CORRECTION TO: REDUCTIONS TO SIMPLE FUSION SYSTEMS

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Abstract. We fill in a gap in the proof of the main theorem in our earlier paper [Ol]. At the same time, we prove a slightly stronger version of the theorem needed for another paper.

The main theorem in our earlier paper [Ol] stated (very roughly) that if $\mathcal{E} \le \mathcal{F}$ are saturated fusion systems such that $\mathcal{E}$ is normal in $\mathcal{F}$ and satisfies certain additional conditions, then there is a sequence of saturated fusion subsystems $\mathcal{E} = \mathcal{F}_0 \trianglelefteq \mathcal{F}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{F}_m = \mathcal{F}$, each normal in the following system and normal in $\mathcal{F}$, such that $\mathcal{F}_i$ has p-power index or index prime to p in $\mathcal{F}_{i+1}$ for each i. We refer to [Ol, Theorem 2.3], or to Theorems 5 and 6 below, for the precise statement.

The theorem was proven by an inductive argument, where we assume that $\mathcal{F}_{i+1}$ has already been constructed with certain properties before constructing $\mathcal{F}_i$. This inductive argument requires that $\mathcal{E}$ be normal in $\mathcal{F}_i$ for each i, a property that was not justified in [Ol]. The missing details are not hard to fill in, but we think it's best to do so formally, especially since the theorem has been applied by various people, either directly as in [HL], or indirectly via Lemma 2.22 in [AO].

Most of the notation and terminology used in [Ol] will be assumed here; we refer to that paper for their definitions. As one exception, since the details of the definition of normal fusion subsystems play an important role here, we begin by recalling them.

Definition 1. Let $\mathcal{E} \le \mathcal{F}$ be saturated fusion systems over finite p-groups $T \le S$. The subsystem $\mathcal{E}$ is *weakly normal* in $\mathcal{F}$ if

- T is strongly closed in $\mathcal{F}$ (in particular, $T \trianglelefteq S$), and
- (strong invariance condition) for each $P \le Q \le T$, each $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P,Q)$, and each $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q,T)$, $\psi\varphi(\psi|_P)^{-1} \in \mathrm{Hom}_{\mathcal{E}}(\psi(P),\psi(Q))$.

The subsystem $\mathcal{E}$ is *normal* in $\mathcal{F}$ ($\mathcal{E} \trianglelefteq \mathcal{F}$) if it is weakly normal and

- (extension condition) each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to some $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[\overline{\alpha}, C_S(T)] \le Z(T)$.

This is different than the definition of a normal fusion subsystem used in [Ol], but the two definitions are equivalent by [AKO, Proposition I.6.4]. This one has the advantage that it simplifies the proof of point (a) in the following proposition.

Proposition 2. *Let $\mathcal{E} \le \mathcal{F}_0 \le \mathcal{F}$ be saturated fusion systems over $T \le S_0 \le S$. Then the following hold.*

(a) *If $\mathcal{E}$ is weakly normal in $\mathcal{F}$, then $\mathcal{E}$ is weakly normal in $\mathcal{F}_0$.*

(b) *If $\mathcal{E} \trianglelefteq \mathcal{F}$ and $\mathcal{E} = O^{p'}(\mathcal{E})$, then $\mathcal{E} \trianglelefteq \mathcal{F}_0$.*

2000 *Mathematics Subject Classification.* Primary 20E25. Secondary 20D20, 20D05, 20D25, 20D45.

Key words and phrases. Fusion systems, Sylow subgroups, finite simple groups, generalized Fitting subgroup, p-solvable groups.

B. Oliver is partially supported by UMR 7539 of the CNRS.

(c) *If* $\mathcal{E} \trianglelefteq \mathcal{F}$ *and* $\mathcal{F}_0$ *has* p*-power index in* $\mathcal{F}$ [AKO, Definition I.7.3]*, then* $\mathcal{E} \trianglelefteq \mathcal{F}_0$*.*

Proof. If T is strongly closed in $\mathcal{F}$, then it is also strongly closed in $\mathcal{F}_0$. If the strong invariance condition holds for $\mathcal{E} \le \mathcal{F}$, then it also holds for $\mathcal{E} \le \mathcal{F}_0$. This proves (a).

Point (b) was shown by David Craven in [Cr, Corollary 8.19].

