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Maps between classifying spaces revisited

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Abstract. In an earlier paper, we developed general techniques which can be used to study the set of homotopy classes of maps between the classifying spaces of two given compact Lie groups. Here, we describe more precisely the general strategy for doing this; and then, as a test of these methods, apply them to determine the existence and uniqueness of (potential) maps $BG \longrightarrow BG'$ studied earlier by Adams and Mahmud. We end with a complete description of the set of homotopy classes of maps from BG_2 to BF_4.

In 1976, Adams & Mahmud [3] published the first systematic study of the problem of determining the homological properties of maps between classifying spaces of compact connected Lie groups. This was continued in later work by one or both authors: Adams [2] extended some of the results to the case of non-connected Lie groups by using complex K-theory; while Adams & Mahmud [4] identified further restrictions which could be made using real or symplectic K-theory.

Recently, in [21], the three of us developed new techniques for studying maps between classifying spaces: techniques based on new decompositions of BG for any compact Lie group G. The main application in [21] was to the problem of determining self maps of BG for any compact connected simple Lie group G. This problem had earlier been studied by several other people (cf. [26], [28], [16], [18], [19], and [23]). The main missing point was to show that the "unstable Adams operations" $\psi^k : BG \to BG$ are unique up to homotopy.

The main tools which now make possible a more precise study of maps between classifying spaces are a series of consequences of the proof of the Sullivan conjecture by Miller (cf. [13]), Carlsson [11], and Lannes [22]. The principal

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result which we use directly is the description, by Dwyer & Zabrodsky [14], Zabrodsky [33], and Notbohm [24], of map(BP, BG) when G is any compact Lie group and P is p-toral (an extension of a torus by a finite p-group). The idea of our approach was to combine these theorems of Dwyer-Zabrodsky and Notbohm with a new decomposition of BG: a decomposition which approximates BG at any prime p as a homotopy direct limit of classifying spaces of p-toral subgroups of G.

In this paper, we show how the same techniques can be used successfully in other situations, to get information about the existence and uniqueness of maps $BG \to BG'$ when G and G' are two distinct compact connected Lie groups. We first describe the general strategy for doing this, taking as our starting point the work of Adams & Mahmud in [3]. To illustrate how these tools work in practice, we then take those examples listed in [3] involving (potential) maps between classifying spaces of distinct rank; and use our methods to determine exactly which ones actually do exist.

We end with a complete description of the set of homotopy classes of maps from BG_2 to BF_4 (Example 3.4). In this case, there are four families of maps, of which two can be constructed (at least away from the prime 3) as composites of the inclusion $G_2 \hookrightarrow F_4$, unstable Adams operations, and Friedlander's "exceptional isogeny" on BG_2. The maps in the other two families come in pairs, where one map from each pair can be constructed in a similar fashion, but by using an inclusion which is defined only between the algebraic groups over $\overline{F_7}$ instead of the inclusion of compact Lie groups. But the remaining map in each pair seems to be completely new, and cannot as far as we can tell be constructed as any composite of algebraically defined maps.

Many of the techniques used here carry over to the case where G is an arbitrary compact Lie group (in particular, a finite group). But since they are based on using Sullivan's arithmetic pullback square for localizations and completions of BG', we do always assume that G' is connected.

Section 1

Throughout the paper, G and G' will denote two compact Lie groups, where G' is connected. We want to study the set $[BG, BG']$ of homotopy classes of maps from BG to BG'. We fix maximal tori $T \subseteq G$ and $T' \subseteq G'$, and let $W = N(T)/T$ and $W' = N(T')/T'$ denote the Weyl groups. We also regard W and W' as groups of automorphisms of T and T', respectively (note, however, that the action of W need not be effective if G is not connected).

Our procedure for studying maps $BG \to BG'$ can be broken up into three steps:

Step 1: Admissible maps

For the purposes of this paper, we define an *admissible map* from G to G' to be a homomorphism $\phi : T \to T'$ such that for every $w \in W$ there exists $w' \in W'$ such that $w' \circ \phi = \phi \circ w$. The motivation for this definition comes from the following result, due in its original form to Adams & Mahmud [3].

THEOREM 1.1. *For any $f : BG \to BG'$, there exists an admissible map $\phi : T \to T'$ such that the following square commutes up to homotopy:*

$$\begin{array}{ccc} BT & \xrightarrow{B\phi} & BT' \\ \big\downarrow{\scriptstyle \mathrm{incl}} & & \big\downarrow{\scriptstyle \mathrm{incl}} \\ BG & \xrightarrow{f} & BG'. \end{array} \tag{1}$$

Furthermore, for any other admissible map $\phi' : T \to T'$ such that $f|BT \simeq B\phi'$, $\phi' = w \circ \phi$ for some $w \in W'$.

PROOF. By [3, Theorem 1.5], there is a homomorphism $\phi : T \to T'$ such that (1) commutes in $\mathbb{Q}$-cohomology. Also, ϕ is admissible by [3, Corollary 1.8], and by [3, Theorem 1.7] is unique up to composition by an element in W'. And by a result of Notbohm [24, Proposition 4.1], two maps $BT \to BG'$ are homotopic if they induce the same map in K-theory, and hence if they induce the same map in $\mathbb{Q}$-cohomology. □

A partial converse to Theorem 1.1 is also proven in [3]: any admissible map $\phi : T \to T'$ is induced by some $f : BG \to BG'[\frac{1}{|W|}]$; i.e., after inverting those primes dividing the order of the Weyl group. In fact, the definition of admissible maps given by Adams & Mahmud in [3] is more general than that given here, in order to allow for admissible maps corresponding to maps $f : BG \to BG'[\frac{1}{n}]$; i.e., maps defined only after inverting some finite number of primes.

We want to determine which admissible maps extend to globally defined maps $f : BG \to BG'$ (and if such maps exist, how many there are). When doing this, it will be useful to write $[BG, BG']_\phi$ to denote the set of homotopy classes of maps which extend a given admissible map ϕ, and similarly for localizations and completions of BG'. The next proposition reduces the problem of describing $[BG, BG']_\phi$ to the case of maps to the p-adic completions of BG'.

PROPOSITION 1.2. *For any set $\mathcal{P}$ of primes and any admissible map $\phi : T \to T'$, the map*

$$[BG, BG'_{(\mathcal{P})}]_\phi \xrightarrow[\cong]{\iota} \prod_{p \in \mathcal{P}} [BG, BG'\hat{{}_p}]_\phi$$

(induced by completion) is a bijection. Also, if $p \nmid |W|$, then $[BG, BG'\hat{{}_p}]_\phi$ has order 1. So in particular (if $W \neq 1$)

$$[BG, BG']_\phi \cong \prod_{p \big| |W|} [BG, BG'\hat{{}_p}]_\phi.$$

PROOF. The bijectivity of ι is shown in [21, Theorem 3.1], using Sullivan's arithmetic pullback square for completions and localizations of the simply-connected space BG'. The fact that $[BG, BG'\hat{{}_p}]_\phi$ has order 1 whenever $p \nmid |W|$

is implicit in [30], although it is stated there only for G finite or connected. Another proof (for arbitrary G) is given in Theorem 1.12 below. □

Note that when G is finite, the last statement in Proposition 1.2 takes the form

$$[BG, BG'] \cong \prod_{p \big| |G|} [BG, BG'^{\wedge}_p].$$

By [3, Theorem 2.21], for any admissible ϕ, there is a homomorphism $\bar{\phi} : W \to W'$ such that ϕ is $\bar{\phi}$-equivariant; i. e., $\bar{\phi}(w) \circ \phi = \phi \circ w$ for each $w \in W$. In general, there can be more than one $\bar{\phi} : W \to W'$ for which ϕ is $\bar{\phi}$-equivariant. But in many cases $\bar{\phi}$ is uniquely determined, and the following lemma gives one condition for this to happen.

Lemma 1.3. *Assume that $\phi : T \to T'$ is a* <u>*regular*</u> *admissible map: an admissible map with either of the following equivalent properties:*
(a) $C_{G'}(\phi(T)) = T'$.
(b) $\mathrm{Im}[\tilde{\phi} : \tilde{T} \to \tilde{T}']$ is not contained in the kernel of any of the roots of G'. ($\tilde{\phi}$ denotes the universal covering map of ϕ.)
Then there is a unique homomorphism $\bar{\phi} : W \to W'$ for which ϕ is $\bar{\phi}$-equivariant.

