THE REPRESENTATION RING OF A COMPACT LIE GROUP REVISITED

BY BOB OLIVER

ABSTRACT. We describe a new construction of the induction homomorphism for representation rings of compact Lie groups: a homomorphism first defined by Graeme Segal. The idea is to first define the induction homomorphism for class functions, and then show that this map sends characters to characters. This requires a detection theorem — a class function of G is a character if its restrictions to certain subgroups of G are characters — which in turn requires a review of the representation theory for nonconnected compact Lie groups.

In his 1968 paper, Segal [Seg] used elliptic operators to construct induction homomorphisms $\mathrm{R}(H) \to \mathrm{R}(G)$ for an arbitrary pair $H \subseteq G$ of compact Lie groups, and then applied this to prove (among other things) a detection result for when a class function on G is a character. In this paper, we give new proofs of these results, but in the reverse order. We begin in Section 1 by showing that a class function on G is a character if its restrictions to all finite subgroups of G are characters. Then, in Section 2, we first define induction homomorphisms $\mathrm{Cl}(H) \to \mathrm{Cl}(G)$ for class functions, and afterwards apply the results of Section 1 to show that they send characters to characters and hence define induction maps between the representation rings. This gives a construction of the induction homomorphisms which is more elementary than that of Segal (though also less elegant), in that it only assumes the standard theory of representations of a compact connected Lie group.

It is the results in Section 3 which, while more technical, provided the original motivation for this work. Let $\mathcal{S}_{\mathcal{P}}(G)$ be the family of all p-toral subgroups of G (for all primes p), where a group is called p-toral if it is an extension of a torus by a finite p-group. Let $\mathrm{R}_{\mathcal{P}}(G)$ be the inverse limit of the representation rings $\mathrm{R}(P)$ for all $P \in \mathcal{S}_{\mathcal{P}}(G)$, where the limit is taken with respect to restriction and conjugation in G. This group $\mathrm{R}_{\mathcal{P}}(G)$ was shown in [JO, Theorem 1.8] to be isomorphic to the Grothendieck group $\mathbb{K}(BG)$ of the monoid of vector bundles over BG; and the "restriction" homomorphism

$$\mathrm{rs}_G : \mathrm{R}(G) \longrightarrow \mathrm{R}_{\mathcal{P}}(G) = \varprojlim_{P \in \mathcal{S}_{\mathcal{P}}(G)} \mathrm{R}(P)$$

is isomorphic to the natural homomorphism $\mathrm{R}(G) \to \mathbb{K}(BG)$ which sends a representation V to the bundle $(EG \times_G V) \downarrow BG$.

1991 *Mathematics Subject Classification.* Primary: 22E99; secondary: 22E46, 55R40, 55R50.
Key words and phrases. Induced representations, class functions.

The main result of Section 3 is a description, for arbitrary G, of the cokernel of the homomorphism rs_G. In particular, we show that it is onto whenever G is finite or $\pi_0(G)$ is a p-group; but that it is not surjective in general. Precise necessary and sufficient conditions for rs_G to be onto are given in Theorem 3.10, and several simpler sufficient conditions are given in Corollary 3.11. Note that rs_G is surjective if and only if bundles over BG have the following property: for each $\xi{\downarrow}BG$ there exist G-representations V, V' such that $\xi \oplus (EG{\times_G}V') \cong (EG{\times_G}V)$ (since by [JO, Theorem 1.8], every bundle over BG is a summand of a bundle coming from a G-representation).

In the above discussion, we have for simplicity dealt only with the complex representation rings. But most of the results are shown below for real as well as complex representations.

I would like to thank in particular Stefan Jackowski for his comments and suggestions about this work. Originally, Sections 1 and 3 were intended to go into our joint paper [JO], but then they grew to the point where we decided to publish them separately. I would also like to thank the colleague who, at the 1996 summer research institute in Seattle, showed me the references [Ta] and [Vo] on representation theory for nonconnected compact Lie groups. (After that conference, I asked several people if they were the ones who had done so, but they all denied it.)

Section 1 Detection of characters

The main results of this section are Propositions 1.2 and 1.5: on detecting characters among class functions. They follow from Proposition 1.4, which describes the representation theory of nonconnected compact Lie groups. The first part of Proposition 1.4 — the bijection between irreducible G-representations and certain irreducible representations of $N_G(T,C)$ — was proven by Takeuchi [Ta, Theorem 4], and is also stated in [Vo, Theorem 1.17]. Since their notation is very different from that used here, we have found it simplest to keep our proof, rather than just refer to [Ta]. Note that the group which we call $N = N_G(T,C)$ is denoted T in [Ta] and [Vo] (and called the Cartan subgroup in [Vo]).

Throughout this section, G denotes a fixed compact Lie group, and G_0 is its identity connected component. Fix a maximal torus $T \subseteq G$, let $W_G = N_G(T)/T$ denote its Weyl group, and let $\mathfrak{t} \subseteq \mathfrak{g}$ denote the Lie algebras of T and G. For any Weyl chamber $C \subseteq \mathfrak{t}$, define

$$N_G(T,C) = \bigl\{g \in N_G(T) \bigm| \mathrm{Ad}(g)(C) = C\bigr\},$$

and

$$N_G(T,\pm C) = \bigl\{g \in N_G(T) \bigm| \mathrm{Ad}(g)(\pm C) = (\pm C)\bigr\}.$$

Here, $\mathrm{Ad}(g)$ denotes the adjoint (conjugation) action of g on $\mathfrak{t}$ and $\mathfrak{g}$. We will see in Proposition 1.1 that $N_G(T,C)$ has exactly one connected component for each connected component of G, and that every element of G is conjugate to an element of $N_G(T,C)$. Then, in Proposition 1.2 below, we show that a (continuous) class function $f \in \mathrm{Cl}(G)$ is a character of G if and only if $f|N_G(T,C)$ is a character, and that f is a real character of G (i.e., the character of a virtual $\mathbb{R}G$-representation) if and only if $f|N_G(T,\pm C)$ is a real character. At the same time, we construct (Proposition 1.4) a one-to-one correspondence between the irreducible representations of G, and those irreducible representations of $N_G(T,C)$ whose weights lie in the dual Weyl chamber C^*. This generalizes the standard relationship, for a connected compact Lie group G, between the irreducible representations of G and those of T.

Afterwards, the detection result is extended to show that an element $f \in \mathrm{Cl}(G)$ is a character (real character) if and only if $f|H$ is a character (real character) of H for each finite subgroup $H \subseteq G$. The classical theorem of Brauer for detecting characters on finite groups can then be applied to further restrict the class of finite subgroups of G which have to be considered.

We first recall the definition and basic properties of the Weyl chambers of a compact connected Lie group G. The set of irreducible representations (or irreducible characters) of T will be identified here with $T^* \stackrel{\text{def}}{=} \mathrm{Hom}(T, S^1)$; which will in turn be regarded as a lattice in $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$.

The *roots* of G (or of G_0) are the characters of the nontrivial irreducible summands of the adjoint representation of T on $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g}$. They occur in pairs $\pm\theta$. Let $R \subseteq T^* \subseteq \mathfrak{t}^*$ denote the set of roots of G. Any element $x_0 \in \mathfrak{t}$ such that $\theta(x_0) \neq 0$ for all $\theta \in R$ determines a choice of positive roots

$$R_+ = \{\theta \in R \,|\, \theta(x_0) > 0\}.$$

And this in turn determines a Weyl chamber

$$C = \{x \in \mathfrak{t} \,|\, \theta(x) \geq 0 \ \forall\theta \in R_+\} \subseteq \mathfrak{t}$$

and a dual Weyl chamber

$$C^* = \{x \in \mathfrak{t}^* \,|\, \langle\theta, x\rangle \geq 0 \ \forall\theta \in R_+\} \subseteq \mathfrak{t}^*.$$

Here, in the definition of C^*, $\langle -, - \rangle$ denotes any G-invariant inner product on $\mathfrak{g}^*$. Note that C^* is independent of the choice of inner product, since a G-invariant inner product is uniquely defined up to scalar on each simple component of G.

Proposition 1.1. *Fix a maximal torus $T \subseteq G$ and a Weyl chamber $C \subseteq \mathfrak{t}$, and set $N = N_G(T,C)$. Then $N \cap G_0 = T$, $N{\cdot}G_0 = G$, and hence $N/T \cong G/G_0$. Also, any element of G is conjugate to an element of N.*

Proof. Recall that the Weyl group $W_{G_0} = N_{G_0}(T)/T$ of G_0 permutes the Weyl chambers of T simply and transitively (cf. [Ad, Lemma 5.13]). Hence each coset of $N_{G_0}(T)$ in $N_G(T)$ contains exactly one connected component of $N = N_G(T,C)$; and so $N \cap G_0 = T$, $N{\cdot}G_0 = G$, and $N/T \cong G/G_0$.

By [Bo, §5.3, Theorem 1(b)], any automorphism of G_0 leaves invariant some maximal torus and some Weyl chamber in G_0. Hence, any element $g{\in}G$ is contained in $N_G(T', C')$ for some maximal torus T' and some Weyl chamber $C' \subseteq T'$; and T' and

T are conjugate in G_0 (cf. [Ad, Corollary 4.23]). Since $N_{G_0}(T)/T$ permutes the Weyl chambers for T transitively, there is $a \in G_0$ such that $T = aT'a^{-1}$ and $C = aC'a^{-1}$; and $aga^{-1} \in N = N_G(T, C)$. $\square$

When dealing with real representations, we need to distinguish between the different types of irreducible representations and characters. As usual, we say that a G-representation V (over $\mathbb{C}$) has *real type* if it has the form $V \cong \mathbb{C}\otimes_{\mathbb{R}} V'$ for some $\mathbb{R}G$-representation V'; and that V has *quaternion type* if it is the restriction of an $\mathbb{H}G$-representation. If V is irreducible and its character is real-valued, then V has real or quaternion type, but not both [Ad, Proposition 3.56]. By a real character will be meant the character of a virtual representation of real type (i.e., the difference of two representations of real type).

Proposition 1.2. *Fix a maximal torus $T \subseteq G$ and a Weyl chamber $C \subseteq \mathfrak{t}$. Then a continuous class function $f : G \to \mathbb{C}$ is a character of G if and only if $f\big|N_G(T, C)$ is a character of $N_G(T, C)$. And a continuous class function $f : G \to \mathbb{R}$ is a real character of G if and only if $f\big|N_G(T, \pm C)$ is a real character.*

The proof of Proposition 1.2 will be given after that of Proposition 1.4 below. We first note some elementary conditions for f to be a character or a real character. In the following lemma, we write as usual $\langle\varphi, \psi\rangle = \int_G \varphi(g)\overline{\psi(g)}$ for any pair of continuous functions $\varphi, \psi : G \to \mathbb{C}$ (where the integral is the Haar integral on G with measure 1).

Lemma 1.3. *(a) A class function $f \in \mathrm{Cl}(G)$ is a character of G if and only if $\langle f, \chi\rangle \in \mathbb{Z}$ for each character χ of G.*

(b) A class function $f : G \to \mathbb{R}$ is a real character of G if and only if f is a character, and $\langle f, \chi_V\rangle \in 2\mathbb{Z}$ for each G-representation V of quaternion type.

(c) A class function $f : G \to \mathbb{R}$ is a real character of G if f is a character, and $f|H$ is a real character of H for some $H \lhd G$ of finite odd index.

Proof. For any pair W, V of complex G-representations,

$$\langle\chi_W, \chi_V\rangle = \dim_{\mathbb{C}}\big((W^* \otimes_{\mathbb{C}} V)^G\big) = \dim_{\mathbb{C}}\big(\mathrm{Hom}_{\mathbb{C}G}(W, V)\big) \in \mathbb{Z}.$$

(Recall that $\chi_{W^*}(g) = \overline{\chi_W(g)}$ for all $g \in G$.) Also, if W has real type and V has quaternion type, then $\mathrm{Hom}_{\mathbb{C}G}(W, V)$ is a quaternion vector space, and so its complex dimension is even. This proves the “only if” parts of points (a) and (b).

