NORMALIZERS OF SETS OF COMPONENTS IN FUSION SYSTEMS

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Abstract. We describe some new ways to construct saturated fusion subsystems, including, as a special case, the normalizer of a set of components of the ambient fusion system. This was motivated in part by Aschbacher's construction of the normalizer of one component, and in part by joint work with three other authors where we had to construct the normalizer of all of the components.

A saturated fusion system over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are injective homomorphisms between the subgroups, and which satisfy certain conditions first formulated by Puig (see [Pg] and Definition 1.1). The motivating examples are the fusion systems $\mathcal{F}_S(G)$ when G is a finite group and $S \in \mathrm{Syl}_p(G)$: the objects of $\mathcal{F}_S(G)$ are the subgroups of S, and the morphisms are those homomorphisms between subgroups induced by conjugation in G.

By analogy with finite groups, Aschbacher, in [A1, Chapter 9], defined a component of a saturated fusion system $\mathcal{F}$ over S to be a subnormal fusion subsystem of $\mathcal{F}$ that is quasisimple (see Definition 3.6). He then showed that the components of $\mathcal{F}$ commute and satisfy other properties satisfied by the components of a finite group. In a later paper [A2, § 2.1], he constructed the normalizer of a component; i.e., the unique largest fusion subsystem that contains the given component as a normal subsystem.

In a recent paper with Carles Broto, Jesper Møller, and Albert Ruiz [BMOR], we needed to construct a normalizer for all of the components; i.e., a largest subsystem of $\mathcal{F}$ that contains each of the components of $\mathcal{F}$ as a normal subsystem. This turned out to be somewhat simpler than Aschbacher's construction of the normalizer of one component, but it also led this author to try to better understand Aschbacher's construction, and to look for ways in which it could be generalized. This has resulted in a slightly more explicit construction of these normalizers, and led to our two main Theorems 2.5 and 3.11. In particular, Theorem 2.5 provides a very general method for constructing saturated fusion subsystems (normal or not) of a given fusion system: one which we hope will have other applications in the future.

As one special case of Theorem 3.11, we get the following theorem about normalizers of components:

Theorem A. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{C}_1, \dots, \mathcal{C}_k \le \mathcal{F}$ be the components of $\mathcal{F}$ over $T_1, \dots, T_k \le S$. Let $J \subseteq \{1, \dots, k\}$ be a nonempty subset such that the subgroup $T_J = \langle T_j \,|\, j \in J \rangle$ is fully normalized in $\mathcal{F}$ (Definition 1.1(c)), and let $\mathcal{C}_J \le \mathcal{F}$ be the central product of the components $\mathcal{C}_j$ for $j \in J$. Set $N_J = N_S(T_J)$ and $W_J = \bigcap_{j \in J} N_S(T_j)$. Then there are saturated fusion subsystems $\mathcal{W}_J \trianglelefteq \mathcal{N}_J \le \mathcal{F}$ over $W_J \trianglelefteq N_J \le S$ such that $\mathcal{N}_J$ is the largest saturated fusion subsystem of $\mathcal{F}$ containing $\mathcal{C}_J$ as a normal subsystem, and $\mathcal{W}_J$ is the largest saturated subsystem containing each $\mathcal{C}_j$ (for $j \in J$) as a normal subsystem.

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In other words, $\mathcal{C}_J \trianglelefteq \mathcal{N}_J$, and for each saturated fusion subsystem $\mathcal{D} \le \mathcal{F}$ such that $\mathcal{C}_J \trianglelefteq \mathcal{D}$ we have $\mathcal{D} \le \mathcal{N}_J$. Similarly, $\mathcal{C}_j \trianglelefteq \mathcal{W}_J$ for each $j \in J$, and for each $\mathcal{D} \le \mathcal{F}$ such that $\mathcal{C}_j \trianglelefteq \mathcal{D}$ for all $j \in J$, we have $\mathcal{D} \le \mathcal{W}_J$. More generally, Theorem 3.11 says that a similar conclusion holds if the components are replaced by an arbitrary set of fusion subsystems satisfying certain conditions listed in Hypotheses 3.3.

We mostly regard Theorems 2.5 and 3.11 as new tools for constructing saturated fusion subsystems of a given fusion system, with Theorem A as one special case of particular interest. But they also have consequences for fundamental groups of linking systems associated to the fusion systems, and the subsystems constructed here can be associated to covering spaces of the geometric realizations of those linking systems. See Remark 2.6 at the end of Section 2 for a more detailed discussion.

In Section 1, we recall some basic definitions and properties of saturated fusion systems, all well known except (perhaps) for the technical Lemma 1.10. In Section 2, we describe a very general way to construct saturated subsystems of a given saturated fusion system $\mathcal{F}$ (see Hypotheses 2.1 and Theorem 2.5). Then, in Section 3, we consider sets of commuting subsystems of a saturated fusion system $\mathcal{F}$, under assumptions (Hypotheses 3.3) that include the case where the subsystems are the components of $\mathcal{F}$ (Proposition 3.7). This leads to Theorem 3.11, a special case of Theorem 2.5, where we construct normalizers of these subsystems. Theorem A then follows as a special case of Theorem 3.11.

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Notation: We write c_x for a homomorphism defined via conjugation by x ($c_x(g) = xgx^{-1}$), and $c_x^P \in \mathrm{Aut}(P)$ for its restriction to a subgroup P when x normalizes P. As usual, when $H \le G$ is a pair of groups and $x \in G$, we write ${}^xH = c_x(H) = xHx^{-1}$ and $H^x = c_x^{-1}(H) = x^{-1}Hx$. When G is a group and $P, Q \le G$, we let $\mathrm{Hom}_G(P,Q) \subseteq \mathrm{Hom}(P,Q)$ be the set of all homomorphisms of the form c_x for $x \in G$ such that ${}^xP \le Q$.

Functions and morphisms are always composed from right to left. Also,

- $\mathscr{S}(G)$ is the set of subgroups of a group G;
- $\underline{\boldsymbol{k}} = \{1, \ldots, k\}$ for $k \ge 1$; and
- $\mathcal{H}_{\le T} = \mathcal{H} \cap \mathscr{S}(T)$ when $\mathcal{H}$ is a set of subgroups of a group S and $T \le S$.

When $\mathcal{F}$ is a fusion system over S and $\mathcal{H} \subseteq \mathscr{S}(S)$, we let $\mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}$ denote the full subcategory whose set of objects is $\mathcal{H}$.

1. Background on fusion and linking systems

We summarize here our basic terminology when working with fusion systems. For a prime p, a *fusion system* over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are injective homomorphisms between subgroups such that for each $P, Q \le S$:

- $\mathrm{Hom}_{\mathcal{F}}(P,Q) \supseteq \mathrm{Hom}_S(P,Q)$; and
- for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P),P)$.

Here, $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ denotes the set of morphisms in $\mathcal{F}$ from P to Q. We also write $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ for the set of isomorphisms, $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Iso}_{\mathcal{F}}(P,P)$, and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$. For

$P \le S$ and $g \in S$, we set

$$P^{\mathcal{F}} = \{\varphi(P) \,|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)\} \qquad \text{and} \qquad g^{\mathcal{F}} = \{\varphi(g) \,|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle g\rangle, S)\}$$

(the sets of subgroups and elements $\mathcal{F}$*-conjugate* to P and to g).

Definition 1.1. *Let $\mathcal{F}$ be a fusion system over a finite p-group S.*

(a) *A subgroup $P \le S$ is* fully automized *in $\mathcal{F}$ if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.*

(b) *A subgroup $P \le S$ is* receptive *in $\mathcal{F}$ if each isomorphism $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$ in $\mathcal{F}$ extends to a morphism $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi^{\mathcal{F}}, S)$, where*

$$N_\varphi^{\mathcal{F}} = \{x \in N_S(Q) \,|\, \varphi c_x^Q \varphi^{-1} \in \mathrm{Aut}_S(P)\}.$$

(c) *A subgroup $P \le S$ is* fully normalized (fully centralized) *in $\mathcal{F}$ if $|N_S(P)| \ge |N_S(Q)|$ ($|C_S(P)| \ge |C_S(Q)|$) for each $Q \in P^{\mathcal{F}}$.*

(d) *The fusion system $\mathcal{F}$ is* saturated *if each $\mathcal{F}$-conjugacy class of subgroups of S contains a member that is fully automized and receptive.*

We will sometimes need to refer to the following criteria for a fusion system to be saturated.

Proposition 1.2 ([RS, Theorem 5.2] or [AKO, Proposition I.2.5])**.** *A fusion system is saturated if and only if it satisfies the following two conditions:*

- (Sylow axiom) *each subgroup $P \le S$ that is fully normalized in $\mathcal{F}$ is also fully automized and fully centralized; and*
- (extension axiom) *each $P \le S$ that is fully centralized in $\mathcal{F}$ is also receptive.*

We often need to refer to the following types or classes of subgroups in a fusion system.

Definition 1.3. *Let $\mathcal{F}$ be a fusion system over a finite p-group S. For a subgroup $P \le S$,*

(a) *P is $\mathcal{F}$-*centric *if $C_S(Q) \le Q$ for each $Q \in P^{\mathcal{F}}$;*

(b) *P is $\mathcal{F}$-*radical *if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$;*

(c) *P is* strongly closed *in $\mathcal{F}$ if for each $x \in P$, $x^{\mathcal{F}} \subseteq P$; and*

(d) *P is* central *in $\mathcal{F}$ if each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ in $\mathcal{F}$ extends to some $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, PR)$ such that $\bar\varphi|_P = \mathrm{Id}_P$.*

Let $\mathcal{F}^{cr} \subseteq \mathcal{F}^{c}$ denote the sets of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups, and $\mathcal{F}$-centric subgroups, respectively. Let

(e) *$Z(\mathcal{F})$ (the* center *of $\mathcal{F}$) be the (unique) largest subgroup central in $\mathcal{F}$; and set*

(f) *$\mathfrak{foc}(\mathcal{F}) = \langle x^{-1}y \,|\, x, y \in S,\ y \in x^{\mathcal{F}}\rangle$ (the* focal subgroup *of $\mathcal{F}$).*

In many cases, to prove saturation, it is not necessary to prove the axioms for all conjugacy classes of subgroups. If $\mathcal{F}$ is a fusion system over a finite p-group S and $\mathcal{H}$ is a set of subgroups of S closed under $\mathcal{F}$-conjugacy, then

- $\mathcal{F}$ is $\mathcal{H}$*-saturated* if each member of $\mathcal{H}$ is $\mathcal{F}$-conjugate to a subgroup that is fully automized and receptive; and
- $\mathcal{F}$ is $\mathcal{H}$*-generated* if each morphism in $\mathcal{F}$ is a composite of restrictions of morphisms between members of $\mathcal{H}$.

Under certain conditions on $\mathcal{H}$, these two conditions suffice to show that $\mathcal{F}$ is saturated.

Proposition 1.4. *Let $\mathcal{F}$ be a fusion system over a finite p-group S, and let $\mathcal{H} \subseteq \mathscr{S}(S)$ be a nonempty set of subgroups closed under $\mathcal{F}$-conjugacy.*

(a) *Assume that $\mathcal{F}$ is $\mathcal{H}$-generated and $\mathcal{H}$-saturated, and that for each $P \in \mathcal{F}^c \smallsetminus \mathcal{H}$ there is $Q \in P^{\mathcal{F}}$ such that $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P)) \neq 1$. Then $\mathcal{F}$ is saturated.*

(b) (Stancu's criterion) *Assume $\mathcal{H}$ is closed under overgroups in $\mathscr{S}(S)$. Assume also that S is fully automized in $\mathcal{F}$, and that each $P \in \mathcal{H}$ that is fully normalized in $\mathcal{F}$ is also receptive in $\mathcal{F}$. Then $\mathcal{F}$ is $\mathcal{H}$-saturated.*

Proof. Point (a) is shown in [BCGLO, Theorem 2.2] (see also the discussion in [AKO, Theorem I.3.10]).

By [AKO, Proposition I.9.3(c⇒a)], Stancu's criterion for saturation [St] implies that in Definition 1.1(d). Point (b), the corresponding implication for $\mathcal{H}$-saturation, holds by the same argument whenever $\mathcal{H}$ is closed under overgroups. □

We will need the following version of Alperin's fusion theorem for fusion systems.

