known of a finite p-group P (for p odd) for which $\mathfrak{X}(P)$ does not contain $J_e(P)$.

3.5. **Obstruction groups at the prime 2.** Since $\varprojlim^1(\mathcal{Z}_G)$ can be nonzero when $p = 2$, it is not surprising that this case is harder. We describe here some of the techniques used to prove, for L simple, that the higher limits of $\mathcal{Y}^{\Gamma}_{\widetilde{L}}$ vanish in degrees ≥ 2 whenever $\widetilde{L}$ is perfect, $\widetilde{L}/Z(\widetilde{L}) \cong L$, and $\Gamma \le \mathrm{Aut}(\widetilde{L})$ contains $\mathrm{Inn}(\widetilde{L})$. To simplify the following discussion, we restrict attention to the case $\widetilde{L} = L$ and $\Gamma = \mathrm{Inn}(L)$; i.e., the case where $\mathcal{Y}^{\Gamma}_{\widetilde{L}} = \mathcal{Z}_L$.

In most cases, one first filters the functor $\mathcal{Z}_L$ by subfunctors

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1} \subseteq F_k = \mathcal{Z}_L,$$

such that for each i there is a p-centric subgroup $P_i \le L$ such that

$$(F_i/F_{i-1})(P) \cong \begin{cases} Z(P) & \text{if } P \text{ conjugate } P_i \\ 0 & \text{otherwise.} \end{cases}$$

We say that "P_i contributes to $\varprojlim^k(\mathcal{Z}_L)$" if

$$\varprojlim^k(F_i/F_{i-1}) \ne 0.$$

Of course, via the long exact sequences which connect the higher limit of F_i, F_{i-1}, and F_i/F_{i-1}, the contribution of one subgroup to $\varprojlim^k(\mathcal{Z}_L)$ can "cancel" the contribution of another subgroup to $\varprojlim^{k\pm1}(\mathcal{Z}_L)$.

For simple groups L (and $p = 2$), there are some cases where a 2-subgroup contributes to $\varprojlim^2(\mathcal{Z}_L)$. For example, this occurs when L is a projective special linear group $PSL_n(2)$ for $n \ge 4$, an alternating group A_n for $n \equiv 0, 1 \pmod 8$, the Mathieu group M_{24}, or the Held group He. In all such cases, this contribution to $\varprojlim^2(\mathcal{Z}_L)$ is cancelled by another subgroup contributing to $\varprojlim^1(\mathcal{Z}_L)$. There do not seem to be any cases in which a 2-subgroup contributes to $\varprojlim^i(\mathcal{Z}_L)$ for $i \ge 3$.

In general, there are graded abelian groups $\Lambda^*(\Gamma; M)$, defined for each finite group Γ and each $\mathbb{Z}_{(p)}[\Gamma]$-module M, such that

$$\varprojlim{}^*(F_i/F_{i-1}) \cong \Lambda^*(N_L(P)/P; Z(P))$$

[JMO, Lemma 5.4]. Thus P contributes to $\varprojlim{}^i(\mathcal{Z}_L)$ if and only if $\Lambda^i(N_L(P)/P; Z(P))$ is nonvanishing. Some of the very nice properties of these functors include:

Proposition 3.5. (a) *If $p \nmid |\Gamma|$, then* $\Lambda^*(\Gamma; M) = \begin{cases} M^\Gamma & \text{if } * = 0 \\ 0 & \text{if } * > 0. \end{cases}$

(b) $\Lambda^*(\Gamma; M) = 0$ *if* $p \big| |\operatorname{Ker}[\Gamma \longrightarrow \operatorname{Aut}(M)]|$

(c) $\Lambda^*(\Gamma; M) = 0$ *if* $O_p(\Gamma) \neq 1$

(d) $\Lambda^*(\Gamma; M) \cong \widetilde{H}^{*-1}_\Gamma(\operatorname{St}(\Gamma); M)$ *(where the Steinberg complex* $\operatorname{St}(\Gamma)$ *is the nerve of the poset of nontrivial p-subgroups of Γ)*

(e) *If $|M| < \infty$ and $\Lambda^n(\Gamma; M) \neq 0$, then there are p-subgroups*

$$1 = P_0 \lneqq P_1 \lneqq \cdots \lneqq P_n \leq \Gamma$$

such that $P_i \lhd P_n$ and $P_i = O_p\big(N(P_1) \cap \cdots \cap N(P_i)\big)$ for all i, such that $P_n \in \operatorname{Syl}_p(N(P_1) \cap \cdots \cap N(P_{n-1}))$*; and such that $\mathbb{F}_p[P_n] \subseteq M$ (as a P_n-module). In particular, if M is finite, then* $\operatorname{rk}_p(M) \geq p^n$.

Proof. See [JMO, Proposition 6.1(i,ii)] for points (a)–(c), [Gr] for point (d), and [O2, Proposition 4.4] for point (e). □

As one simple example, let $V \cong \mathbb{F}_2{}^2$ be the faithful 2-dimensional representation of Σ_3. One can show that

$$\Lambda^2(\Sigma_3 \times \Sigma_3; V^{\otimes 2}) \cong \mathbb{Z}/2,$$

and hence that $P \leq L$ contributes to $\varprojlim{}^2(\mathcal{Z}_L)$ if $N(P)/P \cong \Sigma_3 \times \Sigma_3$ and $Z(P) \cong V^{\otimes 2}$ as a module over $N(P)/P$. This situation does occur in some simple groups, such as $PSL_n(2)$ for $n \geq 4$, and the sporadic groups M_{24} and He. On the other hand, the simplest example where $\Lambda^3 \neq 0$ is the case

$$\Lambda^3(\Sigma_3 \times \Sigma_3 \times \Sigma_3; V^{\otimes 3}) \neq 0,$$

and this does not seem to occur as a pair $(N(P)/P, Z(P))$ in any simple group.

4. Abstract finite fusion and linking systems

The results of the preceding section suggest that it may be possible to formulate a more general context in which to study the homotopy theory of classifying spaces. In other words, we would like to define an algebraic object for which the notion of a classifying space makes sense, and in which the p-local homotopy theory of that classifying space is intrinsic to the algebraic structure and vice versa.

The existence of these algebraic objects was predicted by Dave Benson in the mid-1990's. In the introduction to his paper [Be1], where he relates the Dwyer-Wilkerson space $BDI(4)$ to what he calls the "classifying spaces" of certain "nonexistent finite simple groups" (see Section 5.2), he writes: "This prompts the speculation that there should exist a theory of 'p-local groups' in which one only gives a Sylow p-subgroup and a fusion pattern. The fusion pattern should obey a set of axioms which are strong enough to be able to build a p-completed classifying space.". Later, in some unpublished and privately distributed notes, Benson made this more precise by formulating some of the axioms of what we now call centric linking systems, and predicting that their p-completed realizations should function as classifying spaces of the fusion patterns.

The concept of a fusion system over a finite p-group S was in fact defined in the 1990's by Lluís Puig [Pu3] with an entirely different purpose in mind. Fusion systems of finite groups are particular cases of these more general objects. The version presented here is a modification of his definition, but is completely equivalent to it.

In this section, we first present the definition of abstract (saturated) fusion systems, and then define what it means to be a centric linking system associated to an abstract fusion system. These concepts then lead to the definition of a p-local finite group and its classifying space.

4.1. **Fusion systems.** A *fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following conditions:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P,Q \leq S$.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

The first requirement in the definition is intuitively obvious. It is less clear that the second requirement has to be stated as an axiom. However, it clearly holds when the fusion system is that of a finite group, and it is essential for the theory. Puig calls fusion systems satisfying (b) "divisible" fusion systems.

The next definitions may appear quite mysterious and are quite hard to motivate. The fusion system of a finite group turns out to satisfy an extra set of conditions, which basically makes the theory work. Puig has managed to distil precisely the necessary features in his definition of a *saturated fusion system.* Before we can explain what this means, we must distinguish certain collection of objects in $\mathcal{F}$. By analogy with groups, two objects P and Q which are isomorphic in $\mathcal{F}$ are said to be $\mathcal{F}$-conjugate.

A subgroup $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for every $P' \leq S$ which is $\mathcal{F}$-conjugate to P, and is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.

A fusion system $\mathcal{F}$ is *saturated* if the following two conditions hold:

(I) For each fully normalized subgroup P in $\mathcal{F}$, P is fully centralized, and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that φP is fully centralized, and if we set
$$N_\varphi = \{g \in N_S(P) \,|\, \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$
then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

4.2. **Centric linking systems.** Before we can define a centric linking system associated to a given saturated fusion system, we need to explain what it means to be centric in this context.

Let $\mathcal{F}$ be any fusion system over a finite p-group S. A subgroup $P \leq S$ is $\mathcal{F}$-*centric* if $C_S(P') = Z(P')$ whenever P' is $\mathcal{F}$-conjugate to P. We let $\mathcal{F}^c$ denote the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups.

Let $\mathcal{F}$ be a fusion system over the finite p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c,$$

and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \le S$, which satisfy the following conditions.