Under the hypotheses of (c), $\mathcal{E}$ is weakly normal in $\mathcal{F}_0$ by (a), and it remains to prove that each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}_0}(TC_{S_0}(T))$ such that $[\bar{\alpha}, C_{S_0}(T)] \le Z(T)$. This clearly holds for $\alpha \in \mathrm{Inn}(T)$, and since $\mathrm{Inn}(T) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{E}}(T))$, it suffices to show it for $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ of order prime to p. For such α, since $\mathcal{E} \trianglelefteq \mathcal{F}$, there is $\widehat{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $\widehat{\alpha}|_T = \alpha$ and $[\widehat{\alpha}, C_S(T)] \le Z(T)$, and $\widehat{\alpha}$ restricts to $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_{S_0}(T))$ since $[\widehat{\alpha}, C_S(T)] \le T$. Upon replacing $\bar{\alpha}$ by $\bar{\alpha}^k$ for some appropriate $k \equiv 1 \pmod{|\alpha|}$, we can arrange that $\bar{\alpha}$ has order prime to p (and still $\bar{\alpha}|_T = \alpha$ and $[\bar{\alpha}, C_{S_0}(T)] \le Z(T)$). But then $\bar{\alpha} \in O^p(\mathrm{Aut}_{\mathcal{F}}(TC_{S_0}(T))) \le \mathrm{Aut}_{\mathcal{F}_0}(TC_{S_0}(T))$ since $\mathcal{F}_0$ has p-power index in $\mathcal{F}$, proving the extension condition for $\mathcal{E} \le \mathcal{F}_0$. □

We refer to [Cr, Example 8.18] for an example of saturated fusion systems $\mathcal{E} \le \mathcal{F} \le \widehat{\mathcal{F}}$ where $\mathcal{E} \trianglelefteq \widehat{\mathcal{F}}$, and $\mathcal{E}$ is weakly normal but not normal in $\mathcal{F}$.

Proposition 3 (Compare with [Ol, Proposition 1.8])**.** *Let* $\mathcal{E} \trianglelefteq \mathcal{F}$ *be saturated fusion systems over finite* p*-groups* $T \trianglelefteq S$*. Let* $\chi_0\colon \mathrm{Aut}_{\mathcal{F}}(T) \longrightarrow \Delta$ *be a surjective homomorphism, for some* $\Delta \ne 1$ *of order prime to* p*, such that* $\mathrm{Aut}_{\mathcal{E}}(T) \le \mathrm{Ker}(\chi_0)$*. Then there is a unique proper normal subsystem* $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ *over* S *such that*

$$\mathrm{Aut}_{\mathcal{F}_0}(S) = \big\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \alpha|_T \in \mathrm{Ker}(\chi_0)\big\}, \tag{1}$$

and $\mathrm{Aut}_{\mathcal{F}_0}(T) = \mathrm{Ker}(\chi)$ *and* $\mathcal{E} \trianglelefteq \mathcal{F}_0$*.*

Proof. This was shown in [Ol, Proposition 1.8], except for the statements that $\mathrm{Aut}_{\mathcal{F}_0}(T) = \mathrm{Ker}(\chi)$ and $\mathcal{E}$ is normal in $\mathcal{F}_0$. Since $\mathcal{E}$ is weakly normal in $\mathcal{F}_0$ by Proposition 2(a), normality will follow one we have checked the extension condition.

Set $\mathcal{H}^* = \{P \in \mathcal{F}^c \,|\, P \cap T \in \mathcal{E}^c\}$. Thus $TC_S(T) \in \mathcal{H}^*$. In the proof of [Ol, Proposition 1.8], we construct a map

$$\widehat{\chi}\colon \mathrm{Mor}(\mathcal{F}|_{\mathcal{H}^*}) \longrightarrow \Delta$$

with the property that for each $P \in \mathcal{H}^*$ such that $P \ge T$, and each $\beta \in \mathrm{Aut}_{\mathcal{F}}(P)$, we have $\widehat{\chi}(\beta) = \chi_0(\beta|_T)$ and $\mathrm{Aut}_{\mathcal{F}_0}(P) = \mathrm{Aut}_{\mathcal{F}}(P) \cap \widehat{\chi}^{-1}(1)$. (Note that $\beta|_T \in \mathrm{Aut}_{\mathcal{F}}(T)$ since T is strongly closed in $\mathcal{F}$.) So (1) holds, and

$$\mathrm{Aut}_{\mathcal{F}_0}(TC_S(T)) = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T)) \,|\, \alpha|_T \in \mathrm{Ker}(\chi_0)\}. \tag{2}$$

