A REMARK ON THE CONSTRUCTION OF CENTRIC LINKING SYSTEMS

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Abstract. We give examples to show that it is *not* in general possible to prove the existence and uniqueness of centric linking systems associated to a given fusion system inductively by adding one conjugacy class at a time to the categories. This helps to explain why it was so difficult to prove that these categories always exist, and also helps to motivate the procedure used by Chermak [Ch] when he did prove it.

When S is a finite p-group, a *saturated fusion system* $\mathcal{F}$ over S is a category $\mathcal{F}$ whose objects are the subgroups of S, whose morphisms are injective homomorphisms between the subgroups, and which satisfies certain axioms originally due to Puig [P1, § 2.9] (who calls it a Frobenius S-category). Equivalent sets of axioms can be found, for example, in [BLO2, Definition 1.2] and [AKO, Definition I.2.2]. We omit the details of those axioms here, except to note that if φ is a morphism in $\mathcal{F}$, then all restrictions of φ (obtained by restricting the domain and/or the target) are also in $\mathcal{F}$, and φ^{-1} is in $\mathcal{F}$ if φ is an isomorphism of groups. The motivating example is the fusion system $\mathcal{F}_S(G)$, when G is a finite group and $S \in \mathrm{Syl}_p(G)$. In this case, for $P, Q \leq S$,

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \bigl\{\varphi \in \mathrm{Hom}(P,Q) \bigm| \varphi = c_g = (x \mapsto gxg^{-1}),\ \text{some } g \in G\bigr\}.$$

A *centric linking system* associated to a saturated fusion system $\mathcal{F}$ over S is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S (Definition 1.1), together with a functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}$, which satisfy certain conditions listed in Definition 1.2. One of the central questions in the field has been that of whether each saturated fusion system does admit an associated centric linking system, and if so, whether it is unique up to isomorphism. One obvious way to try to construct a centric linking system associated to $\mathcal{F}$ is to do it one $\mathcal{F}$-conjugacy class (i.e., isomorphism class in $\mathcal{F}$) at a time. One begins with a "linking system" having as unique object S itself (this is not difficult). One then extends the category to also include an isomorphism class of maximal $\mathcal{F}$-centric subgroups of S, and continues adding objects until all $\mathcal{F}$-centric subgroups have been included. In this way, the difficulties in the construction are split up, and one need only work with one small "piece" of the categories at a time.

The main result of this paper is to present some simple examples that show that this procedure is not possible in general. We construct examples of fusion systems $\mathcal{F}$, and sets $\mathfrak{Y} \subseteq \mathfrak{X}$ of $\mathcal{F}$-centric subgroups of S which are closed under $\mathcal{F}$-conjugacy and overgroups (and differ by exactly one $\mathcal{F}$-conjugacy class), such that there is more than one isomorphism class of linking systems associated to $\mathcal{F}$ with object set $\mathfrak{Y}$, only one of which can be extended to a linking system with object set $\mathfrak{X}$. A general framework for doing this is given in Theorem 1.7, and explicit examples satisfying the hypotheses of the theorem are found in Examples 2.1, 2.3, and 2.5.

2000 *Mathematics Subject Classification.* Primary 20D20. Secondary 20D45, 20E45.

Key words and phrases. Finite groups, Sylow subgroups, fusion.

B. Oliver is partially supported by UMR 7539 of the CNRS.

These examples help to explain why a general construction of centric linking systems associated to arbitrary saturated fusion systems was so difficult: one cannot expect to find an inductive construction based on adding one isomorphism class at a time to the set of objects. They also show that the claims by Puig in [P2] (in the introduction and the beginning of § 6) that under the above assumptions on $\mathfrak{Y} \subseteq \mathfrak{X}$, there is up to isomorphism a unique $\mathfrak{Y}$-linking system associated to $\mathcal{F}$ and it always extends to an $\mathfrak{X}$-linking system, are not true. (What we call here an "$\mathfrak{X}$-linking system associated to $\mathcal{F}$" is called a "perfect $\mathcal{F}^{\mathfrak{X}}$-locality" in [P2].)

The existence and uniqueness of centric linking systems was shown by Chermak [Ch, O2] in 2011. His proof used the classification of finite simple groups, but more recent work by Glauberman and Lynd [GbL] has shown that this dependence can be removed. Chermak's construction was also based on an inductive procedure, but he avoided the difficulty raised by the examples constructed here by adding (in general) several $\mathcal{F}$-conjugacy classes at a time, and doing so following a very precise algorithm. This is just one of several remarkable features of his construction.

Notation: We write $C_X(G)$ for the centralizer of an action of G on a set or group X; i.e., the elements fixed by G. Also, c_a denotes *left* conjugation by a: $c_a(x) = {}^a x = axa^{-1}$. When $\mathcal{C}$ is a small category and $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ is a functor to abelian groups, $H^i(\mathcal{C}; F)$ denotes the i-th higher derived functor of the inverse limit of F. Whenever $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor and $c, c' \in \mathrm{Ob}(\mathcal{C})$, we let $F_{c,c'}$ denote the induced map of sets from $\mathrm{Mor}_{\mathcal{C}}(c, c')$ to $\mathrm{Mor}_{\mathcal{D}}(F(c), F(c'))$, and set $F_c = F_{c,c}$ for short.