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \le \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \le S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & Q \\ \big\downarrow{\scriptstyle \delta_P(g)} & & \big\downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{\ f\ } & Q\,. \end{array}$$

Condition (A) means that $\mathcal{F}^c$ is a quotient category of $\mathcal{L}$, which is obtained by dividing out a free action of the centers of source objects. Conditions (B) and (C) ensure compatibility of the different ingredients with each other.

4.3. **p-local finite groups.** A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over the finite p-group S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|^\wedge_p$.

The first thing one wants to make sure of is that genuine finite groups give rise to p-local finite groups, which is indeed the case. For any finite group G and $S \in \mathrm{Syl}_p(G)$ the fusion system $\mathcal{F}_S(G)$ is saturated and $\mathcal{L}^c_S(G)$ is an associated centric linking system. Thus $(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$ is a p-local finite group whose classifying space $|\mathcal{L}^c_S(G)|^\wedge_p$ is homotopy equivalent to $BG^\wedge_p$. It is interesting to point out that another consequence of the proof of the Martino-Priddy conjecture is that the natural centric linking system of a finite group G is (up to equivalence) the only one associated to the fusion system of G (see Section 4.8).

In the examples coming from finite groups it is clear that the nerve of the centric linking system is p-good, so that its p-completion is p-complete. It is one crucial ingredient among what makes it possible to reconstruct both the fusion system and the centric linking system from the classifying space of the corresponding p-local finite group. This is also the case for p-local finite groups; and in fact, we can also describe explicitly the fundamental group of the classifying space.

For any saturated fusion system $\mathcal{F}$ over a finite p-group S, define

$$O^p_{\mathcal{F}}(S) = \bigl\langle g^{-1}\alpha(g) \bigm| g \in P \le S,\ \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \bigr\rangle \lhd S.$$

This subgroup $O^p_{\mathcal{F}}(S)$ is the *hyperfocal subgroup* of $\mathcal{F}$ as defined by Puig [Pu4]. If $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system of a finite group G with respect to $S \in \mathrm{Syl}_p(G)$, then the hyperfocal subgroup theorem [Pu2, §1.1] says that $O^p_{\mathcal{F}}(S) = S \cap O^p(G)$ (where $O^p(G)$ is the smallest normal subgroup of p-power index).

Theorem 4.1. *Let $(S,\mathcal{F},\mathcal{L})$ be any p-local finite group at the prime p. Then $|\mathcal{L}|$ is p-good. Also, the composite*

$$S \xrightarrow{\pi_1(|\theta_S|)} \pi_1(|\mathcal{L}|) \longrightarrow \pi_1(|\mathcal{L}|^\wedge_p),$$

induced by the inclusion $\mathcal{B}(S) \xrightarrow{\theta_S} \mathcal{L}$, is surjective, and induces an isomorphism

$$\pi_1(|\mathcal{L}|^\wedge_p) \cong S/O^p_{\mathcal{F}}(S).$$

Proof. See [BLO2, Proposition 1.11] and [BCGLO, §1]. □

4.4. **Centralizers and mapping spaces.** Our next task is to study mapping spaces from the classifying space of a finite p-group to that of a p-local finite group. The result is a generalization of Theorem 2.3, which we now describe. If G is a finite group and Q a finite p-group, then the set of components $[BQ, BG^\wedge_p]$ is described in terms of homomorphisms from Q to G modulo conjugation in G. This motivates us to define, for any p-local finite group $(S,\mathcal{F},\mathcal{L})$ and any finite p-group Q,

$$\mathrm{Rep}(Q,\mathcal{F}) = \mathrm{Hom}(Q,S)/{\sim},$$

where $\sim$ is the equivalence relation defined by setting $\rho \sim \rho'$ if there is some $\chi \in \mathrm{Iso}_{\mathcal{F}}(\rho(Q),\rho'(Q))$ such that $\rho' = \chi\circ\rho$.

Components of mapping spaces $\mathrm{Map}(BQ, BG^\wedge_p)$ are described in terms of p-completed classifying spaces of centralizers. Thus before we can state the analogous result for p-local finite groups, we need to define what is meant by the centralizer fusion and linking systems. The original definition of the centralizer fusion system is due to Puig.

Fix a p-local finite group $(S,\mathcal{F},\mathcal{L})$ and a subgroup $Q \leq S$ which is fully centralized in $\mathcal{F}$. Define $C_{\mathcal{F}}(Q)$ to be the category whose objects are the subgroups of $C_S(Q)$, and where

$$\mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P,P') = \bigl\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,P') | \exists\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,P'Q), \bar\varphi|_P = \varphi,\ \ \bar\varphi|_Q = 1_Q\bigr\}.$$

Define $C_{\mathcal{L}}(Q)$ to be the category whose objects are the $C_{\mathcal{F}}(Q)$-centric subgroups of $C_S(Q)$, and where $\mathrm{Mor}_{C_{\mathcal{L}}(Q)}(P,P')$ is the set of those $\varphi \in \mathrm{Mor}_{\mathcal{L}}(PQ,P'Q)$ whose underlying homomorphisms are the identity on Q and send P into P'. By [BLO2, Proposition 2.5], the triple $(C_S(Q),C_{\mathcal{F}}(Q),C_{\mathcal{L}}(Q))$ is always itself a p-local finite group in this situation. When Q is fully normalized, there is an analogous construction of a "normalizer" p-local finite group $(N_S(Q),N_{\mathcal{F}}(Q),N_{\mathcal{L}}(Q))$ [BLO2, Lemma 6.2].

If G is a finite group, $S \in \mathrm{Syl}_p(G)$ and $Q \leq S$ is fully centralized in $\mathcal{F} = \mathcal{F}_S(G)$ (i.e., $C_S(Q) \in \mathrm{Syl}_p(C_G(Q))$), then $C_{\mathcal{F}}(Q)$ and $C_{\mathcal{L}}(Q)$ are isomorphic to $\mathcal{F}_{C_S(Q)}(C_G(Q))$ and $\mathcal{L}^c_{C_S(Q)}(C_G(Q))$ respectively. Also $\mathrm{Rep}(Q,\mathcal{F}) \cong \mathrm{Rep}(Q,G)$. With this in mind all that is left is to translate the statement of Theorem 2.3 to the new terminology.

Theorem 4.2. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group, and let $f\colon BS \longrightarrow |\mathcal{L}|^\wedge_p$ be the natural inclusion followed by completion. Then the following hold, for any finite p-group Q.*

(a) *The map*

$$\mathrm{Rep}(Q,\mathcal{F}) \xrightarrow{\ \cong\ } [BQ, |\mathcal{L}|^\wedge_p],$$

defined by sending the class of $\rho\colon Q \longrightarrow S$ to $f\circ B\rho$, is a bijection.

(b) *For any homomorphism $\rho\colon Q \longrightarrow S$ such that ρQ is fully centralized in $\mathcal{F}$,*

$$\Gamma'_{\mathcal{L},\rho Q}\colon |C_{\mathcal{L}}(\rho Q)|^\wedge_p \xrightarrow{\ \simeq\ } \mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)_{B\rho}$$

is a homotopy equivalence.

In particular, $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$ *is p-complete.*

Proof. See [BLO2, Corollary 4.5 & Theorem 6.3]. □

Our original motivation for considering the centric linking system of a finite group G was as a tool for describing the monoid of self homotopy equivalences of $BG^\wedge_p$. The description in Section 2.7 can be extended directly to this more general situation. We refer to Sections 2.6 and 2.7 for definitions of isotypical equivalences, and the category $\mathcal{A}ut(\mathcal{C})$ of self equivalences of a category $\mathcal{C}$.

Theorem 4.3. *For any p-local finite group* $(S, \mathcal{F}, \mathcal{L})$*, the topological monoids* $\mathrm{Aut}(|\mathcal{L}|^\wedge_p)$ *and* $|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|$ *are equivalent in the sense that their classifying spaces are homotopy equivalent. In particular,*

$$\mathrm{Out}(|\mathcal{L}|^\wedge_p) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \qquad \text{and} \qquad \pi_i(\mathrm{Aut}(|\mathcal{L}|^\wedge_p)) \cong \begin{cases} \varprojlim_{\mathcal{F}^c}{}^0(\mathcal{Z}) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

where $\mathcal{Z}(P) = Z(P)$ *for each* $\mathcal{F}$*-centric subgroup* $P \leq S$*.*

Proof. See [BLO2, Theorem 8.1]. □

4.5. **Homology decompositions.** One of the standard techniques when studying maps between p-completed classifying spaces of finite groups is to replace them by (the p-completion of) a homotopy colimit of simpler spaces. There are many ways of decomposing $BG^\wedge_p$ [Dw], of which the two most frequently used are the following:

- The subgroup decomposition: BG is mod p equivalent to the homotopy direct limit, over the orbit category of G (Section 3.2), of the classifying spaces of its p-radical subgroups [JMO]. The same type of decomposition holds for the collections of the p-centric subgroups, or the p-centric and p-radical subgroups of G.
- The centralizer decomposition: BG is mod p equivalent to the homotopy direct limit, over the fusion category of nontrivial elementary abelian p-subgroups $E \leq G$, of the classifying spaces of the centralizers $C_G(E)$ [JM].

Both of these decompositions have analogs for classifying spaces of p-local finite groups.