Since each $\beta \in \mathrm{Aut}_{\mathcal{F}_0}(T)$ extends to some $\bar{\beta} \in \mathrm{Aut}_{\mathcal{F}_0}(TC_S(T))$ by the extension axiom [AKO, Proposition I.2.5] applied to $\mathcal{F}_0$, (2) shows that $\mathrm{Aut}_{\mathcal{F}_0}(T) \le \mathrm{Ker}(\chi_0)$. Similarly, each $\gamma \in \mathrm{Ker}(\chi_0)$ extends to some $\bar{\gamma} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ by the extension axiom for $\mathcal{F}$, and $\bar{\gamma}, \gamma \in \mathrm{Mor}(\mathcal{F}_0)$ by (2) again. Thus $\mathrm{Aut}_{\mathcal{F}_0}(T) = \mathrm{Ker}(\chi_0)$.

For each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$, the extension condition for $\mathcal{E} \trianglelefteq \mathcal{F}$ implies that there exists $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ extending α and with $[\bar{\alpha}, C_S(T)] \le Z(T)$. Then $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}_0}(TC_S(T))$ by (2) and since $\bar{\alpha}|_T = \alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$. So the extension condition holds for $\mathcal{E} \le \mathcal{F}_0$, proving that $\mathcal{E} \trianglelefteq \mathcal{F}_0$. □

Definition 4. Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over finite p-groups $T \trianglelefteq S$, and define

$$C_S(\mathcal{E}) = \{x \in S \,|\, C_{\mathcal{F}}(x) \ge \mathcal{E}\}.$$

We say that $\mathcal{E}$ is *centric* in $\mathcal{F}$ if $C_S(\mathcal{E}) \subseteq T$.

By a theorem of Aschbacher (see Notation 6.1 and (6.7.1) in [As]), for each such $\mathcal{E} \trianglelefteq \mathcal{F}$, $C_S(\mathcal{E})$ is a subgroup of S, and $C_{\mathcal{F}}(C_S(\mathcal{E}))$ contains $\mathcal{E}$. Thus each morphism in $\mathcal{E}$ extends to a morphism in $\mathcal{F}$ between subgroups containing $C_S(\mathcal{E})$ that is the identity on $C_S(\mathcal{E})$.

For each saturated fusion system $\mathcal{F}$ over a finite p-group S, we set

$$\mathrm{Aut}(\mathcal{F}) = \{\beta \in \mathrm{Aut}(S) \,|\, {}^{\beta}\mathcal{F} = \mathcal{F}\}:$$

the group of "fusion preserving" automorphisms of S. (This group was denoted $\mathrm{Aut}(S, \mathcal{F})$ in [Ol].) For $\beta \in \mathrm{Aut}(\mathcal{F})$, let c_β be the automorphism of the category $\mathcal{F}$ that sends P to $\beta(P)$ and sends $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ to $\beta\varphi\beta^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\beta(P), \beta(Q))$.

The next theorem contains most of Theorems 1.14 and 2.3 in [Ol], together with some additional information about automorphisms of the systems.

Theorem 5. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over finite p-groups $T \trianglelefteq S$. Assume that $\mathrm{Aut}_{\mathcal{F}}(T)/\mathrm{Aut}_{\mathcal{E}}(T)$ is p-solvable (equivalently, that $\mathrm{Out}_{\mathcal{F}}(T)$ is p-solvable).*

(a) *In all cases, there is a sequence*

$\mathcal{F}_0 \le \mathcal{F}_1 \le \mathcal{F}_2 \le \cdots \le \mathcal{F}_m = \mathcal{F}$ *of saturated fusion subsystems (for some $m \ge 0$) such that for each $0 \le i < m$, $\mathcal{F}_i$ is normal of p-power index or index prime to p in $\mathcal{F}_{i+1}$ and $\mathcal{E} \trianglelefteq \mathcal{F}_i \trianglelefteq \mathcal{F}$;* (3)

and such that $\mathcal{F}_0$ is a fusion system over $TC_S(T)$ and $\mathrm{Aut}_{\mathcal{F}_0}(T) = \mathrm{Aut}_{\mathcal{E}}(T)$.