1. Higher limits over orbit categories

Recall that when $\mathcal{F}$ is a saturated fusion system over S, two subgroups of S are said to be *$\mathcal{F}$-conjugate* if they are isomorphic in the category $\mathcal{F}$. For example, for a finite group G and $S \in \mathrm{Syl}_p(G)$, two subgroups are $\mathcal{F}_S(G)$-conjugate if and only if they are G-conjugate in the usual sense.

Definition 1.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.*

(a) *A subgroup $P \leq S$ is* $\mathcal{F}$-centric *if for each Q that is $\mathcal{F}$-conjugate to P, $C_S(Q) \leq Q$.*

(b) *Let $\mathcal{F}^c$ denote the set of $\mathcal{F}$-centric subgroups of S, and also (by abuse of notation) the full subcategory of $\mathcal{F}$ with object set $\mathcal{F}^c$.*

(c) *For each set $\mathfrak{X}$ of subgroups of S, let $\mathcal{F}^{\mathfrak{X}} \subseteq \mathcal{F}$ be the full subcategory with* $\mathrm{Ob}(\mathcal{F}^{\mathfrak{X}}) = \mathfrak{X}$.

In general, we write $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ for the set of $\mathcal{F}$-morphisms from P to Q.

When S is a p-group (in fact, any group), we let $\mathcal{T}(S)$ denote the *transporter category* of S: the category whose objects are the subgroup of S, and where

$$\mathrm{Mor}_{\mathcal{T}(S)}(P, Q) = T_S(P, Q) \stackrel{\mathrm{def}}{=} \{ g \in S \,|\, {}^g P \leq Q \}.$$

For any set $\mathfrak{X}$ of subgroups of S, $\mathcal{T}^{\mathfrak{X}}(S)$ denotes the full subcategory of $\mathcal{T}(S)$ with set of objects $\mathfrak{X}$.

Definition 1.2. *Let $\mathcal{F}$ be a fusion system over the p-group S, and let $\mathfrak{X} \subseteq \mathcal{F}^c$ be a nonempty family of subgroups closed under $\mathcal{F}$-conjugacy and overgroups. An $\mathfrak{X}$-*linking system *associated to $\mathcal{F}$ is a category $\mathcal{L}^{\mathfrak{X}}$ with $\mathrm{Ob}(\mathcal{L}^{\mathfrak{X}}) = \mathfrak{X}$, together with functors $\pi\colon \mathcal{L}^{\mathfrak{X}} \longrightarrow \mathcal{F}^{\mathfrak{X}}$ and $\delta\colon \mathcal{T}^{\mathfrak{X}}(S) \longrightarrow \mathcal{L}^{\mathfrak{X}}$ that satisfy the following conditions.*

(A) *Both δ and π are the identity on objects, and π is surjective on morphisms. For each $P,Q\in\mathfrak{X}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P))\le \mathrm{Aut}_{\mathcal{L}^{\mathfrak{X}}}(P)$), and π induces a bijection*

$$\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q)/Z(P) \xrightarrow{\;\cong\;} \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) *For each $P,Q\in\mathfrak{X}$ and each $g\in T_S(P,Q)$, π sends $\delta_{P,Q}(g)\in \mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q)$ to $c_g\in \mathrm{Hom}_{\mathcal{F}}(P,Q)$.*

(C) *For each $P,Q\in\mathfrak{X}$, $f\in \mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q)$, and $g\in P$,*

$$f\circ\delta_P(g)=\delta_Q(\pi(f)(g))\circ f\in \mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q).$$

Two $\mathfrak{X}$-linking systems $\mathcal{L}_1^{\mathfrak{X}}$ and $\mathcal{L}_2^{\mathfrak{X}}$ associated to $\mathcal{F}$ with structural functors $\pi_i\colon \mathcal{L}_i^{\mathfrak{X}}\longrightarrow \mathcal{F}^{\mathfrak{X}}$ and $\delta_i\colon \mathcal{T}^{\mathfrak{X}}(S)\longrightarrow \mathcal{L}_i^{\mathfrak{X}}$, are isomorphic *if there is an isomorphism $\Psi\colon \mathcal{L}_1^{\mathfrak{X}}\xrightarrow{\;\cong\;}\mathcal{L}_2^{\mathfrak{X}}$ of categories that commutes with the π_i and the δ_i.*

Note that the set $\mathcal{F}^c$ is closed under $\mathcal{F}$-conjugacy and overgroups: the first holds by definition, and the second is easily checked. An $\mathcal{F}^c$-linking system is exactly the same as a centric linking system as defined in [AKO, § III.4.1].