Conversely, assume that $f \in \mathrm{Cl}(G)$ is such that $\langle f, \chi\rangle \in \mathbb{Z}$ for each character χ of G. Since the irreducible characters form an orthonormal set, we know that $\langle f, f\rangle \ge \sum_{i=1}^k \langle f, \chi_i\rangle^2$ for any set $\chi_1, \ldots, \chi_k$ of distinct irreducible characters. Since each $\langle f, \chi\rangle \in \mathbb{Z}$, this shows that $\langle f, \chi\rangle = 0$ for all but finitely many irreducible chararcters χ; and so $f = \sum_\chi \langle f, \chi\rangle\cdot\chi$ is a character of G by the Peter-Weyl theorem (cf. [Ad, Theorem 3.47]).

We now consider conditions for a real valued character to be a real character; or equivalently for a self-adjoint representation to be of real type. An irreducible

G-representation (over $\mathbb{C}$) is of *complex type* if its character is not real valued; i.e., if $V \not\cong V^*$. It follows from [Ad, Theorem 3.57] that a G-representation V (over $\mathbb{C}$) is of real type if and only if it is a sum of irreducible representations of real type, of representations $\mathbb{C}\otimes_{\mathbb{R}}W \cong W\oplus W^*$ for W irreducible of complex type, and of representations $\mathbb{C}\otimes_{\mathbb{R}}W \cong W\oplus W$ for W irreducible of quaternion type. If $v = \sum_{i=1}^k n_i[V_i] \in \mathrm{R}(G)$ has real valued character, where the V_i are distinct irreducible G-representations, then $\sum_{i=1}^k n_i[V_i] = \sum_{i=1}^k n_i[(V_i)^*]$, and so each pair V_i, $(V_i)^*$ occurs with the same multiplicity. Hence v has real type if $2|n_i$ for each i such that V_i has quaternion type. Since $n_i = \langle \chi_v, \chi_{V_i}\rangle$, this proves point (b).

It remains to prove point (c): that an element $v \in \mathrm{R}(G)$ with real valued character has real type if $v|H$ has real type for some normal subgroup $H \lhd G$ of finite odd index; we may assume that v is the class of an actual $\mathbb{C}[G]$-representation V. Since all irreducible $\mathbb{C}[G/H]$-representations, aside from the trivial one, have complex type (cf. [Ser, Exercise 13.12]), we can write $\mathbb{C}[G/H] \cong \mathbb{C}\oplus W\oplus W^*$ for some representation W. Since by assumption, $V|H$ has real type and $V^* \cong V$, the isomorphism

$$\mathrm{Ind}_H^G(V|H) \cong \mathbb{C}[G/H] \otimes_{\mathbb{C}} V \cong V \oplus (W\otimes_{\mathbb{C}}V) \oplus (W\otimes_{\mathbb{C}}V)^* \cong V \oplus \mathbb{C}\otimes_{\mathbb{R}}(W\otimes_{\mathbb{C}}V)$$

shows that V has real type. $\square$

By a *weight* of the compact Lie group G is meant an element of the lattice $T^* \subseteq \mathfrak{t}^*$, regarded as an irreducible character of T. If V is any representation of G, then the set of "weights of V" is defined to be the set of characters of irreducible components of $V|T$. Consider the partial ordering of the weights of G, where $\phi_1 \le \phi_2$ if ϕ_1 is contained in the convex hull of the W_G-orbit of ϕ_2 (cf. [Ad, Definition 6.23]). One of the basic theorems of representation theory says that if G is *connected*, then any irreducible G-representation V has a unique W_G-orbit of highest (maximal) weights, each of which occurs with multiplicity one. Furthermore, distinct irreducible representations have distinct orbits of higher weights, and every weight of G can be realized as the highest weight of some irreducible G-representation. Thus, the irreducible representations of any connected G are in one-to-one correspondence with the W_G-orbits of weights of G. And since any given dual Weyl chamber $C^* \subseteq \mathfrak{t}^*$ contains exactly one element in each W_G orbit in $\mathfrak{t}^*$ (cf. [Ad, Corollary 5.16]), the irreducible representations of G are in one-to-one correspondence with the weights in C^*. For more detail, see, e.g., [Ad, Theorem 6.33] or [BtD, Section VI.2].

Now assume that G is not connected. If V is an irreducible G-representation, and if V_0 is any irreducible component of $V|G_0$, then V is an irreducible summand of $\mathrm{Ind}_{G_0}^G(V_0)$. Hence each irreducible summand of $V|G_0$ is obtained from V_0 by conjugation by some element of $\pi_0(G)$; and there is still a uniquely defined W_G-orbit of highest weights for V. In this case, however, the highest weights can occur with multiplicity greater than one; and there can be several irreducible G-representations with the same orbit of highest weights.

In the next proposition, $\mathrm{Irr}(G)$ will denote the set of irreducible representations of G. Also, if $N = N_G(T, C)$ (for any maximal torus $T \subseteq G$ and any Weyl chamber $C \subseteq \mathfrak{t}$), then $\mathrm{Irr}(N, C^*)$ denotes the set of irreducible representations of N whose weights all lie in the dual Weyl chamber C^* of C. For any $V \in \mathrm{Irr}(G)$, $\mathrm{mx}_{C^*}(V) \subseteq C^* \cap T^*$ denotes the set of those maximal weights of the irreducible summands of $V|G_0$ which lie in C^*. And for any N-invariant set of weights $\Phi \subseteq T^*$, $V\langle\Phi\rangle$ denotes the sum of all irreducible summands of $V|T$ with weights in Φ, regarded as an N-representation.

Proposition 1.4. *Fix a maximal torus $T \subseteq G$ and a Weyl chamber $C \subseteq \mathfrak{t}$, and set $N = N_G(T,C)$. For any irreducible G-representation V, the subspace $V\langle\,\mathrm{mx}_{C^*}(V)\rangle$ is always an irreducible summand of $V|N$ having multiplicity one. This induces a bijection*

$$\beta_G : \mathrm{Irr}(G) \xrightarrow{\ \cong\ } \mathrm{Irr}(N, C^*) \qquad \textit{defined by} \qquad \beta_G([V]) = \big[V\langle\,\mathrm{mx}_{C^*}(V)\rangle\big], \quad (1)$$

and an isomorphism

$$\bar\beta_G : \mathrm{R}(G) \xrightarrow{\ \cong\ } \mathrm{R}(N, C^*) \qquad \textit{defined by} \qquad \bar\beta_G([V]) = [V\langle C^*\rangle]. \qquad (2)$$

Proof. Fix an irreducible G_0-representation V_0, and let ϕ be the maximal weight of V_0 lying in C^*. Set $\Phi = (N/T)\cdot\phi \subseteq C^*$, the N/T-orbit of ϕ. Let $(V_0) \subseteq \mathrm{Irr}(G_0)$ denote the G/G_0-orbit of V_0, and let $\mathrm{Irr}(G,(V_0))$ denote the set of all irreducible G-representations with support in (V_0); i.e., the set of those irreducible G-representations V such that all irreducible summands of $V|G_0$ lie in (V_0).

Let V_ϕ denote the (1-dimensional) irreducible representation with weight (character) ϕ; regarded as a subspace of V_0. Since $G/G_0 \cong N/T$, the uniqueness of maximal weights in C^* shows that each irreducible component of $\big(\mathrm{Ind}_{G_0}^G(V_0)\big)|G_0$ contains exactly one weight in $\Phi = (N/T)\cdot\phi$ (and with multiplicity one). Thus,

$$V_0\langle\Phi\rangle = V_\phi \qquad \text{and} \qquad \mathrm{Ind}_{G_0}^G(V_0)\langle\Phi\rangle = \mathrm{Ind}_T^N(V_\phi). \qquad (3)$$

So for any G-representation V' with support in (V_0) (i.e., for any $[V'] \in \mathrm{Irr}(G,(V_0))$), there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{G_0}\big(V_0, V'\big) & \xrightarrow[\cong]{r_1} & \mathrm{Hom}_T\big(V_\phi, V'\langle\Phi\rangle\big) \\ {\scriptstyle F_1}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle F_2}\big\downarrow{\scriptstyle\cong} \\ \mathrm{Hom}_G\big(\mathrm{Ind}_{G_0}^G(V_0), V'\big) & \xrightarrow{r_2} & \mathrm{Hom}_N\big(\mathrm{Ind}_T^N(V_\phi), V'\langle\Phi\rangle\big) \end{array} \qquad (4)$$

where F_1 and F_2 are the Frobenius reciprocity isomorphisms, and r_1 and r_2 are defined by restriction to summands with weights in Φ. The one-to-one correspondence between irreducible G_0-representations and highest weights contained in C^* shows that r_1 is an isomorphism, and thus that r_2 is also an isomorphism.

Now assume that V and V' are two irreducible G-representations with support in (V_0). By Frobenius reciprocity again $(\mathrm{Hom}_G(\mathrm{Ind}_{G_0}^G(V_0), V) \cong \mathrm{Hom}_{G_0}(V_0, V) \neq 0)$, V is a summand of $\mathrm{Ind}_{G_0}^G(V_0)$. So by (3) and (4), for any $[V], [V'] \in \mathrm{Irr}(G,(V_0))$,

$$\mathrm{Hom}_N(V\langle\Phi\rangle, V'\langle\Phi\rangle) \cong \mathrm{Hom}_G(V, V') \cong \begin{cases} \mathbb{C} & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

This shows that $V\langle\Phi\rangle$ is N-irreducible for any $[V]\in \mathrm{Irr}(G,(V_0))$, and that $V\langle\Phi\rangle \cong V'\langle\Phi\rangle$ if and only if $V\cong V'$. And finally, any irreducible N-representation with support in Φ is a summand of $\mathrm{Ind}_T^N(V_\phi)\cong \mathrm{Ind}_{G_0}^G(V_0)\langle\Phi\rangle$, and hence has the form $V\langle\Phi\rangle$ for some $V\in \mathrm{Irr}(G_0,(V_0))$.

We have now shown that $\beta_\Phi : \mathrm{Irr}(G,(V_0)) \xrightarrow{\cong} \mathrm{Irr}(N,\Phi)$, defined by setting $\beta_\Phi([V]) = [V\langle\Phi\rangle]$, is a well defined bijection. Since the restriction to G_0 of any irreducible G-representation is a sum of representations in just one G/G_0-orbit of irreducible G_0-representations, $\beta_G : \mathrm{Irr}(G)\to \mathrm{Irr}(N,C^*)$ is the disjoint union of the β_Φ taken over all N/T-orbits $\Phi\subseteq (C^*\cap T^*)$ and hence also a bijection. This proves point (1). At the same time, it shows that the homomorphism $\bar\beta_G : \mathrm{R}(G)\to \mathrm{R}(N,C^*)$ of (2) is an isomorphism, since its matrix with respect to the bases of irreducible representations is triangular with 1's along the diagonal. □

We are now ready to prove that a class function is a (real) character if its restriction to $N_G(T,C)$ ($N_G(T,\pm C)$) is a (real) character.

Proof of 1.2. **Complex case:** Fix a continuous class function $f : G\to\mathbb{C}$ such that $f|N$ is a character of N. We must show that f is a character of G. Let $v_0\in \mathrm{R}(N)$ be such that $\chi_{v_0} = f|N$, let χ be the character of $\bar\beta_G^{-1}(v_0\langle C^*\rangle)\in \mathrm{R}(G)$ (Proposition 1.4(2)), and set $f' = f-\chi$. By construction, $f'|N$ is the character of an element $v\in \mathrm{R}(N)$ such that $v\langle C^*\rangle = 0$. We will show that $v=0$. It then follows that $f'=0$ (since every element of G is conjugate to an element of N), and hence that $f=\chi$ is a character of G.