For any p-local finite group $(S, \mathcal{F}, \mathcal{L})$, the subgroup decomposition of $|\mathcal{L}|^\wedge_p$ is taken over the *orbit category* of $\mathcal{F}$. This is the category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S, and whose morphisms are defined by

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) = \mathrm{Rep}_{\mathcal{F}}(P, Q) \stackrel{\mathrm{def}}{=} \mathrm{Inn}(Q) \backslash \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

Also, $\mathcal{O}^c(\mathcal{F})$ denotes the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of S. If $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, then $\widetilde{\pi}$ denotes the composite functor

$$\widetilde{\pi} \colon \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F}^c \longrightarrow \mathcal{O}^c(\mathcal{F}).$$

There is a difference between the orbit category of a fusion system and the orbit category of a group. If G is a group and $S \in \mathrm{Syl}_p(G)$, then

$$\mathrm{Mor}_{\mathcal{O}_S(G)}(P, Q) \cong Q \backslash N_G(P, Q), \quad \text{while} \quad \mathrm{Mor}_{\mathcal{O}(\mathcal{F}_S(G))}(P, Q) \cong Q \backslash N_G(P, Q)/C_G(P).$$

In general, of course, these can be very different; but if P is p-centric, they differ only by the action of the group $C'_G(P)$ which is of order prime to p.

Let Top denote the category of spaces. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and let $\mathcal{C} \xrightarrow{\phi} \mathcal{D}$ and $\mathcal{C} \xrightarrow{F} \mathsf{Top}$ be functors. The left homotopy Kan extension of F along ϕ is a functor $L_\phi F : \mathcal{D} \longrightarrow \mathsf{Top}$ with the property that

$$\underrightarrow{\operatorname{hocolim}}_{\mathcal{C}} F \simeq \underrightarrow{\operatorname{hocolim}}_{\mathcal{D}} (L_\phi F).$$

Details on the construction and properties of $L_\phi F$ are given in [HV].

Proposition 4.4. *Fix a saturated fusion system $\mathcal{F}$ and an associated centric linking system $\mathcal{L}$, and let $\widetilde{\pi}\colon \mathcal{L} \longrightarrow \mathcal{O}^c(\mathcal{F})$ be the projection functor. Let*

$$\widetilde{B}\colon \mathcal{O}^c(\mathcal{F}) \longrightarrow \mathsf{Top}$$

be the left homotopy Kan extension along $\widetilde{\pi}$ of the constant functor $\mathcal{L} \xrightarrow{*} \mathsf{Top}$. Then $\widetilde{B}$ is a homotopy lifting of the homotopy functor $P \mapsto BP$, and

$$|\mathcal{L}|^\wedge_p \simeq \Big(\underrightarrow{\operatorname{hocolim}}_{\mathcal{O}^c(\mathcal{F})} (\widetilde{B})\Big)^\wedge_p. \tag{1}$$

Proof. See [BLO2, Proposition 2.2]. The proof is mostly a formality, following from elementary properties and the construction of homotopy Kan extensions. □

The centralizer decomposition of $|\mathcal{L}|$ is also mostly a formality in this context. Recall the definition of centralizer fusion and linking systems in Section 4.4: for any p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and any fully centralized subgroup $Q \leq S$, $(C_S(Q), C_\mathcal{F}(Q), C_\mathcal{L}(Q))$ is again a p-local finite group. Also, for any fusion system $\mathcal{F}$ over a finite p-group S, we let $\mathcal{F}^e$ denote the full subcategory of $\mathcal{F}$ whose objects are the nontrivial elementary abelian p-subgroups of S which are fully centralized in $\mathcal{F}$.

Theorem 4.5. *Fix a p-local finite group $(S, \mathcal{F}, \mathcal{L})$. Then there is a functor*

$$\widetilde{C}\colon \mathcal{F}^e \longrightarrow \mathsf{Top},$$

which is a homotopy lifting of the homotopy functor $E \mapsto |C_\mathcal{L}(E)|$, such that

$$\Big(\underrightarrow{\operatorname{hocolim}}_{E \in \mathcal{F}^e} (\widetilde{C})\Big)^\wedge_p \simeq |\mathcal{L}|^\wedge_p.$$

Proof. See [BLO2, Theorem 2.6]. The functor $\widetilde{C}$ is defined explicitly there as a left Kan extension. □

4.6. **Cohomology of the classifying space.** Arguably one of the most fundamental theorems in group cohomology is the statement that if G is a finite group and $S \in \operatorname{Syl}_p(G)$, then for every p-local G-module M, $H^*(G, M)$ is the module of "stable elements" in $H^*(S, M)$; i.e., the module of elements stable with respect to fusion in G. In our context, if we restrict attention to trivial p-local coefficients, then the same statement holds for classifying space $|\mathcal{L}|^\wedge_p$ of a p-local finite group. As one consequence of this result, $H^*(|\mathcal{L}|^\wedge_p; \mathbb{F}_p)$ is noetherian for any p-local finite group $(S, \mathcal{F}, \mathcal{L})$.

For any fusion system $\mathcal{F}$ over a finite p-group S, let $H^*(\mathcal{F}; \mathbb{F}_p)$ be the subring of $H^*(BS; \mathbb{F}_p)$ consisting of those elements which are stable under all fusion in $\mathcal{F}$; i.e.,

$$H^*(\mathcal{F}; \mathbb{F}_p) = \big\{ x \in H^*(BS; \mathbb{F}_p) \,\big|\, \alpha^*(x) = x|_{BP}, \text{ all } \alpha \in \operatorname{Hom}_\mathcal{F}(P, S) \big\}.$$

The proof that $H^*(|\mathcal{L}|;\mathbb{F}_p) \cong H^*(\mathcal{F};\mathbb{F}_p)$, when $(S,\mathcal{F},\mathcal{L})$ is a p-local finite group, is based on the construction of a certain (S,S)-biset: a set which has left and right actions of S which commute with each other. If $P \le S$ and $\varphi \in \mathrm{Hom}(P,S)$, let $S \times_{(P,\varphi)} S$ denote the biset

$$S \times_{(P,\varphi)} S = (S \times S)/{\sim}, \qquad \text{where} \qquad (x,gy) \sim (x\varphi(g),y) \quad \text{for } x,y \in S,\ g \in P\,,$$

and set

$$[S \times_{(P,\varphi)} S] = \Big(H^*(BS) \xrightarrow{\ \varphi^*\ } H^*(BP) \xrightarrow{\ \mathrm{trf}_P\ } H^*(BS)\Big) \in \mathrm{End}\big(H^*(BS;\mathbb{F}_p)\big).$$

Here, trf_P denotes the transfer map. If B is a disjoint union of bisets B_i of this form, we let $[B]$ be the sum of the endomorphisms $[B_i]$. If B is an (S,S)-biset, then for $P \le S$ and $\varphi \in \mathrm{Inj}(P,S)$, we let $B|_{(P,S)}$ denote the restriction of B to a (P,S)-biset, and let $B|_{(\varphi,S)}$ denote the (P,S)-biset where the left P-action is induced by φ.

Proposition 4.6 ([BLO2, Proposition 5.5]). *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, there is an (S,S)-biset Ω with the following properties:*

(a) *Ω is a disjoint union of bisets of the form $S\times_{(P,\varphi)}S$ for $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$.*

(b) *For each $P \le S$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, $\Omega|_{(P,S)}$ and $\Omega|_{(\varphi,S)}$ are isomorphic as (P,S)-bisets.*

(c) *$|\Omega|/|S| \equiv 1$ (mod p).*

Furthermore, for any biset Ω which satisfies these properties, $[\Omega]$ is an idempotent in $\mathrm{End}(H^*(BS;\mathbb{F}_p))$, and

$$\mathrm{Im}\big[H^*(BS;\mathbb{F}_p) \xrightarrow{\ [\Omega]\ } H^*(BS;\mathbb{F}_p)\big] = H^*(\mathcal{F};\mathbb{F}_p).$$

It was Markus Linckelmann and Peter Webb who first formulated conditions (a), (b), and (c) in the above proposition, and who saw the significance of finding a biset with these properties.

Theorem 4.7. *For any p-local finite group $(S,\mathcal{F},\mathcal{L})$, the natural homomorphism*

$$H^*(|\mathcal{L}|^\wedge_p;\mathbb{F}_p) \xrightarrow{\ \cong\ } H^*(\mathcal{F};\mathbb{F}_p),$$

induced by the inclusion of BS in $|\mathcal{L}|$, is an isomorphism. Furthermore, the ring $H^*(|\mathcal{L}|^\wedge_p;\mathbb{F}_p)$ is noetherian.