Theorem 1.5 ([BLO, Theorem A.10])**.** *If $\mathcal{F}$ be a saturated fusion system over a finite p-group S, then each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms of subgroups that are $\mathcal{F}$-centric and $\mathcal{F}$-radical.*

If $\mathcal{F}$ is a fusion system over a finite p-group S, and $\beta\colon S \longrightarrow T$ is an isomorphism of groups, then ${}^{\beta}\mathcal{F}$ denotes the fusion system over T defined by setting

$$\mathrm{Hom}_{{}^{\beta}\mathcal{F}}(P,Q) = \left\{ \beta\varphi\beta^{-1} \,\middle|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(\beta^{-1}(P), \beta^{-1}(Q)) \right\}$$

for all $P, Q \leq T$. (Recall that we compose from right to left.) In terms of this notation, two fusion systems $\mathcal{F}$ over S and $\mathcal{E}$ over T are *isomorphic* if there is an isomorphism of groups $\beta\colon S \xrightarrow{\cong} T$ such that $\mathcal{E} = {}^{\beta}\mathcal{F}$.

Definition 1.6 ([AKO, Definition I.6.1])**.** *Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S, and let $\mathcal{E} \leq \mathcal{F}$ be a saturated fusion subsystem over $T \leq S$. Then $\mathcal{E}$ is* normal *in $\mathcal{F}$ (denoted $\mathcal{E} \trianglelefteq \mathcal{F}$) if the following four conditions are satisfied:*

- *T is strongly closed in $\mathcal{F}$;*
- (invariance condition) *${}^{\alpha}\mathcal{E} = \mathcal{E}$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$;*
- (Frattini condition) *for each $P \leq T$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,T)$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,T)$ such that $\varphi = \alpha \circ \varphi_0$; and*
- (extension condition) *each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to an automorphism $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[\bar{\alpha}, C_S(T)] \overset{\text{def}}{=} \langle x^{-1}\bar{\alpha}(x) \,|\, x \in C_S(T) \rangle \leq Z(T)$.*

The following elementary property of normal subsystems will be needed.

Lemma 1.7. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over $T \trianglelefteq S$. Let $\mathcal{F}_0 \leq \mathcal{F}$ be a saturated fusion subsystem over $S_0 \leq S$ such that $\mathcal{F}_0 \geq \mathcal{E}$, and assume the extension condition holds for $\mathcal{E} \leq \mathcal{F}_0$. Then $\mathcal{E} \trianglelefteq \mathcal{F}_0$.*

Proof. Since $\mathcal{E} \trianglelefteq \mathcal{F}$, the subgroup T is strongly closed in $\mathcal{F}$, and hence is also strongly closed in $\mathcal{F}_0$. By [AKO, Proposition I.6.4] and since $\mathcal{E} \trianglelefteq \mathcal{F}$, the *strong invariance condition* holds: for each pair of subgroups $P \leq Q \leq T$, each $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P,Q)$, and each $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q,T)$, we have $\psi\varphi(\psi|_P)^{-1} \in \mathrm{Hom}_{\mathcal{E}}(\psi(P), \psi(Q))$.

The strong invariance condition for $\mathcal{E} \le \mathcal{F}_0$ follows immediately from that for $\mathcal{E} \le \mathcal{F}$. Hence by [AKO, Proposition I.6.4] again, $\mathcal{E} \le \mathcal{F}_0$ also satisfies the invariance and Frattini conditions. So if it also satisfies the extension condition, then $\mathcal{E} \trianglelefteq \mathcal{F}_0$. □

We will need to work with quotient fusion systems, but only in the special (and very simple) case where we divide by a central subgroup.

Definition 1.8. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $Z \le Z(\mathcal{F})$ is a central subgroup.*

(a) *Let $\mathcal{F}/Z$ be the fusion system over S/Z where for all $P, Q \le S$ containing Z,*

$$\mathrm{Hom}_{\mathcal{F}/Z}(P/Z, Q/Z) = \bigl\{\varphi/Z \bigm| \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q),\ (\varphi/Z)(gZ) = \varphi(g)Z\bigr\}.$$

(b) *If $\mathcal{E} \le \mathcal{F}$ is a fusion subsystem over $T \le S$, then $Z\mathcal{E} \le \mathcal{F}$ is the fusion subsystem over ZT where for $P, Q \le ZT$,*

$$\mathrm{Hom}_{Z\mathcal{E}}(P,Q) = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \,|\, \varphi|_{P\cap T} \in \mathrm{Hom}_{\mathcal{E}}(P\cap T, Q\cap T)\}.$$

If $\mathcal{E} \trianglelefteq \mathcal{F}$, then $Z\mathcal{E} \le \mathcal{F}$ is a special (and much more elementary) case of a construction of Aschbacher [A1, Theorem 8.20].

Lemma 1.9. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, let $Z \le Z(\mathcal{F})$ be a central subgroup, and let $\mathcal{E} \le \mathcal{F}$ be a saturated fusion subsystem over $T \le S$. Then*

(a) *$\mathcal{F}/Z$ and $Z\mathcal{E}$ are both saturated; and*

(b) *if $Z(\mathcal{E}) \le Z$, then $Z\mathcal{E}/Z \cong \mathcal{E}/Z(\mathcal{E})$.*

Proof. The statement that $\mathcal{F}/Z$ is saturated is a special case of [Cr, Proposition 5.11].

For each $P \le ZT$, $\mathrm{Aut}_{Z\mathcal{E}}(P) \cong \mathrm{Aut}_{\mathcal{E}}(P\cap T)$ and $\mathrm{Aut}_{ZT}(P) \cong \mathrm{Aut}_T(P\cap T)$. So P is fully automized in $Z\mathcal{E}$ if and only if $P\cap T$ is fully automized in $\mathcal{E}$. A similar argument shows that P is receptive in $Z\mathcal{E}$ if and only if $P \cap T$ is receptive in $\mathcal{E}$, and thus $Z\mathcal{E}$ is saturated since $\mathcal{E}$ is saturated.

Assume that $Z(\mathcal{E}) \le Z$, and hence that $Z(\mathcal{E}) = Z \cap T$. Let $\psi\colon T/Z(\mathcal{E}) \xrightarrow{\ \cong\ } ZT/Z$ be the natural isomorphism. By the definitions, a morphism $\varphi/Z \in \mathrm{Hom}(P/Z, Q/Z)$ (for $\varphi \in \mathrm{Hom}(P,Q)$ such that $\varphi(Z) = Z$) lies in $Z\mathcal{E}/Z$ if and only if $\psi^{-1}(\varphi/Z)\psi = (\varphi|_{P\cap T})/Z(\mathcal{E})$ lies in $\mathcal{E}/Z(\mathcal{E})$. So $Z\mathcal{E}/Z = {}^{\psi}(\mathcal{E}/Z(\mathcal{E}))$, and hence $\mathcal{E}/Z(\mathcal{E}) \cong Z\mathcal{E}/Z$. □

Note that Lemma 1.9(b) is a very elementary case of the second isomorphism theorem for fusion systems (see [Cr, Proposition 5.16]).

The following technical lemma will be useful later when proving that certain fusion systems are $\mathcal{H}$-saturated.

Lemma 1.10. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{E} \le \mathcal{F}$ be a fusion subsystem (not necessarily saturated) over $T \le S$. Let $\mathcal{H} \subseteq \mathscr{S}(T)$ be a nonempty set of subgroups closed under $\mathcal{E}$-conjugacy and overgroups in T, and assume that the following two properties hold for all $P \in \mathcal{H}$:*

(i) *for each $\overline{P} \le T$ containing P and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\overline{P}, T)$, if $\varphi|_P \in \mathrm{Hom}_{\mathcal{E}}(P,T)$, then $\varphi \in \mathrm{Hom}_{\mathcal{E}}(\overline{P}, T)$; and*

(ii) *for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_T(P), S)$, there are $R \le S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(R,T)$ such that $R \ge \langle \varphi(N_T(P)), C_S(\varphi(P))\rangle$ and $\psi\varphi \in \mathrm{Hom}_{\mathcal{E}}(N_T(P), T)$.*

Then every subgroup $P \in \mathcal{H}$ that is fully normalized or fully centralized in $\mathcal{E}$ is also receptive (hence fully centralized) in $\mathcal{E}$ and in $\mathcal{F}$. If in addition,

(iii) $\mathrm{Inn}(T) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{E}}(T))$,

then $\mathcal{E}$ is $\mathcal{H}$-saturated.

Proof. Fix $P \in \mathcal{H}$ that is fully normalized or fully centralized in $\mathcal{E}$, and choose $P_2 \in P^{\mathcal{F}}$ that is fully normalized in $\mathcal{F}$. By [AKO, Lemma I.2.6(c)], there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_T(P), S)$ such that $\varphi(P) = P_2$. By (ii), there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(R, T)$ such that $\langle \varphi(N_T(P)), C_S(P_2)\rangle \le R \le S$ and $\psi\varphi \in \mathrm{Hom}_{\mathcal{E}}(N_T(P), T)$.

Set $P_3 = \psi(P_2) \in P^{\mathcal{E}}$; thus $\varphi(N_T(P)) \le N_R(P_2)$ and $\psi(N_R(P_2)) \le N_T(P_3)$. Likewise, $\varphi(C_T(P)) \le C_R(P_2)$ and $\psi(C_R(P_2)) \le C_T(P_3)$, where $C_R(P_2) = C_S(P_2)$ by assumption in (ii). Then $\psi\varphi(C_T(P)) = \psi(C_S(P_2)) = C_T(P_3)$ since P is fully centralized or fully normalized in $\mathcal{E}$, and P and P_3 are both fully centralized in $\mathcal{F}$ and in $\mathcal{E}$ since P_2 is.

We claim that $C_S(P) \le T$. By (ii), applied with the inclusion of $N_T(P)$ into S in the role of φ, there are $R \ge \langle N_T(P), C_S(P)\rangle$ and $\omega \in \mathrm{Hom}_{\mathcal{F}}(R, T)$ such that $\omega|_{N_T(P)} \in \mathrm{Hom}_{\mathcal{E}}(N_T(P), T)$. Then $\omega(C_S(P)) = C_S(\omega(P)) \le T$ since P is fully centralized in $\mathcal{F}$, and $\omega(C_T(P)) = C_T(\omega(P))$ since P is fully centralized in $\mathcal{E}$ and $\omega|_{C_T(P)} \in \mathrm{Mor}(\mathcal{E})$. So $C_S(P) = C_T(P)$.

We have now shown that P is fully centralized and hence receptive in $\mathcal{F}$. It remains to show that it is receptive in $\mathcal{E}$. Let $Q \in P^{\mathcal{E}} \subseteq \mathcal{H}$ and $\rho \in \mathrm{Iso}_{\mathcal{E}}(Q, P)$ be arbitrary, and consider the subgroups

$$N_\rho^{\mathcal{F}} = \{x \in N_S(Q) \mid \rho c_x^Q \rho^{-1} \in \mathrm{Aut}_S(P)\}$$
$$N_\rho^{\mathcal{E}} = \{x \in N_T(Q) \mid \rho c_x^Q \rho^{-1} \in \mathrm{Aut}_T(P)\} \le N_\rho^{\mathcal{F}}.$$

Since P is receptive in $\mathcal{F}$, ρ extends to some $\bar\rho \in \mathrm{Hom}_{\mathcal{F}}(N_\rho^{\mathcal{F}}, S)$. For each $x \in N_\rho^{\mathcal{E}}$, $c_{\bar\rho(x)}^P = \rho c_x^Q \rho^{-1} \in \mathrm{Aut}_T(P)$, and hence there is $y \in T$ such that $\bar\rho(x) \in yC_S(P)$. We just showed that $C_S(P) \le T$, and so $\bar\rho(x) \in T$.

Thus $\bar\rho$ restricts to $\widehat\rho \in \mathrm{Hom}_{\mathcal{F}}(N_\rho^{\mathcal{E}}, T)$. By (i) and since $\widehat\rho|_Q = \rho \in \mathrm{Mor}(\mathcal{E})$, $\widehat\rho \in \mathrm{Hom}_{\mathcal{E}}(N_\rho^{\mathcal{E}}, T)$, extending ρ. Since ρ was arbitrary, this shows that P is receptive in $\mathcal{E}$.