Proof. See [3, Lemma 2.22]. Note, when showing that (b) implies (a), that the centralizer of any torus in the connected group G' is connected (cf. [21, Proposition A.4]). □

We must now consider the problem: given an admissible map $\phi : T \to T'$, is it induced by a map $f : BG \to BG'$ between the classifying spaces? And if so, is the map unique? By Proposition 1.2, ϕ can be extended to a map f if and only if it can be extended to $f_p : BG \to BG'^{\wedge}_p$ for each prime p, and the extension is unique if it is unique for each completion.

Our procedure for answering these questions is based on p-local approximations of BG, by homotopy direct limits over certain orbit categories defined as follows:

Definition 1.4. *A subgroup $P \subseteq G$ is* <u>*p-toral*</u> *if it is an extension of a torus by a finite p-group. A subgroup $P \subseteq G$ is* <u>*p-stubborn*</u> *if it is p-toral, and if $N(P)/P$ is finite and has no nontrivial normal p-subgroups. We let $\mathcal{O}_p(G)$ denote the category of orbits G/P for p-toral $P \subseteq G$ (and $\mathrm{Mor}(G/P_1, G/P_2)$ is the set of all G-maps). And $\mathcal{R}_p(G) \subseteq \mathcal{O}_p(\Gamma)$ denotes the full subcategory of orbits G/P for p-stubborn $P \subseteq G$.*

Our procedure for describing maps from BG to $BG'^{\wedge}_p$ is based on two results which are summarized in the following theorem. The first says that BG can be approximated as a limit of $BP's$ for p-stubborn subgroups $P \subseteq G$, and the second describes the sets $[BP, BG']$ for p-toral P. For any pair of groups G_1, G_2, we let

$$\mathrm{Rep}(G_1, G_2) := \mathrm{Hom}(G_1, G_2)/\,\mathrm{Inn}(G_2)$$

denote the set of G_2-conjugacy classes of homomorphisms $\rho : G_1 \to G_2$.

THEOREM 1.5. *(i)* (Jackowski, McClure, & Oliver [21]) *For any compact Lie group G and any prime p,* $\underset{\longrightarrow\; G/P\in\mathcal{R}_p(G)}{\text{hocolim}}(G/P)$ *is $\mathbb{F}_p$-acyclic. Hence the projection map*

$$\underset{\longrightarrow}{\text{hocolim}}(EG/P) \cong EG\times_G \big(\underset{G/P\in\mathcal{R}_p(G)}{\underset{\longrightarrow}{\text{hocolim}}}(G/P)\big) \xrightarrow{\qquad} BG$$

is an $\mathbb{F}_p$-homology equivalence (and $EG/P\simeq BP$).

(ii) (Dwyer & Zabrodsky [14], Notbohm [24]) *For any p-toral group P, the map*

$$B : \text{Rep}(P, G') \xrightarrow{\;\cong\;} [BP, BG'], \tag{1}$$

which sends ρ to $B\rho$, is a bijection. Also, for any prime p, the completion map $[BP, BG'] \to [BP, BG'{}^\wedge_p]$ is injective, and is bijective if P is a finite p-group.

PROOF. Point (i) is shown in [21, Theorem 1.4]. The bijectivity of (1) is shown in [14] (when P is a finite p-group) and [24] (when P is an arbitrary p-toral group). Note that this does not require the connectivity of G'.

If P is a finite p-group, then $[BP, BG'{}^\wedge_p] \cong [BP, BG']$ by obstruction theory (or the fact that BP is p-complete).

Now assume that P is an arbitrary p-toral group, and let $f, f' : BP \to BG'$ be such that $(f)^\wedge_p \simeq (f')^\wedge_p$. Let $T \subseteq P$ be the identity component. For each $n \geq 1$, let $T_n \subseteq T$ be the subgroup of elements of order dividing p^n. Choose subgroups $P_1 \subseteq P_2 \subseteq \cdots \subseteq P$ such that $P_n \cap T = T_n$ for each n, and such that the union of the P_n is dense in P (cf. [15, Corollary 1.2]). Then for each n, $(f|BP_n)^\wedge_p \simeq (f'|BP_n)^\wedge_p$, and hence $f|BP_n \simeq f'|BP_n$ (since P_n is a finite p-group). Also, $BP^\wedge_p \simeq \underset{\longrightarrow}{\text{hocolim}}((BP_n)^\wedge_p)$ (cf. [15, Proposition 2.3]); and so the obstructions to constructing a homotopy between f and f' lie in

$$\underset{\substack{\longleftarrow \\ n}}{\lim}{}^1 \big(\pi_1(\text{map}(BP_n, BG')_{f|BP_n})\big).$$

By the theorem of Dwyer & Zabrodsky [14], if $\rho_n : P_n \to G'$ is such that $f|BP_n \simeq B\rho_n$, then $\pi_1(\text{map}(BP_n, BG')_{f|BP_n}) \cong \pi_1(BC_{G'}(\text{Im}(\rho_n)))$. In particular, these groups are all finite, so $\underset{\longleftarrow}{\lim}{}^1$ vanishes, and $f \simeq f'$. $\square$

In Steps 2 and 3 below, we fix a prime p, and consider the problem of determining the set of homotopy classes of maps $BG \to BG'{}^\wedge_p$ which extend a given admissible map $\phi : T \to T'$. Fix a maximal p-toral subgroup $N_p(T)\subseteq G$: i. e., a subgroup for which $N_p(T)/T$ is a p-Sylow subgroup of $N(T)/T$. (See, e.g., [21, A.1 & A.2] for more details about maximal p-toral subgroups.)

Step 2: $\mathcal{R}_p$-invariant representations.

An $\mathcal{R}_p$-*invariant representation* on G is an element

$$\rho \in \text{Rep}(N_p(T), G') = \text{Hom}(N_p(T), G')/\,\text{Inn}(G')$$

with the property that the representations $\rho|P$ combine to form an element in the inverse limit:

$$\hat{\rho} = (\rho|P)_{G/P\in\mathcal{R}_p(G)} \in \varprojlim_{G/P\in\mathcal{R}_p(G)} \operatorname{Rep}(P, G').$$

Equivalently, for any p-stubborn $P\subseteq G$ and any two homomorphisms $i_1, i_2 : P \rightarrowtail N_p(T)$ induced by inclusions and conjugation in G, $\rho\circ i_1$ is conjugate (in G') to $\rho\circ i_2$. Analogously, we say that $\rho : N_p(T) \to G'$ is *$\mathcal{O}_p$-invariant* if its restrictions define an element in above limit when taken over all $G/P \in \mathcal{O}_p(G)$ (see Definition 1.4). The next proposition says that these two conditions are equivalent.

PROPOSITION 1.6. *Any $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to G'$ is $\mathcal{O}_p$-invariant. In particular, if ρ is $\mathcal{R}_p$-invariant, and $g, g' \in N_p(T)$ have p-power order and are conjugate in G, then $\rho(g)$ is conjugate to $\rho(g')$.*

PROOF. Fix a p-toral subgroup $P_0\subseteq N_p(T)$, and an element $x\in G$ such that $xP_0x^{-1}\subseteq N_p(T)$. We must show that the homomorphisms

$$s_1 = \rho|P_0,\ s_2 = (\rho|xP_0x^{-1}) \circ \operatorname{conj}(x) : P_0 \longrightarrow G'$$

are conjugate in G'. The last statement then follows from the special case $P_0 = \langle g\rangle$.

Consider the G-complex

$$E\mathcal{R}_p(G) = \operatorname*{hocolim}_{\overrightarrow{G/P\in\mathcal{R}_p(G)}} (G/P).$$

Let $Z_0\subseteq Z\subseteq E\mathcal{R}_p(G)$ be its 0- and 1-skeletons. In other words, Z_0 is the disjoint union of the orbits G/P in $\mathcal{R}_p(G)$, and Z consists of the mapping cylinders of all morphisms in $\mathcal{R}_p(G)$ attached to Z_0. Since ρ is $\mathcal{R}_p$-invariant, the map

$$f_0 = \coprod_{G/P\in\mathcal{R}_p(G)} B(\rho|P) : EG \times_G Z_0 \simeq \coprod_{G/P\in\mathcal{R}_p(G)} BP \longrightarrow BG'$$

can be extended to a map

$$f : EG \times_G Z \to BG'.$$

By [21, Theorems 2.14 & 1.4], $(E\mathcal{R}_p(G))^{P_0}$ is $\mathbb{F}_p$-acyclic, and in particular connected. It is the homotopy colimit of the functor which sends G/P to $(G/P)^{P_0}$, and hence its 1-skeleton Z^{P_0} is also connected. It follows that any two G-maps $G/P_0 \to Z$ are G-homotopic. And if we identify $BP_0 \simeq EG \times_G G/P_0$ in the usual way, then $Bs_i\simeq f \circ (EG \times_G \alpha_i)$, where

$$\alpha_1, \alpha_2 : G/P_0 \longrightarrow G/N_p(T) \subseteq Z_0 \subseteq Z$$

are defined by setting $\alpha_1(gP_0) = gN_p(T)$ and $\alpha_2(gP_0) = gx^{-1}N_p(T)$. Hence $Bs_1\simeq Bs_2$, and s_1 is conjugate to s_2 by Theorem 1.5(ii). $\square$

By Theorem 1.5(ii), $[BP, BG'] \cong \operatorname{Rep}(P, G')$ for all p-toral P. So for any $f : BG \to BG'$, $f|BN_p(T) \simeq B\rho$ for some unique $\mathcal{R}_p$-invariant representation

ρ. Actually, the following slightly stronger version of this observation will be needed.