Aside from differences in requirements for the set of objects, this is the definition of a linking system given in [AKO, Definition III.4.1]), and is equivalent to the definition of a perfect locality in [P2, §§ 2.7–2.8]. It is slightly different from the definition in [BLO2, Definition 1.7], which for the purposes of comparison we call here a "weak $\mathfrak{X}$-linking system".

Definition 1.3. *Let $\mathcal{F}$ be a fusion system over the p-group S, and let $\mathfrak{X}\subseteq\mathcal{F}^c$ be a nonempty family of subgroups closed under $\mathcal{F}$-conjugacy and overgroups. A* weak $\mathfrak{X}$-linking system *associated to $\mathcal{F}$ is a category $\mathcal{L}^{\mathfrak{X}}$ with $\mathrm{Ob}(\mathcal{L}^{\mathfrak{X}})=\mathfrak{X}$, together with a functor $\pi\colon\mathcal{L}^{\mathfrak{X}}\longrightarrow\mathcal{F}^{\mathfrak{X}}$, and monomorphisms $\delta_P\colon P\longrightarrow \mathrm{Aut}_{\mathcal{L}^{\mathfrak{X}}}(P)$ for each $P\in\mathfrak{X}$, such that (A) and (C) in Definition 1.2 both hold and (B) holds when $P=Q$ and $g\in P$. Two weak $\mathfrak{X}$-linking systems $\mathcal{L}_1^{\mathfrak{X}}$ and $\mathcal{L}_2^{\mathfrak{X}}$ associated to $\mathcal{F}$, with structural functors $\pi_i\colon\mathcal{L}_i^{\mathfrak{X}}\longrightarrow\mathcal{F}^{\mathfrak{X}}$ and monomorphisms $(\delta_i)_P\colon P\longrightarrow \mathrm{Aut}_{\mathcal{L}_i^{\mathfrak{X}}}(P)$ for $P\in\mathfrak{X}$, are* isomorphic *if there is an isomorphism of categories $\Psi\colon\mathcal{L}_1^{\mathfrak{X}}\xrightarrow{\;\cong\;}\mathcal{L}_2^{\mathfrak{X}}$ that commutes with the π_i and with the $(\delta_i)_P$.*

Most of the time, we just write "$\mathcal{L}^{\mathfrak{X}}$ is a (weak) $\mathfrak{X}$-linking system", and the functors π and δ (or functions δ_P) are understood. When we need to be more explicit, we write "$(\mathcal{L}^{\mathfrak{X}},\pi,\delta)$ is an $\mathfrak{X}$-linking system", or "$(\mathcal{L}^{\mathfrak{X}},\pi,\{\delta_P\})$ is a weak $\mathfrak{X}$-linking system" to include the structural functors (or functions) in the notation.

For $\mathcal{F}$ and $\mathfrak{X}$ as above, an $\mathfrak{X}$-linking system $(\mathcal{L}^{\mathfrak{X}},\pi,\delta)$ restricts in an obvious way to a weak $\mathfrak{X}$-linking system $(\mathcal{L}^{\mathfrak{X}},\pi,\{(\delta_0)_P\})$: just let $(\delta_0)_P\colon P\longrightarrow \mathrm{Aut}_{\mathcal{L}_0^{\mathfrak{X}}}(P)$ be the restriction of $\delta_P\colon \mathrm{Aut}_{\mathcal{T}^{\mathfrak{X}}(S)}(P)=N_S(P)\longrightarrow \mathrm{Aut}_{\mathcal{L}^{\mathfrak{X}}}(P)$ for each $P\in\mathfrak{X}$. Note that for each $P\in\mathfrak{X}$, $\mathrm{Ker}((\delta_0)_P)\le \mathrm{Ker}(\pi_P\circ(\delta_0)_P)=Z(P)$ by (B), so $(\delta_0)_P$ is a monomorphism by (A) ($Z(P)$ acts freely on $\mathrm{Aut}_{\mathcal{L}^{X}}(P)$). We also say that the $\mathfrak{X}$-linking system $(\mathcal{L}^{\mathfrak{X}},\pi,\delta)$ extends $(\mathcal{L}^{\mathfrak{X}},\pi,\{(\delta_0)_P\})$ in this situation.

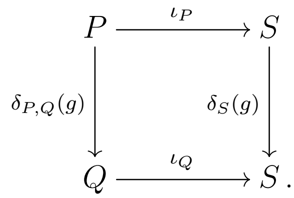
Proposition 1.4. *Let $\mathcal{F}$ be a fusion system over the p-group S, and let $\mathfrak{X}\subseteq\mathcal{F}^c$ be a nonempty family of subgroups closed under $\mathcal{F}$-conjugacy and overgroups. Then each weak $\mathfrak{X}$-linking system $(\mathcal{L}^{\mathfrak{X}},\pi,\{(\delta_0)_P\})$ extends to an $\mathfrak{X}$-linking system $(\mathcal{L}^{\mathfrak{X}},\pi,\delta)$, and any two such extensions are isomorphic as linking systems.*

Proof. The following property of (weak) linking systems is used repeatedly in the proof.