Fix any $\phi\in T^*$, and let $N_\phi\subseteq N$ denote the subgroup of elements fixing ϕ. Choose any $\psi\in \mathrm{interior}(C^*)^N$ (N/T acts linearly on $\mathbf{t}^*$ and leaves the dual Weyl chamber C^* invariant). Then $\phi+\mathbb{R}\psi$ is not contained in the wall of any dual Weyl chamber (since ψ is not); and so there is a dual Weyl chamber C_1^* such that $\phi+\epsilon\psi\in \mathrm{interior}(C_1^*)$ for small $\epsilon>0$. Let $w\in W_G$ be any element such that $w(C_1^*) = C^*$ (W_{G_0} permutes the Weyl chambers transitively). Then $w\phi\in C^*$, since $\phi\in C_1^*$. Also, for any $a\in N_\phi$, $a(\psi)=\psi$ and $a(\phi)=\phi$ by assumption, so a leaves C_1^* invariant, and hence waw^{-1} leaves $C^* = w(C_1^*)$ invariant. Thus $wN_\phi w^{-1}\subseteq N$; and so $v\langle w\phi\rangle = 0\in \mathrm{R}(wN_\phi w^{-1})$ since $v\langle C^*\rangle = 0\in \mathrm{R}(N)$. Since χ_v is constant on G-conjugacy classes (it is the restriction of a class function on G), it now follows that $v\langle\phi\rangle = 0\in \mathrm{R}(N_\phi)$.

Let $\phi_1,\ldots,\phi_k\in T^*$ be N/T-orbit representatives for the support of v, and write $N_i = N_{\phi_i}$ (the subgroup of elements which fix ϕ_i). Then $v=\sum_{i=1}^k \mathrm{Ind}_{N_i}^N(v\langle\phi_i\rangle)$. We have just seen that $v\langle\phi_i\rangle = 0\in \mathrm{R}(N_i)$ for each i, and hence $v=0$.

Real case: Write $N_\pm = N_G(T,\pm C)$, for short. Fix a class function $f : G\to\mathbb{C}$ such that $f|N_\pm$ is a real character. Then f is a character by Step 2, and $f(G)\subseteq\mathbb{R}$ since any element of G is conjugate to an element of $N\subseteq N_\pm$ (Proposition 1.1). By Lemma 1.3(b), we can assume (after replacing f by its sum with an appropriate real character) that $f=\chi_V$, where $V=\sum_{i=1}^k V_i$, the V_i are distinct irreducible G-representations of quaternion type, and $V|N_\pm$ is a representation of real type. We claim that $V=0$ (i.e., that $k=0$).

Assume otherwise: that $k>0$. Choose a W_G-orbit Ψ of maximal weights in one of the V_i — say V_1 — which does not occur in any of the others except possibly as maximal weights. Set $\Phi = \Psi \cap C^*$ and $\Phi_\pm = \Psi \cap (\pm C^*)$. By Proposition 1.4 (and the original assumption on Ψ), $V_1\langle\Phi\rangle$ is irreducible as an N-representation, and does not occur as a summand of $V_i|N$ for any $i \neq 1$. So the $N_\pm$-representation $V_1' \stackrel{\text{def}}{=} V_1\langle\Phi_\pm\rangle$ is irreducible — since

$$V_1'|N \cong V_1\langle\Phi\rangle \oplus V_1\langle\Phi_\pm \smallsetminus \Phi\rangle$$

— and V_1' does not occur as a summand of $V_i|N_\pm$ for any $i \neq 1$. Also, since V_1 is self-conjugate, the elements of Ψ, and hence of $\Phi_\pm$, occur in pairs $\pm\phi$. This shows that $V_1' = V_1\langle\Phi_\pm\rangle$ is invariant under the conjugate linear automorphism $j : V_1 \to V_1$, and hence that it also has quaternion type. Thus, $V|N_\pm$ contains with multiplicity one the irreducible summand V_1' of quaternion type, and this contradicts the assumption that $V|N_\pm$ is a representation of real type. □

It remains to extend this criterium to a result which detects characters by restriction to finite subgroups of G. As usual, a finite group is called elementary if it is the product of a p-group (for some prime p) and a cyclic group. A finite group G is called $\mathbb{R}$-elementary if it is elementary, or if it contains a normal cyclic subgroup $C \lhd G$ of 2-power index with the property that for any $g \in G$, conjugation by g acts on C via the identity or via $(x \mapsto x^{-1})$.

Proposition 1.5. *For any class function $f : G \to \mathbb{C}$, f is a character of G if and only if its restriction to any finite elementary subgroup of G is a character; and f is a real character of G if and only if its restriction to each finite $\mathbb{R}$-elementary subgroup of G is a real character.*

Proof. When G is finite, the proposition holds by the classical Brauer theorems for detecting characters of finite groups (cf. [Ser, Theorem 21 and Proposition 36]). So it will suffice to show that f is a (real) character of G if and only if its restrictions to all finite subgroups of G are (real) characters. By Proposition 1.2, it suffices to prove this when the connected component G_0 of G is a torus.

Assume now that $G_0 = T$ is a torus. We can choose a sequence $H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots$ of subgroups of G such that each H_i intersects all connected components of G, and such that the union of the H_i is dense in G. The simplest way to see this is to set $n = |G/T|$, let ${}_nT \subseteq T$ denote the n-torsion subgroup, and note that the homomorphism $H^2(G/T; {}_nT) \to H^2(G/T; T)$ is surjective since $n{\cdot}H^2(G/T; T) = 0$. Hence there is a subgroup $H_0 \subseteq G$ such that $H_0 \cap T = {}_nT$ and $\langle H_0, T\rangle = G$; and we can define $H_k = \langle H_0, {}_{n\cdot 2^k}T\rangle$ for each $k > 0$.

Let $f \in \mathrm{Cl}(G)$ be any class function whose restriction to each H_i is a character. For each character χ of G,

$$\langle f, \chi\rangle_G \stackrel{\text{def}}{=} \int_G f{\cdot}\overline{\chi} = \lim_{i\to\infty}\Big(\frac{1}{|H_i|}\sum_{g\in H_i} f(g){\cdot}\overline{\chi(g)}\Big) = \lim_{i\to\infty}\langle f, \chi\rangle_{H_i}$$

(by definition of the Riemann integral); and $\langle f, \chi\rangle_{H_i} \in \mathbb{Z}$ for each i since $f|H_i$ is a character of H_i. Thus, $\langle f, \chi\rangle_G \in \mathbb{Z}$ for each χ, and so f is a character of G by

Lemma 1.3(a). And if $f|H_i$ is a real character for each i, then f is real valued (the union of the H_i being dense in G), $\langle f,\chi\rangle_G = \lim_{i\to\infty} \langle f,\chi\rangle_{H_i} \in 2\mathbb{Z}$ for each character χ of quaternion type by Lemma 1.3(b), and so f is a real character by Lemma 1.3(b). □

Section 2 Induction for representations of compact Lie groups

Again, throughout the section, G denotes a fixed compact Lie group. We construct an induction homomorphism $\mathrm{R}(H) \to \mathrm{R}(G)$, for an arbitrary closed subgroup $H \subseteq G$, by first defining it between the groups of class functions, and then using the results of Section 1 to show that it sends characters to characters.

The following lemma is useful for constructing continuous functions on G, and on certain closed subsets of G.

Lemma 2.1. *Let $\mathcal{F}$ be any set of closed subgroups of G, closed under conjugation and closed in the space of all subgroups (with the Hausdorff topology). Set $G_{\mathcal{F}} = \cup_{H\in\mathcal{F}} H$: the union of the subgroups in $\mathcal{F}$. Then for any function $f : G_{\mathcal{F}} \to \mathbb{C}$ invariant under conjugation, f is continuous on $G_{\mathcal{F}}$ if $f|H$ is continuous for all $H \in \mathcal{F}$.*

Proof. Fix any conjugation invariant function $f : G_{\mathcal{F}} \to \mathbb{C}$ such that $f|H$ is continuous for all $H \in \mathcal{F}$. It will suffice to show, for any sequence $g_i \to g$ in $G_{\mathcal{F}}$, that some subsequence of the $f(g_i)$ converges to $f(g)$. Since if f is not continuous at g, then there is $\epsilon > 0$ and a sequence $\{g_i\}$ in $G_{\mathcal{F}}$ converging to g such that $|f(g_i) - f(g)| > \epsilon$ for all i.

Fix such g_i and g; and for each i choose $H_i \in \mathcal{F}$ such that $g_i \in H_i$. Since $\mathcal{F}$ is closed in the space of closed subgroups of G, and since this space is compact (cf. [tD, Proposition IV.3.2(i)]), we can replace the g_i by a subsequence and assume that the H_i converge to some subgroup $H \in \mathcal{F}$. By [tD, Theorem I.5.9], there exist elements $a_i \to e$ such that $a_i H_i a_i^{-1} \subseteq H$ for i sufficiently large. And hence

$$\lim_{i\to\infty} f(g_i) = \lim_{i\to\infty} f(a_i g_i a_i^{-1}) = f(g)$$

since $f|H$ is continuous. □

The next lemma is also rather technical, and will be used later to show that the induction homomorphism we define for class functions is well defined.

Lemma 2.2. *Fix a closed subgroup $H \subseteq G$ and an element $g \in G$, and let $(G/H)^g$ be the fixed point set of the action of g on G/H. A coset $aH \in G/H$ lies in $(G/H)^g$ if and only if $a^{-1}ga \in H$. And if a_1H and a_2H lie in the same connected component of $(G/H)^g$, then $a_2H = xa_1H$ for some $x \in C_G(g)$. In particular, in this situation, $a_1^{-1}ga_1$ is conjugate in H to $a_2^{-1}ga_2$.*

Proof. For any $a \in G$, $aH \in (G/H)^g$ if and only if $gaH = aH$, if and only if $a^{-1}ga \in H$. Also, if $a_2H = xa_1H$ for any $a_1, a_2 \in G$ and any $x \in C_G(g)$, then $a_1^{-1}ga_1$ and $a_2^{-1}ga_2$ are conjugate by an element of H.

Now fix an element $aH \in (G/H)^g$. Let $C_G(g)_0$ be the identity connected component of the centralizer of g. We must show that the connected component of aH in $(G/H)^g$ is $C_G(g)_0{\cdot}aH$. Equivalently, via translation by a^{-1}, we must show that the connected component of eH in $(G/H)^{a^{-1}ga}$ is $C_G(a^{-1}ga){\cdot}eH$. So upon replacing $a^{-1}ga$ by g, we are reduced to the case where $a = e$ and $g \in H$.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ denote the Lie algebras of $H \subseteq G$. For all $x \in G$, $xH \in (G/H)^g$ if and only if $xH = gxH = gxg^{-1}H$. In particular, $C_G(g){\cdot}H \subseteq (G/H)^g$; and the tangent plane at eH to the manifold $(G/H)^g$ is $(\mathfrak{g}/\mathfrak{h})^{\mathrm{Ad}(g)}$ (the fixed point set of the adjoint action of g on $\mathfrak{g}/\mathfrak{h}$). Also, the projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{h}$ is split, equivariantly with respect to the action of the compact group H, and so $\mathfrak{g}^{\mathrm{Ad}(g)}$ surjects onto $(\mathfrak{g}/\mathfrak{h})^{\mathrm{Ad}(g)}$. Since $\mathfrak{g}^{\mathrm{Ad}(g)}$ is the Lie algebra of $C_G(g)$, this shows that the two submanifolds $C_G(g)_0{\cdot}H \subseteq (G/H)^g$ have the same dimension, and hence that $C_G(g)_0{\cdot}H$ is the connected component of eH in $(G/H)^g$. □

We can now define the induction homomorphism for class functions, motivated by the formula given by Segal [Seg, p. 119].

Propositition 2.3. *Let $H \subseteq G$ be any closed subgroup. Then there is a homomorphism*

$$\mathrm{Ind}_H^G : \mathrm{Cl}(H) \longrightarrow \mathrm{Cl}(G)$$

determined (uniquely) by the following formula. Fix any $g \in G$, let $F_1, \dots, F_k$ be the connected components of $(G/H)^g$, and choose elements $a_iH \in F_i$. Then for any $f \in \mathrm{Cl}(H)$,

$$\mathrm{Ind}_H^G(f)(g) = \sum_{i=1}^{k} \chi(F_i){\cdot}f(a_i^{-1}ga_i). \tag{1}$$

Proof. Fix any $f \in \mathrm{Cl}(H)$. By Lemma 2.2, for each $g \in G$, $\mathrm{Ind}_H^G(f)(g)$ is independent of the choice of representatives a_iH for the components of $(G/H)^g$. Also, $\mathrm{Ind}_H^G(f)$ is conjugation invariant by definition; and it only remains to check that it is continuous.