PROPOSITION 1.7. *For any $f : BG \to BG'^{\wedge}_p$ extending $\phi : T \to T'$ (i.e., $f \in [BG, BG'^{\wedge}_p]_\phi$), $f|BN_p(T) \simeq (B\rho)^{\wedge}_p$ for some unique $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to G'$.*

PROOF. Choose finite p-subgroups $P_1 \subseteq P_2 \subseteq \cdots \subseteq N_p(T)$, as in the proof of Theorem 1.5(ii) above, such that the union of the P_n is dense in $N_p(T)$. By Theorem 1.5(ii) (the Dwyer-Zabrodsky theorem), for each n, $f|BP_n \simeq B\rho_n$ for some $\rho_n : P_n \to G'$. We may choose the ρ_n such that $\rho_n|T_n = \phi|T_n$ for all n. Also, for each n, $\rho_{n+1}|P_n$ and ρ_n are conjugate in G', and hence by an element of $C_{G'}(T_n)$. And since $C_{G'}(T_n) = C_{G'}(T)$ for n sufficiently large, the ρ_n can be successively chosen such that $\rho_{n+1}|P_n = \rho_n$ for all n.

Now let $\rho : N_p(T) \to G'$ be the unique continuous extension of $\cup_{n=1}^{\infty}\rho_n$. In particular, $\rho|T = \phi$. And since $BN_p(T)^{\wedge}_p \simeq \operatorname{hocolim}_{\longrightarrow}((BP_n)^{\wedge}_p)$ (cf. [15, Proposition 2.3]), the same argument (involving $\lim\limits_{\longleftarrow}{}^1$) used in Theorem 1.5(ii) shows that $f|BN_p(T) \simeq (B\rho)^{\wedge}_p$.

The $\mathcal{R}_p$-invariance of ρ now follows immediately from Theorem 1.5(ii): for any p-toral $P \subseteq G$ and any $\sigma_1, \sigma_2 : P \to G'$, $(B\sigma_1)^{\wedge}_p \simeq (B\sigma_2)^{\wedge}_p$ if and only if σ_1 and σ_2 are conjugate in G'. $\square$

The question is now: given an admissible map $\phi : T \to T'$, can it be extended to an $\mathcal{R}_p$-invariant representation, and if so to how many? As will be seen below, in many concrete cases, this can easily be determined using ad hoc methods. But we have no general techniques for doing this. In particular, this is the missing step if we want to construct the unstable Adams operations for the exceptional Lie groups using this procedure.

The following proposition gives one simple (but very useful) tool for showing that certain admissible maps do not extend to maps between classifying spaces.

PROPOSITION 1.8. *Let $\phi : T \to T'$ be an admissible map such that*

(i) $\mathrm{Ker}(\phi)$ *is finite (i.e., $\tilde{\phi} : \tilde{T} \to \tilde{T}'$ is injective), and*

(ii) there exists an element $t \in \mathrm{Ker}(\phi)$ of p-power order, which is conjugate in G to some element in $N_p(T) \smallsetminus T$.

Then ϕ does not extend to any $\mathcal{R}_p$-invariant representation from G to G'. In particular, $B\phi$ is not the restriction of any map $BG \to BG'^{\wedge}_p$.

PROOF. Assume that ϕ extends to an $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to G'$. Since any p-toral subgroup of G' is conjugate to a subgroup of $N(T')$, we may assume that $\mathrm{Im}(\rho) \subseteq N(T')$ (this can be done without changing $\rho|T$, but that is not necessary here). By assumption, t is conjugate in G to some element $t_1 \in N_p(T) \smallsetminus T$. The conjugation action of t_1 on T is nontrivial, and ρ sends T into T' with finite kernel. Thus, the conjugation action of $\rho(t_1)$ on T' must be nontrivial. But since ρ is $\mathcal{O}_p$-invariant by Proposition 1.6, $\rho(t_1)$ is conjugate to $\rho(t) = 1$; and this is a contradiction.

The last statement follows from Proposition 1.7. □

Step 3: Computation of higher limits.

An $\mathcal{R}_p$-invariant representation ρ determines a family of maps, compatible up to homotopy, from the $BP \simeq EG/P$ to $BG'{}^{\wedge}_p$. The obstructions to extending these to a map from $\operatorname{hocolim}_{\longrightarrow}(EG/P)$ to $BG'{}^{\wedge}_p$ — and hence from BG to $BG'{}^{\wedge}_p$ — were described (in a much more general context) by Wojtkowiak in [29].

We use Wojtkowiak's notation, and write $\mathcal{H}gr$ to denote the category whose objects are groups, and where $\operatorname{Mor}_{\mathcal{H}gr}(G_1, G_2) = \operatorname{Rep}(G_1, G_2)$. Also, $\mathcal{G}r$ will denote the usual category of groups; and $p\text{-}\mathcal{H}gr \subseteq \mathcal{H}gr$ and $p\text{-}\mathcal{G}r \subseteq \mathcal{G}r$ the subcategories of finite p-groups. Then $\varprojlim^2_{\mathcal{C}}(F)$ is defined for any functor $F : \mathcal{C} \to \mathcal{H}gr$, and $\varprojlim^1_{\mathcal{C}}(F)$ is defined for any $F : \mathcal{C} \to \mathcal{G}r$ (cf. [29]).

THEOREM 1.9. *Fix an $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to G'$, and define functors*

$$\Pi_1^\rho : \mathcal{R}_p(G) \to p\text{-}\mathcal{H}gr \qquad \text{and} \qquad \Pi_n^\rho : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}\text{-}mod \quad (n \geq 2)$$

by setting

$$\Pi_n^\rho(G/P) = \pi_n\left(\operatorname{map}(BP, BG'{}^{\wedge}_p)_{B\rho|P}\right) \cong \begin{cases} \pi_1(BC_{G'}(\operatorname{Im}(\rho))) & \text{if } n = 1 \\ [\pi_n(BC_{G'}(\operatorname{Im}(\rho)))]^{\wedge}_p & \text{if } n \geq 2. \end{cases}$$

Then $B\rho$ extends to a map $f : BG \to BG'{}^{\wedge}_p$ if the higher limits $\varprojlim^{n+1}(\Pi_n^\rho)$ vanish for all $n \geq 1$. If there is such an extension f, then Π_1^ρ lifts to a functor $\mathcal{R}_p(G) \to p\text{-}\mathcal{G}r$, and f is unique up to homotopy if the limits $\varprojlim^n(\Pi_n^\rho)$ vanish for all $n \geq 1$.

PROOF. The formula for the homotopy groups of $\operatorname{map}(BP, BG'{}^{\wedge}_p)_{B\rho|P}$ is implicit in Notbohm [24]; and is shown explicitly in [21, Theorem 3.2(iii)] using the results in [24]. Note in particular that for any p-toral $P \subseteq G'$, $\pi_1(BC_{G'}(P)) \cong \pi_0(C_{G'}(P))$ is a p-group (cf. [21, Proposition A.4]).

By Theorem 1.5(i),

$$[BG, BG'{}^{\wedge}_p] \cong \Big[\operatorname*{hocolim}_{\substack{\longrightarrow \\ G/P \in \mathcal{R}_p(G)}} (EG/P), BG'{}^{\wedge}_p \Big].$$

So by Wojtkowiak [29], the obstructions to extending $B\rho$ to a map $f : BG \to BG'{}^{\wedge}_p$ lie in the higher limits $\varprojlim^{n+1}(\Pi_n^\rho)$, and the obstructions to uniqueness in the $\varprojlim^n(\Pi_n^\rho)$. Note that once we have found one map f which extends $B\rho$, then it can be used to consistently define base points for the $\operatorname{map}(BP, BG'{}^{\wedge}_p)_{f|BP}$, and hence to lift Π_1^ρ to a functor to $p\text{-}\mathcal{G}r$ (so that $\varprojlim^1(\Pi_1^\rho)$ is defined). □

In fact, it is not hard to extend the arguments in [29] to construct a second quadrant spectral sequence which converges to the homotopy of $\operatorname{map}(BG, BG'{}^{\wedge}_p)_\rho$ — the space of maps which extend $B\rho$. See also [9] & [10]. Problems with nonabelian values for Π_1^ρ do not occur in any of the concrete examples we consider in Sections 2 and 3; but they do have to be considered in the proof of Corollary 1.11 below.