Let $P,Q,R\in\mathfrak{X}$, $\psi\in\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,R)$, $\psi_2\in\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(Q,R)$, and $\varphi_1\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ be such that $\pi_{Q,R}(\psi_2)\circ\varphi_1=\pi_{P,R}(\psi)$. Then there is a unique morphism $\psi_1\in\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q)$ such that $\pi_{P,Q}(\psi_1)=\varphi_1$ and $\psi_2\circ\psi_1=\psi$. (1)

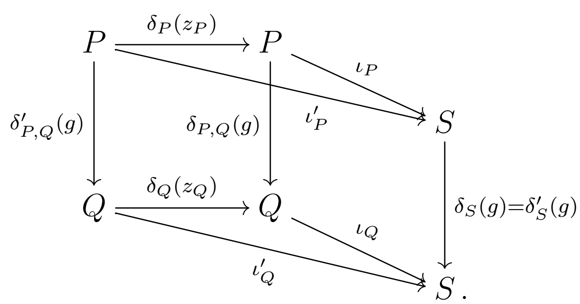
This is an easy consequence of condition (A), and is shown in [BLO2, Lemma 1.10(a)].

The existence of an $\mathfrak{X}$-linking system $(\mathcal{L}^{\mathfrak{X}},\pi,\delta)$ that extends $(\mathcal{L}^{\mathfrak{X}},\pi,\{(\delta_0)_P\})$ is shown in [BLO2, Lemma 1.11] (at least, when $\mathfrak{X}=\mathcal{F}^c$). We recall the construction here. For each $P\in\mathfrak{X}$, choose an "inclusion morphism" $\iota_P\in\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,S)$ such that $\pi_{P,S}(\iota_P)=\mathrm{incl}_P^S$ (the inclusion of P in S), and such that $\iota_S=\mathrm{Id}_S$. For each $P,Q\in\mathfrak{X}$ and each $g\in T_S(P,Q)$, there is a unique element $\delta_{P,Q}(g)\in\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q)$ such that the following square commutes in $\mathcal{L}^{\mathfrak{X}}$:



This is immediate by (1), applied with P,Q,S in the role of P,Q,R and $c_g\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ in the role of φ_1. From the uniqueness in (1), we also see that these morphisms combine to define a functor $\delta\colon\mathcal{T}^{\mathfrak{X}}(S)\longrightarrow\mathcal{L}^{\mathfrak{X}}$. By condition (C) (and the uniqueness in (1)), $(\delta_0)_P(g)=\delta_P(g)$ for each $P\in\mathfrak{X}$ and each $g\in P$. Thus $(\mathcal{L}^{\mathfrak{X}},\pi,\delta)$ is an $\mathfrak{X}$-linking system that extends $(\mathcal{L}_0^{\mathfrak{X}},\pi_0,\{(\delta_0)_P\})$. Note also that $\iota_P=\delta_{P,S}(1)$ for each $P\in\mathfrak{X}$.

Now let δ' be another functor such that $(\mathcal{L}^{\mathfrak{X}},\pi,\delta')$ is an $\mathfrak{X}$-linking system that extends $(\mathcal{L}_0^{\mathfrak{X}},\pi_0,\{(\delta_0)_P\})$. For each $P\in\mathfrak{X}$, set $\iota'_P=\delta'_{P,S}(1)$. Then $\pi_{P,S}(\iota'_P)=\mathrm{incl}_P^S=\pi_{P,S}(\iota_P)$ by condition (B), so by (A), there is $z_P\in Z(P)$ such that $\iota'_P=\iota_P\circ(\delta_0)_P(z_P)=\iota_P\circ\delta_P(z_P)$. For each $P,Q\in\mathfrak{X}$ and $\psi\in\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q)$, consider the following diagram:



Here, $\delta_S(g)=\delta'_S(g)$ since both are equal to $(\delta_0)_S(g)$ by assumption. The two parallelograms commute since δ and δ' are functors, and the two triangles commute by choice of z_P and z_Q. Hence the square on the left commutes by the uniqueness in (1). So if we define a functor $\Theta\colon\mathcal{L}^{\mathfrak{X}}\longrightarrow\mathcal{L}^{\mathfrak{X}}$ by setting $\Theta(P)=P$ for $P\in\mathfrak{X}$ and $\Theta(\psi)=\delta_Q(z_Q)\circ\psi\circ\delta_P(z_P)^{-1}$ for $\psi\in\mathrm{Mor}_{\mathcal{L}^{\mathfrak{X}}}(P,Q)$, then $\Theta\circ\delta'=\delta$ and $\pi\circ\Theta=\pi$. Thus Θ is an isomorphism from $(\mathcal{L}^{\mathfrak{X}},\pi,\delta')$ to $(\mathcal{L}^{\mathfrak{X}},\pi,\delta)$. □

Since isomorphic linking systems clearly restrict to isomorphic weak linking systems, Proposition 1.4 shows that for $\mathcal{F}$ and $\mathfrak{X}$ as above, there is a natural bijection between the set of isomorphism classes of $\mathfrak{X}$-linking systems associated to $\mathcal{F}$ and the set of isomorphism classes of weak $\mathfrak{X}$-linking systems associated to $\mathcal{F}$. In particular, the obstruction theory set up in [BLO2, § 3] for the existence and uniqueness of weak linking systems also applies to that for linking systems in the sense of Definition 1.2 (see Proposition 1.6).