(b) *If $\mathcal{E}$ is centric in $\mathcal{F}$, then there is a sequence of subsystems satisfying (3) such that $\mathcal{F}_0 = \mathcal{E}$.*

In either case, the subsystems can be chosen so that for each $1 \le j \le m$, and each $\beta \in \mathrm{Aut}(\mathcal{F}_j)$ with $c_\beta(\mathcal{E}) = \mathcal{E}$, we have $c_\beta(\mathcal{F}_i) = \mathcal{F}_i$ for all $0 \le i < j$.

Proof. We outline here the proof as given in [Ol]: enough to explain how Propositions 2(c) and 3 are used to prove that $\mathcal{E} \trianglelefteq \mathcal{F}_i$ for each i, and explain why the last statement is true. We refer frequently to the following transitivity result for normality (see [As, 7.4]):

If $\mathcal{F}_2 \trianglelefteq \mathcal{F}_1 \trianglelefteq \mathcal{F}$ are saturated fusion systems over finite p-groups $S_2 \trianglelefteq S_1 \trianglelefteq S$ such that $c_\alpha(\mathcal{F}_2) = \mathcal{F}_2$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_1)$, then $\mathcal{F}_2 \trianglelefteq \mathcal{F}$. (4)

(a) Set $G = \mathrm{Aut}_{\mathcal{F}}(T)$ and $G_0 = \mathrm{Aut}_{\mathcal{E}}(T)$. Since G/G_0 is p-solvable, there are subgroups $G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m = G$ (some $m \ge 0$) such that for each $0 \le i < m$, either $G_i = O^p(G_{i+1})G_0$ (hence G_{i+1}/G_i is a p-group), or $G_i = O^{p'}(G_{i+1})G_0$ (hence G_{i+1}/G_i has order prime to p). In particular, the G_i are all normal in G since G_0 is. For each i, set $S_i = N_S^{G_i}(T) \overset{\mathrm{def}}{=} \{x \in S \,|\, c_x \in G_i\}$. Thus $S_i \trianglelefteq S$ and $\mathrm{Aut}_{S_i}(T) = G_i \cap \mathrm{Aut}_S(T) \in \mathrm{Syl}_p(G_i)$ for each i, $S_m = S$, and $S_0 = TC_S(T)$. We will construct successively subsystems $\mathcal{F} = \mathcal{F}_m \ge \mathcal{F}_{m-1} \ge \cdots \ge \mathcal{F}_0$ in $\mathcal{F}$ such that for each $0 \le i \le m-1$, $\mathcal{F}_i$ is a fusion system over S_i, $\mathrm{Aut}_{\mathcal{F}_i}(T) = G_i$, and the conditions on $\mathcal{F}_i$ in (3) all hold.

Assume, for some $0 \le i < m$, that $\mathcal{F}_{i+1} \trianglelefteq \mathcal{F}$ has been constructed satisfying these conditions. Thus $\mathrm{Aut}_{\mathcal{F}_{i+1}}(T) = G_{i+1}$. If G_{i+1}/G_i has order prime to p, then by Proposition 3, applied with G_{i+1}/G_i in the role of Δ, there is a unique saturated subsystem $\mathcal{F}_i \trianglelefteq \mathcal{F}_{i+1}$ of index prime to p over $S_{i+1} = S_i$ such that $\mathrm{Aut}_{\mathcal{F}_i}(S_{i+1}) = \{\alpha \in \mathrm{Aut}_{\mathcal{F}_{i+1}}(S_{i+1}) \,|\, \alpha|_T \in G_i\}$, and also $\mathcal{E} \trianglelefteq \mathcal{F}_i$ and $\mathrm{Aut}_{\mathcal{F}_i}(T) = G_i$.