Let $\mathcal{F}$ be the family of abelian subgroups of G. Clearly, $\mathcal{F}$ is closed in the Hausdorff topology, and its union is all of G. By Lemma 2.1, it will suffice to show that $f|A$ is continuous for each $A \in \mathcal{F}$. Let X be a connected component of some subgroup $A \in \mathcal{F}$; we can assume that X generates $\pi_0(A)$. For any $g \in X$, $A/\langle g\rangle$ is connected (where $\langle g\rangle$ is the closure of the subgroup generated by g); and hence is a torus (or trivial). If $(G/H)^g = \coprod_{i=1}^k F_i$, where the F_i are connected components, then $(G/H)^A = \coprod_{i=1}^k (F_i)^{A/\langle g\rangle}$ and $\chi((F_i)^{A/\langle g\rangle}) = \chi(F_i)$ for each i. Thus, if we write $(G/H)^A = \coprod_{j=1}^m E_j$ (where the E_j are the connected components), and choose elements $b_jH \in E_j$, then

$$\mathrm{Ind}_H^G(f)(g) = \sum_{j=1}^{m} \chi(E_j){\cdot}f(b_j^{-1}gb_j).$$

This formula holds for all $g \in X$, and shows that $\mathrm{Ind}_H^G(f)$ is continuous on X. □

The following double coset formula for induction and restriction of class functions is analogous to that shown by Feshbach [Fe] for equivariant cohomology theories. It was shown for representations by Snaith [Sn, Theorem 2.4], using Segal's definition. We prove it here for class functions, using directly the definition in Proposition 2.3.

Lemma 2.4. *Fix closed subgroups $H, K \subseteq G$, and write*

$$K\backslash G/H = \coprod_{i=1}^{k} U_i$$

where each U_i is a connected component of one orbit type for the action of K on G/H. Fix elements $a_1, \ldots, a_k \in G$ such that $Ka_iH \in U_i$. For each i, let $\varphi_i : \mathrm{Cl}(H) \to \mathrm{Cl}(K)$ denote the composite

$$\varphi_i : \mathrm{Cl}(H) \xrightarrow{\mathrm{Res}} \mathrm{Cl}(a_i^{-1}Ka_i \cap H) \xrightarrow{\mathrm{conj}(a_i^{-1})} \mathrm{Cl}(K \cap a_iHa_i^{-1}) \xrightarrow{\mathrm{Ind}} \mathrm{Cl}(K).$$

Then, as functions from $\mathrm{Cl}(H)$ to $\mathrm{Cl}(K)$,

$$\mathrm{Res}^G_K \circ \mathrm{Ind}^G_H = \sum_{i=1}^{k} \chi^\sharp(U_i) \cdot \varphi_i; \tag{1}$$

where for each i,

$$\chi^\sharp(U_i) = \chi(\overline{U_i}, \overline{U_i} \smallsetminus U_i) = \chi(\overline{U_i}) - \chi(\overline{U_i} \smallsetminus U_i).$$

Proof. Fix elements $f \in \mathrm{Cl}(H)$ and $g \in K$. We will compare the two maps in (1), when evaluated on a given class function f and a given element g.

Let $\widetilde{U}_i \subseteq G/H$ denote the inverse image of U_i under the projection to $K\backslash G/H$. Let $F_1, \ldots, F_m$ be the connected components of $(G/H)^g$. Thus, $G/H = \coprod_{i=1}^k \widetilde{U}_i$ and $(G/H)^g = \coprod_{j=1}^m F_j$. For each i, j, set

$$V_{ij} = (K \cdot a_iH) \cap F_j \subseteq \widetilde{U}_i \cap F_j \subseteq G/H$$

(note that V_{ij} need not be connected). Then the $V_{ij} \to \widetilde{U}_i \cap F_j \to U_i$ are fibration sequences, and so

$$\chi(F_j) = \sum_{i=1}^{k} \chi^\sharp(\widetilde{U}_i \cap F_j) = \sum_{i=1}^{k} \chi(V_{ij}) \cdot \chi^\sharp(U_i).$$

for each j. Fix elements $b_{ij} \in K$, for each i, j, such that $b_{ij}a_iH \in V_{ij}$. Then by definition of the induction map (and Lemma 2.1),

$$(\mathrm{Res}^G_K \circ \mathrm{Ind}^G_H)(f)(g) = \mathrm{Ind}^G_H(f)(g) = \sum_{i=1}^{k} \sum_{j=1}^{m} \chi(V_{ij}) \cdot \chi^\sharp(U_i) \cdot f(a_i^{-1} b_{ij}^{-1} g b_{ij} a_i)$$

$$= \sum_{i=1}^{k} \chi^\sharp(U_i) \cdot \sum_{j=1}^{m} \chi(V_{ij}) \cdot \big(f \circ \mathrm{conj}(a_i^{-1})\big)(b_{ij}^{-1} g b_{ij}).$$

And for each i, if we set $K_i = K \cap a_i H a_i^{-1}$ (the isotropy subgroup of the action of K on $a_i H \in G/H$), then $(K/K_i)^g \cong (K \cdot a_i H)^g = \coprod_{j=1}^m V_{ij}$ $(\subseteq G/H)$; and so

$$\sum_{j=1}^{m} \chi(V_{ij}) \cdot \big(f \circ \operatorname{conj}(a_i^{-1})\big)(b_{ij}^{-1} g b_{ij}) = \operatorname{Ind}_{K_i}^{K} \big(f \circ \operatorname{conj}(a_i^{-1})\big)(g) = \varphi_i(f)(g). \qquad \square$$

When G is finite, the formula given in Proposition 2.3 is just the usual formula for the induction of characters (cf. [Ser, Theorem 12]). Hence by the double coset formula in Lemma 2.4, for each character (real character) χ of H and each finite subgroup $K \subseteq G$, $\big(\operatorname{Ind}_H^G(\chi)\big)|K$ is a character (or real character) of K. The detection result of Proposition 1.5 now applies to show:

Theorem 2.5. *The homomorphism* Ind_H^G *of Proposition 2.3 sends characters to characters, and sends real characters to real characters. It thus restricts to homomorphisms*

$$\operatorname{Ind}_H^G : \mathrm{R}(H) \longrightarrow \mathrm{R}(G) \qquad \text{and} \qquad \operatorname{Ind}_H^G : \mathrm{RO}(H) \longrightarrow \mathrm{RO}(G). \qquad \square$$

These induction homomorphisms are in fact functorial; i.e., they compose in the expected way.

Lemma 2.6. *For any closed subgroups* $K \subseteq H \subseteq G$,

$$\operatorname{Ind}_K^G = \operatorname{Ind}_H^G \circ \operatorname{Ind}_K^H : \mathrm{Cl}(K) \longrightarrow \mathrm{Cl}(G),$$

and hence

$$\operatorname{Ind}_K^G = \operatorname{Ind}_H^G \circ \operatorname{Ind}_K^H : \mathrm{R}(K) \longrightarrow \mathrm{R}(G).$$

Proof. Fix any element $g \in G$, and consider the projection $(G/K)^g \xrightarrow{\ \mathrm{pr}\ } (G/H)^g$. For any $aH \in (G/H)^g$,

$$\mathrm{pr}^{-1}(aH) = \{ahK \mid h \in H,\ h^{-1}(a^{-1}ga)h \in K\} = a \cdot (H/K)^{a^{-1}ga}.$$

If aH and $a'H$ lie in the same connected component of $(G/H)^g$, then $a'H = xaH$ for some $x \in C_G(g)$ (Lemma 2.2), and so $\mathrm{pr}^{-1}(a'H) = x \cdot \mathrm{pr}^{-1}(aH)$. It follows that pr is a fibration (fiber bundle) over each connected component of $(G/H)^g$. The result now follows from the definition of the induction homomorphisms (Proposition 2.3), together with the multiplicativity of Euler characteristics in a fibration. $\square$

We leave it as an exercise to check that this induction homomorphism is the same as that defined by Segal in [Seg] (use the formula given in [Seg, p. 119]).

It is not hard to prove Frobenius reciprocity for induction and restriction of representations, using the definition given here. And that in turn implies, for example, that the induction map $\operatorname{Ind}_{N(T)}^G : \mathrm{R}(N(T)) \to \mathrm{R}(G)$ is always surjective, and split by the restriction map. See also [Sn, Section 2.3] for the proofs of these results using Segal's definition of induction.

Section 3 Representations supported by p-toral subgroups

Again, throughout this section, G will be a fixed compact Lie group, and G_0 will denote its identity connected component. Let $\mathcal{S}_\mathcal{P}(G)$ denote the family of p-toral subgroups of G, for all primes p. We now consider the groups

$$\mathrm{R}_\mathcal{P}(G) = \varprojlim_{P\in\mathcal{S}_\mathcal{P}(G)} \mathrm{R}(P) \qquad \text{and} \qquad \mathrm{RO}_\mathcal{P}(G) = \varprojlim_{P\in\mathcal{S}_\mathcal{P}(G)} \mathrm{RO}(P),$$

where the limits are taken with respect to inclusion and conjugation; and the natural "restriction" maps

$$\mathrm{rs}^{\mathrm{U}}_G : \mathrm{R}(G) \longrightarrow \mathrm{R}_\mathcal{P}(G) \qquad \text{and} \qquad \mathrm{rs}^{\mathrm{O}}_G : \mathrm{RO}(G) \longrightarrow \mathrm{RO}_\mathcal{P}(G).$$

These groups were shown in [JO] to be naturally isomorphic to the Grothendieck groups $\mathbb{K}(BG)$ and $\mathbb{KO}(BG)$, respectively, of vector bundles over BG (and $\mathrm{rs}^{\mathrm{U}}_G$ and $\mathrm{rs}^{\mathrm{O}}_G$ are isomorphic to the natural homomorphisms $\mathrm{R}(G) \to \mathbb{K}(BG)$ and $\mathrm{RO}(G) \to \mathbb{KO}(BG)$).

The homomorphisms rs_G are shown here to split as a direct sums of homomorphisms between finitely generated groups, one for each G/G_0-orbit of irreducible G_0-representations, and the cokernel of each summand is computed (Theorem 3.9). In particular, this yields necessary and sufficient conditions for $\mathrm{rs}^{\mathrm{U}}_G$ to be onto (Theorem 3.10 and Corollary 3.11). The orthogonal case seems to be much more complicated; but we do at least show that $\mathrm{rs}^{\mathrm{O}}_G$ is onto whenever G is finite or $\pi_0(G)$ has prime power order (Propositions 3.2 and 3.4), and then give some examples which show that $\mathrm{rs}^{\mathrm{O}}_G$ can fail to be onto even when $\mathrm{rs}^{\mathrm{U}}_G$ is onto.

It will be useful to define the "character" of an element of $\mathrm{R}_\mathcal{P}(G)$. For any compact Lie group G, let $G_\mathcal{P}$ denote the union of the connected components in G of prime power order in $\pi_0(G)$. Let $\mathrm{Cl}(G_\mathcal{P})$ denote the space of continuous functions $G_\mathcal{P} \to \mathbb{C}$ invariant on conjugacy classes (i.e., the "class functions" on $G_\mathcal{P}$).

Lemma 3.1. *There is a (unique) character homomorphism*

$$\chi : \mathrm{R}_\mathcal{P}(G) \longrightarrow \mathrm{Cl}(G_\mathcal{P}),$$

such that for any $v = \big(v_P\big)_{P\in\mathcal{S}_\mathcal{P}(G)} \in \mathrm{R}_\mathcal{P}(G)$, $\chi(v)|P = \chi_{v_P}$ for all P in $\mathcal{S}_\mathcal{P}(G)$. Also, χ sends $\mathrm{R}_\mathcal{P}(G)$ ($\mathrm{RO}_\mathcal{P}(G)$) isomorphically to the subgroup of those class functions on $G_\mathcal{P}$ whose restriction to each p-toral subgroup $P \subseteq G$, for all primes p, is a character of P (a real character of P).