We next define orbit categories, since they play an important role here. In fact, we need to consider two different types of orbit categories: those for fusion systems and those for groups.

Definition 1.5. (a) *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. The* orbit *category $\mathcal{O}(\mathcal{F})$ of $\mathcal{F}$ is the category with the same objects (the subgroups of S), and such* *that for each* $P, Q \le S$,

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q) = Q\backslash\mathrm{Hom}_{\mathcal{F}}(P,Q).$$

Here, $g \in Q$ *acts on* $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ *by post-composition with* $c_g \in \mathrm{Inn}(Q)$. *Thus a morphism in* $\mathcal{O}(\mathcal{F})$ *is a conjugacy class of morphisms in* $\mathcal{F}$. *Also, let* $\mathcal{O}(\mathcal{F}^c) \subseteq \mathcal{O}(\mathcal{F})$ *be the full subcategory with object set* $\mathcal{F}^c$.

(b) *Let* G *be a finite group, and fix* $S \in \mathrm{Syl}_p(G)$. *Let* $\mathcal{O}_S(G)$ *be the category where* $\mathrm{Ob}(\mathcal{O}_S(G))$ *is the set of subgroups of* S, *and where*

$$\mathrm{Mor}_{\mathcal{O}_S(G)}(P,Q) = \mathrm{map}_G(G/P, G/Q):$$

the set of G*-equivariant maps from the transitive* G*-set* G/P *to the* G*-set* G/Q. *Note that each* $\varphi\colon G/P \longrightarrow G/Q$ *has the form* $\varphi(gP) = gaQ$ *(for all* $g \in G$*) for some fixed* $a \in G$ *such that* $P \le {}^aQ$.

Note that for a finite group G and $S \in \mathrm{Syl}_p(G)$, there is a natural surjective functor

$$\mathcal{O}_S(G) \longrightarrow \mathcal{O}(\mathcal{F}_S(G)):$$

this is the identity on objects, and sends a morphism $(gP \mapsto gaQ)$ (from G/P to G/Q) to the class of $c_a^{-1} \in \mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q)$.

We refer to [AKO, § III.5.1] for a very brief summary of some basic properties of "higher limits": higher derived functors of inverse limits. We also refer to [JMO, §§ 5–6] for more details about higher limits over orbit categories of groups, to [BLO2, § 3] for those over orbit categories of fusion systems, and to [AKO, § III.5.4] for both.

When $\mathcal{F}$ is a saturated fusion system over a finite p-group S, and $\mathfrak{X} \subseteq \mathcal{F}^c$ is closed under $\mathcal{F}$-conjugacy and overgroups, define

$$\mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathsf{Ab} \qquad \text{by setting} \qquad \mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}(P) = \begin{cases} Z(P) = C_S(P) & \text{if } P \in \mathfrak{X} \\ 0 & \text{if } P \notin \mathfrak{X}. \end{cases}$$

When $P, Q \in \mathfrak{X}$, $\mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}$ sends a morphism $(P \xrightarrow{[\varphi]} Q)$ to $\big(Z(P) \xrightarrow{\varphi^{-1}} Z(Q)\big)$ (where $[\varphi] \in \mathrm{Mor}(\mathcal{O}(\mathcal{F}^c))$ is the class of $\varphi \in \mathrm{Mor}(\mathcal{F}^c)$). If $\mathfrak{Y} \subseteq \mathfrak{X} \subseteq \mathcal{F}^c$ are both closed under $\mathcal{F}$-conjugacy and overgroups, it is not hard to see that $\mathcal{Z}_{\mathcal{F}}^{\mathfrak{Y}}$ is a quotient functor of $\mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}$.

Proposition 1.6. *Let* $\mathcal{F}$ *be a saturated fusion system over a finite* p*-group* S. *Let* $\mathfrak{X} \subseteq \mathcal{F}^c$ *be a nonempty family of subgroups that is closed under* $\mathcal{F}$*-conjugacy and overgroups, and let* $\mathfrak{Link}_{\mathcal{F}}^{\mathfrak{X}}$ *be the set of all isomorphism classes of* $\mathfrak{X}$*-linking systems associated to* $\mathcal{F}$.

(a) *The set* $\mathfrak{Link}_{\mathcal{F}}^{\mathfrak{X}}$ *is nonempty if and only if a certain obstruction in* $H^3(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}})$ *is zero. In particular,* $\mathfrak{Link}_{\mathcal{F}}^{\mathfrak{X}} \neq \varnothing$ *whenever* $H^3(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}) = 0$.