One might expect the computation of these higher limits to be quite hard in general. We did, however, succeed in [21] in developing some very powerful algorithms which are successful in making these computations in many cases. They are based on some functors $\Lambda^*(\Gamma; M)$, defined (for fixed p) for any finite group Γ and any $\mathbb{Z}_{(p)}[\Gamma]$-module M.

More precisely, for any such Γ and M, let $\mathcal{O}_p(\Gamma)$ denote the category of orbits Γ/P for p-subgroups $P \subseteq \Gamma$. Let $F_M : \mathcal{O}_p(\Gamma) \to \mathbb{Z}_{(p)}$-*mod* be the contravariant functor defined by setting $F_M(\Gamma/1) = M$, and $F_M(\Gamma/P) = 0$ for $1 \neq P \subseteq \Gamma$. We then define

$$\Lambda^*(\Gamma; M) := \varprojlim_{\mathcal{O}_p(\Gamma)}{}^* (F_M).$$

The following theorem explains the significance of these functors.

Theorem 1.10. *Consider a contravariant functor* $F : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}$*-mod.*

(i) Assume, for some p-stubborn subgroup $P \subseteq G$*, that* F *vanishes except on the orbit type* G/P*. Then* $\varprojlim^*(F) \cong \Lambda^*(N(P)/P; F(G/P))$*.*

(ii) Assume, for some $n \geq 0$*, that* $\Lambda^n(N(P)/P; F(G/P)) = 0$ *for all p-stubborn* $P \subseteq G$*. Then* $\varprojlim^n(F) = 0$*.*

(iii) If $p \,\big|\, |\operatorname{Ker}[\Gamma \to \operatorname{Aut}(M)]|$*, then* $\Lambda^*(\Gamma; M) = 0$*. If* $p \nmid |\Gamma|$*, then* $\Lambda^0(\Gamma; M) \cong M^\Gamma$ *and* $\Lambda^n(\Gamma; M) = 0$ *for all* $n \geq 1$*. If* $p \,\big|\, |\Gamma|$*, then* $\Lambda^0(\Gamma; M) = 0$*. If* $p^2 \nmid |\Gamma|$*, then* $\Lambda^n(\Gamma; M) = 0$ *for all* $n \geq 2$*.*

Proof. Point (i) is shown in [21, Lemma 5.4].

To see point (ii), let $N_p(T) = P_0, P_1, \ldots, P_k$ be conjugacy class representatives for all p-stubborn subgroups of G (there are finitely many by [21, Proposition 1.6]). Assume the P_i are arranged from smallest to largest; i. e., such that $(P_i) \subseteq (P_j)$ implies $i \leq j$. For each $0 \leq i \leq k$, define F_i by setting $F_i(G/P) = F(G/P)$ if $(P) = (P_j)$ for some $j \leq i$, and $F_i(G/P) = 0$ otherwise. Then $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k = F$. By (i), for each i,

$$\varprojlim{}^n (F_i/F_{i-1}) \cong \Lambda^n(NP_i/P_i; F(G/P_i)) = 0.$$

The long exact sequence of higher limits for a short exact sequence of functors (cf, [21, Proposition 5.1]), can now be used to show that $\varprojlim^n(F) = 0$.

Finally, point (iii) is shown in [21, Propositions 5.5, 6.1, and 6.2]. □

Note that if we only are interested in the existence of maps extending ρ, then we need only compute higher limits $\varprojlim^n$ over $\mathcal{R}_p(G)$ for $n \geq 2$. The following corollary to Theorem 1.10 gives some simple sufficient conditions for these to vanish.

Corollary 1.11. *Fix contravariant functors*

$$F : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}\text{-}mod, \quad F_1 : \mathcal{R}_p(G) \to p\text{-}\mathcal{G}r \quad \text{and} \quad F_2 : \mathcal{R}_p(G) \to p\text{-}\mathcal{H}gr.$$

(i) Assume for each p-stubborn subgroup $P \subseteq G$ that either $p \nmid |N(P)/P|$, or $p \big| |\operatorname{Ker}[N(P)/P \to \operatorname{Aut}(F(G/P))]|$. Then $\varprojlim^1(F_1) = 0$, and $\varprojlim^n(F) = 0$ for all $n \geq 1$.

(ii) Assume for each p-stubborn subgroup $P \subseteq G$ that either $p^2 \nmid |N(P)/P|$, or $p \big| |\operatorname{Ker}[N(P)/P \to \operatorname{Aut}(F(G/P))]|$. Then $\varprojlim^2(F_2) = 0$, and $\varprojlim^n(F) = 0$ for all $n \geq 2$.

Proof. For the functor F, which takes values in $\mathbb{Z}_{(p)}$-*mod*, these claims follow immediately from Theorem 1.10.

Now let F_i be one of the functors F_1 or F_2. Let G/P be minimal (i.e., P is minimal) among those objects upon which F_i is nonvanishing. Let $F_i' \subseteq F_i$ be the subfunctor such that $F_i'(G/P) = Z(F_i(G/P))$, and such that F_i' vanishes on all other orbit types. Then F_i' is a functor to the category of finite abelian p-groups. So by Theorem 1.10, $\varprojlim^1(F_i') = 0$ if $p \nmid |N(P)/P|$, and $\varprojlim^2(F_i') = 0$ if $p^2 \nmid |N(P)/P|$. We may assume inductively that $\varprojlim^i(F_i/F_i') = 0$; and an examination of the definitions of the nonabelian $\varprojlim^1$ and $\varprojlim^2$ in [29] now shows that $\varprojlim^i(F_i) = 0$.

As a first illustration of the use of Theorems 1.5 and 1.10, we show how they apply in the case where $p \nmid |W|$.

Theorem 1.12. (Adams & Mahmud, Wojtkowiak) *If $p \nmid |W|$, then any admissible map $\phi : T \to T'$ extends to a unique map $f : BG \to BG'^{\wedge}_p$.*

Proof. The existence of an extension is shown in [3, Theorem 1.10], at least when G is connected; and existence and uniqueness when G is finite or connected is shown in [30]. We show here how the result follows from the theory just presented. When $p \nmid |W|$, then the only p-stubborn subgroups of G are the maximal tori (any p-toral subgroup $P \subseteq G$ is contained in a maximal torus, and $N(P)/P$ is finite). Thus, $\mathcal{R}_p(G)$ is equivalent to the category with one object G/T, with $\operatorname{End}(G/T) \cong W$. In particular, $N_p(T) = T$, and any admissible map ϕ is itself $\mathcal{R}_p$-invariant. Finally, for each $i, j \geq 1$,

$$\varprojlim{}^i(\Pi_j^\phi) \cong H^i(W; \Pi_j^\phi(G/T)) = 0$$

since $p \nmid |W|$ and $\Pi_j^\phi(G/T)$ is p-local. $\square$

We next consider the cases where $p^2 \nmid |W|$, or where G' is a matrix group. These conditions hold for several of the examples from [3] which we study in Sections 2 and 3. In these cases, we again get some very strong results about the existence and uniqueness of maps $BG \to BG'^{\wedge}_p$.

Proposition 1.13. *Fix a prime p. Assume that $\pi_0(G)$ is a p-group, and that $p^2 \nmid |W|$. Then $p^2 \nmid |N(P)/P|$ for all p-stubborn subgroups $P \subseteq G$. Also, any $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to G'$ extends to a map $BG \to BG'^{\wedge}_p$.*

Proof. Fix a p-stubborn subgroup $P\subseteq G$. If P is contained in a maximal torus, then P is a maximal torus ($N(P)/P$ is finite); and so $p^2\nmid|N(P)/P|$. So we are left with the case where P is not contained in any maximal torus.