If in addition, (iii) holds, and T is fully automized in $\mathcal{E}$, then $\mathcal{E}$ is $\mathcal{H}$-saturated by Proposition 1.4(b). □

We end the section with two group theoretic lemmas which are included for convenient reference. The first is very elementary.

Lemma 1.11. *Let $\chi\colon G \longrightarrow K$ be a homomorphism of finite groups, and let $P \le G$ be a p-subgroup. Then $P \in \mathrm{Syl}_p(G)$ if and only if $\chi(P) \in \mathrm{Syl}_p(\chi(G))$ and $\mathrm{Ker}(\chi|_P) \in \mathrm{Syl}_p(\mathrm{Ker}(\chi))$.*

Proof. Just note that $|G : P| = |\chi(G) : \chi(P)| \cdot |\mathrm{Ker}(\chi) : \mathrm{Ker}(\chi|_P)|$. □

The following well known result about automorphisms of p-groups is useful when identifying elements of $O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ in a fusion system $\mathcal{F}$ over $S \ge P$.

Lemma 1.12. *Let P be a finite p-group, and let $1 = P_0 \le P_1 \le \cdots \le P_k = P$ be a sequence of subgroups, all normal in P. Let $\varGamma \le \mathrm{Aut}(P)$ be a group of automorphisms that normalizes each of the P_i. Then for each $\alpha \in \varGamma$ such that $x^{-1}\alpha(x) \in P_{i-1}$ for each $1 \le i \le k$ and each $x \in P_i$, we have $\alpha \in O_p(\varGamma)$.*

Proof. Let $\varGamma_0 \le \varGamma$ be the subgroup of all $\alpha \in \varGamma$ that induce the identity on each P_i/P_{i-1}. Then $\varGamma_0 \trianglelefteq \varGamma$ since $\varGamma$ normalizes each of the P_i. It remains only to prove that each element of $\varGamma_0$ has p-power order, and this is shown, for example, in [G, Lemma 5.3.3]. □

2. Maps from fusion systems to groups

We give in this section a very general setup for constructing saturated fusion subsystems: one that includes Theorem A as a special case. Recall that when $\mathcal{F}$ is a fusion system over S and $\mathcal{H} \subseteq \mathscr{S}(S)$ is a set of subgroups of S, we let $\mathcal{F}^{\mathcal{H}}$ denote the full subcategory of $\mathcal{F}$ whose set of objects is $\mathcal{H}$. Thus $\mathrm{Mor}(\mathcal{F}^{\mathcal{H}})$ is the set of all morphisms in $\mathcal{F}$ between subgroups in $\mathcal{H}$.

Hypotheses 2.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Fix a subgroup $T \trianglelefteq S$, a set of subgroups $\mathcal{H} \subseteq \mathscr{S}(S)$ such that $T \in \mathcal{H}$, and a map $\chi\colon \mathrm{Mor}(\mathcal{F}^{\mathcal{H}}) \longrightarrow G$ for some finite group G. Assume the following hold:*

- (i) *$\mathcal{H}$ is closed under $\mathcal{F}$-conjugacy and overgroups;*
- (ii) *$\chi(\varphi\psi) = \chi(\varphi)\chi(\psi)$ whenever $\varphi, \psi \in \mathrm{Mor}(\mathcal{F}^{\mathcal{H}})$ are composable, and $\chi(\mathrm{incl}_P^Q) = 1$ for each $P \leq Q$ in $\mathcal{H}$;*
- (iii) *$\chi(\mathrm{Inn}(T)) = 1$ and $\chi(\mathrm{Aut}_{\mathcal{F}}(T)) = G$; and*
- (iv) *for each $P \leq T$ not in $\mathcal{H}$, there is $x \in N_T(P) \smallsetminus P$ such that $c_x^P \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$.*

We could instead define χ as a functor from $\mathcal{F}^{\mathcal{H}}$ to $\mathcal{B}(G)$, where $\mathcal{B}(G)$ is a category with one object and endomorphism group G, and then remove the first part of condition (ii). We could also have defined χ to be a homomorphism from $\pi_1(|\mathcal{F}^{\mathcal{H}}|)$ to G, where $|\mathcal{F}^{\mathcal{H}}|$ is the geometric realization of the category $\mathcal{F}^{\mathcal{H}}$, and then removed (ii) completely. (See, e.g., Section III.2.2 and Proposition III.2.8 in [AKO] for more detail.) But it seems simplest to work with a map χ as above.

Notation 2.2. *Assume Hypotheses 2.1. Set $U = \chi(\mathrm{Aut}_S(T)) \leq G$, and let $\widehat{\chi}\colon S \longrightarrow U$ be the homomorphism that sends $x \in S$ to $\chi(c_x^T) \in \chi(\mathrm{Aut}_S(T)) = U$. For each $V \leq U$, define*

$$S_V = \widehat{\chi}^{-1}(V) = \{x \in S \,|\, \widehat{\chi}(x) \in V\} \quad \text{and} \quad \mathcal{H}_{\leq S_V} = \mathcal{H} \cap \mathscr{S}(S_V) = \{P \in \mathcal{H} \,|\, P \leq S_V\}.$$

For each $H \leq G$, define the fusion subsystem $\mathcal{F}_H \leq \mathcal{F}$ over $S_{U\cap H}$ by setting

$$\mathcal{F}_H = \big\langle \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \,\big|\, P, Q \in \mathcal{H}_{\leq S_{U\cap H}},\ \chi(\varphi) \in H \big\rangle.$$

In particular, we set $S_{\mathrm{Id}} = \mathrm{Ker}(\widehat{\chi})$, and let $\mathcal{F}_{\mathrm{Id}}$ be the fusion system over S_{Id} generated by morphisms in $\chi^{-1}(1)$.

Note that $S_U = S$ in the situation of Hypotheses 2.1 and Notation 2.2. We will see later that $\mathcal{H} = \mathcal{H}_{\leq S_U} \supseteq \mathcal{F}^{cr}$, and hence that $\mathcal{F}_G = \mathcal{F}$ by Theorem 1.5 (Alperin's fusion theorem).

We first list some of the basic properties that always hold in the situation of Hypotheses 2.1.

Lemma 2.3. *Assume Hypotheses 2.1 and Notation 2.2. Then the following hold:*

- (a) *For all $P \in \mathcal{H}$ and $x \in N_S(P)$, we have $\widehat{\chi}(x) = \chi(c_x^P)$. For $P, Q \in \mathcal{H}$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, we have ${}^{\chi(\varphi)}\widehat{\chi}(x) = \widehat{\chi}(\varphi(x))$ for all $x \in P$.*
- (b) *For each $Q, R \leq U$, we have $\chi(\mathrm{Hom}_{\mathcal{F}}(S_Q, S_R)) = \{g \in G \,|\, {}^{g}Q \leq R\}$. Also, $U \in \mathrm{Syl}_p(G)$.*
- (c) *If $P \in \mathcal{H}$, then $C_S(P) \leq S_{\mathrm{Id}}$.*
- (d) *If $\mathcal{D} \leq \mathcal{F}$ is a fusion subsystem over $D \leq S$ with $D \geq T$, then $\mathcal{D}^{cr} \subseteq \mathcal{H}$. More precisely, for each subgroup $P \in \mathcal{D}^c \smallsetminus \mathcal{H}$,*

$$\mathrm{Out}_D(P) \cap O_p(\mathrm{Out}_{\mathcal{D}}(P)) \neq 1.$$

- (e) *If $V \leq U$ is strongly closed in U with respect to G, then S_V is strongly closed in $\mathcal{F}$.*

Proof. **(a)** For $P \in \mathcal{H}$ and $x \in N_S(P)$, we have $\widehat{\chi}(x) = \chi(c_x^T) = \chi(c_x^{PT}) = \chi(c_x^P)$, where the last two equalities hold since $\chi(\mathrm{incl}_T^{PT}) = 1 = \chi(\mathrm{incl}_P^{PT})$.

For $P, Q \in \mathcal{H}$, $x \in P$, and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, we have $\varphi c_x^P = c_{\varphi(x)}^Q \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, so ${}^{\chi(\varphi)}\widehat{\chi}(x) = \widehat{\chi}(\varphi(x))$.

(b) Fix $Q, R \le U$. Then $\chi(\mathrm{Hom}_{\mathcal{F}}(S_Q, S_R)) \subseteq \{g \in G \,|\, {}^gQ \le R\}$ by (a), and it remains to show the opposite inclusion. Fix $g \in G$ such that ${}^gQ \le R \le U$: we must find $\varphi \in \mathrm{Hom}_{\mathcal{F}}(S_Q, S_R)$ such that $\chi(\varphi) = g$. Choose $\beta \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\chi(\beta) = g$.

Since $\chi(\mathrm{Aut}_{S_Q}(T)) = Q$,

$$\chi({}^{\beta}\mathrm{Aut}_{S_Q}(T)) = {}^{\chi(\beta)}Q = {}^gQ \le R = \chi(\mathrm{Aut}_{S_R}(T)),$$

and hence ${}^{\beta}\mathrm{Aut}_{S_Q}(T) \le \mathrm{Aut}_{S_R}(T)\mathrm{Aut}_{\mathcal{F}_{\mathrm{Id}}}(T)$. Since $\mathrm{Aut}_{S_{\mathrm{Id}}}(T) = \mathrm{Aut}_S(T) \cap \mathrm{Aut}_{\mathcal{F}_{\mathrm{Id}}}(T) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_{\mathrm{Id}}}(T))$ (recall $\mathrm{Aut}_{\mathcal{F}_{\mathrm{Id}}}(T) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(T)$), and $\mathrm{Aut}_{S_R}(T)$ normalizes $\mathrm{Aut}_{\mathcal{F}_{\mathrm{Id}}}(T)$ and contains $\mathrm{Aut}_{S_{\mathrm{Id}}}(T)$, there is $\gamma \in \mathrm{Aut}_{\mathcal{F}_{\mathrm{Id}}}(T)$ such that ${}^{\gamma\beta}\mathrm{Aut}_{S_Q}(T) \le \mathrm{Aut}_{S_R}(T)$. Also, T is receptive in $\mathcal{F}$ since it is normal in S, so $\gamma\beta$ extends to $\varphi \in \mathrm{Hom}_{\mathcal{F}}(S_Q, S_R)$, and $\chi(\varphi) = \chi(\gamma)\chi(\beta) = 1 \cdot g = g$.

Since $\mathcal{F}$ is saturated and T is fully normalized (normal in S), we have $\mathrm{Aut}_S(T) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(T))$. Hence $U = \chi(\mathrm{Aut}_S(T)) \in \mathrm{Syl}_p(G)$ (Lemma 1.11).

(c) If $P \in \mathcal{H}$ and $x \in C_S(P)$, then $\widehat{\chi}(x) = \chi(c_x^P) = \chi(\mathrm{Id}_P) = 1$ by (a), so $x \in S_{\mathrm{Id}}$. Thus $C_S(P) \le S_{\mathrm{Id}}$.

(d) Let $\mathcal{D} \le \mathcal{F}$ be a fusion subsystem over $D \le S$ with $D \ge T$, and assume $P \in \mathcal{D}^c \smallsetminus \mathcal{H}$. Set $P_0 = P \cap T$, and let $Q = \{t \in N_T(P_0) \,|\, c_t \in O_p(\mathrm{Aut}_{\mathcal{F}}(P_0))\}$. Then $P \le N_S(Q)$, and $Q \nleq P$ by Hypotheses 2.1(iv) and since $P_0 \notin \mathcal{H}$. So $N_{PQ}(P) > P$, and we can choose $x \in N_Q(P) \smallsetminus P$. Then $c_x^P|_{P_0} \in O_p(\mathrm{Aut}_{\mathcal{F}}(P_0))$ and c_x^P induces the identity on P/P_0, so $c_x^P \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ by Lemma 1.12.