(b) *If* $\mathfrak{Link}_{\mathcal{F}}^{\mathfrak{X}} \neq \varnothing$, *then the group* $H^2(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}})$ *acts freely and transitively on* $\mathfrak{Link}_{\mathcal{F}}^{\mathfrak{X}}$, *and hence has the same cardinality as* $\mathfrak{Link}_{\mathcal{F}}^{\mathfrak{X}}$.

Proof. This follows with exactly the same proof as that of [BLO2, Proposition 3.1] (the case where $\mathfrak{X} = \mathcal{F}^c$). □

Now fix a finite group $\varGamma$ and a $\mathbb{Z}_{(p)}\varGamma$-module M. Choose $T \in \mathrm{Syl}_p(\varGamma)$, and let

$$F_M\colon \mathcal{O}_T(\varGamma)^{\mathrm{op}} \longrightarrow \mathsf{Ab} \qquad \text{be defined by} \qquad F_M(P) = \begin{cases} M & \text{if } P = 1 \\ 0 & \text{if } P \neq 1. \end{cases} \tag{2}$$

Here, $\mathrm{Aut}_{\mathcal{O}_T(\Gamma)}(1) \cong \Gamma$ has the given action on $F_M(1) = M$. For each $i \geq 0$, set

$$\Lambda^i(\Gamma; M) = H^i(\mathcal{O}_T(\Gamma); F_M).$$

Theorem 1.7. *Fix a finite group Γ, and an $\mathbb{F}_p\Gamma$-module M on which Γ acts faithfully. Set $G = M \rtimes \Gamma$, choose $T \in \mathrm{Syl}_p(\Gamma)$, and set $S = M \rtimes T \in \mathrm{Syl}_p(G)$. Set*

$$\mathfrak{X} = \{P \leq S \,|\, P \geq M\} \qquad \text{and} \qquad \mathfrak{Y} = \{P \leq S \,|\, P > M\} = \mathfrak{X} \smallsetminus \{M\}\,.$$

Then

(a) *there is a unique isomorphism class of $\mathfrak{X}$-linking systems associated to $\mathcal{F}_S(G)$; and*

(b) *the set of isomorphism classes of $\mathfrak{Y}$-linking systems associated to $\mathcal{F}_S(G)$ is in bijective correspondence with $\Lambda^3(\Gamma; M)$.*

Thus if $\Lambda^3(\Gamma; M) \neq 0$, then there is (up to isomorphism) more than one $\mathfrak{Y}$-linking system associated to $\mathcal{F}$, only one of which can be extended to an $\mathfrak{X}$-linking system.

Proof. By [O2, Lemma 1.6(b)],

$$H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}) = 0 \qquad \text{for all } i > 0. \tag{3}$$

In particular, by Proposition 1.6, there is up to isomorphism a unique $\mathfrak{X}$-linking system $\mathcal{L}^{\mathfrak{X}}$ associated to $\mathcal{F}$. This also shows that there is at least one $\mathfrak{Y}$-linking system: the full subcategory of $\mathcal{L}^{\mathfrak{X}}$ with object set $\mathfrak{Y}$.

Let $\mathcal{Z}_0 \subseteq \mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}$ be the subfunctor

$$\mathcal{Z}_0(P) = \begin{cases} 0 & \text{if } P \in \mathfrak{Y} \\ \mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}(P) = M & \text{if } P = M. \end{cases}$$

Thus $\mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}}/\mathcal{Z}_0 \cong \mathcal{Z}_{\mathcal{F}}^{\mathfrak{Y}}$. By [BLO2, Proposition 3.2], for each $i \geq 0$,

$$H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_0) \cong \Lambda^i(\mathrm{Out}_{\mathcal{F}}(M); \mathcal{Z}_0(M)) \cong \Lambda^i(\Gamma; M).$$

So from (3) and the long exact sequence of higher limits for the extension

$$0 \longrightarrow \mathcal{Z}_0 \longrightarrow \mathcal{Z}_{\mathcal{F}}^{\mathfrak{X}} \longrightarrow \mathcal{Z}_{\mathcal{F}}^{\mathfrak{Y}} \longrightarrow 0$$

(see, e.g., [JMO, Proposition 5.1(i)] or [O2, Lemma 1.7]), we get that

$$H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^{\mathfrak{Y}}) \cong \Lambda^{i+1}(\Gamma; M) \qquad \text{for all } i > 0. \tag{4}$$

By Proposition 1.6 again, the $\mathfrak{Y}$-linking systems associated to $\mathcal{F}$ are in bijective correspondence with $\Lambda^3(\Gamma; M)$. □

2. Some explicit examples

We now give some concrete examples of pairs (Γ, M) such that $\Lambda^3(\Gamma; M) \neq 0$, using three independent methods.

In general, if M is an $\mathbb{F}_p\Gamma$-module such that $\Lambda^k(\Gamma; M) \neq 0$ for some $k \geq 1$, then $\dim_{\mathbb{F}_p}(M) \geq p^k$ (see [BLO1, Proposition 6.3] or [AKO, Lemma III.5.27]). This helps to explain why the examples given below (for $k = 3$) are fairly large: there are no examples when $\dim(M) < p^3$. In fact, in the examples of 2.3 and 2.5, M has dimension exactly p^3.