Proof. Let $\mathcal{F}$ be the set of p-toral subgroups of G (for all primes p), whose identity connected component is a maximal torus of G. Clearly, $\mathcal{F}$ is closed in the Hausdorff topology (note that for $P \in \mathcal{F}$, the order of $\pi_0(P)$ is bounded by $|N_G(T)/T|$). And by Proposition 1.1, $G_\mathcal{P}$ is the union of the $P \in \mathcal{F}$.

Now, for any $v = \big(v_P\big)_{P\in\mathcal{S}_\mathcal{P}(G)} \in \mathrm{R}_\mathcal{P}(G)$, define $\chi(v) : G_\mathcal{P} \to \mathbb{C}$ to be the union of the characters χ_{v_P}. This is well defined, and invariant under conjugation, by definition of the inverse limit. Also, $\chi(v)$ is continuous by Lemma 2.1, applied to the family $\mathcal{F}$; and so $\chi(v) \in \mathrm{Cl}(G_{\mathcal{P}})$.

The character homomorphism χ is clearly a monomorphism, and the descriptions of the images of $\mathrm{R}_{\mathcal{P}}(G)$ and $\mathrm{RO}_{\mathcal{P}}(G)$ are immediate from the construction. □

We are now ready to study the groups $\mathrm{R}_{\mathcal{P}}(G)$ and $\mathrm{RO}_{\mathcal{P}}(G)$, beginning with the following case.

Proposition 3.2. *If $\pi_0(G)$ has prime power order, then*

$$rs_G^{\mathrm{U}} : \mathrm{R}(G) \xrightarrow{\cong} \mathrm{R}_{\mathcal{P}}(G) \qquad \textit{and} \qquad rs_G^{\mathrm{O}} : \mathrm{RO}(G) \xrightarrow{\cong} \mathrm{RO}_{\mathcal{P}}(G)$$

are isomorphisms. Furthermore, for any G,

$$\mathrm{R}_{\mathcal{P}}(G) \cong \varprojlim_{P \in \mathcal{F}_{\mathcal{P}}(G)} \mathrm{R}(P) \qquad \textit{and} \qquad \mathrm{RO}_{\mathcal{P}}(G) \cong \varprojlim_{P \in \mathcal{F}_{\mathcal{P}}(G)} \mathrm{RO}(P); \tag{1}$$

where $\mathcal{F}_{\mathcal{P}}(G)$ denotes the family of subgroups $H \subseteq G$ of finite index such that H/G_0 has prime power order (and the limits are taken with respect to inclusion and conjugation).

Proof. If $\pi_0(G)$ is a p-group for any prime p, then $G = G_{\mathcal{P}}$, and so $\mathrm{rs}_G^{\mathrm{U}}$ and $\mathrm{rs}_G^{\mathrm{O}}$ are both monomorphisms by Lemma 3.1. To prove that they are isomorphisms, we must show that a class function $f \in \mathrm{Cl}(G)$ is a (real) character if its restriction to all p-toral subgroups of G is a (real) character.

Fix a maximal torus T and a Weyl chamber $C \subseteq \mathfrak{t}$, and set $N = N_G(T, C)$. Then N is p-toral by Proposition 1.1; and by Proposition 1.2 a class function $f \in \mathrm{Cl}(G)$ is a character of G if $f|N$ is a character of N. This shows that $\mathrm{rs}_G^{\mathrm{U}}$ is an isomorphism. If $\pi_0(G)$ is a 2-group, then $N_G(T, \pm C)$ is 2-toral, and the same argument shows that $\mathrm{rs}_G^{\mathrm{O}}$ is an isomorphism. Finally, if $\pi_0(G)$ is a p-group for an odd prime p, then for any $v \in \mathrm{RO}_{\mathcal{P}}(G)$, $\chi(v)$ is a real valued character of G whose restriction to G_0 is a real character of G_0 (since $\pi_0(G_0) = 1$ is a 2-group); and so $\chi(v)$ is a real character of G by Lemma 1.3(c).

This finishes the proof of the first statement above. The formulas in (1) now follow immediately (by the transitivity of inverse limits). □

The importance of the formulas in (1) above is that they show that the groups $\mathrm{R}_{\mathcal{P}}(G)$ and $\mathrm{RO}_{\mathcal{P}}(G)$, and also the maps $\mathrm{rs}_G^{\mathrm{U}}$ and $\mathrm{rs}_G^{\mathrm{O}}$, split as sums of groups and maps indexed by the irreducible representations of the identity component G_0. This will be made more explicit in Theorem 3.9 below.

The next proposition describes how standard induction techniques apply to study $\mathrm{R}_{\mathcal{P}}(G)$ and $\mathrm{rs}_G^{\mathrm{U}}$. Recall that a finite group Γ is p-elementary if it is a product of a p-group and a cyclic group, and is elementary if it is p-elementary for some prime p. Also, Γ is 2-$\mathbb{R}$-elementary if it contains a normal cyclic subgroup C_m of 2-power index such that any element of G either centralizes C_m or acts on it via $(a \mapsto a^{-1})$; and is $\mathbb{R}$-elementary if it is elementary or 2-$\mathbb{R}$-elementary.

Proposition 3.3. *(a) For any subgroup $H \subseteq G$ of finite index, there is an induction homomorphism*

$$\mathrm{Ind}_H^G : \mathrm{R}_{\mathcal{P}}(H) \longrightarrow \mathrm{R}_{\mathcal{P}}(G),$$

with the property that for any $v \in \mathrm{R}_{\mathcal{P}}(H)$ and any $g \in G_{\mathcal{P}}$,

$$\chi(\mathrm{Ind}_H^G(v))(g) = \bigl(\mathrm{Ind}_H^G(\chi(V))\bigr)(g) = \sum_{aH \in (G/H)^g} \chi(v)(a^{-1}ga). \tag{1}$$

(b) Let $\mathcal{E}(G)$ and $\mathcal{E}_{\mathbb{R}}(G)$ denote the sets of subgroups $E \subseteq G$ of finite index such that E/G_0 is elementary or $\mathbb{R}$-elementary, respectively. Then restriction induces isomorphisms

$$\mathrm{Coker}(rs_G^{\mathrm{U}}) \xrightarrow{\ \cong\ } \varprojlim_{E \in \mathcal{E}(G)} \mathrm{Coker}(rs_E^{\mathrm{U}}) \quad \textit{and} \quad \mathrm{Coker}(rs_G^{\mathrm{O}}) \xrightarrow{\ \cong\ } \varprojlim_{E \in \mathcal{E}_{\mathbb{R}}(G)} \mathrm{Coker}(rs_E^{\mathrm{O}});$$

where the limits are taken with respect to inclusion and conjugation in G.

Proof. We regard Ind_H^G as a homomorphism $\mathrm{Cl}(H_{\mathcal{P}}) \to \mathrm{Cl}(G_{\mathcal{P}})$, defined via formula (1). Note that this is just the restriction to $G_{\mathcal{P}}$ of the formula given in Proposition 2.3 (though only in the case where $[G{:}H] < \infty$). In particular, the double coset formula of Lemma 2.4 applies in this situation.

(a) Fix any $v \in \mathrm{R}_{\mathcal{P}}(H)$, and let $\chi = \chi(v) \in \mathrm{Cl}(H_{\mathcal{P}})$ be its character. We must show that $\mathrm{Ind}_H^G(\chi)$ is the character of an element of $\mathrm{R}_{\mathcal{P}}(G)$; or equivalently (by Lemma 3.1) that $\mathrm{Ind}_H^G(\chi)|P$ is a character for all p-toral subgroups $P \subseteq G$ (for all primes p). And for any such P, $gPg^{-1} \cap H$ is p-toral for each $g \in G$, so $\chi|\bigl(gPg^{-1} \cap H\bigr)$ is a character, and hence $\mathrm{Ind}_H^G(\chi)|P$ is a character of P by the double coset formula.

(b) Let $\mathcal{F}(G)$ be the class of subgroups of G of finite index. The functor $H \mapsto \mathrm{R}(H/G_0)$ satisfies the double coset formula and Frobenius reciprocity relations for induction and restriction, and hence is a Green ring over $\mathcal{F}(G)$ in the sense of Dress [Dr]. Also, the double coset formula of Lemma 2.4 says that $H \mapsto \mathrm{R}_{\mathcal{P}}(H)$ and $H \mapsto \mathrm{Coker}(\mathrm{rs}_H^{\mathrm{U}})$ are both Mackey functors over $\mathcal{F}(G)$ (again in the sense of Dress); and both are modules over $\mathrm{R}(-/G_0)$ satisfying Frobenius reciprocity. Since $\mathrm{R}(G/G_0)$ is generated by induction from the $\mathrm{R}(E/G_0)$ for $E \in \mathcal{E}(G)$ [Ser, §10.5, Theorem 19], the "fundamental theorem" of Mackey functors and Green rings says that $F(G) \cong \varprojlim_{E \in \mathcal{E}(G)}(F(E))$ for any such module over $\mathrm{R}(-/G_0)$. This is shown in [Dr, Propositions 1.1′ and 1.2], and a more direct proof is given in [Ol, Theorem 11.1].

Similarly, $H \mapsto \mathrm{RO}_{\mathcal{P}}(H)$ and $H \mapsto \mathrm{Coker}(\mathrm{rs}_H^{\mathrm{O}})$ are Mackey functors over $\mathcal{F}(G)$, and modules over $\mathrm{RO}(-/G_0)$ satisfying Frobenius reciprocity. Since $\mathrm{RO}(G/G_0)$ is generated by induction from the $\mathrm{RO}(E/G_0)$ for $E \in \mathcal{E}_{\mathbb{R}}(G)$ [Ser, §12.6, Theorem 27], the same argument applies to show that $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{O}}) \cong \varprojlim_{E \in \mathcal{E}_{\mathbb{R}}(G)} \mathrm{Coker}(\mathrm{rs}_E^{\mathrm{O}})$. □

In fact, the induction map $\mathrm{Ind}_H^G : \mathrm{R}_{\mathcal{P}}(H) \to \mathrm{R}_{\mathcal{P}}(G)$ is defined for any closed subgroup $H \subseteq G$, using the formula for induction of characters in Proposition 2.3.

To see this, one must check, for any $f \in \mathrm{Cl}(H)$, that $\mathrm{Ind}_H^G(f)|G_{\mathcal{P}} = 0$ if $f|H_{\mathcal{P}} = 0$. This would be immediate if we knew that $H \cap G_{\mathcal{P}} \subseteq H_{\mathcal{P}}$; but that is not the case in general. The existence of the induction map is thus slightly more tricky than in the case where $[G:H] < \infty$, but is not difficult.

We now turn to the case of finite groups.

Proposition 3.4. *If G is finite, then $\mathrm{rs}_G^{\mathrm{U}}$ and $\mathrm{rs}_G^{\mathrm{O}}$ are both surjective.*

Proof. By Proposition 3.3(b), it suffices to show that $\mathrm{rs}_G^{\mathrm{U}}$ is onto when G is elementary, and that $\mathrm{rs}_G^{\mathrm{O}}$ is onto when G is $\mathbb{R}$-elementary. We do this in the orthogonal case only; the unitary case is similar (but simpler).

Assume that G is $\mathbb{R}$-elementary, and fix an element $v = \big(v_P\big)_{P \in \mathcal{S}_{\mathcal{P}}(G)} \in \mathrm{RO}_{\mathcal{P}}(G)$. In other words, $v_P \in \mathrm{RO}(P)$ for each p-subgroup $P \subseteq G$ (for each prime $p\big||G|$); and by subtracting a constant character we can assume that $\chi_{v_P}(1) = 0$ for each P. For each $p\big||G|$, write $v_p = v_{\mathrm{Syl}_p(G)} \in \mathrm{RO}(\mathrm{Syl}_p(G))$. It will suffice to show that each v_p extends to an element $v_p' \in \mathrm{R}(G)$ whose character vanishes on all elements of order prime to p (then $v = \mathrm{rs}_G^{\mathrm{O}}(\sum v_p')$). This is clear if $\mathrm{Syl}_p(G)$ has a normal complement, since in that case v_p' can be taken to be the composite of v_p with a surjection $G \twoheadrightarrow \mathrm{Syl}_p(G)$.