Step 1: We start with a general observation. Let $H \lhd P$ be any normal subgroup of finite index, let $S \lhd P$ be the identity component, and let $\mathrm{Aut}(P,H)$ be the group of automorphisms of P leaving H invariant. Then

$$\mathrm{Ker}\left[\frac{\mathrm{Aut}(P,H)}{\mathrm{Inn}(S)} \xrightarrow{(\mathrm{Res},\ \mathrm{Quot})} \frac{\mathrm{Aut}(H)}{\mathrm{Inn}(S)} \times \mathrm{Aut}(P/H)\right]$$

is a finite p-group. When P is finite, this is shown in [17, Corollary 5.3.3]. If $H=S$, then this holds since the kernel can be identified with $H^1(P/S;S)$ (any automorphism α induces the 1-cocycle $(g \mapsto g^{-1}\alpha(g))$). And the general case now follows since the kernel of each of the maps

$$\frac{\mathrm{Aut}(P,H)}{\mathrm{Inn}(S)} \longrightarrow \mathrm{Aut}(S)\times\mathrm{Aut}(P/S,H/S) \longrightarrow \mathrm{Aut}(S)\times\mathrm{Aut}(H/S)\times\mathrm{Aut}(P/H)$$

is a finite p-group.

Assume in addition that H is normalized by $N(P)$ (i.e., $H \lhd N(P)$). We claim that in this case,

$$\mathrm{Ker}\left[N(P) \to \mathrm{Aut}(H) \times \mathrm{Aut}(P/H)\right] \subseteq P. \tag{1}$$

Let K denote this kernel. Since $N(P)/P$ has no nontrivial normal p-subgroups (by definition of "p-stubborn"), it will suffice to show that the image of K in $N(P)/P$ is a p-group. For any $g \in K$, $\mathrm{conj}(g)$ has p-power order in $\mathrm{Aut}(P,H)/\mathrm{Inn}(S)$, so $\mathrm{conj}(g^{p^k}) = \mathrm{conj}(x)$ for some k and some $x \in T$. Hence $g^{p^k}x^{-1} \in C_G(P) \subseteq P$ (see [21, Lemma 1.5]), and so $g^{p^k} \in P$.

Step 2: Assume that $p\big||N(P)/P|$ (otherwise we are done). Let P'/P be a p-Sylow subgroup of $N(P)/P$. Since P' is p-toral, we may assume that $P'\subseteq N_p(T)$. Since $P \not\subseteq T$ (and $p^2\nmid|W|$), we must have $P' \subseteq \langle P,T\rangle$. If $P\cap T\lhd N(P)$, then

$$P'\cap T \subseteq \mathrm{Ker}\Big[N(P) \longrightarrow \mathrm{Aut}(P\cap T)\times\mathrm{Aut}\big(P/(P\cap T)\big)\Big];$$

and this contradicts (1).

Thus, $P\cap T$ is not normal in $N(P)$. Consider the Frobenius subgroup $\Phi(P)$: the subgroup generated by all commutators and p-th powers in P (cf. [17, §5.1]). Then $C_P(\Phi(P))\lhd N(P)$, and $C_P(\Phi(P))\supseteq P\cap T$. Since $P\cap T$ has index p in P and is not normal in $N(P)$, this shows that $C_P(\Phi(P)) = P$; i. e., that $\Phi(P)\subseteq Z(P)$.

In particular, the conjugation actions of P on $\Phi(P)$ and $P/\Phi(P)$ are trivial — the first since $\Phi(P)$ is central and the second since $[P,P] \subseteq \Phi(P)$. So there is a well defined map

$$(\kappa_1,\kappa_2) : N(P)/P \to \mathrm{Aut}(\Phi(P)) \times \mathrm{Aut}(P/\Phi(P)).$$

By (1) again, (κ_1,κ_2) is injective. Furthermore, $P'/P\subseteq\mathrm{Ker}(\kappa_1)$ ($P' \subseteq \langle P,T\rangle$ and $\Phi(P)\subseteq T$), P'/P is a Sylow p-subgroup of $N(P)/P$; and so $\mathrm{Im}(\kappa_1)$ has order prime to p. And $p\nmid|\mathrm{Ker}(\kappa_2)|$, since (κ_1,κ_2) is injective.

Step 3: Identify $P/\Phi(P) = V \cong (\mathbb{F}_p)^n$, set $\Gamma = \mathrm{Im}(\kappa_2)$, and regard Γ as a subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$. This subgroup has the following properties:

(a) any Sylow p-subgroup of Γ acts as the identity on some codimension one subspace of V. It suffices to check this on one Sylow p-subgroup of $\Gamma = \kappa_2(N(P)/P)$; e.g., on $\kappa_2(P'/P)$. And this acts via the identity on $(P\cap T)/\Phi(P)$, since $P' \subseteq \langle P, T\rangle$. Conversely, any two distinct Sylow p-subgroups of Γ fix distinct codimension one subspaces, since the stabilizer in $\Gamma \subseteq \mathrm{GL}_n(\mathbb{F}_p)$ of any codimension one subspace has a normal (and hence unique) Sylow p-subgroup.

(b) Γ contains no nontrivial normal p-subgroup. Since if $1 \neq Q \lhd \Gamma$ is a nontrivial normal p-subgroup, then $(\kappa_1, \kappa_2)^{-1}(1 \times Q)$ is a nontrivial normal p-subgroup of $N(P)/P$ (recall that $p\nmid|Im(\kappa_1)|$).

By (b), either $p\nmid|\Gamma|$ (and hence $p\nmid|N(P)/P|$), or Γ contains at least 2 distinct Sylow p-subgroups A, B. In the second case, let $H \subseteq \Gamma$ be the subgroup generated by A and B. Since V^A and V^B are distinct and have codimension 1 in V, V^H has codimension 2. Also, the conjugation map $H \to \mathrm{Aut}(V/V^H) \cong \mathrm{GL}_2(\mathbb{F}_p)$ must be injective, since its kernel is a normal p-subgroup (Step 1 again); is therefore contained in $A\cap B$, and hence fixes all of V. Since $p^2\nmid|\mathrm{GL}_2(\mathbb{F}_p)|$, $p^2\nmid|H|$; and so $p^2\nmid|\Gamma|$ since H contains a Sylow subgroup of Γ. And finally, since $p\nmid|\mathrm{Ker}(\kappa_2)|$, this shows that $p^2\nmid|N(P)/P|$.

Step 4: It remains to prove the last statement. By Corollary 1.11(ii), $\varprojlim^n(\Pi^\rho_m) = 0$ for any $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to G'$, any $m \geq 1$, and any $n \geq 2$ (with $n = 2$ if $m = 1$). In particular, ρ lifts to a map $BG \to BG'$. $\square$

The following corollary lists some more consequences of Proposition 1.13, and also of character theory for representations.

Corollary 1.14. *Fix a prime p. Assume that G is connected.*

(i) Assume that $G' \cong \mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{SO}(2n+1)$, or $\mathrm{Sp}(n)$. Then a homomorphism $\rho : N_p(T) \to G'$ is $\mathcal{R}_p$-invariant if and only if for any pair $g, g' \in N_p(T)$ of elements conjugate in G, $\rho(g)$ and $\rho(g')$ are conjugate in G'. Also, an admissible map $\phi : T \to T'$ has (up to conjugacy) at most one extension to an $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to G'$.

(ii) Now assume that $p^2\nmid|W|$; and that $G' \cong \mathrm{SU}(n)$ or $\mathrm{U}(n)$, or p is odd and $G' \cong \mathrm{SO}(2n+1)$ or $\mathrm{Sp}(n)$. Then any given admissible map $\phi : T \to T'$ lifts to at most one map $BG \to BG'^{\wedge}_p$.

Proof. (i) When G' is one of the groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, or $\mathrm{SO}(2n+1)$, character theory applies to show that two homomorphisms $\rho_1, \rho_2 : P \to G'$ are conjugate if and only if $\rho_1(g)$ and $\rho_2(g)$ have the same trace for each $g \in G$, if and only if $\rho_1(g)$ and $\rho_2(g)$ are conjugate for each $g \in G$. When $G' \cong \mathrm{SO}(2n+1)$, this property holds because $O(2n+1) \cong \mathrm{SO}(2n+1) \times \mathbb{Z}/2$. Note that it does not hold for $G' \cong \mathrm{SO}(2n)$.

By definition (and Proposition 1.6), a homomorphism $\rho : N_p(T) \to G'$ is $\mathcal{R}_p$-invariant if and only if for each p-toral $P \subseteq N_p(T)$, and each $x \in G$ such that

$xPx^{-1} \subseteq N_p(T)$, the homomorphisms

$$\rho|P,\ (\rho|(xPx^{-1})) \circ \mathrm{conj}(x) : P \longrightarrow G'$$

are conjugate in G'. So by the remarks on character theory, ρ is $\mathcal{R}_p$-invariant if and only if $\rho(g)$ is conjugate to $\rho(g')$ for all pairs $g, g' \in N_p(T)$ of elements conjugate in G such that either the closures of $\langle g \rangle$ and $\langle g' \rangle$ are p-toral, or both lie in T.