Example 2.1. Let p be any prime, and let Γ be a finite group of Lie type of Lie rank 3 in defining characteristic p. Let $\mathrm{St}(\Gamma)$ be the Steinberg module (over $\mathbb{F}_p$) for Γ. Then $\Lambda^3(\Gamma; \mathrm{St}(\Gamma)) \cong \mathbb{F}_p$.

Proof. Fix $U \in \mathrm{Syl}_p(\varGamma)$. By a theorem of Grodal [Gr, Theorem 4.1], and since $\varGamma$ has Lie rank 3,

$$\Lambda^3(\varGamma;\mathrm{St}(\varGamma)) = H^3(\mathcal{O}_U(\varGamma);F_{\mathrm{St}(\varGamma)}) \cong \mathrm{Hom}_{\varGamma}(\mathrm{St}(\varGamma),\mathrm{St}(\varGamma)) \cong \mathbb{F}_p,$$

where the last isomorphism holds since $\mathrm{St}(\varGamma)$ is absolutely irreducible (see [Ca, Proposition 6.2.2]). □

For example, in Example 2.1, we can take $\varGamma = SL_4(p)$, and let M be its p^6-dimensional Steinberg module [Ca, Corollary 6.4.3].

We next list some of the elementary properties of the $\Lambda^*(\varGamma;M)$ that will be used in the other two examples.

Proposition 2.2. *Fix a finite group $\varGamma$ and a $\mathbb{Z}_{(p)}\varGamma$-module M.*

(a) $\Lambda^0(\varGamma;M) = 0$ *if* $p \,\big|\, |\varGamma|$, *and* $\Lambda^0(\varGamma;M) \cong C_M(\varGamma)$ *otherwise.*

(b) *If* $p \,\big|\, |C_\varGamma(M)|$, *or if* $O_p(\varGamma) \neq 1$, *then* $\Lambda^i(\varGamma;M) = 0$ *for all* $i \geq 0$.

(c) (Künneth formula) *If $\varGamma_1$ and $\varGamma_2$ are two finite groups, and M_i is a finitely generated $\mathbb{F}_p\varGamma_i$-module for $i = 1,2$, then for each $k \geq 0$,*

$$\Lambda^k(\varGamma_1 \times \varGamma_2;M_1 \otimes_{\mathbb{F}_p} M_2) \cong \bigoplus_{j=0}^{k} \Lambda^j(\varGamma_1;M_1) \otimes_{\mathbb{F}_p} \Lambda^{k-j}(\varGamma_2;M_2).$$

(d) *If* $T \in \mathrm{Syl}_p(\varGamma)$ *has order* p, *then* $\Lambda^1(\varGamma;M) \cong C_M(N_\varGamma(T))/C_M(\varGamma)$, *and* $\Lambda^i(\varGamma;M) = 0$ *for* $i \neq 1$.

Proof. See [JMO, Propositions 6.1(i,ii,v) & 6.2(i)], respectively. □

As one easy application of Proposition 2.2(d), if $V \cong (\mathbb{F}_p)^p$ is the natural module for Σ_{p+1} over $\mathbb{F}_p$ (i.e., the $(p+1)$-dimensional permutation module modulo the diagonal), then

$$\Lambda^i(\Sigma_{p+1};V) \cong \begin{cases} \mathbb{F}_p & \text{if } i=1 \\ 0 & \text{if } i \neq 1. \end{cases} \tag{5}$$

This will be used in each of the next two examples below.

Example 2.3. For each prime p,

$$\Lambda^3(\Sigma_{p+1} \times \Sigma_{p+1} \times \Sigma_{p+1}\,;\, V \otimes V \otimes V) \cong \mathbb{F}_p,$$

where $V \cong (\mathbb{F}_p)^p$ is the natural module for Σ_{p+1}.

Proof. This follows from (5) and the Künneth formula (Proposition 2.2(c)). □

The last example is based on taking wreath products with C_p, using the following formula.

Lemma 2.4. *Let $\varGamma$ be a finite group such that $p \,\big|\, |\varGamma|$. Then for each $\mathbb{F}_p\varGamma$-module M and each $i \geq 1$,*

$$\Lambda^i(\varGamma \wr C_p;M^p) \cong \Lambda^{i-1}(\varGamma;M)\,.$$

Proof. Set $G = \varGamma \wr C_p$ for short. Let $G_0 \trianglelefteq \varGamma$ and $x \in G \smallsetminus G_0$ be such that $G_0 = \varGamma^p$ (a fixed identification), $x^p = 1$, and ${}^x(g_1,\ldots,g_p) = (g_2,\ldots,g_p,g_1)$. For $g = (g_1,\ldots,g_p) \in G_0$, $(gx)^p = 1$ if and only if $g_1g_2\cdots g_p = 1$, in which case gx is G-conjugate to x.