Thus $c_x^P \in O_p(\mathrm{Aut}_{\mathcal{F}}(P)) \cap \mathrm{Aut}_{\mathcal{D}}(P) \le O_p(\mathrm{Aut}_{\mathcal{D}}(P))$. Also, $P \ge C_D(P)$ since P is $\mathcal{D}$-centric, and $[c_x^P] \ne 1$ in $\mathrm{Out}(P)$ since $x \in T \le D$. So $c_x^P \notin \mathrm{Inn}(P)$, and $[c_x^P] \ne 1$ in $\mathrm{Out}_D(P) \cap O_p(\mathrm{Out}_{\mathcal{D}}(P))$. In particular, P is not $\mathcal{D}$-radical, and since this holds for all $P \in \mathcal{D}^c \smallsetminus \mathcal{H}$, we conclude that $\mathcal{D}^{cr} \subseteq \mathcal{H}$.

(e) Assume $V \le U$ is strongly closed in U with respect to G. If S_V is not strongly closed in $\mathcal{F}$, then by Theorem 1.5 (Alperin's fusion theorem), there are $P \in \mathcal{F}^{cr}$, $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, and $x \in P \cap S_V$ such that $\alpha(x) \notin S_V$. Also, $P \in \mathcal{H}$ by (d), applied with $\mathcal{F}$ in the role of $\mathcal{D}$. Then $\widehat{\chi}(x) \in V$ and $\widehat{\chi}(\alpha(x)) \notin V$ are conjugate in G by (a), contradicting the assumption that V is strongly closed. □

We next look at some of the properties of the fusion subsystems $\mathcal{F}_H$.

Lemma 2.4. *Assume Hypotheses 2.1 and Notation 2.2, fix a subgroup $H \le G$, and set $V = H \cap U$.*

(a) *If $P \in \mathcal{H}_{\le S_V}$ is fully normalized or fully centralized in $\mathcal{F}_H$, then P is receptive and fully centralized in $\mathcal{F}_H$ and in $\mathcal{F}$.*

(b) *We have $\mathcal{F}_H^{cr} \subseteq \mathcal{F}_H^c \cap \mathcal{H} \subseteq \mathcal{F}^c$.*

Proof. **(a)** By Lemma 1.10, this implication is true if the following two conditions hold.

(i) *For each $P \le \overline{P}$ in $\mathcal{H}$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\overline{P}, T)$, if $\varphi|_P \in \mathrm{Hom}_{\mathcal{F}_H}(P,T)$, then $\varphi \in \mathrm{Hom}_{\mathcal{F}_H}(\overline{P}, T)$.*

This holds since for each such φ, $\chi(\varphi) = \chi(\varphi)\chi(\mathrm{incl}_P^{\bar{P}}) = \chi(\varphi|_P) \in H$.

(ii) *For each $P \in \mathcal{H}$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_{S_V}(P), S)$, there are $R \le S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(R, S_V)$, where $R \ge \langle \varphi(N_{S_H}(P)), C_S(\varphi(P))\rangle$ and $\psi\varphi \in \mathrm{Hom}_{\mathcal{F}_H}(N_{S_V}(P), S_V)$.*

To see this, set $g = \chi(\varphi)$: then $\chi(\varphi(N_{S_V}(P))) \le {}^{g}V \cap U$ by Lemma 2.3(a). Set $R = S_{{}^{g}V\cap U}$: by Lemma 2.3(b), there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(R, S_V)$ such that $\chi(\psi) = g^{-1}$. Thus $\psi\varphi \in \mathrm{Hom}_{\mathcal{F}_H}(N_{S_V}(P), S_V)$. Also, $\varphi(N_{S_V}(P)) \le R$ by construction, and $C_S(\varphi(P)) \le S_{\mathrm{Id}} \le R$ by Lemma 2.3(c).

(b) If $P \in \mathcal{F}_H^{cr}$, then $P \in \mathcal{H}$ by Lemma 2.3(d), while $P \in \mathcal{F}_H^{c}$ by definition. So $\mathcal{F}_H^{cr} \subseteq \mathcal{F}_H^{c} \cap \mathcal{H}$. If $P \in \mathcal{F}_H^{c} \cap \mathcal{H}$, then P is fully centralized in $\mathcal{F}_H$ since it is $\mathcal{F}_H$-centric, and is fully centralized in $\mathcal{F}$ by (a). Also, $C_S(P) = C_{S_V}(P) \le P$ by Lemma 2.3(c), and hence $P \in \mathcal{F}^c$. □

We are now ready to determine under what conditions the fusion subsystems $\mathcal{F}_H$ are saturated.

Theorem 2.5. *Assume Hypotheses 2.1 and Notation 2.2. For each subgroup $H \le G$,*

(a) *the fusion subsystem $\mathcal{F}_H$ is saturated if and only if $H \cap U \in \mathrm{Syl}_p(H)$; and*

(b) *$\mathcal{F}_H$ is saturated and normal in $\mathcal{F}$ if and only if $H \trianglelefteq G$.*

Proof. Set $V = H \cap U$ for short. If $\mathcal{F}_H$ is saturated, then $\mathrm{Aut}_{S_V}(T) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_H}(T))$, and hence $V = \chi(\mathrm{Aut}_{S_V}(T)) \in \mathrm{Syl}_p(H)$ by Lemma 1.11.

Conversely, assume now that $V \in \mathrm{Syl}_p(H)$; we must show that $\mathcal{F}_H$ is saturated. In Step 1, we reduce to the case where S_V is fully normalized in $\mathcal{F}$. In Step 2, we prove that $\mathcal{F}_H$ is $\mathcal{H}_{\le S_V}$-saturated, and in Step 3, finish the proof that it is saturated. Point (b) is shown in Step 4.

Step 1: We first show that it suffices to prove saturation of $\mathcal{F}_H$ when S_V is fully normalized in $\mathcal{F}$. Let $H \le G$ be arbitrary, assuming only that $V \in \mathrm{Syl}_p(H)$. Choose $Q \in (S_V)^{\mathcal{F}}$ that is fully normalized in $\mathcal{F}$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(S_V, Q)$, and set $g = \chi(\varphi)$. Then $Q \ge S_{\mathrm{Id}}$ since S_{Id} is strongly closed (Lemma 2.3(e)), so $Q = S_W$ where $W = \widehat{\chi}(Q) = {}^{g}V$ (Lemma 2.3(a)). Then ${}^{\varphi}\mathcal{F}_H = \mathcal{F}_{{}^{g}H}$: by inspection, c_φ sends the defining generators of $\mathcal{F}_H$ (see Notation 2.2) to those of $\mathcal{F}_{{}^{g}H}$. In particular, $\mathcal{F}_H$ is saturated if $\mathcal{F}_{{}^{g}H}$ is saturated, and so it suffices to prove the latter. Note that $W = {}^{g}V \le {}^{g}H \cap U$, with equality since ${}^{g}V \in \mathrm{Syl}_p({}^{g}H)$.

We assume from now on that $S_V = S_W$ is fully normalized in $\mathcal{F}$.

Step 2: We now show that $\mathcal{F}_H$ is $\mathcal{H}_{\le S_V}$-saturated. By Lemma 2.4(a), if $P \in \mathcal{H}_{\le S_V}$ is fully normalized in $\mathcal{F}_H$, then it is also receptive in $\mathcal{F}_H$. So by Proposition 1.4(b), it remains only to prove that S_V is fully automized in $\mathcal{F}_H$; i.e., that $\mathrm{Inn}(S_V) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_H}(S_V))$.

By Lemma 2.3(b), the homomorphism

$$\chi_V\colon \mathrm{Aut}_{\mathcal{F}}(S_V) \longrightarrow N_G(V)/V \qquad \text{defined by} \qquad \chi_V(\alpha) = \chi(\alpha)V$$

is well defined and surjective. Also, $\mathrm{Aut}_{\mathcal{F}_H}(S_V) = \chi_V^{-1}(N_H(V)/V)$ by definition, and $N_H(V)/V$ has order prime to p since $V \in \mathrm{Syl}_p(H)$. Thus $\mathrm{Ker}(\chi_V)$ has index prime to p in $\mathrm{Aut}_{\mathcal{F}_H}(S_V)$. Furthermore, $\mathrm{Aut}_S(S_V) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S_V))$ since S_V is fully normalized in $\mathcal{F}$, so by Lemma 1.11, $\mathrm{Ker}(\chi_V|_{\mathrm{Aut}_S(S_V)})$ is a Sylow p-subgroup of $\mathrm{Ker}(\chi_V)$ and hence of $\mathrm{Aut}_{\mathcal{F}_H}(S_V)$. For $x \in N_S(S_V)$,

$$c_x^{S_V} \in \mathrm{Ker}(\chi_V) \iff \chi(c_x^{S_V}) \in V \iff \widehat{\chi}(x) \in V \iff x \in S_V,$$

so that $\mathrm{Ker}(\chi_V|_{\mathrm{Aut}_S(S_V)}) = \mathrm{Inn}(S_V)$, finishing the proof that $\mathrm{Inn}(S_V) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_H}(S_V))$.

Step 3: By Lemma 2.3(d), we have $\mathrm{Out}_{S_V}(P)\cap O_p(\mathrm{Out}_{\mathcal{F}_H}(P))\neq 1$ for each $P\in\mathcal{F}_H^c\smallsetminus\mathcal{H}$. Also, $\mathcal{F}_H$ is $\mathcal{H}_{\le S_V}$-saturated by Step 2, and is $\mathcal{H}_{\le S_V}$-generated by definition of $\mathcal{F}_H$ (see Notation 2.2). So $\mathcal{F}_H$ is saturated by Proposition 1.4.

Step 4: If H is normal in G, then $V=H\cap U\in\mathrm{Syl}_p(H)$ since $U\in\mathrm{Syl}_p(G)$, so $\mathcal{F}_H$ is saturated by Step 3.

By Lemma 2.3(e), S_V is strongly closed in $\mathcal{F}$. Each $\alpha\in\mathrm{Aut}_{\mathcal{F}}(S_V)$ permutes the generating set used to define $\mathcal{F}_H$, and hence induces an automorphism of $\mathcal{F}_H$. So the invariance condition for $\mathcal{F}_H\le\mathcal{F}$ holds. The extension condition also holds since $C_S(S_V)\le S_{\mathrm{Id}}\le S_V$ by Lemma 2.3(c).

Now, $G=N_G(V)H$ by the Frattini argument. Fix $P\in\mathcal{H}_{\le S_V}$ and $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,S_V)$, and let $h\in H$ and $g\in N_G(V)$ be such that $\chi(\varphi)=gh$. By Lemma 2.3(b), there is $\beta\in\mathrm{Aut}_{\mathcal{F}}(S_V)$ such that $\chi(\beta)=g$. Set $\varphi_0=\varphi\beta^{-1}$; then $\chi(\varphi_0)=h\in H$ and hence $\varphi_0\in\mathrm{Hom}_{\mathcal{F}_H}(P,S_V)$. This proves the Frattini condition for $\mathcal{F}_H\le\mathcal{F}$ for morphisms between members of $\mathcal{H}$, and the general case follows by Theorem 1.5 and since $(\mathcal{F}_H)^{cr}\subseteq\mathcal{H}$ (Lemma 2.3(d)). Thus $\mathcal{F}_H\trianglelefteq\mathcal{F}$.

Conversely, assume $\mathcal{F}_H$ is saturated and normal in $\mathcal{F}$. Then $T\le S_V$ for each $V\le U$ since $\widehat{\chi}(T)=1$ by Hypotheses 2.1(iii), and hence T is an object in $\mathcal{F}_H$. (This holds for all $H\le G$.) Since $\mathcal{F}_H\trianglelefteq\mathcal{F}$, we have $\mathrm{Aut}_{\mathcal{F}_H}(T)\trianglelefteq\mathrm{Aut}_{\mathcal{F}}(T)$ (see, e.g., [AKO, Proposition I.6.4(c)]), and hence $H=\chi(\mathrm{Aut}_{\mathcal{F}_H}(T))\trianglelefteq\chi(\mathrm{Aut}_{\mathcal{F}}(T))=G$. □

Remark 2.6. For readers familiar with linking systems associated to fusion systems and their geometric realizations (see Sections III.2 and III.4 in [AKO]), we note here that in the situation of Hypotheses 2.1 and Notation 2.2, if $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, then it is straightforward to define a linking subsystem $\mathcal{L}_H\le\mathcal{L}$ associated to $\mathcal{F}_H$ for each $H\le G$ such that $H\cap U\in\mathrm{Syl}_p(H)$. More precisely, if $\pi\colon\mathcal{L}\longrightarrow\mathcal{F}$ is the structure functor for $\mathcal{L}$, then we set $\widetilde{\chi}=\chi\circ\pi\colon\mathrm{Mor}(\mathcal{L}^{\mathcal{H}})\longrightarrow G$, where $\mathcal{L}^{\mathcal{H}}$ is the full subcategory of $\mathcal{L}$ with objects in $\mathcal{H}\cap\mathcal{F}^c$, and let $\mathcal{L}_H\le\mathcal{L}$ be the subcategory with objects $\mathcal{F}^c\cap\mathcal{H}_{\le S_V}$ ($V=H\cap U$) and morphisms in $\widetilde{\chi}^{-1}(H)$. Using the fact that $(\mathcal{F}_H)^{cr}\subseteq\mathcal{F}^c$ (see Lemma 2.4(b)), one easily checks that this is a linking system.