The only case left to consider is that where p is odd, G is 2-$\mathbb{R}$-elementary, and $\mathrm{Syl}_p(G)$ has no normal complement. Set $p^k = |\mathrm{Syl}_p(G)|$; then there is a surjection $G \twoheadrightarrow D(2p^k)$, where $D(2p^k)$ is dihedral of order $2p^k$. One easily checks that any $v_p \in \mathrm{RO}(C_{p^k})$ such that $\chi_{v_p}(1) = 0$ extends to an element $v_p'' \in \mathrm{RO}(D(2p^k))$ such that $\chi_{v_p''}(g) = 0$ for all g of order prime to p. And hence if $v_p' \in \mathrm{RO}(G)$ is the composite of v_p'' with the surjection $G \twoheadrightarrow D(2p^k)$, then $v_p'|\mathrm{Syl}_p(G) = v_p$ and $\chi_{v_p'}$ vanishes on all elements of order prime to p. □

Recall that for any torus T, we let $\mathfrak{t}$ denote the Lie algebra of T, and regard the group $T^* = \mathrm{Hom}(T, S^1)$ of irreducible characters of T as a lattice in $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$. The following definitions establish some of the notation which will be used when dealing with irreducible characters and representations of groups with torus identity component.

Definition 3.5. *If G is a compact Lie group with identity component T, then the* support *of a G-representation V is the (G/T-invariant) subset* $\mathrm{Supp}(V) \subseteq T^*$ *of all characters of irreducible summands of $V|T$. More generally, for any $v \in \mathrm{R}(G)$,* $\mathrm{Supp}(v) \in T^*$ *is the union of the supports of the irreducible G-representations which occur in the decomposition of v. For any G/T-invariant subset $\Phi \subseteq T^*$,* $\mathrm{Irr}(G,\Phi)$ *denotes the set of irreducible G-representations with support in Φ, and* $\mathrm{R}(G,\Phi) \subseteq \mathrm{R}(G)$ *denotes the subgroup of elements with support in Φ. For $\phi \in T^*$, we write (ϕ) for the G/T-orbit of ϕ (and write* $\mathrm{Irr}(G,\phi)$*, etc., if ϕ is G/T-invariant). Finally, if V is any G-representation, then $V\langle\Phi\rangle$ and $V\langle\phi\rangle$ denote the largest summands of V with support in Φ or ϕ, respectively.*

The descriptions of $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})$ in Lemma 3.8 and Theorem 3.9 below will be given in terms of a certain function $\delta(G)$, defined for compact Lie groups whose identity component is a torus and central.

Definition 3.6. *Assume that G lies in a central extension $1 \to T \to G \to \Gamma \to 1$, where T is a torus and Γ is a finite group. For each $\phi \in T^*$, define*

$$\delta(G,\phi) = \gcd\bigl\{\dim(V) \bigm| V \in \mathrm{Irr}(G,\phi)\bigr\};$$

and set

$$\delta(G) = \mathrm{lcm}\bigl\{\delta(G,\phi) \bigm| \phi \in T^*\bigr\}.$$

The next lemma gives a partial description of this function, independantly of representations; and also lists some of its more technical properties which will be needed in later proofs.

Lemma 3.7. *Assume that $G_0 \subseteq Z(G)$; i.e., that G lies in a central extension $1 \to T \to G \to \Gamma \to 1$, where T is a torus and Γ is finite. Set $e = \mathrm{expt}\bigl(T \cap [G,G]\bigr)$. For each prime $p \bigm| |\Gamma|$, let G_p be a maximal p-toral subgroup of G: the extension of T by a Sylow p-subgroup of Γ. Then*

- *(a)* $\delta(G) = 1$ *if and only if* $e = 1$, *if and only if* $G \cong T \times \Gamma$
- *(b)* $e \bigm| \delta(G)$ *and* $\delta(G)^2 \bigm| |\Gamma|$
- *(c)* $\delta(G) = \prod_{p \mid |\Gamma|} \delta(G_p)$, *and* $\delta(G,\phi) = \prod_{p \mid |\Gamma|} \delta(G_p,\phi)$ *for all* $\phi \in T^*$
- *(d)* $\delta(G,\phi') = \delta(G,\phi)$ *for all* $\phi',\phi \in T^*$ *with* $\phi' \equiv \phi \pmod{e}$
- *(e)* $\delta(G,n\phi) = \delta(G,\phi)$ *for all* $\phi \in T^*$, *and all* $n \in \mathbb{Z}$ *with* $(n,e) = 1$.

Proof. Note first that for any $H \subseteq G$ of finite index, and any $\phi \in T^*$,

$$\delta(H,\phi) \bigm| \delta(G,\phi) \bigm| [G:H]{\cdot}\delta(H,\phi). \tag{1}$$

The first relation holds since each G-representation with support in ϕ can be regarded as an H-representation; and the second since $\mathrm{Ind}_H^G(V)$ has support in ϕ for any H-representation V with support in ϕ.

(b) Fix any $\phi \in T^*$, and choose $a \in T \cap [G,G]$ such that $\phi(a)$ generates $\phi(T \cap [G,G])$. Then for any G-representation V with support in ϕ, a acts on V via multiplication by $\phi(a)$; and since $a \in [G,G]$, $\phi(a){\cdot}\mathrm{Id}_V$ has determinant $\phi(a)^{\dim(V)} = 1$. Thus, $|\phi(a)| \bigm| \dim(V)$ for all such V, and so

$$|\phi(a)| = |\phi(T \cap [G,G])| \bigm| \delta(G,\phi). \tag{2}$$

In particular, $e = \mathrm{expt}(T \cap [G,G])$ divides $\delta(G)$.

Now fix any $\phi \in T^*$, and let V_ϕ be the 1-dimensional irreducible T-representation with character ϕ. Let $V_1,\ldots,V_k$ be the irreducible G-representations with support in ϕ. For each i, the multiplicity of V_i in $\mathrm{Ind}_T^G(V_\phi)$ is

$$\dim_{\mathbb{C}}\bigl(\mathrm{Hom}_G(\mathrm{Ind}_T^G(V_\phi),V_i)\bigr) = \dim_{\mathbb{C}}\bigl(\mathrm{Hom}_T(V_\phi,V_i)\bigr) = \dim_{\mathbb{C}} V_i.$$

Thus, $|\Gamma| = \dim(\mathrm{Ind}_T^G(V_\phi)) = \sum_{i=1}^k \dim(V_i)^2$. And so $\delta(G,\phi)$, the greatest common divisor of the $\dim(V_i)$, is such that $\delta(G,\phi)^2 \big| |\Gamma|$.

(a) We prove here the slightly more general equivalence that

$$\delta(G,\phi)=1 \iff \phi(T\cap[G,G])=1 \iff G/\operatorname{Ker}(\phi)\cong T/\operatorname{Ker}(\phi)\times\Gamma. \qquad (3)$$

The third statement clearly implies the first, and the first implies the second by (2).

By the universal coefficient theorem, $H^2(\Gamma;T)\cong \mathrm{Hom}(H_2(\Gamma),T)$; and $T\cap[G,G]$ is the image of the homomorphism $\eta_G : H_2(\Gamma)\to T$ which corresponds to $[G]$ as an element if $H^2(\Gamma;T)$. So $G\cong T\times\Gamma$ if $T\cap[G,G]=1$, and $G/\operatorname{Ker}(\phi)\cong T/\operatorname{Ker}(\phi)\times\Gamma$ if $\phi(T\cap[G,G])=1$.

(c) This formula follows immediately from (1), and the fact that $\delta(G_p,\phi)\big||G_p/T|$ is a power of p for each p.

(d) If $\phi\equiv 0 \pmod e$, then $\phi(T\cap[G,G])=1$, and so $\delta(G,\phi)=1$ by (3). If $\phi'\equiv\phi\not\equiv 0 \pmod e$, then the two composites

$$H_2(\Gamma) \xrightarrow{\ \eta_G\ } T \underset{\phi'}{\overset{\phi}{\rightrightarrows}} S^1$$

are equal. Hence $(G/\operatorname{Ker}(\phi),\phi)\cong(G/\operatorname{Ker}(\phi'),\phi')$ as pairs, and $\delta(G,\phi)=\delta(G,\phi')$.

(e) For any $n\in\mathbb{Z}$ and any G-representation V with support ϕ, $\psi^n(V)$ is a virtual representation with support $n\phi$: since $\chi_{\psi^n V}(gt)=\chi_V(g^nt^n)=\chi_{\psi^n V}(g)\cdot\phi(t)^n$ for any $g\in G$ and $t\in T$. Cf. [Ad, Lemma 3.61] for details. Also, V and $\psi^n(V)$ have the same (virtual) dimension, and hence $\delta(G,n\phi)\big|\delta(G,\phi)$. So by (d), $\delta(G,n\phi)=\delta(G,\phi)$ if n is invertible mod e. □

Ian Leary has pointed out to me that $\delta(G)$ is the greatest common divisor of the indices $[G:H]$ of those subgroups $H\subseteq G$ of finite index such that H splits as a product $H\cong T\times(H/T)$.

Whenever $G_0=T$ is a torus, $\mathrm{R}(G)$ splits as the direct sum, taken over all G/T-orbits $(\phi)\subseteq T^*$, of the subgroups $\mathrm{R}(G,(\phi))$ of finite rank. In a similar fashion, $\mathrm{rs}_G^{\mathrm{U}}$ splits as the direct sum over all $(\phi)\subseteq T^*$ of homomorphisms

$$\mathrm{rs}_{G,(\phi)} : \mathrm{R}(G,(\phi)) \longrightarrow \mathrm{R}_{\mathcal{P}}(G,(\phi)).$$

We are now ready to describe the cokernel of each of these summands for such G. The key case to consider is that when $T=G_0$ is central and ϕ is faithful.

Lemma 3.8. *Assume that G lies in a central extension $1\to T\to G\xrightarrow{\sigma}\Gamma\to 1$, where $T\cong S^1$, and where Γ is finite. Fix a faithful (injective) character $\phi\in T^*$. Let S be the set of all conjugacy classes of elements $g\in\Gamma$ such that no two elements in $\sigma^{-1}g$ are conjugate; and let $S_{\mathcal{P}}\subseteq S$ be the set of conjugacy classes of elements of prime power order. For each $g\in S_{\mathcal{P}}$, let $\eta(g)$ be the largest divisor of $\delta(C_G(g),\phi)$ which is prime to the order of g. Then*

$$\mathrm{R}(G,\phi)\cong\mathbb{Z}^{|S|},\quad \mathrm{R}_{\mathcal{P}}(G,\phi)\cong\mathbb{Z}^{|S_{\mathcal{P}}|},\quad \textit{and}\quad \mathrm{Coker}(rs_{G,\phi})\cong\bigoplus_{1\neq g\in S_{\mathcal{P}}}\mathbb{Z}/\eta(g).$$

Proof. A character χ of G has support in ϕ if and only if it satisfies the relation $\chi(gt) = \chi(g)\phi(t)$ for all $g \in G$ and $t \in T$. In particular, since ϕ is injective, $\chi(g) = 0$ for any g which is conjugate to gt for some $1 \neq t \in T$. Thus, $\mathrm{Cl}(G, \phi)$ is a complex vector space of dimension $|S|$; and by the Peter-Weyl theorem (and the independence of irreducible characters) $\mathrm{R}(G, \phi)$ is a free abelian group of rank $|S|$. Also, $\mathrm{R}_{\mathcal{P}}(G, \phi)$ is torsion free (it is detected by characters defined on $G_{\mathcal{P}}$), and $\mathrm{Ker}(\mathrm{rs}_{G,\phi})$ is the set of elements of $\mathrm{R}(G, \phi)$ whose characters vanish on $G_{\mathcal{P}}$. So the image of $\mathrm{rs}_{G,\phi}$ is free of rank $|S_{\mathcal{P}}|$; and once we have shown that $\mathrm{rs}_{G,\phi}$ has finite cokernel it will follow that $\mathrm{R}_{\mathcal{P}}(G, \phi)$ is a free abelian group of the same rank.