If G_{i+1}/G_i is a p-group, then the argument in the proof of [Ol, Theorem 1.14] shows that there is a unique $\mathcal{F}_i \trianglelefteq \mathcal{F}_{i+1}$ over S_i of p-power index such that $\mathrm{Aut}_{\mathcal{F}_i}(T) = G_i$ and $\mathcal{E} \le \mathcal{F}_i$. Also, $\mathcal{E} \trianglelefteq \mathcal{F}_i$ by Proposition 2(c). Since $\mathrm{Aut}_{\mathcal{F}_i}(T)$ has p-power index in

$\mathrm{Aut}_{\mathcal{F}_{i+1}}(T)$, we have

$$\mathrm{Aut}_{\mathcal{F}_i}(T) \geq O^p(\mathrm{Aut}_{\mathcal{F}_{i+1}}(T))\mathrm{Aut}_{S_i}(T) \leq O^p(G_{i+1})G_i = G_i,$$

where the first inclusion is an equality since $\mathrm{Aut}_{S_i}(T) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_i}(T))$ and the second is an equality since $\mathrm{Aut}_{S_i}(T) \in \mathrm{Syl}_p(G_i)$. Thus $\mathrm{Aut}_{\mathcal{F}_i}(T) = G_i$.

(b) By (a) and (4), it suffices to prove this when $S = TC_S(T)$. By [Ol, Corollary 2.2], $S/T = TC_S(T)/T$ is abelian.

Set $\mathcal{H} = \{P \leq S \mid P \geq C_S(T)\}$, and let $\mathcal{F}^* \subseteq \mathcal{F}$ be the full subcategory with $\mathrm{Ob}(\mathcal{F}^*) = \mathcal{H}$. Define

$$\chi \colon \mathrm{Mor}(\mathcal{F}^*) \longrightarrow \mathrm{Aut}_{\mathcal{F}/T}(S/T)$$

by sending $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ to the induced automorphism of

$$S/T = PT/T = QT/T \cong P/(P\cap T) \cong Q/(Q \cap T).$$

Here, $PT = QT = S$ since $P,Q \in \mathcal{H}$ and $S = TC_S(T)$, and $\varphi(P\cap T) \leq Q\cap T$ since T is strongly closed. Thus each $\varphi \in \mathrm{Mor}(\mathcal{F}^*)$ factors through some $\bar\varphi \in \mathrm{Aut}(S/T)$. In the notation of Craven [Cr, Definition 5.5], $\bar\varphi \in \mathrm{Aut}_{\overline{\mathcal{F}}_T}(S/T)$, and so $\bar\varphi \in \mathrm{Aut}_{\mathcal{F}/T}(S/T)$ by [Cr, Theorem 5.14]. See also [As, Theorem 12.5] for a different proof that $\bar\varphi \in \mathrm{Mor}(\mathcal{F}/T)$.

We now apply [Ol, Lemma 1.6], whose hypotheses (i)–(v) are shown to hold in the proof of [Ol, Theorem 2.3]. By that lemma, $\mathcal{F}_2 \overset{\text{def}}{=} \langle \chi^{-1}(1)\rangle$ is a saturated fusion subsystem over S normal of index prime to p in $\mathcal{F}$ such that $\mathrm{Aut}_{\mathcal{F}_2}(S) = \mathrm{Ker}(\chi|_{\mathrm{Aut}_{\mathcal{F}}(S)})$.

By Proposition 2(a), $\mathcal{E}$ is weakly normal in $\mathcal{F}_2$. If $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$, then since $\mathcal{E} \trianglelefteq \mathcal{F}$, α extends to $\bar\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $[\bar\alpha, C_S(T)] \leq Z(T)$. Since $S = TC_S(T)$, this implies that $\chi(\bar\alpha) = 1$, and hence that $\bar\alpha \in \mathrm{Aut}_{\mathcal{F}_2}(S)$. So the extension condition holds, and $\mathcal{E} \trianglelefteq \mathcal{F}_2$.

The construction of $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$ of p-power index such that $\mathcal{E} \trianglelefteq \mathcal{F}_1$ and has index prime to p follows from exactly the same argument as used in [Ol], except that $\mathcal{E}$ is normal in $\mathcal{F}_1$ by Proposition 2(c).