Proof. See [BLO2, Theorem 5.8]. The idea is to use a certain decomposition theorem for unstable algebras over the Steenrod algebra, due to Dwyer and Wilkerson [DW1, Theorem 1.2]. Their theorem implies both that $H^*(\mathcal{F};\mathbb{F}_p)$ is the inverse limit of the cohomology rings $H^*(|C_{\mathcal{L}}(E)|;\mathbb{F}_p)$ as E runs over the elementary abelian p-subgroups $1 \ne E \le S$, and also that higher derived functors of this inverse system vanish. Since $|\mathcal{L}|^\wedge_p$ is the homotopy direct limit of spaces homotopy equivalent to $|C_{\mathcal{L}}(E)|^\wedge_p$ (Theorem 4.5), we can then conclude from the spectral sequence of a homotopy colimit that $H^*(\mathcal{F};\mathbb{F}_p) \cong H^*(|\mathcal{L}|;\mathbb{F}_p)$. One of the requirements when applying the Dwyer-Wilkerson theorem is that the homomorphism

$$H^*(\mathcal{F};\mathbb{F}_p) \longrightarrow H^*(BS;\mathbb{F}_p)$$

be a split monomorphism, and the splitting is provided by Proposition 4.6.

The argument just sketched is carried out inductively, since we need to assume the theorem holds for the centralizers. This means that a separate argument is needed for fusion systems $\mathcal{F}$ with nontrivial center; i.e., those for which $\mathcal{F} = C_{\mathcal{F}}(E)$ (and hence $\mathcal{L} = C_{\mathcal{L}}(E)$) for some $1 \ne E \le S$. □

The (S,S)-biset Ω of Proposition 4.6, associated to a given saturated fusion system $\mathcal{F}$ over S, can also be used to construct a spectrum associated to $\mathcal{F}$ which is equivalent to the suspension spectrum of any classifying space for $\mathcal{F}$ (if any exists). Any finite (S,S)-biset induces, via inclusions and transfer maps, a stable map from the suspension spectrum $\Sigma^\infty BS$ to itself. The properties of Ω listed in Proposition 4.6 imply that the induced map $[\Omega]$ is an idempotent in the ring of homotopy classes of maps from $\Sigma^\infty BS$ to itself, and that the image of $H^*([\Omega];\mathbb{F}_p)$ is equal to $H^*(\mathcal{F};\mathbb{F}_p)$. The infinite mapping telescope of $[\Omega]$ is thus a stable summand of BS whose mod p cohomology is isomorphic to $H^*(\mathcal{F};\mathbb{F}_p)$. Hence by Theorem 4.7, if $|\mathcal{L}|^\wedge_p$ is any classifying space for $\mathcal{F}$, then this stable summand of $\Sigma^\infty(BS)$ is homotopy equivalent as a spectrum to $\Sigma^\infty(|\mathcal{L}|^\wedge_p)$.

4.7. **Fusion and linking systems determined by their classifying space.** A priori, one might think that the classifying space of a p-local finite group $(S,\mathcal{F},\mathcal{L})$ should contain only part of the information given by the fusion and linking systems $\mathcal{F}$ and $\mathcal{L}$. But in fact, one can recover both categories from the homotopy type of $|\mathcal{L}|^\wedge_p$. This is done with the help of the functors from spaces to categories described in Section 2.5.

Theorem 4.8. *Fix a p-local finite group $(S,\mathcal{F},\mathcal{L})$, and let $f\colon BS \longrightarrow |\mathcal{L}|^\wedge_p$ be the natural inclusion. Then there are equivalences of categories*

$$\mathcal{F} \cong \mathcal{F}_{S,f}(|\mathcal{L}|^\wedge_p) \qquad \text{and} \qquad \mathcal{L} \cong \mathcal{L}^c_{S,f}(|\mathcal{L}|^\wedge_p).$$

Proof. See [BLO2, Proposition 7.3]. The proof is very similar to that of Theorem 2.4. □

In other words, if $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ are two p-local finite groups and $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}'|^\wedge_p$, then $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ are isomorphic as triples, via isomorphisms of groups and of categories which commute with all of the structures which link them. To see how this follows from Theorem 4.8, note that by Theorem 4.2, for any homotopy equivalence $|\mathcal{L}|^\wedge_p \xrightarrow[\simeq]{\psi} |\mathcal{L}'|^\wedge_p$, there is an isomorphism $S \xrightarrow[\cong]{\alpha} S'$ such that the following square commutes up to homotopy:

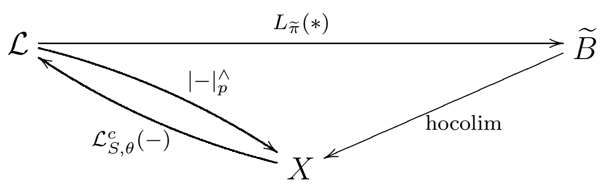
$$\begin{array}{ccc} BS & \xrightarrow{\quad f \quad} & |\mathcal{L}|^\wedge_p \\ {\scriptstyle B\alpha}\downarrow & & \downarrow{\scriptstyle \psi} \\ BS' & \xrightarrow{\quad f' \quad} & |\mathcal{L}'|^\wedge_p\,. \end{array}$$

Here, f and f' are the natural inclusions.

We also note here that for any given fusion system $\mathcal{F}$ over a finite p-group S, there are bijective correspondences

$$\left\{\begin{matrix}\text{linking}\\ \text{systems}\\ \text{assoc. to } \mathcal{F}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{classifying}\\ \text{spaces}\\ \text{for } \mathcal{F}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{liftings of the}\\ \text{homotopy functor}\\ P \mapsto BP\end{matrix}\right\}$$

which are given as follows:



More precisely, the first set (from the left) consists of all linking systems associated to $\mathcal{F}$ up to isomorphism (isomorphisms of categories which commute with the projections to $\mathcal{F}$ and the distinguished monomorphisms). The second set contains all classifying spaces (p-completed nerves of linking systems associated to $\mathcal{F}$), modulo the relation that $|\mathcal{L}|_p^\wedge$ and $|\mathcal{L}'|_p^\wedge$ are equivalent if there is a homotopy equivalence between them which commutes (up to homotopy) with the natural inclusions of BS. The third set consists of all functors from the orbit category $\mathcal{O}^c(\mathcal{F})$ to spaces which lift the homotopy functor $P \mapsto BP$, modulo natural homotopy equivalences of functors to Top. The bijection between the first two sets follows from Theorem 4.8, and the commutativity of the triangle involving Kan extension, homotopy colimit, and $|-|_p^\wedge$ was shown in Proposition 4.4. It remains to check that each lifting of $(P \mapsto BP)$ is the left Kan extension of some linking system, and this is shown in the proof of [BLO2, Proposition 2.3], where an explicit procedure is given for constructing a linking system associated to any given homotopy lifting.

It is also worth noting that the obstructions to lifting a given saturated fusion system $\mathcal{F}$ to centric linking system $\mathcal{L}$ described in Section 4.8 below coincide with those for lifting homotopy functors on the orbit category associated to $\mathcal{F}$ described by Dwyer and Kan [DK]. It was this observation which first suggested to us the above bijections, and in fact which first suggested to us the idea that two groups have equivalent p-completed classifying spaces if and only if their linking systems are equivalent.

4.8. **Existence and uniqueness of linking systems or classifying spaces.** One of the main open questions in this subject is that of the existence and uniqueness of linking systems associated to a given saturated fusion system. The obstructions for these problems are well understood, and are closely related to those for the Martino-Priddy conjecture as discussed in Section 3.

Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. Define a functor

$$\mathcal{Z}_\mathcal{F} \colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

by setting $\mathcal{Z}_\mathcal{F}(P) = Z(P) = C_S(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$. The obstruction to the existence of a centric linking system associated to $\mathcal{F}$ lies in $\varprojlim^3(\mathcal{Z}_\mathcal{F})$, and the obstruction to its uniqueness lies in $\varprojlim^2(\mathcal{Z}_\mathcal{F})$ [BLO2, Proposition 3.1]. This is completely analogous to the obstructions to the existence and uniqueness of an extension of a group G by another group K which acts on G via outer automorphisms: obstructions which lie in $H^3(K; Z(G))$ and $H^2(K; Z(G))$, respectively.

One way to see the obstruction to the existence of associated linking systems is to construct directly a "category" which satisfies all of the conditions for a linking system, *except* that the composition of morphisms need not be associative. This can be done in such a way that the failure of associativity assigns to each triple of composable morphisms $P_0 \to P_1 \to P_2 \to P_3$ in $\mathcal{O}^c(\mathcal{F})$ an element of $Z(P_0)$, and these elements combine to form a 3-cocycle with coefficients in $\mathcal{Z}_\mathcal{F}$.

We do have some results about the existence and uniqueness of associated linking systems in various special cases. For example,

Proposition 4.9. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then there exists a linking system associated to $\mathcal{F}$ if* $\mathrm{rk}(S) < p^3$*, and the linking system is unique if* $\mathrm{rk}(S) < p^2$.

Proof. [BLO2, Corollary 3.4]. The point of this proof is that the corresponding higher limit obstruction groups for existence and uniquness vanish under the hypotheses of the proposition. □

Note that the Martino-Priddy conjecture is a special case of the problem of uniqueness of centric linking systems. The proof of this conjecture, as sketched in Section 3, shows in fact that there is exactly one linking system associated to the fusion system of a finite group. We do still hope to find a proof of this conjecture which is independent of the classification of finite simple groups: not only for esthetic reasons, but also because it seems likely that any such proof could extend to a proof of the existence and uniqueness of linking systems (hence classifying spaces) associated to any given saturated fusion system.