The computation of the cokernel of $\mathrm{rs}_{G,\phi}$ will be carried out in two steps.

Step 1 Assume first that Γ is p-elementary for some prime p. Then we can write $G = C_n \times P$, where C_n is cyclic of order n prime to p, and where P is p-toral. In particular, $\mathrm{R}(G) \cong \mathrm{R}(C_n) \otimes \mathrm{R}(P)$ and $\mathrm{R}(G, \phi) \cong \mathrm{R}(C_n) \otimes \mathrm{R}(P, \phi)$. Let $\mathrm{IR}(-)$ denote the augmentation ideal of $\mathrm{R}(-)$, and similarly for $\mathrm{IR}_{\mathcal{P}}(-)$. Consider the following commutative diagram with split short exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{IR}(C_n) \otimes \mathrm{R}(P, \phi) & \longrightarrow & \mathrm{R}(G, \phi) & \longrightarrow & \mathrm{R}(P, \phi) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \mathrm{rs}_{C_n} \otimes \mathrm{augm.}} & & \Big\downarrow {\scriptstyle \mathrm{rs}_{G,\phi}} & & \Big\downarrow {\scriptstyle =} & & \\
0 & \longrightarrow & \mathrm{IR}_{\mathcal{P}}(C_n) \otimes \mathbb{Z} & \longrightarrow & \mathrm{R}_{\mathcal{P}}(G, \phi) & \longrightarrow & \mathrm{R}(P, \phi) & \longrightarrow & 0.
\end{array}$$

Here, $\mathrm{IR}_{\mathcal{P}}(C_n)$ is the product of the $\mathrm{IR}(\mathrm{Syl}_q(C_n))$ for $q|n$, and any $v \in \mathrm{IR}(\mathrm{Syl}_q(C_n))$ lifts to an element of $\mathrm{IR}(C_n)$ whose character vanishes on other Sylow subgroups. Hence $\mathrm{IR}(C_n)$ surjects onto $\mathrm{IR}_{\mathcal{P}}(C_n)$, and so

$$\mathrm{Coker}(\mathrm{rs}_{G,\phi}) \cong \mathrm{Coker}(\mathrm{rs}_{C_n} \otimes \mathrm{augm.}) \cong \mathrm{IR}_{\mathcal{P}}(C_n) \otimes \mathrm{Coker}\big[\mathrm{R}(P, \phi) \xrightarrow{\mathrm{augm.}} \mathbb{Z}\big].$$

The cokernel of this augmentation map is by definition $\mathbb{Z}/\delta(P, \phi)$, and so

$$\mathrm{Coker}(\mathrm{rs}_{G,\phi}) \cong \mathrm{IR}_{\mathcal{P}}(C_n) \otimes \big(\mathbb{Z}/\delta(P, \phi)\big). \tag{1}$$

Step 2 Now assume that G is arbitrary. Let $\mathcal{E}(G)$ be the set of subgroups of G of finite index such that E/T is elementary, and (for each prime $p\big||\Gamma|$) let $\mathcal{E}_p(G)$ be the set of those $E \in \mathcal{E}(G)$ such that E/T is p-elementary. By Proposition 3.3, $\mathrm{Coker}(\mathrm{rs}_{G,\phi})$ is the inverse limit of the groups $\mathrm{Coker}(\mathrm{rs}_{E,\phi})$, taken over all $E \in \mathcal{E}(G)$. By (1), $\mathrm{Coker}(\mathrm{rs}_{E,\phi})$ is a finite p-group for all $E \in \mathcal{E}_p(G)$. Hence $\mathrm{Coker}(\mathrm{rs}_{G,\phi})$ is finite; and (for each p) $\mathrm{Coker}(\mathrm{rs}_{G,\phi})_{(p)}$ is the inverse limit of the $\mathrm{Coker}(\mathrm{rs}_{E,\phi})$ for $E \in \mathcal{E}_p(G)$.

Fix a prime $p\big||\Gamma|$; we want to determine the p-power torsion in $\mathrm{Coker}(\mathrm{rs}_{G,\phi})$. If $K' \subseteq K$ are finite cyclic subgroups of order prime to p, then the composite

$$\mathrm{IR}(K')_{(p)} \xrightarrow{\mathrm{Ind}} \mathrm{IR}(K)_{(p)} \xrightarrow{\mathrm{Res}} \mathrm{IR}(K')_{(p)} \tag{2}$$

is multiplication by $[K{:}K']$, and hence an isomorphism. Thus, if K is cyclic of order prime to p, we can split

$$\mathrm{IR}_{\mathcal{P}}(K)_{(p)} = \bigoplus_{q\big||K|} \mathrm{IR}(\mathrm{Syl}_q(K)) \cong \bigoplus_{1 \neq K' \subseteq K_{\mathcal{P}}} \widetilde{\mathrm{IR}}(K')_{(p)}$$

(i.e., taking the second sum over subgroups of prime power order). Here, $\widetilde{\mathrm{IR}}(K') \subseteq \mathrm{IR}(K')$ is the kernel of the map given by restriction to the subgroup of prime index, and is free with rank equal to the number of generators of K'.

For each $n\big||\Gamma|$ prime to p, let Cyc_n be the set of all cyclic subgroups $K \subseteq \Gamma$ of order n if n is a prime power, and set $\mathrm{Cyc}_n = \emptyset$ otherwise. By Lemma 3.7(c), for any maximal p-toral subgroup $P \subseteq H$, $\delta(P,\phi)$ is the largest power of p dividing $\delta(H,\phi)$. So with the help of (1) we now get

$$\begin{aligned}\mathrm{Coker}(\mathrm{rs}_{G,\phi})_{(p)} &\cong \varprojlim_{E\in\mathcal{E}_p(G)} \mathrm{Coker}(\mathrm{rs}_{E,\phi}) \\ &\cong \bigoplus_{p\nmid n\big||\Gamma|} \Big(\varprojlim_{K\in\mathrm{Cyc}_n} \big(\widetilde{\mathrm{IR}}(K)\otimes \mathbb{Z}/\delta(\sigma^{-1}(C_\Gamma(K)),\phi)\big)_{(p)}\Big). \qquad (3)\end{aligned}$$

For each $n = q^k$ (where $q \neq p$ is prime), set

$$\mathrm{Cyc}'_n = \big\{K{=}\langle g\rangle{\in}\,\mathrm{Cyc}_n \;\big|\, \text{no two elts. in } \sigma^{-1}g \text{ conjugate in } G\big\}.$$

Fix some $K \in \mathrm{Cyc}_{q^k} \smallsetminus \mathrm{Cyc}'_{q^k}$, and let $K' \subseteq K$ be the subgroup of index q. Then there exists $x \in N_G(\sigma^{-1}K)$ such that for each $g \in \sigma^{-1}(K\smallsetminus K')$, $xgx^{-1} = gt$ for some $1 \neq t \in T$. The character of any element $v \in \widetilde{\mathrm{IR}}(K) \cong \widetilde{\mathrm{IR}}(\sigma^{-1}K,\phi)$ vanishes on $\sigma^{-1}K'$; and hence (since $\chi_v(gt) = \chi_v(g){\cdot}\phi(t)$) v is fixed by the action of x only if $v = 0$. Thus, x acts on $\widetilde{\mathrm{IR}}(K)$ with trivial fixed point set; and in particular such terms contribute nothing to the limit in (3).

Formula (3) thus reduces to a sum, over conjugacy class representatives for all $K \in \mathrm{Cyc}'_n$, of the groups

$$H^0\big(N_G(\sigma^{-1}K);\widetilde{\mathrm{IR}}_{\mathcal{P}}(K)\big) \otimes \big(\mathbb{Z}/\delta(C_G(\sigma^{-1}K),\phi)\big)_{(p)}.$$

The first factor here is free of rank equal to the number of Γ-conjugacy classes of generators of K. The formula for $\mathrm{Coker}(\mathrm{rs}_{G,\phi})$ now follows upon taking the product over all primes $p\big||\Gamma|$. $\square$

As an example, consider the group $G = C_n \times (S^1 \times_{C_2} Q(8))$, where n is odd, $Q(8)$ is a quaternion group of order 8, and the second product is taken while identifying the central elements of order 2 in S^1 and $Q(8)$. By Lemma 3.8, if $\phi \in T^*$ is a generator, then $\mathrm{rs}_{G,k\phi}$ is onto for k even, while $\mathrm{Coker}(\mathrm{rs}_{G,k\phi}) \cong \mathbb{Z}/2 \otimes \mathrm{IR}_{\mathcal{P}}(C_n) \neq 0$ if k is odd.

The groups dealt with in Lemma 3.8 seem quite specialized, but we are now ready to show that the general case — for an arbitrary compact Lie group G — can always be reduced to the cases handled there.

Theorem 3.9. *Let G be any compact Lie group. Fix a maximal torus $T \subseteq G$ and a Weyl chamber $C \subseteq \mathfrak{t}$, and set $N = N(T,C) \subseteq G$. Then rs^{U}_G splits as a direct sum of homomorphisms*

$$rs_{G,(V_0)} : \mathrm{R}(G,(V_0)) \longrightarrow \mathrm{R}_{\mathcal{P}}(G,(V_0)),$$

taken over all G/G_0-orbits $(V_0) \subseteq \mathrm{Irr}(G_0)$.

For any $V_0 \in \mathrm{Irr}(G_0)$, let ϕ be the maximal weight of V_0 in C^*, let $N_\phi \subseteq N$ be the subgroup of elements which fix ϕ, and set $K_\phi = \mathrm{Ker}(\phi) \subseteq T$. Then the assignment $\big([V] \mapsto [V\langle\phi\rangle]\big)$ induces isomorphisms

$$\mathrm{R}(G,(V_0)) \xrightarrow{\;\cong\;} \mathrm{R}(N_\phi/K_\phi,\phi), \qquad \mathrm{R}_{\mathcal{P}}(G,(V_0)) \xrightarrow{\;\cong\;} \mathrm{R}_{\mathcal{P}}(N_\phi/K_\phi,\phi),$$

and

$$\mathrm{Coker}(rs_{G,(V_0)}) \xrightarrow{\;\cong\;} \mathrm{Coker}(rs_{N_\phi/K_\phi,\phi}).$$

Proof. By Lemma 3.2, $\mathrm{R}_{\mathcal{P}}(G)$ is the inverse limit of the representation rings $\mathrm{R}(H)$, taken over all $H \subseteq G$ of finite index such that H/G_0 has prime power order. Since each $\mathrm{R}(H)$ splits as a sum of finitely generated groups indexed by the G/G_0-orbits $(V_0) \in \mathrm{Irr}(G_0)$, we now see that $\mathrm{R}_{\mathcal{P}}(G)$ also splits as such a sum. And hence $\mathrm{rs}_G^{\mathrm{U}}$ also splits as a direct sum of homomorphisms $\mathrm{rs}_{G,(V_0)}$.

Now fix $V_0 \in \mathrm{Irr}(G_0)$ and let ϕ be its maximal weight in C^*. Write $\Phi = (\phi)$ for short: the N/T-orbit of $\phi \in C_T^*$. By Proposition 1.4, the assignment $[V] \mapsto [V\langle\Phi\rangle]$ defines a bijection from $\mathrm{Irr}(G,(V_0))$ to $\mathrm{Irr}(N,\Phi)$, and hence an isomorphism $\mathrm{R}(G,(V_0)) \xrightarrow{\cong} \mathrm{R}(N,\Phi)$. Similarly, it induces isomorphisms $\mathrm{R}(H,(V_0)) \xrightarrow{\cong} \mathrm{R}(H \cap N,\Phi)$ for each $H \subseteq G$ of finite index, and upon taking the inverse limit over all such H for which H/G_0 has prime power order we get an isomorphism $\mathrm{R}_{\mathcal{P}}(G,(V_0)) \xrightarrow{\cong} \mathrm{R}_{\mathcal{P}}(N,\Phi)$. And this in turn induces an isomorphism between the cokernels of $\mathrm{rs}_{G,(V_0)}$ and $\mathrm{rs}_{N,\Phi}$.