This setup also gives information about the fundamental groups of the geometric realizations $|\mathcal{F}^{\mathcal{H}}|$ and $|\mathcal{L}|$. As noted earlier, the map $\chi\colon\mathrm{Mor}(\mathcal{F}^{\mathcal{H}})\longrightarrow G$ can be regarded as a surjective homomorphism from $\pi_1(|\mathcal{F}^{\mathcal{H}}|)$ onto G. Since $|\mathcal{L}|$ and $|\mathcal{L}^{\mathcal{H}}|$ are homotopy equivalent, where $\mathcal{L}^{\mathcal{H}}\subseteq\mathcal{L}$ is the full subcategory with object set $\mathcal{H}$ (see [BCGLO, Theorem 3.5]), $\widetilde{\chi}$ induces a surjection from $\pi_1(|\mathcal{L}|)\cong\pi_1(|\mathcal{L}^{\mathcal{H}}|)$ onto G, and the spaces $|\mathcal{L}_H|$ (for $H\le G$ as above) are equivalent to covering spaces of $|\mathcal{L}|$ (see [AKO, Propsition III.2.9]).

3. The normalizer of a set of components

We adopt here the notation and terminology used in [Ol] for morphisms of fusion systems and sets of commuting subsystems. Recall that we set $\underline{\boldsymbol{k}}=\{1,\dots,k\}$ for $k\ge 1$.

A sequence $\mathcal{E}_1,\dots,\mathcal{E}_k\le\mathcal{F}$ of fusion subsystems over finite p-groups $T_1,\dots,T_k\le S$ *commutes in* $\mathcal{F}$ if there is a morphism of fusion systems from $\mathcal{E}_1\times\cdots\times\mathcal{E}_k$ to $\mathcal{F}$ that extends the inclusions of the $\mathcal{E}_i$ into $\mathcal{F}$. In other words, there is a homomorphism of groups $I\colon T_1\times\cdots\times T_k\longrightarrow S$ that sends $(t_1,\dots,t_k)$ to $t_1\cdots t_k$, and which induces a functor $\widehat{I}$ from the category $\mathcal{E}_1\times\cdots\times\mathcal{E}_k$ to $\mathcal{F}$. Equivalently:

Lemma 3.1 ([Ol, Lemma 2.8])**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{E}_1,\dots\mathcal{E}_k\le\mathcal{F}$ be fusion subsystems over subgroups $T_1,\dots,T_k\le S$. Then $\mathcal{E}_1,\dots\mathcal{E}_k$ commute in $\mathcal{F}$ if and only if $[T_i,T_j]=1$ for each $i\neq j$ in $\underline{\boldsymbol{k}}$, and for each k-tuple of morphisms*

$\varphi_i \in \mathrm{Hom}_{\mathcal{E}_i}(P_i, Q_i)$ (for $i \in \underline{\boldsymbol{k}}$), there is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_1 \cdots P_k, Q_1 \cdots Q_k)$ such that $\varphi|_{P_i} = \varphi_i$ for each i.

When $\mathcal{E}_1, \ldots, \mathcal{E}_k \leq \mathcal{F}$ commute in $\mathcal{F}$, and $\widehat{I} \colon \mathcal{E}_1 \times \cdots \times \mathcal{E}_k \longrightarrow \mathcal{F}$ is as above, we define the *central product* of the $\mathcal{E}_i$ in $\mathcal{F}$ to be the fusion subsystem

$$\begin{aligned} \mathcal{E}_1 \cdots \mathcal{E}_k &= \widehat{I}(\mathcal{E}_1 \times \cdots \times \mathcal{E}_k) \\ &= \big\langle \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q) \,\big|\, P = P_1 \cdots P_k,\ Q = Q_1 \cdots Q_k,\ \varphi|_{P_i} \in \mathrm{Hom}_{\mathcal{E}_i}(P_i, Q_i)\ \forall i \in \underline{\boldsymbol{k}} \big\rangle. \end{aligned} \tag{3.1}$$

The central product of commuting saturated fusion subsystems is always saturated: we leave it as an exercise to show that $\mathcal{E}_1 \cdots \mathcal{E}_k \cong (\mathcal{E}_1 \times \cdots \times \mathcal{E}_k)/Z$ for some central subgroup $Z \leq Z(\mathcal{E}_1 \times \cdots \times \mathcal{E}_k)$. (The image of an arbitrary morphism between saturated fusion systems is also saturated, but this is a much deeper theorem first shown by Puig: see, e.g., Proposition 5.11 and Corollary 5.15 in [Cr].)

Recall the fusion subsystems $Z\mathcal{E} \leq \mathcal{F}$, for $\mathcal{E} \leq \mathcal{F}$ and $Z \leq Z(\mathcal{F})$, of Definition 1.8(b).

Lemma 3.2. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{E}_1, \ldots, \mathcal{E}_k \leq \mathcal{F}$ be commuting saturated fusion subsystems over $T_1, \ldots, T_k \leq S$. Set $T = T_1 \cdots T_k$ and $\mathcal{E} = \mathcal{E}_1 \cdots \mathcal{E}_k$, and set $Z = Z(\mathcal{E})$. Then*

(a) *for each $i \in \underline{\boldsymbol{k}}$, $\mathcal{E}_i$ is the full subcategory of $\mathcal{E}$ with objects the subgroups of T_i;*

(b) *if $t_i \in T_i$ for all $i \in \underline{\boldsymbol{k}}$ and $t = t_1 \cdots t_k$, then $t^{\mathcal{E}} = \{u_i \cdots u_k \,|\, u_i \in t_i^{\mathcal{E}_i}\}$;*

(c) *$T/Z = (T_1Z/Z) \times \cdots \times (T_kZ/Z)$; and*

(d) *$\mathcal{E}/Z = (Z\mathcal{E}_1/Z) \times \cdots \times (Z\mathcal{E}_k/Z)$ where $Z\mathcal{E}_i/Z \cong \mathcal{E}_i/Z(\mathcal{E}_i)$ for each i.*

Proof. By assumption, there is a functor $\widehat{I}$ from $\mathcal{E}_1 \times \cdots \times \mathcal{E}_k$ into $\mathcal{F}$ that restricts to the inclusion on each factor. Since each $\mathcal{E}_i$ is a full subcategory of the direct product $\prod_{i=1}^k \mathcal{E}_i$, its image under $\widehat{I}$ is a full subcategory of $\widehat{I}(\mathcal{E}_1 \times \cdots \times \mathcal{E}_k) = \mathcal{E}$. This proves (a).

Point (b) follows immediately from (3.1). Since the center of a saturated fusion system is the set of elements whose conjugacy class has order 1 (see [AKO, Lemma I.4.2]), (b) implies that

$$\text{for } t_i \in T_i \text{ (all } i \in \underline{\boldsymbol{k}}\text{)},\ t_1 \cdots t_k \in Z \text{ if and only if } t_i \in Z(\mathcal{E}_i) \text{ for all } i. \tag{3.2}$$

Let $I \colon T_1 \times \cdots \times T_k \longrightarrow T$ be the homomorphism $I(t_1, \ldots, t_k) = t_1 \cdots t_k$. Thus I is surjective, and $I^{-1}(Z) = Z(\mathcal{E}_1) \times \cdots \times Z(\mathcal{E}_k)$ by (3.2). This proves (c), and also shows that I induces an isomorphism from $\prod_{i=1}^k (T_i/Z(\mathcal{E}_i))$ to T/Z. It also proves that $Z \cap T_i = Z(\mathcal{E}_i)$ for each i, and hence that the inclusion of T_i into ZT_i induces an isomorphism $T_i/Z(\mathcal{E}_i) \cong ZT_i/Z \leq T/Z$. Thus T/Z is the direct product of its subgroups ZT_i/Z, and the independence of the factors (T_iZ/Z) implies that $T_iZ \cap T_jZ = Z$ for any pair $i \neq j$ of distinct indices in $\underline{\boldsymbol{k}}$.

The equality $\mathcal{E}/Z = \prod_{i=1}^k (Z\mathcal{E}_i/Z)$ now follows immediately from the description of $\mathcal{E} = \mathcal{E}_1 \cdots \mathcal{E}_k$ in (3.1). □

We now describe the setup needed to state our main theorem on normalizers.

Hypotheses 3.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{E}_1, \ldots, \mathcal{E}_k \leq \mathcal{F}$ (for some $k \geq 1$) be saturated fusion subsystems over $T_1, \ldots, T_k \leq S$ that commute in $\mathcal{F}$. Set $T = T_1 \cdots T_k$ and $\mathcal{E} = \mathcal{E}_1 \cdots \mathcal{E}_k$, and assume*

(i) *$\mathcal{E} \trianglelefteq \mathcal{F}$;*

(ii) *$T_i \nleq Z(\mathcal{E})$ for each $i \in \underline{\boldsymbol{k}}$; and*

(iii) *each element of $\mathrm{Aut}_{\mathcal{F}}(T)$ permutes the subgroups T_i for $i \in \underline{\boldsymbol{k}}$.*

By Lemma 3.2(a), each $\mathcal{E}_i$ is the full subcategory of $\mathcal{E}$ with objects the subgroups of T_i. Since each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ permutes the subgroups T_i by (iii), this means that α also permutes the fusion subsystems $\mathcal{E}_i$.

We also need some more notation.

Notation 3.4. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_k \le \mathcal{F}$ be as in Hypotheses 3.3. Let $\chi_0\colon \mathrm{Aut}_{\mathcal{F}}(T) \longrightarrow \Sigma_k$ be the homomorphism defined by (iii): $\chi_0(\alpha) = \sigma$ (for $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\sigma \in \Sigma_k$) if $\alpha(T_iZ(\mathcal{E})) = T_{\sigma(i)}Z(\mathcal{E})$ for each $i \in \underline{\boldsymbol{k}}$. Define also*

$$Z = Z(\mathcal{E}), \qquad G = \chi_0(\mathrm{Aut}_{\mathcal{F}}(T)) \le \Sigma_k,$$
$$\mathcal{H} = \{P \le S \mid P \cap T_i \nleq Z \text{ for each } i \in \underline{\boldsymbol{k}}\}, \qquad U = \chi_0(\mathrm{Aut}_S(T)) \le G.$$

In the next proposition, we give some other criteria that imply condition (iii) in Hypotheses 3.3. A fusion system is *indecomposable* if it is not the direct product of two or more proper fusion subsystems. Recall also the focal subgroup $\mathfrak{foc}(-)$ from Definition 1.3(f).

Proposition 3.5. *Assume $\mathcal{E}_1, \ldots, \mathcal{E}_k \le \mathcal{F}$ are saturated fusion systems over finite p-groups $T_1, \ldots, T_k \le S$ that commute in $\mathcal{F}$ and satisfy conditions (i) and (ii) in Hypotheses 3.3. Then either of the conditions*

(iii′) *for each $i \in \underline{\boldsymbol{k}}$, $\mathcal{E}_i/Z(\mathcal{E}_i)$ is indecomposable and $\mathfrak{foc}(\mathcal{E}_i) = T_i$, or*

(iii″) *for each $i \in \underline{\boldsymbol{k}}$, $\mathcal{E}_i/Z(\mathcal{E}_i)$ is indecomposable, $T_i \ge Z(\mathcal{E}_1 \cdots \mathcal{E}_k)$, and $Z(\mathcal{E}_i/Z(\mathcal{E}_i)) = 1$*

implies condition (iii).