If ρ is $\mathcal{R}_p$-invariant and $g \in N_p(T)$ is arbitrary, then $g = \lim(g_i)$ for some sequence of elements $g_i \in N_p(T)$ of p-power order, each g_i is conjugate to some $g'_i \in T$, and after restricting to a subsequence the g'_i converge to some $g' \in T$ conjugate to g. Then $\rho(g) = \rho(g')$ (ρ is continuous); and hence $\rho(g) = \rho(g'')$ for any $g'' \in N_p(T)$ conjugate to g.

By the same argument, if $\rho, \rho' : N_p(T) \to G'$ are both $\mathcal{R}_p$-invariant and $\rho|T = \rho'|T$, then $\rho(g)$ is conjugate to $\rho(g')$ for all $g \in N_p(T)$ (since every element is conjugate to an element of T); and hence ρ is conjugate to ρ'.

(ii) If $p^2 \nmid |W|$, then $\underleftarrow{\lim}^n(\Pi_n^\rho) = 0$ for any $\mathcal{R}_p$-invariant representation ρ and any $n \geq 2$ (see Proposition 1.13).

If $G' = \mathrm{U}(n)$, then the centralizer of any subgroup is a product of unitary groups, and hence connected. If $G' = \mathrm{Sp}(n)$ or $\mathrm{SO}(2n+1)$ and p is odd, then the centralizer of any p-toral subgroup is a product of unitary groups and (possibly) one symplectic or special orthogonal group, and is again connected. And if $G' = \mathrm{SU}(n)$, then the centralizer of any subgroup is generated by its connected component and $Z(G')$. Thus, in all of these cases, $N(P)/P$ acts trivially on $\Pi_1^\rho(G/P)$; and $\underleftarrow{\lim}^1(\Pi_1^\rho) = 0$ by Corollary 1.11(i).

This shows that any $\mathcal{R}_p$-invariant representation lifts to a unique map $BG \to BG'^{\wedge}_p$. And an admissible map $\phi : T \to T'$ extends to at most one $\mathcal{R}_p$-invariant representation (up to conjugacy in G'): since any two extensions must have the same character. $\square$

Section 2

We now want to apply these procedures to the examples of admissible maps given by Adams & Mahmud in [3, Section 2]. For simplicity, we concentrate in this section on the question of which admissible maps can be realized as maps $BG \to BG'$ (and at which primes p) — and pay less attention to the uniqueness question.

The first three examples in [3, §2] involve cases where all admissible maps are zero: and these maps can clearly be realized. Their next four examples involve unstable Adams operations $\psi^k : BG \to BG$ constructed by Wilkerson [28], and the "exceptional isogenies" of Friedlander [16].

An unstable Adams operation of degree k is a self-map $\psi^k : BG \to BG$ which extends the admissible map $\phi_k : T \to T$ defined by $\phi_k(t) = t^k$. The following theorem combines the results of several authors.

THEOREM 2.1. *For any compact connected group G and any integer k, there exists an unstable Adams operation of degree k on BG if and only if $(k, |W|) = 1$. Also, any two unstable Adams operations of the same degree k are homotopic.*

PROOF. For $(k, |W|) = 1$, the unstable Adams operations were first constructed by Sullivan [26] when $G = \mathrm{U}(n)$, and then by Wilkerson [28] for arbitrary connected G. The necessity of the condition $(k, |W|) = 1$ was shown by Ishiguro [19]. And the uniqueness of the maps was one of our main results in [21]. $\square$

It is thus the remaining examples (2.8 to 2.11) which provide the main interest here. Those are the examples involving maps between simple groups of different rank. In all cases, we refer to [3] for more discussion.

In all of these examples, for any admissible map $\phi : T \to T'$, we let $\widetilde{\phi} : \widetilde{T} \to \widetilde{T}'$ denote its universal covering map. As will be seen, the only primes dividing the order of W (the Weyl group of G) are 2 and 3 in all cases. So any admissible map can be realized as a map $BG \to BG'\hat{{}_p}$ for $p \geq 5$ (see Theorem 1.12 above).

In the first three examples, G is one of the classical groups SU(3) or Sp(3). The p-stubborn subgroups of all of the classical groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $O(n)$, $\mathrm{SO}(n)$, and $\mathrm{Sp}(n)$ can be described explicitly (see [25]). For the results shown here, however, Proposition 1.13 can be used instead, to avoid having to list all p-stubborn subgroups.

As one example, given here without proof, consider the group $G = \mathrm{SU}(3)$. Set $\zeta = \exp(2\pi i/3)$, and consider the elements

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \qquad \text{and} \qquad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then the only 3-stubborn subgroups of SU(3) are (up to conjugacy) the groups

$$N_3(T) = \langle T, A_1 \rangle, \quad \text{and} \quad \Gamma_3 = \langle A, A_1 \rangle.$$

Note that Γ_3 is a group of order 27 with center of order 3, and that $N(\Gamma_3)/\Gamma_3 \cong SL_2(\mathbb{F}_3)$.

In what follows, $(x_1, \ldots, x_n)$ denotes the usual coordinates in $\widetilde{T}$ when G is one of the groups $\mathrm{SU}(n) \subseteq \mathrm{U}(n) \subseteq \mathrm{Sp}(n)$:

$$\exp(x_1, \ldots, x_n) = \mathrm{diag}\left(\exp(2\pi i \cdot x_1), \ldots, \exp(2\pi i \cdot x_n)\right).$$

EXAMPLE 2.2. [3, Example 2.8] $G = \mathrm{SU}(3)$ *and* $G' = \mathrm{SU}(6)$*. For any* $k, m \in \mathbb{Z}$*, define* $\phi_{k,m} : T \to T'$ *by setting*

$$\widetilde{\phi}_{k,m}(x_1, x_2, x_3) = (kx_1, kx_2, kx_3, mx_1, mx_2, mx_3).$$

Then

(i) for $p = 2, 3$*,* $\phi_{k,m}$ *extends to* $(f_{k,m})\hat{{}_p} : B\,\mathrm{SU}(3) \to B\,\mathrm{SU}(6)\hat{{}_p}$ *if and only if each of* k, m *is* 0 *or prime to* p

(ii) $\phi_{k,m}$ *extends to a map* $f_{k,m} : B\,\mathrm{SU}(3) \to B\,\mathrm{SU}(6)$ *if and only if each of* k, m *is* 0 *or prime to* 6.
In all cases, the extension is unique.

Proof. Point (ii) follows from point (i) and Proposition 1.2. And the uniqueness follows from Corollary 1.14(ii).

Fix $p = 2$ or 3. If each of k, m is 0 or prime to p, then there are unstable Adams operations $\psi_k, \psi_m : B\,\mathrm{SU}(3) \to B\,\mathrm{SU}(3)\hat{_p}$ of degrees k and m, respectively (see [28]). By definition, $\psi_k|BT{\simeq}B(t \mapsto t^k)$, and similarly for ψ_m. So if we let $f_{k,m}$ be the composite

$$f_{k,m} : B\,\mathrm{SU}(3) \xrightarrow{(\psi_k,\psi_m)} B\,\mathrm{SU}(3)\hat{_p} \times B\,\mathrm{SU}(3)\hat{_p} \xrightarrow{\oplus} B\,\mathrm{SU}(6)\hat{_p},$$

then $f_{k,m}|BT{\simeq}B\phi_{k,m}$.

We now prove the converse. If $k \equiv m \equiv 0 \pmod p$, then all order p elements of T lie in $\mathrm{Ker}(\phi_{k,m})$. So by Proposition 1.8, $\phi_{k,m}$ extends to a map between the classifying spaces only if $\widetilde{\phi}_{k,m}$ is noninjective, only if $k = m = 0$.

It remains to consider the case where $p{\nmid}m$, and $p \mid k \neq 0$ (or vice versa). In this case, ϕ is regular ($C_{G'}(\phi(T)) = T'$), and so by Lemma 1.3 there is a unique choice of homomorphism $\bar{\phi} : W \to W'$ for which ϕ is $\bar{\phi}$-equivariant (namely, $\bar{\phi}(w) = w{\oplus}w$). Assume that $B\phi$ extends to a map $f\hat{_p} : BG \to BG'\hat{_p}$; then ϕ extends to an $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to \mathrm{SU}(6)$ by Proposition 1.7.

If $p = 3$, then set $\zeta = \exp(2\pi i/3)$, and consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \qquad \text{and} \qquad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

as before. Then $\rho(A_1){\equiv}A_1{\oplus}A_1 \pmod{T'}$; and in particular $\mathrm{Tr}(\rho(A_1)) = 0$. On the other hand, since $3|k$ but $3{\nmid}m$,

$$\rho(A) = \phi(A) = \mathrm{diag}(1, 1, 1, 1, \zeta^m, \zeta^{2m}),$$

and so $\mathrm{Tr}(\rho(A)) = 3$. Thus, A and A_1 are conjugate in $\mathrm{SU}(3)$ but $\rho(A)$ and $\rho(A_1)$ are not conjugate in $\mathrm{SU}(6)$. And since ρ is $\mathcal{R}_p$-invariant, this contradicts Proposition 1.6.