5. Examples and methods of construction

One of the weak points in our investigations into the theory of p-local finite groups is the problem of constructing "exotic" examples: examples which do not come from finite groups. The key difficulty seems to be that of showing that newly constructed fusion systems are saturated. Some attempts to develop general techniques for constructing saturated fusion systems are described in Section 5.1. We then describe some which arise more "naturally", notably those of the type studied by Solomon and Benson, and those which come from block theory. We finish this section with a discussion of "extensions" of p-local finite groups: a subject which is still to a large extent under development.

We first note the following general result which is very useful when constructing saturated fusion systems.

Proposition 5.1 ([BCGLO, Theorem 2.3])**.** *Let $\mathcal{F}$ be a fusion system over a finite p-group S. Assume the following hold:*

(I) *If $P \leq S$ is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, then* $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(II) *For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, if we set*

$$N_\varphi = \{x \in N_S(P) \mid \varphi c_x \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$

then φ extends to some $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$.

Let $\mathcal{F}' \subseteq \mathcal{F}$ be the subcategory with the same objects, and whose morphisms are the composites of restrictions of morphisms in $\mathcal{F}$ between $\mathcal{F}$-centric subgroups. Then $\mathcal{F}'$ is saturated.

In fact, [BCGLO, Theorem 2.3] is formulated more generally, and deals with fusion systems which satisfy the axioms of saturation only on subgroups which are both $\mathcal{F}$-centric and $\mathcal{F}$-radical (though an extra axiom is then needed). But the case formulated above seems to be the most important one.

5.1. **Construction of exotic p-local finite groups.**

The following theorem is the only result we know which gives a geometric criterion for showing that a fusion system is saturated. If S is a finite p-group, then we say that a map $BS \xrightarrow{f} X$ is *Sylow* if every map $BP \longrightarrow X$, for a finite p-group P, factors through f up to homotopy.

A map $Y \xrightarrow{f} X$ is *centric* if composition with f induces a homotopy equivalence $\mathrm{Map}(Y,Y)_{\mathrm{Id}} \simeq \mathrm{Map}(Y,X)_f$ between the connected component of Id_Y and the connected component of f.

Theorem 5.2. *Fix a space X, a finite p-group S, and a map $f\colon BS \longrightarrow X$. Assume that*

(a) *f is Sylow, and*

(b) *$f|_{BP}$ is a centric map for each $\mathcal{F}_{S,f}(X)$-centric subgroup $P \le S$.*

Let $\mathcal{F}' \subseteq \mathcal{F}_{S,f}(X)$ be the subcategory with the same objects, and whose morphisms are the composites of restrictions of morphisms in $\mathcal{F}_{S,f}(X)$ between $\mathcal{F}$-centric subgroups. Then $\mathcal{F}'$ is saturated, and the triple $\big(S, \mathcal{F}', \mathcal{L}^c_{S,f}(X)\big)$ is a p-local finite group.

Proof. See [BLO4]. One uses direct geometric arguments to show that $\mathcal{F}_{S,f}(X)$ satisfies the axioms of saturation on centric subgroups, and then applies Proposition 5.1. □

One might expect that it is difficult to find interesting examples of maps $BS \longrightarrow X$ which satisfy the centricity condition (b) above. But in fact, with the help of [BLO2, Proposition 4.2], which says that homotopy colimits commute with mapping spaces $\mathrm{Map}(BP,-)$ under certain conditions, spaces X can be constructed which satisfy the above conditions without being completed classifying spaces of finite groups.

The following is one example of how Theorem 5.2 can be applied. If $\mathcal{F}_0$ is a fusion system over a finite p-group S, and for each $i = 1, \ldots, m$ we are given subgroups $Q_i \le S$ and groups of automorphisms $\Delta_i \le \mathrm{Out}(Q_i)$, then we let

$$\mathcal{F} \overset{\text{def}}{=} \langle \mathcal{F}_0; \mathcal{F}_{Q_1}(\Delta_1), \ldots, \mathcal{F}_{Q_m}(\Delta_m)\rangle$$

be the fusion system over S defined as follows. For each pair of subgroups $P, P' \le S$, $\mathrm{Hom}_{\mathcal{F}}(P,P')$ is the set of composites

$$P = P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} P_2 \longrightarrow \cdots \longrightarrow P_{k-2} \xrightarrow{\varphi_{k-1}} P_{k-1} \xrightarrow{\varphi_k} P_k = P',$$

where for each i, either $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_0}(P_{i-1}, P_i)$; or $P_{i-1}, P_i \le Q_j$ for some j, and $\varphi_i = \alpha|_{P_{i-1}}$ for some $\alpha \in \mathrm{Aut}(Q_j)$ with $[\alpha] \in \Delta_j$.

Proposition 5.3. *Fix a finite group G, a Sylow p-subgroup $S \le G$, and subgroups $Q_1, \ldots, Q_m \le S$ such that no Q_i is conjugate to a subgroup of Q_j for $i \ne j$. Set $K_i = \mathrm{Out}_G(Q_i)$, and fix subgroups $\Delta_i \le \mathrm{Out}(Q_i)$ which contain K_i. Assume for each i that*

(1) *$p \nmid [\Delta_i{:}K_i]$;*

(2) *Q_i is p-centric in G, but for each $P \lneqq Q_i$ there is $\alpha \in \Delta_i$ such that $\alpha(P)$ is not p-centric in G; and*

(3) *for all $\alpha \in \Delta_i \smallsetminus K_i$, $K_i \cap \alpha K_i \alpha^{-1}$ has order prime to p.*

Then the fusion system $\mathcal{F} \overset{\text{def}}{=} \langle \mathcal{F}_S(G); \mathcal{F}_{Q_1}(\Delta_1), \ldots, \mathcal{F}_{Q_m}(\Delta_m)\rangle$ is saturated, and has an associated centric linking system.

Proof. See [BLO4]. One first checks that for each i, there is a unique extension G_i of Q_i by Δ_i which contains $N_G(Q_i)$ (up to isomorphism). Let X be the union of BG and the BG_i, where each BG_i is attached to BG along their common subspace $BN_G(Q_i)$. The proposition then follows from Theorem 5.2, applied to the space $X^\wedge_p$. Condition

(2) guarantees that the fusion system of $X_p^\wedge$ is generated by its restriction to centric subgroups; i.e., that $\mathcal{F}' = \mathcal{F}_{S,f}(X_p^\wedge)$ in the notation of Theorem 5.2. □

Proposition 5.3 is a generalization of [BLO2, Proposition 9.1]. Some more concrete applications of that result, for primes $p \geq 5$, are given in [BLO2, §9]: constructions which can be thought of (very roughly) as mixing features of the fusion systems of two different groups.

The above proposition can also be applied when $p = 3$, to construct exotic 3-local finite groups with Sylow subgroup the groups

$$S_\pm = \langle a, b, x \,|\, [a, b] = 1,\ xax^{-1} = ab,\ xbx^{-1} = ba^{\mp 3} \rangle$$

of order 81. We do not yet know whether it can be used to construct any exotic 2-local finite groups.

In all of our examples, except those described in the next section, the proof that a saturated fusion system is not the fusion system of a finite group always involves using the classification of finite simple groups.

5.2. **Fusion systems of a type studied by Solomon.** Well before Puig formulated the axioms defining a saturated fusion system, Ron Solomon had essentially found one which does not arise from the p-fusion system of any finite group. This was a bi-product of one step in the classification of finite simple groups [Sol]. Solomon considered the problem of classifying all finite simple groups whose Sylow 2-subgroups are isomorphic to those of the Conway group Co_3. The end result of his paper was that Co_3 is the only such group. In the process of proving this, he needed to consider groups G in which all involutions are conjugate, and such that the centralizer of each involution contains a normal subgroup isomorphic to $\mathrm{Spin}_7(q)$ with odd index, where q is an odd prime power. Solomon showed that such a group G does not exist. However, the 2-local structure that he found turned out to be perfectly consistent. It was only by analyzing its interaction with the p-local structure (where p is the prime of which q is a power) that he found a contradiction.

In a later paper [Be1], Dave Benson, inspired by Solomon's work, constructed certain spaces which can be thought of as the 2-completed classifying spaces which the groups studied by Solomon would have if they existed. Benson's construction was arguably the first indication of the existence of spaces which "behave" like p-completed classifying spaces of finite groups, but which are themselves not of this form. To construct these spaces he started with the spaces $BDI(4)$ constructed by Dwyer and Wilkerson having the property that

$$H^*(BDI(4); \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3, x_4]^{GL_4(2)}$$

(the rank four Dickson algebra at the prime 2). He then considered, for each odd prime power q, the homotopy fixed point set of the $\mathbb{Z}$-action on $BDI(4)$ generated by an "Adams operation" ψ^q constructed by Dwyer and Wilkerson. Denote this homotopy fixed point set by $BDI_4(q)$.