(a,b) It remains to prove the last statement (invariance under automorphisms), and show that $\mathcal{F}_i \trianglelefteq \mathcal{F}$ for all i (not only for $i = m-1$). To see this, choose $1 \leq j \leq m$ and $\beta \in \mathrm{Aut}(\mathcal{F}_j) \leq \mathrm{Aut}(S_j)$ such that $c_\beta(\mathcal{E}) = \mathcal{E}$. Then $\beta(T) = T$ and ${}^\beta\mathrm{Aut}_{\mathcal{E}}(T) = \mathrm{Aut}_{\mathcal{E}}(T)$. In (b), we have $c_\beta(\mathcal{F}_i) = \mathcal{F}_i$ for $0 \leq i < j$ by the uniqueness of choices of subsystems at each stage. In (a), ${}^\beta\mathrm{Aut}_{\mathcal{F}_i}(T) = \mathrm{Aut}_{\mathcal{F}_i}(T)$ for each $0 \leq i < j$ by construction of $G_i = \mathrm{Aut}_{\mathcal{F}_i}(T)$, and hence by the uniqueness of the choices (depending only on the G_i), we have $c_\beta(\mathcal{F}_i) = \mathcal{F}_i$.

In particular, if $0 \leq i < m$ is such that $\mathcal{F}_{i+1} \trianglelefteq \mathcal{F}$, this says that $c_\beta(\mathcal{F}_i) = \mathcal{F}_i$ for each $\beta \in \mathrm{Aut}_{\mathcal{F}}(S_{i+1}) \leq \mathrm{Aut}(\mathcal{F}_{i+1})$, and together with (4), it implies that $\mathcal{F}_i \trianglelefteq \mathcal{F}$. It now follows inductively that $\mathcal{F}_i \trianglelefteq \mathcal{F}$ for each i. □

For each saturated fusion system $\mathcal{F}$ over a finite p-group S, set $\mathcal{F}^\infty = \mathcal{F}_0$ for any sequence $\mathcal{F}_0 \trianglelefteq \mathcal{F}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{F}_m = \mathcal{F}$ of saturated subsystems such that $O^p(\mathcal{F}_0) = O^{p'}(\mathcal{F}_0) = \mathcal{F}_0$, and such that $\mathcal{F}_i \trianglelefteq \mathcal{F}$ and $\mathcal{F}_i$ has index prime to p or p-power index in $\mathcal{F}_{i+1}$ for each $0 \leq i < m$. By [Ol, Lemma 1.13], $\mathcal{F}^\infty$ is independent of the choice of the $\mathcal{F}_i$.

Theorem 6 ([Ol, Theorem 2.3])**.** *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over finite p-groups $T \trianglelefteq S$ such that $\mathcal{E}$ is centric in $\mathcal{F}$. Assume either*

(a) $\mathrm{Aut}_{\mathcal{F}}(T)/\mathrm{Aut}_{\mathcal{E}}(T)$ *is p-solvable; or*

(b) $\mathrm{Out}(\mathcal{E}) \stackrel{\text{def}}{=} \mathrm{Aut}(\mathcal{E})/\mathrm{Aut}_{\mathcal{E}}(T)$ *is* p*-solvable.*

Then $\mathcal{F}^\infty = \mathcal{E}^\infty$.

Proof. Since $\mathrm{Aut}_{\mathcal{F}}(T) \le \mathrm{Aut}(\mathcal{E})$ (since $\mathcal{E} \trianglelefteq \mathcal{F}$), we have $\mathrm{Aut}_{\mathcal{F}}(T)/\mathrm{Aut}_{\mathcal{E}}(T) \le \mathrm{Out}(\mathcal{E})$. So (b) implies (a). It thus suffices to prove the theorem when (a) holds, and this follows immediately from Theorem 5(b) and [Ol, Lemma 1.13]. □

References

[As] M. Aschbacher, The generalized Fitting subsystem of a fusion system, Memoirs Amer. Math. Soc. 209 (2011), nr. 986

[AKO] M. Aschbacher, R. Kessar, & B. Oliver, Fusion systems in algebra and topology, Cambridge Univ. Press (2011)

[AO] M. Aschbacher & B. Oliver, Fusion systems, Bull. Amer. Math. Soc. 53 (2016), 555–615

[Cr] D. Craven, The theory of fusion systems, an algebraic approach, Cambridge Studies in Advanced Mathematics 131, Cambridge Univ. Press (2011)

[HL] E. Henke & J. Lynd, Fusion systems with Benson-Solomon components, arXiv:1806.01938

[Ol] B. Oliver, Reductions to simple fusion systems, Bull. London Math. Soc. 48 (2016), 923–934

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