If $p = 2$, then set

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$S^1_\alpha = \left\{ \begin{pmatrix} z^{-2} & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix} : z \in S^1 \right\}.$$

Then $P = \langle S^1_\alpha, B, B_1\rangle$ is 2-stubborn in G, since

$$N(P) \subseteq C_G(S^1_\alpha) \cong \mathrm{U}(2)$$

and

$$\begin{aligned} N_G(P)/P &\cong N_{\mathrm{U}(2)}(P)/P = N_{\mathrm{SU}(2)}(P\cap \mathrm{SU}(2))/(P\cap\mathrm{SU}(2)) \\ &\cong N_{\mathrm{SU}(2)}(Q(8))/Q(8) \cong \Sigma_3. \end{aligned}$$

Since B is conjugate to B_1 in $\mathrm{U}(2)$, there is an element in $N(P)$ which sends B to B_1 and centralizes S^1_α. Let x be such an element. Since ρ is $\mathcal{R}_p$-invariant, there is an element y of $\mathrm{SU}(6)$ for which the composites $\rho\circ\mathrm{conj}(x)$ and $\mathrm{conj}(y)\circ\rho$ are equal. Then y centralizes $\phi(S^1_\alpha)$; i.e., $\rho(B_1)$ is conjugate to $\rho(B) = \phi(B)$ in

$$C_{\mathrm{SU}(6)}(\phi(S^1_\alpha)) = \mathrm{SU}(6) \cap \big(\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(1)\times\mathrm{U}(2)\big).$$

Also, $\rho(B_1)\equiv B_1\oplus B_1 \pmod{T'}$; and one now checks (by comparing traces) that this is impossible. $\square$

Example 2.3. [3, Example 2.9] *$G = \mathrm{SU}(3)$ and $G' = \mathrm{SU}(6)$. For any $k, m \in \mathbb{Z}$, define $\phi_{k,m} : T \to T'$ by setting*

$$\begin{aligned} &\widetilde{\phi}_{k,m}(x_1,x_2,x_3) \\ &\qquad = (kx_1{+}mx_2, kx_2{+}mx_3, kx_3{+}mx_1, kx_1{+}mx_3, kx_2{+}mx_1, kx_3{+}mx_2). \end{aligned}$$

Assume that $k, m \neq 0$, and that $k{\neq}m$ (otherwise $\phi_{k,m}$ is one of the maps in [3, Example 2.8]*). Then*

(i) $\phi_{k,m}$ does not extend to any map $(f_{k,m})\hat{_2} : B\,\mathrm{SU}(3) \to B\,\mathrm{SU}(6)\hat{_2}$.

(ii) $\phi_{k,m}$ extends to a map $(f_{k,m})\hat{_3} : B\,\mathrm{SU}(3) \to B\,\mathrm{SU}(6)\hat{_3}$ if and only if $(k{+}m, 3) = 1$; in which case the extension is unique.

Proof. Uniqueness follows from Corollary 1.14(ii): any admissible map extends to at most one homotopy class of maps between the classifying spaces.

Under the given assumptions on k and m, ϕ is always regular: $C_{G'}(\phi(T)) = T'$. So by Lemma 1.3, $\bar{\phi} : W \to W' \cong \Sigma_6$ is uniquely defined. And one easily checks that the image of any element of $W\smallsetminus 1$ is fixed-point free as a permutation on 6 objects. In other words, for any p, any extension of ϕ to $\rho : N_p(T) \to G'$, and any $g{\in}N_p(T){\smallsetminus}T$, $\rho(g)$ has no nonzero diagonal entries, and hence has trace zero.

$p = 2$: Consider the elements

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

in $N_p(T)$. Then for any $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to \mathrm{SU}(6)$ extending $\phi = \phi_{k,m}$,

$$\rho(B) = \phi(B) = \mathrm{diag}\left((-1)^{k+m}, (-1)^k, (-1)^m, (-1)^k, (-1)^{k+m}, (-1)^m\right).$$

Under the conditions on k and m, this cannot have trace zero. On the other hand, we noted above that $Tr(\rho(B_1))=0\neq Tr(\rho(B))$. Since B_1 and B are conjugate in $G=\mathrm{SU}(3)$, this contradicts the assumption that ρ is $\mathcal{R}_2$-invariant (see Proposition 1.6).

$p=3$: Set $\zeta=\exp(2\pi i/3)$ again, and set

$$A=\begin{pmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^2\end{pmatrix}\qquad\text{and}\qquad A_1=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

in $N_p(T)$. By definition,

$$\phi(A)=\mathrm{diag}(\zeta^m,\zeta^{k-m},\zeta^{-k},\zeta^{-m},\zeta^k,\zeta^{m-k}).$$

Then $Tr(\phi(A))=0$ if and only if exactly one of the numbers $k,m,k-m$ is a multiple of 3, if and only if $3\nmid(k+m)$.

If ϕ lifts to a map $BG\to BG'\hat{_3}$, then it extends to some $\mathcal{R}_p$-invariant representation $\rho: N_3(T)\to G'$. We saw earlier that $\mathrm{Tr}(\phi(A_1))=0$, and so $\phi(A)$ must also have trace 0. And we just saw that this implies $3\nmid k+m$.

Conversely, if $3\nmid k+m$, then define $\rho: N_p(T)=\langle T,A_1\rangle\to G'$ by setting

$$\rho|T=\phi,\qquad \rho(A_1)=A_1\oplus A_1.$$

This is easily seen to be a well defined homomorphism, which is $\mathcal{R}_p$-invariant by Corollary 1.14(i). So by Proposition 1.13, ρ extends to a map $BG\to BG'\hat{_3}$. $\square$

As was shown by Adams & Mahmud [3, Proposition 2.16 and other remarks in Section 2], Examples 2.2 and 2.3 here give a complete list of all non-nullhomotopic maps $BSU(3)\to BSU(6)$ or $BSU(3)\to BSU(6)\hat{_p}$.

Example 2.4. [3, Example 2.10] $G=\mathrm{Sp}(3)$ *and* $G'=\mathrm{Sp}(4)$. *Define* $\phi: T\to T'$ *by setting*

$$\widetilde{\phi}(x_1,x_2,x_3)=(x_1+x_2+x_3,-x_1+x_2+x_3,x_1-x_2+x_3,x_1+x_2-x_3).$$

Then ϕ *extends to a unique map* $f\hat{_p}: B\,\mathrm{Sp}(3)\to B\,\mathrm{Sp}(4)\hat{_p}$ *for all odd* p, *but does not extend to any map when* $p=2$. *In particular,* $B\phi$ *is not the restriction of any map* $B\,\mathrm{Sp}(3)\to B\,\mathrm{Sp}(4)$.

Proof. If $p\geq 5$, then $p\nmid|W|$, and so ϕ extends to a (unique) map $f\hat{_p}: B\,\mathrm{Sp}(3)\to B\,\mathrm{Sp}(4)\hat{_p}$ by Theorem 1.12.

If $p=2$, then note first that $\mathrm{diag}(-1,-1,1)\in\mathrm{Ker}(\phi)$, and that this element is conjugate to elements in $N(T)\smallsetminus T$. Hence, by Proposition 1.8, ϕ does not extend to any map $B\,\mathrm{Sp}(3)\to B\,\mathrm{Sp}(4)\hat{_2}$.

Now set $p = 3$. We can take $N_p(T) = \langle T, A_1 \rangle$, where $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.
Define $\rho : N_p(T) \to G' = \mathrm{Sp}(4)$ by setting $\rho|T = \phi$, and

$$\rho(A_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (1) \oplus A_1.$$

Using Corollary 1.14(i), it is not hard to check that ρ is $\mathcal{R}_p$-invariant. And by Proposition 1.13, ρ lifts to a map $f_{\hat{3}} : B\,\mathrm{Sp}(3) \to B\,\mathrm{Sp}(4)_{\hat{3}}$; which is unique by Corollary 1.14(ii). $\square$

Section 3

The last example (Example 2.11 in [3]) involves the groups G_2 and F_4, and illustrates some of the techniques which can be used when working with exceptional Lie groups. This is the most interesting case of those we consider here. It is the only case where we find new maps $BG \to BG'$ (without finite localization); i. e., maps which are not composites of maps constructed earlier. It is also the only case where we give a complete classification of the homotopy classes in $[BG, BG']$.