Fix $T \in \mathrm{Syl}_p(\varGamma)$, and set $S = T^p\langle x\rangle \in \mathrm{Syl}_p(G)$. Define $\mathfrak{N}\colon \mathcal{O}_S(G)^{\mathrm{op}} \longrightarrow \mathbb{F}_p\text{-}\mathsf{mod}$ by setting

$$\mathfrak{N}(P) = \big\{ \textstyle\sum_{g\in P} g\xi \,\big|\, \xi \in M^p \big\}.$$

If $\varphi \in \mathrm{Mor}_{\mathcal{O}_S(G)}(P,Q) = \mathrm{map}_G(G/P, G/Q)$ has the form $\varphi(gP) = gaQ$ for some $a \in G$ such that $P \leq {}^aQ$, then $\mathfrak{N}(\varphi)\colon \mathfrak{N}(Q) \longrightarrow \mathfrak{N}(P)$ is defined by setting $\mathfrak{N}(\varphi)(\xi) = a\xi$. By [JMO, Proposition 5.2] (recall that $\widehat{H}^0(P;M) \cong C_M(P)/\mathfrak{N}(P)$), or (more explicitly) by [O1, Proposition 1.7],

$$H^i(\mathcal{O}_S(G);\mathfrak{N}) \cong \begin{cases} \mathfrak{N}(G) \stackrel{\text{def}}{=} \bigl\{\sum_{g\in G} g\xi \,\big|\, \xi \in M^p\bigr\} = 0 & \text{if } i=0 \\ 0 & \text{if } i \geq 1. \end{cases} \tag{6}$$

(Recall that $p \,\big|\, |\Gamma|$ and $pM=0$ when checking that $\mathfrak{N}(G)=0$.)

Let F_{M^p} be as in (2), regarded as a subfunctor of $\mathfrak{N}$, and set $\mathfrak{N}_0 = \mathfrak{N}/F_{M^p}$. Thus $\mathfrak{N}_0(P) = \mathfrak{N}(P)$ for $1 \neq P \leq S$ and $\mathfrak{N}_0(1) = 0$. By (6) and the long exact sequence for the extension $0 \to F_{M^p} \longrightarrow \mathfrak{N} \longrightarrow \mathfrak{N}_0 \to 0$ of functors, for each $i>0$,

$$\Lambda^i(G;M^p) = H^i(\mathcal{O}_S(G);F_{M^p}) \cong H^{i-1}(\mathcal{O}_S(G);\mathfrak{N}_0). \tag{7}$$

Now fix $1 \neq P \leq S$, and set $P_0 = P \cap G_0$. Assume that $\Lambda^*(N_G(P)/P;\mathfrak{N}_0(P)) \neq 0$. For $1 \leq i \leq p$, let P_i be the image of P_0 under projection to the i-th factor of $G_0 = \Gamma^p$, and set $\widehat{P} = P_1 \times \cdots \times P_p \in G_0$. Each element in $N_G(P)$ normalizes P_0 and hence normalizes $\widehat{P}$, so $P\widehat{P} \leq G$, and $N_{P\widehat{P}}(P) \trianglelefteq N_G(P)$. If $\widehat{P} > P_0$, then $P\widehat{P} > P$, and $1 \neq N_{P\widehat{P}}(P)/P \leq O_p(N_G(P)/P)$. This contradicts Proposition 2.2(b), and thus $P_0 = \widehat{P} = P_1 \times \cdots \times P_p$.

If two or more of the P_i are nontrivial, then $\mathfrak{N}(P)=0$. Otherwise, we can assume (up to conjugacy in G) that $P_i = 1$ for $2 \leq i \leq p$. If $P_1 \neq 1$, then $P = P_0$, $\mathfrak{N}(P) \leq M \times 0 \times \cdots 0$, and hence $1 \times \Gamma^{p-1} \leq N_G(P)$ acts trivially on $\mathfrak{N}(P)$. Since $p \,\big|\, |\Gamma|$, this again contradicts Proposition 2.2(b). Hence $P_0 = 1$, $P \neq 1$, and P is G-conjugate to $\langle x\rangle$ by the earlier remarks.

Thus for $P \leq S$, $\Lambda^*(N_G(P)/P;\mathfrak{N}_0(P)) = 0$ except when P is G-conjugate to $\langle x\rangle$. So by (7) and [AKO, Corollary III.5.21(b)],

$$\Lambda^i(G;M^p) \cong H^{i-1}(\mathcal{O}_S(G);\mathfrak{N}_0) \cong \Lambda^{i-1}(N_G(\langle x\rangle)/\langle x\rangle;\mathfrak{N}_0(\langle x\rangle)) \cong \Lambda^{i-1}(\Gamma;M). \qquad \square$$

The third example now follows immediately from (5) and Lemma 2.4.

Example 2.5. For each prime p,

$$\Lambda^3(\Sigma_{p+1} \wr C_p \wr C_p\,;\, V^{p^2}) \cong \mathbb{F}_p,$$

where $V \cong (\mathbb{F}_p)^p$ is the natural module for Σ_{p+1}. $\square$

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