In [LO], the second and third authors have shown that for each odd prime power q, there is a saturated fusion system $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ over a Sylow 2-subgroup $S = S(q) \leq \mathrm{Spin}_7(q)$, with the properties that all involutions in S are $\mathcal{F}$-conjugate, and that when $z \in Z(S)$ is the generator, then $C_{\mathcal{F}}(z)$ is the fusion system of $\mathrm{Spin}_7(q)$. The obstruction theory described in Section 4.8 applies to show that there is a unique centric linking system $\mathcal{L}^c_{\mathrm{Sol}}(q)$ associated to $\mathcal{F}_{\mathrm{Sol}}(q)$. We thus get a 2-local finite group $(S(q), \mathcal{F}_{\mathrm{Sol}}(q), \mathcal{L}^c_{\mathrm{Sol}}(q))$, which by Solomon's theorem (together with some other group theoretic results) cannot be associated to any finite group.

The classifying spaces $B\mathrm{Sol}(q) \stackrel{\text{def}}{=} |\mathcal{L}^c_{\mathrm{Sol}}(q)|^\wedge_2$ turn out to be equivalent to the spaces constructed by Benson in [Be1]. For fixed q, one can take the union of the spaces $B\mathrm{Sol}(q^n)$ for all n (strictly speaking this is not true as stated, but it can be done up to homotopy), and the 2-completion of this union is homotopy equivalent to $BDI(4)$. Conversely, if $\psi^q \in \mathrm{Aut}(BDI(4))$ is an "Adams map" (its restriction to a maximal torus is induced by the Frobenius automorphism ($x \mapsto x^q$) on the algebraic closure $\bar{\mathbb{F}}_q$), then $B\mathrm{Sol}(q)$ is homotopy equivalent to $BDI_4(q)$ — the homotopy fixed point set of ψ^q when regarded as an action of the monoid $\mathbb{N}$. For more details, see [LO, §4].

Other "exotic" p-local finite groups have been constructed by the first author together with Møller [BM]. Their method resembles Benson's construction of $BDI_4(q)$. Namely, they first take homotopy fixed point sets of actions of Adams maps on certain p-compact groups, and then show that these are the classifying spaces of p-local finite groups.

5.3. **The fusion system of a block.** Part of the motivation for constructing classifying spaces for fusion systems, and part of the reason in general for looking at abstract fusion and linking systems, comes from representation theory. Fix a finite group G, a prime p, and an algebraically closed field k of characteristic p. A *block* in $k[G]$ is a factor in the maximal decomposition of $k[G]$ as a product of rings, or equivalently a minimal central idempotent. Alperin and Broué [AB] defined, for any block b, inclusion and conjugacy relations among the Brauer pairs associated to b (the "b-subpairs"); and Puig [Pu3] showed that these satisfy the axioms of a saturated fusion system over the defect group of the block. We refer to [AB] or [Alp] for more details (including the definitions of defect groups and Brauer pairs). The existence of a (unique or canonical) linking system associated to the fusion system of a block would thus imply the existence of a classifying space for the block, which might in turn have implications in representation theory (see [Li]).

5.4. **Extensions of p-local finite groups.** We describe here some work in progress which will appear in [BCGLO]; work which in certain specialized situations allows us to study extensions of p-local finite groups. It is still unclear how to define such extensions in general.

One of the reasons for this difficulty is that an extension of (finite) groups need not induce a fibration sequence of their p-completed classifying spaces. For example, if $1 \to H \to G \to K \to 1$ is an extension where K has order prime to p, then $BK^\wedge_p$ is contractible, while $BH^\wedge_p$ and $BG^\wedge_p$ need not be homotopy equivalent (since H and G need not have the same cohomology mod p). Two cases where an extension does induce a fibration sequence of p-completed classifying spaces are those where the quotient group is a finite p-group, and the case of central extensions. It is this last case which is the simplest to consider.

If $\mathcal{F}$ is a fusion system over a finite p-group S, then a subgroup $A \leq S$ is called *central* if $C_{\mathcal{F}}(A) = \mathcal{F}$; i.e., if each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ extends to a morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PA,QA)$ which is the identity on A. Clearly, if A is central in $\mathcal{F}$, then $A \leq Z(S)$. In such a situation, there is an obvious way to define a quotient fusion system $\mathcal{F}/A$ over S/A, by letting $\mathrm{Hom}_{\mathcal{F}/A}(P/A,Q/A)$ be the image in $\mathrm{Hom}(P/A,Q/A)$ of $\mathrm{Hom}_{\mathcal{F}}(P,Q)$, and this is saturated by [BLO2, Lemma 5.6]. If, furthermore, $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, then there is a canonical way to construct a centric linking system $\mathcal{L}/A$ associated to $\mathcal{F}/A$ ([BLO2, Lemma 5.6] again). A central extension of a p-local finite group $(S,\mathcal{F},\mathcal{L})$ by an abelian group A can now be defined as a p-local finite group $(\widetilde{S}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{L}})$, together with an isomorphism $A \cong A' \leq \widetilde{S}$ where A' is central in $\widetilde{\mathcal{F}}$, such that $(\widetilde{S}/A, \widetilde{\mathcal{F}}/A, \widetilde{\mathcal{L}}/A) \cong (S, \mathcal{F}, \mathcal{L})$.

Theorem 5.4. *Central extensions of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ by a finite abelian p-group A are in one-to-one correspondence with principal fibrations*

$$BA \to X \to |\mathcal{L}|^\wedge_p,$$

and also in natural one-to-one correspondence with $H^2(|\mathcal{L}|^\wedge_p; A)$.

Proof. See [BCGLO, §8]. □

We next look at extensions where the quotient group is a finite p-group.

Theorem 5.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Let $\lambda\colon \pi_1(|\mathcal{L}|^\wedge_p) \twoheadrightarrow \pi$ be any surjection of groups, and set $S_0 = \mathrm{Ker}[S \twoheadrightarrow \pi_1(|\mathcal{L}|^\wedge_p) \twoheadrightarrow \pi]$. Let $f\colon |\mathcal{L}|^\wedge_p \longrightarrow B\pi$ be the classifying map for λ, and let X be its homotopy fiber. Then there is a p-local finite group $(S_0, \mathcal{F}_0, \mathcal{L}_0)$, where $\mathcal{F}_0$ is a subcategory of $\mathcal{F}$, such that $|\mathcal{L}_0|^\wedge_p \simeq X$.*

Conversely, assume that $|\mathcal{L}_0|^\wedge_p \longrightarrow X \longrightarrow B\pi$ is a fibration sequence, where the fiber is the classifying space of a p-local finite group $(S_0, \mathcal{F}_0, \mathcal{L}_0)$. Then $X \simeq |\mathcal{L}|^\wedge_p$ for some p-local finite group $(S, \mathcal{F}, \mathcal{L})$, where $S_0 \lhd S$, $\mathcal{F}_0 \subseteq \mathcal{F}$, and $S/S_0 \cong \pi$.

Proof. See [BCGLO, §5]. □

Note that in the above proposition, we do *not* say that the centric linking system $\mathcal{L}_0$ is a subcategory of $\mathcal{L}$. In fact, it is not in general a subcategory, since $\mathcal{F}_0$-centric subgroups of S_0 need not be $\mathcal{F}$-centric. This helps to illustrate another of the key problems in this subject: the lack of a good concept of subobjects (or morphisms), since even group inclusions and group homomorphisms do not in general send p-centric subgroups to p-centric subgroups.

We can also describe "extensions" where the quotient is a group of order prime to p. For the purpose of the following theorem, for any saturated fusion system $\mathcal{F}$ over a finite p-group S, we say that a saturated fusion subsystem $\mathcal{F}' \subseteq \mathcal{F}$ over S has index prime to p if

$$\mathrm{Aut}_{\mathcal{F}'}(P) \geq O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$$

for each $P \leq S$. Here, for any group G, $O^{p'}(G)$ is the largest normal subgroup of index prime to p; i.e., the subgroup generated by the Sylow p-subgroups of G.

Theorem 5.6. *Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. Then there is a normal subgroup $\mathrm{Out}^0_{\mathcal{F}}(S) \lhd \mathrm{Out}_{\mathcal{F}}(S)$ of index prime to p, and a map*

$$\widehat{\theta}\colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$$

with the following properties:

(a) *$\widehat{\theta}(\beta \circ \alpha) = \widehat{\theta}(\beta)\cdot\widehat{\theta}(\alpha)$ for each composable pair of morphisms β, α in $\mathcal{F}^c$.*

(b) *The restriction of $\widehat{\theta}$ to $\mathrm{Aut}_{\mathcal{F}}(S)$ is the natural surjection.*

(c) *$\widehat{\theta}$ sends inclusions to the identity.*

(d) *For each subgroup $T \leq \mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$, the subcategory $\mathcal{F}_T \subseteq \mathcal{F}$ with the same objects and whose morphisms are generated by restrictions of morphisms in $\widehat{\theta}^{-1}(T)$, is a saturated fusion subsystem of $\mathcal{F}$ of index prime to p.*

(e) *Each saturated fusion subsystem $\mathcal{F}' \subseteq \mathcal{F}$ of index prime to p is equal to $\mathcal{F}_T$ for some subgroup $T \leq \mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$.*

(f) *If $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$ with projection functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}$, and if $T \le \operatorname{Out}_{\mathcal{F}}(S)/\operatorname{Out}^0_{\mathcal{F}}(S)$ is a subgroup, then $\mathcal{L}_T \overset{\text{def}}{=} \pi^{-1}(\mathcal{F}_T) \subseteq \mathcal{L}$ is a centric linking system associated to $\mathcal{F}_T$.*

Proof. See [BCGLO, §6], where the subgroup $\operatorname{Out}^0_{\mathcal{F}}(S)$ is defined explicitly. (In fact, it is the smallest subgroup of $\operatorname{Out}_{\mathcal{F}}(S)$ for which a map $\widehat{\theta}$ exists satisfying conditions (a–c).) Note, in point (f), that if $T \lhd \operatorname{Out}_{\mathcal{F}}(S)/\operatorname{Out}^0_{\mathcal{F}}(S)$ is a normal subgroup with quotient group σ, then $|\mathcal{L}_T|^\wedge_p \longrightarrow |\mathcal{L}|^\wedge_p \longrightarrow B\sigma$ is *not* in general a fibration sequence. □

The map $\widehat{\theta}$ of Theorem 5.6 cannot be extended to arbitrary morphisms in $\mathcal{F}$, and still satisfy conditions (a–c). For example, any such extension would have to send the inclusion $1 \longrightarrow S$ to the identity in $\operatorname{Out}_{\mathcal{F}}(S)/\operatorname{Out}^0_{\mathcal{F}}(S)$ — and Condition (a) would then imply that all automorphisms of S also get sent to the identity.