Proof. Set $\mathcal{E} = \mathcal{E}_1 \cdots \mathcal{E}_k$, $T = T_1 \cdots T_k$, and $Z = Z(\mathcal{E}) \le Z(T)$. By Lemma 3.2(c,d), we have $\mathcal{E}/Z = \prod_{i=1}^k (Z\mathcal{E}_i/Z)$ and $T/Z = \prod_{i=1}^k (ZT_i/Z)$, where for each i, $Z\mathcal{E}_i/Z \cong \mathcal{E}_i/Z(\mathcal{E}_i)$ and hence is indecomposable under either condition (iii′) or (iii″). Also, $\mathfrak{foc}(\mathcal{E}/Z) = T/Z$ (if (iii′) holds) or $Z(\mathcal{E}/Z) = 1$ (if (iii″) holds), and in either case, the factorization of $\mathcal{E}/Z$ as a product of the indecomposable fusion subsystems $Z\mathcal{E}_i/Z$ is unique by [Ol, Corollary 5.3].

Now assume $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T) \le \mathrm{Aut}(\mathcal{E})$. Then α induces an automorphism $\bar{\alpha} \in \mathrm{Aut}(\mathcal{E}/Z)$, and by the uniqueness of the factorization, there is $\sigma \in \Sigma_k$ such that $\bar{\alpha}(T_iZ/Z) = T_{\sigma(i)}Z/Z$ and hence $\alpha(T_iZ) = T_{\sigma(i)}Z$ for each i. If (iii″) holds, then $T_iZ = T_i$ for each i, so α permutes the T_i. If (iii′) holds, then α permutes the fusion subsystems $Z\mathcal{E}_i$ (since they are full subcategories of $\mathcal{E}$ by Lemma 3.2(a)), and hence permutes the focal subgroups $\mathfrak{foc}(Z\mathcal{E}_i) = \mathfrak{foc}(\mathcal{E}_i) = T_i$. In either case, $\alpha(T_i) = T_{\sigma(i)}$, finishing the proof of (iii). □

The following definition of a component of a fusion system is taken from [A1].

Definition 3.6. *Let $\mathcal{E} \le \mathcal{F}$ be saturated fusion systems over finite p-groups $T \le S$. The subsystem $\mathcal{E}$ is* subnormal *in $\mathcal{F}$ (denoted $\mathcal{E} \trianglelefteq\trianglelefteq \mathcal{F}$) if there is a sequence of subsystems $\mathcal{E} = \mathcal{E}_0 \trianglelefteq \mathcal{E}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{E}_m = \mathcal{F}$ each normal in the following one. The subsystem $\mathcal{E}$ is* quasisimple *if $\mathcal{E}/Z(\mathcal{E})$ is simple and $\mathfrak{foc}(\mathcal{E}) = T$. The* components *of $\mathcal{F}$ are the subnormal saturated fusion subsystems of $\mathcal{F}$ that are quasisimple.*

A saturated fusion system $\mathcal{F}$ is *constrained* if it has a normal subgroup $Q \trianglelefteq \mathcal{F}$ that is $\mathcal{F}$-centric. By (9.9.1) and (9.12.3) in [A1], $\mathcal{F}$ is constrained if and only if it has no components. The assumption in the following example that $\mathcal{F}$ not be constrained is made to ensure that it has at least one component.

Example 3.7. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $\mathcal{F}$ is not constrained. Then $\mathcal{F}$ satisfies Hypotheses 3.3, with the set $\mathrm{Comp}(\mathcal{F})$ of components of $\mathcal{F}$ in the role of $\{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$ and with their central product in the role of $\mathcal{E}$.*

Proof. All of these statements are shown in [A1, Chapter 9]. Since $\mathcal{F}$ is not constrained, $C_S(O_p(\mathcal{F})) \nleq O_p(\mathcal{F})$, so $O_p(\mathcal{F})$ is strictly contained in the generalized Fitting subsystem $F^*(\mathcal{F})$ by [A1, 9.11], and hence $\mathrm{Comp}(\mathcal{F}) \neq \varnothing$ by [A1, 9.9]. Also, $E(\mathcal{F})$ (defined in [A1] to be the smallest normal subsystem containing all components) is the central product of the components by [A1, 9.9.1], and is normal in $\mathcal{F}$ by [A1, 9.8.1]. Thus the components of $\mathcal{F}$ commute and satisfy condition (i), and satisfy (ii) and (iii′) since they are quasisimple. □

We now start to look at some of the consequences of these hypotheses. Condition (a) in the next lemma is stated in two forms: a simpler form which suffices here in most cases, and a longer, more technical form needed in one of the proofs below.

Lemma 3.8. *Assume Hypotheses 3.3 and Notation 3.4. For $I \subseteq \underline{\boldsymbol{k}}$, set $T_I = \langle T_i \,|\, i \in I\rangle$.*

(a) *For each $P, Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, there is an element $\sigma \in G \leq \Sigma_k$ such that $\varphi(P \cap T_i) \leq T_{\sigma(i)}$ for each $i \in \underline{\boldsymbol{k}}$. If $P \in \mathcal{H}$, then $Q \in \mathcal{H}$, and this element σ is unique.*

(a′) *Fix $P, Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Then there are $\sigma \in G \leq \Sigma_k$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\varphi(P \cap T_I) \leq T_{\sigma(I)} = \alpha(T_I)$ for each $I \subseteq \underline{\boldsymbol{k}}$. In particular, $\varphi(P \cap T_i) \leq T_{\sigma(i)} \in (T_i)^{\mathcal{F}}$ for each $i \in \underline{\boldsymbol{k}}$. If $j \in \underline{\boldsymbol{k}}$ is such that $P \cap T_j \nleq Z$, then $\varphi(P \cap T_j) \nleq T_\ell$ for $\ell \neq \sigma(j)$.*

Define $\chi\colon \mathrm{Mor}(\mathcal{F}^{\mathcal{H}}) \longrightarrow G$ by setting $\chi(\varphi) = \sigma$ whenever $P, Q \in \mathcal{H}$ and φ and σ are as in (a). Then

(b) *$\mathcal{F}$, T, $\mathcal{H}$, χ, G, and U together satisfy Hypotheses 2.1.*

Proof. Recall that $Z = Z(\mathcal{E})$ and $\mathcal{E} \trianglelefteq \mathcal{F}$, and $\mathcal{H} = \{P \leq S \,|\, P \cap T_i \nleq Z(\mathcal{E}) \;\forall\, i \in \underline{\boldsymbol{k}}\}$.

(a,a′) Fix $P, Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Then $\varphi(P \cap T) \leq T$ since T is strongly closed in $\mathcal{F}$ (recall $\mathcal{E} \trianglelefteq \mathcal{F}$). By the Frattini condition for $\mathcal{E} \trianglelefteq \mathcal{F}$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P \cap T, T)$ such that $\varphi|_{P\cap T} = \alpha\varphi_0$. By Hypotheses 3.3(iii), there is $\sigma = \chi_0(\alpha) \in G \leq \Sigma_k$ such that $\alpha(T_i) = T_{\sigma(i)}$ for each i. Hence for each $I \subseteq \underline{\boldsymbol{k}}$,

$$\varphi(P \cap T_I) = \alpha(\varphi_0(P \cap T_I)) \leq \alpha(T_I) = T_{\sigma(I)}. \tag{3.3}$$

Now let $j, \ell \in \underline{\boldsymbol{k}}$ be such that $P \cap T_j \nleq Z$ and $\varphi(P \cap T_j) \leq T_\ell$. Choose $x \in (P \cap T_j) \smallsetminus Z$. Then $\varphi(x)Z \in (T_{\sigma(j)}Z/Z) \cap (T_\ell Z/Z)$, and $\varphi(x) \notin Z$ since $Z = Z(\mathcal{E})$ is strongly closed in $\mathcal{F}$. So $\ell = \sigma(j)$ by Lemma 3.2(c).

In particular, if $P \in \mathcal{H}$, then $P \cap T_i \nleq Z$ for each $i \in \underline{\boldsymbol{k}}$, so σ is the unique permutation satisfying $\varphi(P \cap T_i) \leq T_{\sigma(i)}$ for each i. Also, $\varphi(P) \cap T_{\sigma(i)} \geq \varphi(P \cap T_i) \nleq Z$ for each i since Z is strongly closed, so $Q \geq \varphi(P) \in \mathcal{H}$.

(b) We must prove the following four properties:

(i) $\mathcal{H}$ is closed under $\mathcal{F}$-conjugacy and overgroups;

(ii) $\chi(\varphi\psi) = \chi(\varphi)\chi(\psi)$ whenever $\varphi, \psi \in \mathrm{Mor}(\mathcal{F}^{\mathcal{H}})$ are composable, and $\chi(\mathrm{incl}_P^Q) = 1$ for each $P \leq Q$ in $\mathcal{H}$;

(iii) $\chi(\mathrm{Inn}(T)) = 1$ and $\chi(\mathrm{Aut}_{\mathcal{F}}(T)) = G$; and

(iv) for each $P \leq T$ not in $\mathcal{H}$, there is $x \in N_T(P) \smallsetminus P$ such that $c_x^P \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

Point (i) holds since $\mathcal{H}$ is closed under $\mathcal{F}$-conjugacy by (a), and is closed under overgroups by definition. Point (ii) and the first statement in (iii) hold by definition of χ, and $\chi(\mathrm{Aut}_{\mathcal{F}}(T)) = G$ by definition of G (Notation 3.4) and since $\chi|_{\mathrm{Aut}_{\mathcal{F}}(T)} = \chi_0$.

It remains to show (iv). Let $Y \le T$ be such that $Y \ge Z$ and $Y/Z = Z(T/Z)$. Thus $Y \trianglelefteq T$. For each $i \in \underline{\boldsymbol{k}}$, $(Y \cap T_i Z)/Z = Z(T_i Z/Z) \ne 1$ (recall that $T/Z = \prod_{i=1}^k (T_i Z/Z)$ by Lemma 3.2(c)), so $Y \cap T_i Z \nleq Z$, and $Y \cap T_i \nleq Z$ since $Y \ge Z$. Hence $Y \in \mathcal{H}$.

If $P \le T$ and $P \notin \mathcal{H}$, then $P \ngeq Y$ since $\mathcal{H}$ is closed under overgroups. Hence $YP > P$ and $N_{YP}(P) > P$. Choose $x \in N_Y(P) \smallsetminus P$; then $[x, Z \cap P] = 1$ and $[x, P] \le Z \cap P$. So c_x^P induces the identity on $P \cap Z$ and on $P/(P \cap Z)$, and since Z is strongly closed in $\mathcal{F}$, we have $c_x^P \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ by Lemma 1.12. □

We need some more notation. Note that as a special case of Hypotheses 3.3(iii), the conjugation action of each $x \in S$ permutes the subgroups $T_i \le T$ for $i \in \underline{\boldsymbol{k}}$, thus inducing an action of S on $\underline{\boldsymbol{k}}$.

Notation 3.9. *Assume Hypotheses 3.3 and Notation 3.4. For each nonempty subset* $J = \{j_1, \dots, j_\ell\} \subseteq \underline{\boldsymbol{k}}$, *define*

$$T_J = T_{j_1} \cdots T_{j_\ell} = \langle T_j \,|\, j \in J \rangle, \qquad N_J = N_S(J) = N_S(T_J),$$

$$\mathcal{E}_J = \mathcal{E}_{j_1} \cdots \mathcal{E}_{j_\ell}, \qquad W_J = C_S(J) = \bigcap\nolimits_{j \in J} N_S(T_j).$$

Also, define

$$\mathcal{H}^{(J)} = \{P \le S \,|\, P \cap T_j \nleq Z \text{ for each } j \in J\} \supseteq \mathcal{H},$$

$$\mathcal{N}_J = \bigl\langle \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \,\big|\, P, Q \in \mathcal{H}^{(J)}_{\le N_J},\ \varphi(P \cap T_J) \le T_J \bigr\rangle, \quad \text{and}$$

$$\mathcal{W}_J = \bigl\langle \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \,\big|\, P, Q \in \mathcal{H}^{(J)}_{\le W_J},\ \varphi(P \cap T_j) \le T_j \text{ for each } j \in J \bigr\rangle.$$

These fusion subsystems $\mathcal{W}_J \le \mathcal{N}_J$ over $W_J \le N_J$ are the main focus of our attention in the rest of the section. We will show in Theorem 3.11 that they are saturated, and are unchanged if we replace $\mathcal{H}^{(J)}$ by $\mathcal{H}$ in their definitions.