The homomorphism $\mathrm{R}(N,\Phi) \to \mathrm{R}(N_\phi,\phi) \cong \mathrm{R}(N_\phi/K_\phi,\phi)$, defined by sending $[V]$ to $[V\langle\phi\rangle]$, is an isomorphism: its inverse is the induction map $[V] \mapsto [\mathrm{Ind}_{N_\phi}^N(V)]$. This same assignment also defines an isomorphism $\mathrm{R}_{\mathcal{P}}(N,\Phi) \xrightarrow{\cong} \mathrm{R}_{\mathcal{P}}(N_\phi/K_\phi,\phi)$ (whose inverse is again the induction map); and hence defines an isomorphism between the cokernels of $\mathrm{rs}_{N,\Phi}$ and $\mathrm{rs}_{N_\phi/K_\phi,\phi}$. $\square$

The above general description of $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})$ is rather complicated. In contrast, the conditions for the map $\mathrm{rs}_G^{\mathrm{U}}$ to be onto can be formulated more simply.

Theorem 3.10. *Let G be any compact Lie group. Fix a maximal torus $T \subseteq G$ and a Weyl chamber $C \subseteq \mathfrak{t}$, and set $N = N(T,C) \subseteq G$. Let $\mathcal{E}'(N)$ denote the set of subgroups $E \subseteq N$ of finite index such that E/T is elementary but not of prime power order. Then*

$$\mathrm{expt}\big(\mathrm{Coker}(rs_G^{\mathrm{U}})\big) = \mathrm{lcm}\big\{\delta(E/[E,T]) \,\big|\, E \in \mathcal{E}'(N)\big\}. \tag{1}$$

In particular, $\mathrm{rs}_G^{\mathrm{U}}$ is surjective if and only if $\mathrm{rs}_N^{\mathrm{U}}$ is surjective, if and only if $T \cap [E,E] = [E,T]$ for all $E \in \mathcal{E}'(N)$.

Proof. It is clear from part (c) that the exponent of $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})$ divides the number given in (1). To show that these are equal, fix any prime p, and choose $E \subseteq N$

of finite index such that E/T is p-elementary but not a p-group. We must show that $\delta(E/[E,T])\big|\operatorname{expt}\big(\operatorname{Coker}(\mathrm{rs}_G^{\mathrm{U}})\big)$. Choose any $\phi' \in (T/[E,T])^* \subseteq T^*$ such that $\delta(E/[E,T],\phi') = \delta(E/[E,T])$. Since N/T acts linearly on $\mathfrak{t}^*$ and leaves C^* invariant, the fixed set $(C^*)^E$ is a cone shaped subspace of $(\mathfrak{t}^*)^E$ with nonempty interior. Hence, we can choose $\phi \in C^* \cap (T/[E,T])^* = (C_T^*)^E$ such that $\phi \equiv \phi'$ modulo the exponent of $\frac{T\cap[E,E]}{[E,T]}$. If $q \neq p$ is any other prime dividing $|E/T|$, then

$$\delta(E/[E,T],q\phi) = \delta(E/[E,T],\phi) = \delta(E/[E,T],\phi') = \delta(E/[E,T])$$

by Lemma 3.7(d,e). And finally, if $gT \in E/T$ is the element of order q, then $gT \in S$ in the notation of Lemma 3.8: no two elements in $gT/\operatorname{Ker}(q\phi)$ are conjugate. Thus,

$$\delta(E/[E,T]) = \delta(E/[E,T],q\phi)\big|\operatorname{expt}\big(\operatorname{Coker}(\mathrm{rs}_{E,q\phi})\big)\big|\operatorname{expt}\big(\operatorname{Coker}(\mathrm{rs}_G^{\mathrm{U}})\big)$$

by Lemma 3.8; and this finishes the proof of formula (1). The necessary and sufficient conditions for $\mathrm{rs}_G^{\mathrm{U}}$ to be surjective now follow from Lemma 3.7(a). □

Since the general condition for $\mathrm{rs}_G^{\mathrm{U}}$ to be surjective is still somewhat complicated, we now list some special cases which are simpler to formulate.

Corollary 3.11. *For any compact Lie group G, $\operatorname{Coker}(rs_G^{\mathrm{U}})$ has finite exponent, and*

$$\operatorname{expt}\big(\operatorname{Coker}(rs_G^{\mathrm{U}})\big)^2\big||\pi_0(G)|. \tag{1}$$

Furthermore, rs_G^{U} is surjective if G satisfies any of the following conditions:

(a) G is finite or connected.

(b) All elements of $\pi_0(G)$ have prime power order.

(c) $\pi_0(G)$ is a periodic group: all of its Sylow subgroups are cyclic or quaternion.

(d) $Z(G_0) = 1$.

(e) G is a semidirect product of the form $G = G_0 \rtimes \Gamma$, where $\Gamma \subseteq G$ normalizes some maximal torus T and leaves invariant some Weyl chamber in T.

Proof. Fix a maximal torus $T \subseteq G_0$, and a Weyl chamber C. Set $N = N(T,C)$. As in Theorem 3.10, let $\mathcal{E}'(N)$ be the set of subgroups $H \subseteq N$ of finite index such that H/T is elementary but not of prime power order.

By Lemma 3.7(b), $\delta(H/[T,H])^2\big||H/T|\big||\pi_0(G)|$ for each $H \subseteq N$ of finite index. So (1) follows from Theorem 3.10.

(a) $\mathrm{rs}_G^{\mathrm{U}}$ is onto by Lemma 3.4 if G is finite, and by (1) if G is connected.

(b) If all elements of $\pi_0(G) = \pi_0(N)$ have prime power order, then $\mathcal{E}'(N) = \emptyset$, and so $\mathrm{rs}_G^{\mathrm{U}}$ is onto by Theorem 3.10.

(c) Note that $H_2(\Gamma) = 0$ for any finite periodic group Γ. Hence, if $\pi_0(G)$ is periodic, then for any $H \in \mathcal{E}'(N)$, $H/[H,T] \cong T/[H,T] \times H/T$. So $\mathrm{rs}_N^{\mathrm{U}}$ and $\mathrm{rs}_G^{\mathrm{U}}$ are onto by Theorem 3.10.

(e) The conditions on Γ imply that N is a semidirect product of T with Γ, and hence that $\mathrm{rs}_G^{\mathrm{U}}$ is onto by Theorem 3.10.

(d) By [Bo, §4.10, Corollaire], the surjection $\mathrm{Aut}(G_0) \twoheadrightarrow \mathrm{Out}(G_0)$ is split by outer automorphisms which fix T and C. Let $\Gamma \subseteq G$ be the subgroup of elements whose conjugation action lies in the image of any given splitting map. Then $G = G_0 \rtimes \Gamma$ (since $G_0 \cap \Gamma = Z(G_0) = 1$); and so $\mathrm{rs}_G^{\mathrm{U}}$ is onto by (e). $\square$

We remark here that G being a semidirect product $G_0 \rtimes \Gamma$ does not in itself imply that $\mathrm{rs}_G^{\mathrm{U}}$ is onto. As an example, set

$$G = C_3 \times \big(\mathrm{SU}(2) {\times_{C_2}} Q(8)\big),$$

where C_3 is cyclic of order 3, $Q(8)$ is a quaternion group of order 8, and the product is taken by identifying the central subgroups of order 2 in $\mathrm{SU}(2)$ and $Q(8)$. Then Theorem 3.10 applies to show that $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})$ has exponent 2. But $\mathrm{SU}(2){\times_{C_2}}Q(8)$ is also a semidirect product of $\mathrm{SU}(2)$ with $C_2{\times}C_2$: the splitting comes from the diagonal subgroup

$$\langle (i,i)\rangle \times \langle (j,j)\rangle \subseteq Q(8){\times_{C_2}}Q(8) \subseteq \mathrm{SU}(2){\times_{C_2}}Q(8).$$

So far, we have dealt mostly with the case of unitary representations. The general conditions for $\mathrm{rs}_G^{\mathrm{O}}$ to be surjective seem to be much more complicated. For example, with a little more work, one can show that if G is a central extension of a torus by a finite group, then $\mathrm{rs}_G^{\mathrm{O}}$ is onto if and only if $\mathrm{rs}_G^{\mathrm{U}}$ is onto. In contrast, the following example provides a simple way of constructing groups G for which $\mathrm{rs}_G^{\mathrm{O}}$ is not onto but $\mathrm{rs}_G^{\mathrm{U}}$ is onto.

Example 3.12. *Fix any pair (G',V'), where G' is a compact connected Lie group, and V' an irreducible G'-representation of real type having the additional property that some central element $z{\in}Z(G')$ of order 2 acts on V' by $(-\mathrm{Id})$. Choose any odd prime power $n>1$, and set $G = G'{\times_{C_2}}Q(4n)$: the central product of G' with the quaternion group of order $4n$, where z is identified with the central element of $Q(4n)$. Then $\mathrm{rs}_G^{\mathrm{O}}$ is not onto.*

Proof. Let W be any effective irreducible representation of $Q(4n)$, and set $V = V'{\otimes_{\mathbb{C}}}W$. Then V is an irreducible G-representation of quaternion type, but its restriction to any p-toral subgroup of G (for any prime p) has real type. In particular, $[V]$ represents an element of $\mathrm{RO}_{\mathcal{P}}(G)$; but since $\mathrm{rs}_G^{\mathrm{O}}$ and $\mathrm{rs}_G^{\mathrm{U}}$ are injective (all elements of $\pi_0(G) \cong D(2n)$ have prime power order), it does not lie in the image of $\mathrm{rs}_G^{\mathrm{O}}$. $\square$

For example, we can take $G' = \mathrm{SO}(2m)$ for any $m \geq 2$, and let V' be the standard G'-representation on $\mathbb{C}^{2m}$. Set $G = G'{\times_{C_2}}Q(4n)$, for some odd prime power $n \geq 3$. Then $\mathrm{rs}_G^{\mathrm{O}} : \mathrm{RO}(G) \to \mathrm{RO}_{\mathcal{P}}(G)$ fails to be onto, while $\mathrm{rs}_G^{\mathrm{U}}$ is onto (in fact, an isomorphism) by Corollary 3.11(b) (all elements of $\pi_0(G)$ have prime power order).

References

[Ad] J. F. Adams, Lectures on Lie groups, Benjamin (1969)

[Bo] N. Bourbaki, Groupes et algèbres de Lie, Chapitre 9, Hermann (1982)

[BtD] T. Bröcker & T. tom Dieck, Representations of compact Lie groups, Springer-Verlag (1985)

[tD] T. tom Dieck, Transformation groups, de Gruyter (1987)

[Dr] A. Dress, Induction and structure theorems for orthogonal representations of finite groups, Annals of Math. 102 (1975), 291–325

[Fe] M. Feshbach, The transfer and compact Lie groups, Trans. A. M. S. 251 (1979), 139–169

[JO] S. Jackowski & B. Oliver, Vector bundles over classifying spaces of compact Lie groups, Acta math. (to appear)

[Ol] R. Oliver, Whitehead groups of finite groups, London Math. Soc. Lecture Notes Series 132, Cambridge Univ. Press (1988)

[Seg] G. Segal, The representation ring of a compact Lie group, Publ. math. I. H. E. S. 34 (1968), 113–128

[Ser] J.-P. Serre, Linear representations of finite groups, Springer-Verlag (1977)

[Sn] V. Snaith, Explicit Brauer induction, Inventiones math. 94 (1988), 455–478

[Ta] M. Takeuchi, A remark on the character ring of a compact Lie group, J. Math. Soc. Japan 23 (1971), 662–675

[Vo] D. Vogan, Unitary representations of reductive Lie groups, Annals of math. studies 118, Princeton Univ. Press (1987)