6. p-local compact groups

In very recent work still in progress, we have begun to extend the results on p-local finite groups to a more general situation, one which includes p-completed classifying spaces of compact Lie groups and classifying spaces of p-compact groups. One of our hopes is that this will provide a new tool to study certain maps between these spaces, in particular self equivalences of these spaces.

6.1. **Discrete p-toral groups.** A p-toral group is a compact Lie group P whose identity component is a torus T, such that $P/T \cong \pi_0(P)$ is a finite p-group. By contrast, a *discrete p-toral group* is a discrete group P, with normal subgroup $P_0 \lhd P$, such that P_0 is isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$ $(= \mathbb{Z}[\frac{1}{p}]/\mathbb{Z})$ and P/P_0 is a finite p-group. Any discrete p-toral group P contains a unique minimal subgroup P_0 of finite index, which we call its connected component. Also, $P_0 \cong (\mathbb{Z}/p^\infty)^r$ for some $r \overset{\text{def}}{=} \operatorname{rk}(P) = \operatorname{rk}(P_0)$. We define $|P| = (\operatorname{rk}(P), |P/P_0|) \in \mathbb{N}\times\mathbb{N}$, and order the elements of $\mathbb{N}\times\mathbb{N}$ lexicographically. This ordering of the "orders" of discrete p-toral groups is what allows us to adapt with very little change the definition of a saturated fusion system over a finite p-group to this new situation.

The motivation for studying fusion and linking systems over discrete p-toral groups comes from regarding them as "discrete approximations" of p-toral groups. The p-toral subgroups of compact Lie groups play the same role as p-subgroups of finite groups (more on this below). Note that by a "discrete p-toral subgroup" of a p-toral group (or of any other compact Lie group) we mean a subgroup which *as a discrete group* is discrete p-toral, even though its topology as a subgroup might not be discrete.

Proposition 6.1. *Each p-toral group P contains a dense subgroup $P_\delta \le P$ which is a discrete p-toral group of the same rank; and any two such subgroups are conjugate in P. Furthermore, the inclusion induces an isomorphism $H^*(BP;\mathbb{F}_p) \cong H^*(BP_\delta;\mathbb{F}_p)$, and hence a homotopy equivalence $(BP_\delta)^\wedge_p \simeq BP^\wedge_p$.*

Proof. See [DW3, Proposition 6.9]. Let $T \le P$ be the identity connected component, and let $T_\delta \le T$ be the subgroup of elements of p-power order. Clearly, T_δ is the unique discrete p-toral subgroup of T of the same rank, and there is a bijective correspondence between dense discrete p-toral subgroups of P which contain T_δ and splittings of the group extension $1 \to T/T_\delta \to P/T_\delta \to P/T \to 1$. Since P/T is a finite p-group and T/T_δ is uniquely p-divisible, $H^i(P/T; T/T_\delta) = 0$ for all $i > 0$, and hence the above group extension has a splitting which is unique up to conjugation.

The proof that $H^*(BP;\mathbb{F}_p) \cong H^*(BP_\delta;\mathbb{F}_p)$ is easily reduced to the case where $P \cong S^1$ and $P_\delta \cong \mathbb{Z}/p^\infty$, and this result is classical. □

One advantage in working with discrete p-toral groups rather than p-toral groups is that all subgroups of discrete p-toral groups are again discrete p-toral groups, unlike the case for compact p-toral groups. It is also simpler in general to work with discrete groups, without worrying about their topology.

6.2. **Fusion and linking systems over discrete p-toral groups.** A *fusion system* $\mathcal{F}$ over a discrete p-toral group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following conditions:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P,Q \leq S$.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

A subgroup $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P, and is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P. By establishing upper bounds on numbers of components in centralizers and normalizers, one can show that each subgroup $P \leq S$ is $\mathcal{F}$-conjugate to a fully centralized subgroup, and to a fully normalized subgroup.

The fusion system $\mathcal{F}$ is *saturated* if the following three conditions hold.

(I) For each $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$, $\mathrm{Out}_{\mathcal{F}}(P)$ is finite, and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$.

(II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that φP is fully centralized, and if we set

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$

then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

(III) If $P_1 \leq P_2 \leq P_3 \leq \cdots$ is a sequence of subgroups of S, $P_\infty = \bigcup_{n=1}^\infty P_n$, and $\varphi \in \mathrm{Hom}(P_\infty, S)$ is such that $\varphi|_{P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all n, then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$.

We note here that $\mathrm{Out}_S(P)$ is a finite p-group for any pair $P \leq S$ of discrete p-toral groups.

Since every subgroup $P \leq S$ is $\mathcal{F}$-conjugate to a subgroup P' which is fully normalized in $\mathcal{F}$, condition (I) implies that $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathcal{F}}(P')$ is finite for all $P \leq S$. The only real difference between this new definition and the definition of a saturated fusion system over a finite p-group is condition (III), which can be thought of as a "continuity" condition.

When S is a discrete p-toral group and $\mathcal{F}$ is a saturated fusion system over S, then the definitions of $\mathcal{F}$-centric subgroups of S, and of centric linking systems associated to $\mathcal{F}$, are identical to the definitions given in Section 4 above when S is a finite p-group. A *p-local compact group* is now defined to be a triple $(S,\mathcal{F},\mathcal{L})$, where S is a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of such a triple $(S,\mathcal{F},\mathcal{L})$ is the p-completed nerve $|\mathcal{L}|_p^\wedge$. Thus, as in the case for fusion and linking systems over finite p-groups, a centric linking system for a fusion system $\mathcal{F}$ over a discrete p-toral group S can be thought of as a means of associating a classifying space to $\mathcal{F}$.

6.3. **Reduction to finite subcategories.** The main difficulty when generalizing results about p-local finite groups to the discrete p-toral case is that the categories are no longer finite. It is, however, possible to replace any centric linking system $\mathcal{L}$ associated to a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S, by a finite full subcategory $\mathcal{L}_0 \subseteq \mathcal{L}$ such that $|\mathcal{L}_0|_p^\wedge \simeq |\mathcal{L}|_p^\wedge$. This is done by restricting $\mathcal{L}$ to those $\mathcal{F}$-centric subgroups which are also *$\mathcal{F}$-radical*; i.e., those $P \leq S$ for which $\operatorname{Out}_{\mathcal{F}}(P)$ contains no nontrivial normal p-subgroup.

Proposition 6.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $\mathcal{L}$ be an associated centric linking system to $\mathcal{F}$. Then S contains only finitely many $\mathcal{F}$-conjugacy classes of subgroups which are both $\mathcal{F}$-centric and $\mathcal{F}$-radical. There is a finite full subcategory $\mathcal{L}_0 \subseteq \mathcal{L}$ whose objects include $\mathcal{F}$-conjugacy class representatives for all $\mathcal{F}$-radical $\mathcal{F}$-centric subgroups of S, such that the inclusion $|\mathcal{L}_0|_p^\wedge \subseteq |\mathcal{L}|_p^\wedge$ is a homotopy equivalence.*

Proof. See [BLO3, Corollary 3.5 & Proposition 5.5]. □

6.4. **A homology decomposition of the classifying space.** The *orbit category* of a fusion system $\mathcal{F}$ over a discrete p-toral group S is defined exactly as before: it has the same objects as $\mathcal{F}$, and

$$\operatorname{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q) = \operatorname{Rep}_{\mathcal{F}}(P,Q) \overset{\text{def}}{=} \operatorname{Hom}_{\mathcal{F}}(P,Q)/\operatorname{Inn}(Q).$$

Also, we write $\mathcal{O}^c(\mathcal{F})$ to denote the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of S.

When $\mathcal{F}$ is a saturated fusion system over a discrete p-toral group S, then we have the same correspondence between centric linking systems associated to $\mathcal{F}$, classifying spaces associated to $\mathcal{F}$, and liftings over $\mathcal{O}^c(\mathcal{F})$ of the homotopy functor $(P \mapsto BP)$ as we do when S is finite (Section 4.7). In particular, any classifying space for $\mathcal{F}$ has a decomposition as the homotopy direct limit of the corresponding homotopy lifting, as described by the next proposition.