Lemma 3.10. *Assume Hypotheses 3.3 and Notation 3.4 and 3.9. Then for each* $\varnothing \ne J \subseteq \underline{\boldsymbol{k}}$,

(a) *if* $\chi_0(\alpha)(J) = J$ *for each* $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$, *then* $\mathcal{E}_J \trianglelefteq \mathcal{F}$; *and*

(b) *if* $\mathcal{D} \le \mathcal{F}$ *is a fusion subsystem over* $D \le S$ *such that* $\mathcal{E}_J \trianglelefteq \mathcal{D}$, *then* $\mathcal{D}^{cr} \subseteq \mathcal{H}^{(J)}$.

Proof. **(a)** We first show that $\mathcal{E}_J \trianglelefteq \mathcal{E}$ (for all $\varnothing \ne J \subseteq \underline{\boldsymbol{k}}$). Lemma 3.2(b) implies that T_J is strongly closed in $\mathcal{E}$. The invariance and Frattini conditions for $\mathcal{E}_J \le \mathcal{E}$ hold since $\mathcal{E}_J$ is the full subcategory of $\mathcal{E}$ with objects the subgroups of T_J (Lemma 3.2(a)). By Lemma 3.1, for each $\alpha \in \mathrm{Aut}_{\mathcal{E}_J}(T_J)$, there is $\bar\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ such that $\bar\alpha|_{T_J} = \alpha$ and $\bar\alpha|_{T_i} = \mathrm{Id}_{T_i}$ for each $i \in \underline{\boldsymbol{k}} \smallsetminus J$, and hence $[\bar\alpha, C_T(T_J)] = [\alpha, Z(T_J)] \le Z(T_J)$. The extension condition thus holds, and so $\mathcal{E}_J \trianglelefteq \mathcal{E}$.

Thus $\mathcal{E}_J \trianglelefteq \mathcal{E} \trianglelefteq \mathcal{F}$ (recall $\mathcal{E} \trianglelefteq \mathcal{F}$ by Hypotheses 3.3(i)). If $\mathrm{Aut}_{\mathcal{F}}(T) \le \Sigma_k$ sends J to itself, then each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ permutes the T_j for $j \in J$ by Hypotheses 3.3(iii), hence normalizes T_J, and normalizes $\mathcal{E}_J$ since it is the full subcategory of $\mathcal{E}$ with objects the subgroups of T_J (Lemma 3.2(a) again). So $\mathcal{E}_J \trianglelefteq \mathcal{F}$ by [A1, 7.4].

(b) Let $J \subseteq \underline{\boldsymbol{k}}$ be arbitrary. Set $Z_J = Z(\mathcal{E}_J)$, and let $Y_J \le T_J$ be such that $Y_J \ge Z_J$ and $Y_J/Z_J = Z(T_J/Z_J)$. For each $j \in J$, $(Y_J \cap T_j Z_J)/Z_J = Z(T_j Z_J/Z_J) \ne 1$ since $T_j Z_J/Z_J \cong T_j Z/Z \ne 1$ by Hypotheses 3.3(ii) (and since the T_j commute pairwise). Let $t_j \in T_j$ and $z_j \in Z_J$ be such that $t_j z_j \in Y_J \smallsetminus Z_J$. Then $t_j \in Y_J \smallsetminus Z_J$ since $z_j \in Z_J \le Y_J$, and $t_j \in (Y_J \cap T_j) \smallsetminus Z$ since $T_j \cap Z \le Z_J$. Thus $Y_J \cap T_j \nleq Z$ for each $j \in J$, and so $Y_J \in \mathcal{H}^{(J)}$.

Let $\mathcal{D} \le \mathcal{F}$ be a fusion subsystem over $D \le S$ such that $\mathcal{E}_J \trianglelefteq \mathcal{D}$, and in particular, $T_J \trianglelefteq D$. Let $P \le D$ be such that $P \notin \mathcal{H}^{(J)}$. Then $P \ngeq Y_J$ since $\mathcal{H}^{(J)}$ is closed under overgroups. Also, P normalizes Y_J since it normalizes T_J and Z_J, so $PY_J > P$ and $N_{PY_J}(P) > P$.

Choose $x \in N_{Y_J}(P) \smallsetminus P$. Then $[x, T_J] \le [Y_J, T_J] \le Z_J$ and $[x, Z_J] \le [Y_J, Z_J] = 1$ (recall $Z_J = Z(\mathcal{E}_J) \le Z(T_J)$), so c_x^P acts trivially on $P \cap Z_J$ and on $(P \cap T_J)/(P \cap Z_J)$. Also, $[x, P] \le [N_{Y_J}(P), P] \le P \cap T_J$ since $Y_J \le T_J \trianglelefteq D$ and $P \le D$, so c_x^P also acts trivially on $P/(P \cap T_J)$. Since $\mathcal{E}_J \trianglelefteq \mathcal{D}$, the subgroups T_J and Z_J are both strongly closed in $\mathcal{D}$, and hence $c_x^P \in O_p(\mathrm{Aut}_{\mathcal{D}}(P))$ by Lemma 1.12.

Recall that $x \in Y_J \le T_J \le D$. If $c_x^P \notin \mathrm{Inn}(P)$, then $O_p(\mathrm{Out}_{\mathcal{D}}(P)) \ne 1$ and P is not $\mathcal{D}$-radical. If $c_x^P \in \mathrm{Inn}(P)$, then $x \in PC_D(P) \smallsetminus P$, so $C_D(P) \nleq P$, and P is not $\mathcal{D}$-centric. In either case, $P \notin \mathcal{D}^{cr}$, and hence $\mathcal{D}^{cr} \subseteq \mathcal{H}^{(J)}$. □

We are now ready to prove the "normalizing" properties of the fusion subsystems $\mathcal{W}_J \le \mathcal{N}_J \le \mathcal{F}$ defined in Notation 3.9.

Theorem 3.11. *Assume Hypotheses 3.3 and Notations 3.4 and 3.9, and let $J \subseteq \underline{\boldsymbol{k}}$ be a nonempty subset such that T_J is fully normalized in $\mathcal{F}$. Then $\mathcal{W}_J$ and $\mathcal{N}_J$ are saturated fusion subsystems of $\mathcal{F}$, $\mathcal{W}_J$ is normal in $\mathcal{N}_J$, and the following hold.*

(a) *For all $P, Q \in \mathcal{H}^{(J)}$ contained in N_J,*
$$\mathrm{Hom}_{\mathcal{N}_J}(P, Q) = \bigl\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q) \bigm| \varphi(P \cap T_J) \le T_J\bigr\}. \tag{3.4}$$
Also, $\mathcal{E}_J \trianglelefteq \mathcal{N}_J$. If $\mathcal{D} \le \mathcal{F}$ is a saturated fusion subsystem such that $\mathcal{E}_J \trianglelefteq \mathcal{D}$, then $\mathcal{D} \le \mathcal{N}_J$.

(b) *For all $P, Q \in \mathcal{H}^{(J)}$ contained in W_J,*
$$\mathrm{Hom}_{\mathcal{W}_J}(P, Q) = \bigl\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q) \bigm| \varphi(P \cap T_j) \le T_j \text{ for each } j \in J\bigr\}. \tag{3.5}$$
Also, $\mathcal{E}_j \trianglelefteq \mathcal{W}_J$ for each $j \in J$. If $\mathcal{D} \le \mathcal{F}$ is a saturated fusion subsystem such that $\mathcal{E}_j \trianglelefteq \mathcal{D}$ for each $j \in J$, then $\mathcal{D} \le \mathcal{W}_J$.

Proof. We first check (3.4) and (3.5). Each of the morphisms appearing in the definition of $\mathcal{N}_J$ (see Notation 3.9) sends elements of T_J to elements of T_J. So T_J is strongly closed in $\mathcal{N}_J$. Hence for all $P, Q \in \mathcal{H}^{(J)}_{\le N_J}$, the set $\mathrm{Hom}_{\mathcal{N}_J}(P, Q)$ is contained in the right hand side of (3.4), while the opposite inclusion holds by definition of $\mathcal{N}_J$. This proves (3.4), and (3.5) follows by a similar argument.

Define
$$\mathcal{N}_J^0 = \bigl\langle \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q) \bigm| P, Q \in \mathcal{H}_{\le N_J},\ \varphi(P \cap T_J) \le T_J \bigr\rangle \le \mathcal{N}_J,$$
$$\mathcal{W}_J^0 = \bigl\langle \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q) \bigm| P, Q \in \mathcal{H}_{\le W_J},\ \varphi(P \cap T_j) \le T_j \text{ for each } j \in J \bigr\rangle \le \mathcal{W}_J.$$
Thus the only difference between $\mathcal{N}_J^0$ and $\mathcal{N}_J$ or between $\mathcal{W}_J^0$ and $\mathcal{W}_J$ lies in the set of objects used in the definition.

We prove in Step 1 that $\mathcal{W}_J^0$ and $\mathcal{N}_J^0$ are saturated and $\mathcal{W}_J^0 \trianglelefteq \mathcal{N}_J^0$. In Step 2, we prove (a) and (b) when $J = \underline{\boldsymbol{k}}$ (in which case $\mathcal{H} = \mathcal{H}^{(J)}$ and hence $\mathcal{N}_J^0 = \mathcal{N}_J$ and $\mathcal{W}_J^0 = \mathcal{W}_J$). We then prove $\mathcal{N}_J^0 = \mathcal{N}_J$ and (a) for arbitrary $J \subseteq \underline{\boldsymbol{k}}$ in Steps 3 and 4, respectively, and prove $\mathcal{W}_J^0 = \mathcal{W}_J$ and (b) in the general case in Step 5. Throughout the proof, we assume that T_J is fully normalized in $\mathcal{F}$.

Step 1: Let $\chi_0 \colon \mathrm{Aut}_{\mathcal{F}}(T) \longrightarrow G$ and $U = \chi_0(\mathrm{Aut}_S(T)) \le G$ be as in Notation 3.4, and let $\widehat{\chi} \colon S \longrightarrow U$ be the surjective homomorphism that sends $x \in S$ to $\chi_0(c_x^T) \in \chi_0(\mathrm{Aut}_S(T)) = U$. By Lemma 3.8(b), we are in the situation of Hypotheses 2.1.

Consider the action of $G \le \Sigma_k$ on $\underline{\boldsymbol{k}}$, and set
$$H = N_G(J), \quad H_0 = C_G(J), \quad V = H \cap U = N_U(J), \quad V_0 = H_0 \cap U = C_U(J).$$
Note that $\mathcal{N}_J^0 = \mathcal{F}_H$ and $\mathcal{W}_J^0 = \mathcal{F}_{H_0}$ under Notation 2.2.

Recall that $N_J = N_S(T_J) = N_S(J)$ under the action of S on $\underline{\boldsymbol{k}}$ induced by its conjugation action on T. For each $\sigma \in G$, there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\chi_0(\alpha) = \sigma$, and $T_{\sigma(J)} = \alpha(T_J) \in (T_J)^{\mathcal{F}}$. Since T_J is fully normalized in $\mathcal{F}$, we have $|N_S(T_{\sigma(J)})| \leq |N_S(T_J)|$ and hence $|N_S(\sigma(J))| \leq |N_S(J)|$ for each $\sigma \in G$. The map $N_S(\sigma(J)) \longrightarrow N_U(\sigma(J))$ induced by $\widehat{\chi}$ is surjective and its kernel $\mathrm{Ker}(\widehat{\chi}) = C_S(\underline{\boldsymbol{k}})$ is independent of σ, so $|N_U(\sigma(J))| \leq |N_U(J)|$ for all $\sigma \in G$. Since $N_U(\sigma(J)) = U \cap N_G(\sigma(J))$ where $N_G(\sigma(J)) = {}^{\sigma}N_G(J) = {}^{\sigma}H$ for each σ, this shows that $|U \cap H| \geq |U \cap {}^{\sigma}H| = |U^{\sigma} \cap H|$ for each $\sigma \in G$, and hence that $V = U \cap H \in \mathrm{Syl}_p(H)$. Since $H_0 \trianglelefteq H$, we also have $V_0 = V \cap H_0 \in \mathrm{Syl}_p(H_0)$.

The hypotheses of Theorem 2.5 thus hold, and so $\mathcal{N}_J^0 = \mathcal{F}_H$ and $\mathcal{W}_J^0 = \mathcal{F}_{H_0}$ are both saturated. By the same theorem applied with $\mathcal{N}_J^0$ in the role of $\mathcal{F}$ (and since $H_0 \trianglelefteq H$), we have $\mathcal{W}_J^0 \trianglelefteq \mathcal{N}_J^0$.

Step 2: Assume $J = \underline{\boldsymbol{k}}$. In particular, $\mathcal{H}^{(J)} = \mathcal{H}$, $\mathcal{N}_J^0 = \mathcal{N}_J$, and $\mathcal{W}_J^0 = \mathcal{W}_J$. Point (a) holds since $\mathcal{E}_{\underline{\boldsymbol{k}}} = \mathcal{E} \trianglelefteq \mathcal{F} = \mathcal{N}_{\underline{\boldsymbol{k}}}$ by Hypotheses 3.3(i).

It remains to prove (b). We already checked (3.5), and $\mathcal{E}_i \trianglelefteq \mathcal{W}_{\underline{\boldsymbol{k}}}$ for each $i \in \underline{\boldsymbol{k}}$ by Lemma 3.10(a), applied with $\mathcal{W}_{\underline{\boldsymbol{k}}}$ in the role of $\mathcal{F}$ (hence with $G = 1$) and $\{i\}$ in the role of J.

Let $\mathcal{D} \leq \mathcal{F}$ be a saturated fusion subsystem over $D \leq S$ such that $\mathcal{E}_i \trianglelefteq \mathcal{D}$ for each $i \in \underline{\boldsymbol{k}}$. Then $T_i \trianglelefteq D$ for each $i \in \underline{\boldsymbol{k}}$, so $D \leq W_{\underline{\boldsymbol{k}}}$ and $D \geq T_1 \cdots T_k = T$. Also, for each $P, Q \in \mathcal{H}_{\leq D}$,

$$\mathrm{Hom}_{\mathcal{D}}(P,Q) \leq \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \,|\, \varphi(P \cap T_i) \leq T_i \ \forall\, i \in \underline{\boldsymbol{k}}\} = \mathrm{Hom}_{\mathcal{W}_{\underline{\boldsymbol{k}}}}(P,Q)$$

(recall T_i is strongly closed in $\mathcal{D}$). Since $\mathcal{D}^{cr} \subseteq \mathcal{H}$ by Lemma 3.10(b), we now have $\mathcal{D} \leq \mathcal{W}_{\underline{\boldsymbol{k}}}$ by Theorem 1.5 (Alperin's fusion theorem).

Step 3: Throughout the rest of the proof, we let $\varnothing \neq J \subseteq \underline{\boldsymbol{k}}$ be arbitrary, subject to the condition that T_J be fully normalized in $\mathcal{F}$. We prove in this step that $\mathcal{N}_J^0 = \mathcal{N}_J$.

We first check that $\mathcal{N}_J$ is $\mathcal{H}^{(J)}_{\leq N_J}$-saturated. By Lemma 1.10, it suffices to show, for all $P \in \mathcal{H}^{(J)}_{\leq N_J}$, that the following three points hold.

(i) *If $P \leq \overline{P} \leq N_J$, and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\overline{P}, N_J)$ is such that $\varphi|_P \in \mathrm{Hom}_{\mathcal{N}_J}(P, N_J)$, then $\varphi \in \mathrm{Mor}(\mathcal{N}_J)$.*

By Lemma 3.8(a′), there is $\sigma \in \Sigma_k$ such that $\varphi(\overline{P} \cap T_i) \leq T_{\sigma(i)}$ for each $i \in \underline{\boldsymbol{k}}$ and $\varphi(\overline{P} \cap T_J) \leq T_{\sigma(J)}$. Since $P \in \mathcal{H}^{(J)}$, there is $x_j \in (P \cap T_j) \smallsetminus Z$ for each $j \in J$, and since $\varphi|_P \in \mathrm{Mor}(\mathcal{N}_J)$, we have $\varphi(x_j) \in (T_{\sigma(j)} \cap T_J) \smallsetminus Z$. So $\sigma(j) \in J$ by Lemma 3.2(c), and $\sigma(J) = J$. Hence $\varphi(\overline{P} \cap T_J) \leq T_J$, and so $\varphi \in \mathrm{Mor}(\mathcal{N}_J)$.

(ii) *For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_{N_J}(P), S)$, there are $R \leq S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(R, N_J)$ such that $R \geq \langle \varphi(N_{N_J}(P)), C_S(\varphi(P)) \rangle$ and $\psi\varphi \in \mathrm{Hom}_{\mathcal{N}_J}(N_{N_J}(P), N_J)$.*

For such φ, by Lemma 3.8(a′), there is $\sigma \in G \leq \Sigma_k$ such that $\varphi(N_{N_J}(P) \cap T_i) \leq T_{\sigma(i)}$ for each $i \in \underline{\boldsymbol{k}}$ and $\varphi(N_{N_J}(P) \cap T_J) \leq T_{\sigma(J)} \in (T_J)^{\mathcal{F}}$. Set $J' = \sigma(J)$. Since T_J is fully normalized in $\mathcal{F}$, there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(T_{J'}), S)$ such that $\psi(T_{J'}) = T_J$ (see [AKO, Lemma I.2.6(c)]) and hence $\psi(N_S(T_{J'})) \leq N_S(T_J) = N_J$. Then $\psi\varphi(N_{N_J}(P) \cap T_J) \leq T_J$, and so $\psi\varphi \in \mathrm{Hom}_{\mathcal{N}_J}(N_{N_J}(P), N_J)$. Also, $C_S(\varphi(P)) \leq C_S(J') \leq N_S(J') = N_S(T_{J'})$ since $\varphi(P) \in \mathcal{H}^{(J')}$.

It remains to show that $\varphi(N_{N_J}(P)) \leq N_S(T_{J'})$. Fix $x \in N_{N_J}(P)$. Then c_x^T permutes the subgroups $P \cap T_j$ for $j \in J$, and $P \cap T_j \nleq Z$ since $P \in \mathcal{H}^{(J)}$. So $c_{\varphi(x)}^T$ permutes the subgroups $\varphi(P \cap T_j) \leq T_{\sigma(j)}$. Hence $\widehat{\chi}(\varphi(x)) \in N_U(J')$, and $\varphi(x) \in N_S(J') = N_S(T_{J'})$. Thus $\varphi(N_{N_J}(P)) \leq N_S(T_{J'})$.

(iii) *The Sylow group N_J is fully automized in $\mathcal{N}_J$.*

This holds since $\mathrm{Aut}_{\mathcal{N}_J}(N_J) = \mathrm{Aut}_{\mathcal{N}_J^0}(N_J)$ by definition, and $\mathcal{N}_J^0$ (also over N_J) has already been shown to be saturated.

Thus $\mathcal{N}_J$ is $\mathcal{H}^{(J)}_{\le N_J}$-saturated. Since $(\mathcal{N}_J)^{cr} \subseteq \mathcal{H}$ by Lemma 2.3(d), and $\mathcal{N}_J$ is defined so as to be $\mathcal{H}^{(J)}_{\le N_J}$-generated, $\mathcal{N}_J$ is saturated by Proposition 1.4(a). Also, $\mathcal{N}_J$ and $\mathcal{N}_J^0$ are equal after restriction to subgroups in $\mathcal{H}_{\le N_J} \supseteq (\mathcal{N}_J)^{cr}$, so $\mathcal{N}_J^0 = \mathcal{N}_J$ by Theorem 1.5 (Alperin's fusion theorem).

Step 4: We now prove point (a) in the general case. We first check that $\mathcal{E} \trianglelefteq \mathcal{N}_J$. By the extension condition for $\mathcal{E} \trianglelefteq \mathcal{F}$, each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to some $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[\alpha, C_S(T)] \le Z(T)$. Also, $C_S(T) \le C_S(\underline{\boldsymbol{k}}) \le N_S(J) = N_J$, so $TC_S(T) = TC_{N_J}(T)$. Since $\mathcal{E}$ is a central product of the $\mathcal{E}_i$, we have $\alpha(T_i) = T_i$ for each $i \in \underline{\boldsymbol{k}}$, so in particular, $\bar{\alpha}(T_J) = \alpha(T_J) = T_J$, and $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{N}_J}(TC_{N_J}(T))$. The extension condition thus holds for $\mathcal{E} \le \mathcal{N}_J$, and so $\mathcal{E} \trianglelefteq \mathcal{N}_J$ by Lemma 1.7.

Thus Hypotheses 3.3 hold with $\mathcal{E} \trianglelefteq \mathcal{N}_J$ in the place of $\mathcal{E} \trianglelefteq \mathcal{F}$. So $\mathcal{E}_J \trianglelefteq \mathcal{N}_J$ by Lemma 3.10(a).

Now assume $\mathcal{D} \le \mathcal{F}$ is a saturated fusion subsystem over $D \le S$ such that $\mathcal{E}_J \trianglelefteq \mathcal{D}$. In particular, $T_J \le D$ and is strongly closed in $\mathcal{D}$, and hence $D \le N_S(T_J) = N_J$. By (3.4) and since T_J is strongly closed in $\mathcal{D}$, for each $P \in \mathcal{H}^{(J)}_{\le D}$, we have

$$\mathrm{Aut}_{\mathcal{D}}(P) \le \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \,|\, \alpha(P \cap T_J) \le T_J\} = \mathrm{Aut}_{\mathcal{N}_J}(P).$$

Also, $\mathcal{D}$ is $\mathcal{H}^{(J)}_{\le D}$-generated by Theorem 1.5 and since $\mathcal{D}^{cr} \subseteq \mathcal{H}^{(J)}$ (Lemma 3.10(b)), and hence $\mathcal{D} \le \mathcal{N}_J$.

Step 5: By construction, $\mathcal{W}_J^0 \le \mathcal{W}_J \le \mathcal{N}_J = \mathcal{N}_J^0$, where $\mathcal{N}_J^0$ and $\mathcal{W}_J^0$ are saturated by Step 1 and the equality holds by Step 3. By Step 2, applied with $\mathcal{E}_J \trianglelefteq \mathcal{N}_J$ in the role of $\mathcal{E} \trianglelefteq \mathcal{F}$ and $\mathcal{H}^{(J)}_{\le W_J}$ in the role of $\mathcal{H}$, the subsystem $\mathcal{W}_J$ is saturated, $\mathcal{E}_j \trianglelefteq \mathcal{W}_J$ for each $j \in J$, and $\mathcal{D} \le \mathcal{W}_J$ for each $\mathcal{D} \le \mathcal{F}$ such that $\mathcal{E}_j \trianglelefteq \mathcal{D}$ for all $j \in J$.

To prove (b), it remains only to show that $\mathcal{W}_J = \mathcal{W}_J^0$. Both are saturated, and they are equal after restriction to subgroups in $\mathcal{H}_{\le W_J}$. Since $(\mathcal{W}_J)^{cr} \subseteq \mathcal{H}$ by Lemma 2.3(d), we have $\mathcal{W}_J = \mathcal{W}_J^0$ by Theorem 1.5 (Alperin's fusion theorem). □

More generally, for each set $\mathscr{J}$ of pairwise disjoint subsets of $\underline{\boldsymbol{k}}$ with union $\widehat{J} \subseteq \underline{\boldsymbol{k}}$, if $T_{\widehat{\mathscr{J}}}$ is fully normalized in $\mathcal{F}$ and certain "Sylow conditions" hold (those needed to apply Theorem 2.5), then one can construct a largest saturated fusion subsystem that normalizes $\mathcal{E}_J$ for each $J \in \mathscr{J}$. However, the precise statement of such a result seems much more complicated than in the special cases considered in Theorem 3.11, so we won't show that here.

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