Proposition 6.3. *Fix a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be any finite full subcategory whose objects include $\mathcal{F}$-conjugacy class representatives for all $\mathcal{F}$-radical $\mathcal{F}$-centric subgroups of S. Let $\mathcal{L}_0$ be any centric linking system associated to $\mathcal{F}_0$, and let $\widetilde{\pi}_0\colon \mathcal{L}_0 \longrightarrow \mathcal{O}(\mathcal{F}_0)$ be the projection functor. Let*

$$\widetilde{B}\colon \mathcal{O}(\mathcal{F}_0) \longrightarrow \mathsf{Top}$$

be the left homotopy Kan extension over $\widetilde{\pi}_0$ of the constant functor $\mathcal{L}_0 \xrightarrow{*} \mathsf{Top}$. Then $\widetilde{B}$ is a homotopy lifting of the homotopy functor $P \mapsto BP$, and

$$|\mathcal{L}|_p^\wedge \simeq |\mathcal{L}_0|_p^\wedge \simeq \Big(\underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\operatorname{hocolim}}}(\widetilde{B})\Big)_p^\wedge. \tag{1}$$

Proof. See [BLO3, Proposition 5.4]. □

6.5. **Reduced linking systems.** Constructing centric linking systems over discrete p-toral groups is not as straightforward as it sometimes is in the case of finite p-groups. For this reason, we define another type of category which is intermediate between fusion systems and linking systems.

Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be any full subcategory. A *reduced linking system* associated to $\mathcal{F}_0$ consists of a category $\bar{\mathcal{L}}_0$, together with a functor $\bar{\pi}\colon \bar{\mathcal{L}}_0 \longrightarrow \mathcal{F}_0$ and distinguished *homomorphisms* $P \xrightarrow{\bar{\delta}_P} \mathrm{Aut}_{\bar{\mathcal{L}}_0}(P)$, which satisfy axioms (A), (B), and (C) in Section 4.2, *except* that $\mathrm{Ker}(\bar{\delta}_P) = Z(P)_0$, and that the group $Z(P)/Z(P)_0 \cong \pi_0(Z(P))$ (not $Z(P)$ itself) acts freely on morphism sets $\mathrm{Mor}_{\bar{\mathcal{L}}}(P,Q)$ and induces bijections

$$\mathrm{Mor}_{\bar{\mathcal{L}}}(P,Q)/\pi_0(Z(P)) \xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

When $\mathcal{F}_0 = \mathcal{F}^c$ (i.e., when all $\mathcal{F}$-centric subgroups are objects of $\mathcal{F}_0$ and $\bar{\mathcal{L}}_0$), we call $\bar{\mathcal{L}}_0$ a reduced centric linking system associated to $\mathcal{F}$.

Fortunately, we do know that each reduced linking system lifts to a unique linking system.

Proposition 6.4. *Let $\mathcal{F}$ be a fusion system over the discrete p-toral group S. Then any reduced linking system $\bar{\mathcal{L}}$ associated to $\mathcal{F}^c$ lifts to a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$ which is unique up to isomorphism.*

Proof. See [BLO3, Corollary 5.7]. □

6.6. **Compact Lie groups.** It was immediately obvious that every finite group can be regarded as a p-local finite group. In fact, we were already studying the fusion system and centric linking system for finite groups long before we considered the more abstract definitions. The definitions of fusion systems, and of reduced centric linking systems, for compact Lie groups are also straightforward, but the definition of their linking systems is less obvious.

Fix a compact Lie group G and a prime p. Let $T \le G$ be a maximal torus, and fix a Sylow p-subgroup $\bar{S}/T \in \mathrm{Syl}_p(N_G(T)/T)$. Then $\bar{S}$ is a maximal p-toral subgroup of G, and any other maximal p-toral subgroup of G is conjugate to $\bar{S}$. These subgroups play the role of "Sylow p-subgroups" when working with compact Lie groups. For example, an arbitrary p-toral subgroup $P \le G$ is maximal p-toral if and only if $\chi(G/P)$ is prime to p, where χ denotes as usual the Euler characteristic.

Now let $S \le \bar{S}$ be any discrete p-toral subgroup which is dense in $\bar{S}$ and has the same rank. By Proposition 6.1, there is such a subgroup, and any two such subgroups are conjugate in $\bar{S}$. Since the closure of any discrete p-toral subgroup of G is p-toral, we now see that S is a maximal discrete p-toral subgroup of G, and that any other maximal discrete p-toral subgroup is G-conjugate to S.

Define the fusion category $\mathcal{F}_S(G)$ exactly as was done for finite G: its objects consist of all subgroups of S, and $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q)$ is the set of all group homomorphisms which are induced by conjugation in G. A subgroup $P \le S$ is $\mathcal{F}_S(G)$-centric if and only if $Z(P)$ is a maximal discrete p-toral subgroup of $C_G(P)$, or equivalently if and only if $C_G(P) = \overline{Z(P)} \times C'_G(P)$ for some (unique) subgroup $C'_G(P)$ which is finite of order prime to p.

We now define the reduced centric linking system $\bar{\mathcal{L}}_S^c(G)$ to be the category whose objects are the $\mathcal{F}_S(G)$-centric subgroups of S, and where

$$\mathrm{Mor}_{\bar{\mathcal{L}}_S^c(G)}(P,Q) = N_G(P,Q)\big/\big(\overline{Z(P)}_0 \times C'_G(P)\big).$$

By Proposition 6.4, there is a unique centric linking system $\mathcal{L}_S^c(G)$ associated to $\bar{\mathcal{L}}_S^c(G)$.

Theorem 6.5. *Fix a compact Lie group G and a maximal discrete p-toral subgroup $S \le G$. Then $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p-local compact group, with classifying space $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.*

Proof. See [BLO3, §8]. The proofs that $\mathcal{F}_S(G)$ is a fusion system and $\mathcal{L}_S^c(G)$ an associated centric linking system are straightforward. The proof that $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$ is based on the homology decomposition of $BG_p^\wedge$ in [JMO]. □

6.7. **p-compact groups.** A p-compact group is a triple (X, BX, e), where BX is a pointed, connected, p-complete space, $e\colon X \longrightarrow \Omega(BX)$ is a homotopy equivalence, and $H^*(X;\mathbb{F}_p)$ is finite. These objects were introduced in the 1980's by Dwyer and Wilkerson [DW3], and extensively studied by them and many other authors. The concept of a p-compact group was designed to be a homotopy theoretic analog of a classifying space of a compact Lie group. For instance, if G is a compact Lie group whose group of components is a p-group, then $BG_p^\wedge$ is the classifying space of a p-compact group. In contrast, if $\pi_0(G)$ is not a p-group, then the mod p cohomology of the loop space $\Omega(BG_p^\wedge)$ is not in general finite.

One of our original motivations for defining p-local compact groups was to provide a wider framework for studying p-compact groups. This requires first checking that every p-compact group X can be considered as a p-local compact group; in particular, that $BX \simeq |\mathcal{L}|_p^\wedge$ for some centric linking system $\mathcal{L}$ associated to a saturated fusion system $\mathcal{F}$ over a discrete p-toral group.

By [DW3, §8–9], for any p-compact group X, there is a unique maximal p-toral subgroup, hence a unique maximal discrete p-toral subgroup $f\colon BS \longrightarrow BX$ (Proposition 6.1); and any other map from the classifying space of a discrete p-toral group to BX factors through f. We set

$$\mathcal{F}_{S,f}(X) \stackrel{\text{def}}{=} \mathcal{F}_{S,f}(BX) \qquad \text{and} \qquad \bar{\mathcal{L}}_{S,f}^c(X) \stackrel{\text{def}}{=} \mathcal{L}_{S,f}^c(BX),$$

where the fusion and linking systems of the space BX are defined as in Section 2.5. Using the properties of the mapping spaces $\mathrm{Map}(BP, BX)$ described in [DW3, §5–6] (for a discrete p-toral group P), we check that $\mathcal{F}_{S,f}(X)$ is a saturated fusion system over S, and that $\bar{\mathcal{L}}_{S,f}^c(X)$ is a reduced centric linking system associated to $\mathcal{F}_{S,f}(X)$. Let $\mathcal{L}_{S,f}^c(X)$ be the centric linking system associated to $\bar{\mathcal{L}}_{S,f}^c(X)$ (and hence to $\mathcal{F}_{S,f}(X)$) by Proposition 6.4. We then show:

Theorem 6.6. *For any p-compact group X and any maximal discrete p-toral subgroup $f\colon BS \longrightarrow BX$, $(S, \mathcal{F}_{S,f}(X), \mathcal{L}_{S,f}^c(X))$ is a p-local compact group with classifying space*

$$|\mathcal{L}_{S,f}^c(X)|_p^\wedge \simeq BX\,.$$

Proof. See [BLO3, §9]. The proof that $|\mathcal{L}_{S,f}^c(X)|_p^\wedge \simeq BX$ is based on the decomposition, shown in [CLN], of BX as a homotopy direct limit of BP's for p-toral (or discrete p-toral) subgroups $P \le X$. □

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