This result can be loosely described as follows. Following the notation of Adams & Mahmud, we say that a map $f : BG_2 \to BF_4$ has type (k, m) if the following diagram commutes in rational cohomology:

$$\begin{array}{ccc} B\,\mathrm{SU}(3) & \xrightarrow{(\psi_k, \psi_m)} & B(\mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3)) \\ {\scriptstyle \mathrm{incl}} \big\downarrow & & {\scriptstyle \mathrm{incl}} \big\downarrow \\ BG_2 & \xrightarrow{f} & BF_4. \end{array}$$

Here, the vertical maps are induced by inclusions of subgroups (see Lemmas 3.2 and 3.3 below), and ψ_k and ψ_m are the unstable Adams operations on $B\,\mathrm{SU}(3)$. Also, type (k, m) is equivalent to type $(-k, -m)$, but not to type (m, k) (the two $\mathrm{SU}(3)$ factors are embedded asymmetrically in F_4). Then every map from BG_2 to BF_4 has type (k, m) for some k and m. Aside from the null homotopic maps ($k = m = 0$), there are for each $k > 0$ prime to 6 unique homotopy classes of maps of type $(k, 0)$ and $(k, -k)$, and two homotopy classes each of maps of types (k, k) and $(k, 2k)$.

The maps of type $(1, 0)$ are induced by an inclusion $\iota : G_2 \to F_4$, and the $(k, 0)$ are the composites of $B\iota$ with unstable Adams operations on BG_2. Also, the maps of type $(k, -k)$ can be obtained, though only away from the prime 3, as composites of maps of type $(k, 0)$ with the "exceptional isogeny" on BG_2 [16].

Of the remaining homotopy classes of maps, it is the two classes of type $(1, 1)$ which are the most fundamental: the pairs of maps of type (k, k) and $(k, 2k)$ are obtained by composing the maps of type $(1,1)$ with unstable Adams operations and (away from the prime 3) the exceptional isogeny on BG_2. In [32], Testerman constructs a homomorphism $\sigma : G_2(\overline{\mathbb{F}_7}) \hookrightarrow F_4(\overline{\mathbb{F}_7})$, which is remarkable because its image is a maximal connected subgroup of $F_4(\overline{\mathbb{F}_7})$. Via homotopy equivalences of Friedlander and Mislin [31], σ induces maps $(BG_2)^\wedge_p \to (BF_4)^\wedge_p$ of type $(1,1)$ for all primes $p \neq 7$ (this is discussed in more detail at the end of the section). In other words, the homomorphism between algebraic groups provides a different construction of one of the two homotopy classes of maps of type $(1,1)$ (and a way of distinguishing the two classes).

The following lemma contains two general facts which are very useful when making computations in the exceptional Lie groups.

Lemma 3.1. *Let G be any compact connected Lie group.*

(i) (Borel) *If G is simply connected, then the centralizer of any element in G — or the fixed point set of any automorphism — is connected.*

(ii) Fix a maximal torus $T \subseteq G$ and an element $g \in T$. Let $W = N_G(T)/T$ and $W_g = N_{C(g)}(T)/T$ be the Weyl groups of G and of the centralizer $C_G(g)$, respectively. Then the number of elements in T conjugate (in G) to g is just the Weyl group index $[W : W_g]$.

Proof. (i) See [5, Theorem 3.4 and Corollary 3.5] (or [8, p.48, Théorème 1]).

(ii) By [1, Lemma 4.33], two elements of T are conjugate in G if and only if they are conjugate by an element of W. The result is then immediate. □

The groups G_2 and F_4 each comes in only one "version": the simply connected groups are center-free (cf. Bourbaki [7, §VI.4], where the center of the simply connected group is denoted $P(R)/Q(R)$). From now on, we fix $G = G_2$ and $G' = F_4$; let $T \subseteq G$ and $T' \subseteq G'$ be maximal tori, and let W, W' be the Weyl groups. As before, $\widetilde{T}$ and $\widetilde{T}'$ denote the universal covers of T and T', respectively. We will also need to study the integral lattices of G and G':

$$\Lambda = \operatorname{Ker}[\widetilde{T} \to T] \qquad \text{and} \qquad \Lambda' = \operatorname{Ker}[\widetilde{T}' \to T'].$$

The next two lemmas collect for later use information about some of the subgroups of G_2 and F_4. Note (cf. [8, p.40, Prop. 15]) that any isomorphism between the root systems and integral lattices of two compact connected Lie groups extends to an isomorphism between the groups themselves. This fact is very useful when working with the exceptional Lie groups, and in particular when identifying the centralizers of elements.

Lemma 3.2. *We can identify $\widetilde{T} = \widetilde{T}(G_2) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sum x_i = 0\}$ (with the usual inner product); such that the set of roots of G_2 is*

$$R = \left\{ \pm(x_i - x_j), \pm\tfrac{2x_1 - x_2 - x_3}{3}, \pm\tfrac{2x_2 - x_1 - x_3}{3}, \pm\tfrac{2x_3 - x_1 - x_2}{3} \right\} \subseteq \widetilde{T}^*$$

(where $1 \le i < j \le 3$); and such that the integral lattice is

$$\Lambda = \left\{ (n_1, n_2, n_3) \in \mathbb{Z}^3 : \sum n_i = 0 \right\}.$$

(i) All elements of order 2 in G_2 are conjugate to each other, and the centralizer of any such element is isomorphic to $\mathrm{SO}(4) \cong \mathrm{Sp}(1) \times_{C_2} \mathrm{Sp}(1)$.

(ii) G_2 contains a unique conjugacy class of subgroups isomorphic to $(C_2)^3$. For any such $A \cong (C_2)^3$ in G_2, $C_{G_2}(A) = A$ and $N_{G_2}(A)/A \cong \mathrm{GL}_3(\mathbb{F}_2)$.

(iii) G_2 contains exactly 6 conjugacy classes of 2-stubborn subgroups, with representatives $P_1, \ldots, P_6$ as listed below. They are all presented as subgroups of $\mathrm{Sp}(1) \times_{C_2} \mathrm{Sp}(1) \subseteq G_2$. *Also, $Q \subseteq N \subseteq \mathrm{Sp}(1)$ are the subgroups $Q = \langle i, j \rangle$ (the quaternion group of order 8), and $N = \langle S^1, j \rangle$.*

P	$N(P)/P$
$P_1 = N \times_{C_2} N = N_2(T)$	1
$P_2 = N \times_{C_2} Q$	$1 \times \Sigma_3$
$P_3 = Q \times_{C_2} N$	$\Sigma_3 \times 1$
$P_4 = Q \times_{C_2} Q$	$\Sigma_3 \times \Sigma_3$
$P_5 = \langle S^1 \times_{C_2} S^1, (j,j) \rangle \cong T \rtimes C_2$	Σ_3
$P_6 = \langle (i,i), (j,j), (1,-1) \rangle \cong (C_2)^3$	$\mathrm{GL}_3(\mathbb{F}_2)$.

Furthermore, all morphisms in $\mathcal{R}_2(G_2)$ between the G_2/P_i listed here are composites of automorphisms of the orbits, and of the maps induced by the given inclusions.

(iv) There is a unique conjugacy class of subgroup $\mathrm{SU}(3) \subseteq G_2$. *Also, there is a semidirect product* $\mathrm{SU}(3) \rtimes C_2 \subseteq G_2$, *where C_2 acts on* $\mathrm{SU}(3)$ *by complex conjugation.*

(v) G_2 contains exactly two conjugacy classes of 3-stubborn subgroups, presented in the following list as subgroups of $\mathrm{SU}(3)$*:*

P	$N(P)/P$
$\langle T, A_1 \rangle = N_3(T)$	$C_2 \times C_2$
$\langle A, A_1 \rangle$	$\mathrm{GL}_2(\mathbb{F}_3)$.

Here, $A, A_1 \in \mathrm{SU}(3)$ again denote the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \qquad \text{and} \qquad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Proof. The description of the root system of G_2 is given in Bourbaki [7, §VI.4.13] (the description there is slightly different, but is clearly equivalent). Note that the last three roots as given above are equal (as functions on $\widetilde{T}$) to the coordinate functions $\pm x_1$, $\pm x_2$, and $\pm x_3$ — but the form given above makes clearer their lengths and angles relative to the other roots.

The description of Λ follows easily from the fact that the integral lattice is the group of elements in $\widetilde{T}$ whose value on each root is integral. This property of the integral lattice holds in general for connected center-free groups (cf. [1, Proposition 5.3]).

(i) Set $\widetilde{g} = (\frac{1}{2}, -\frac{1}{2}, 0) \in \widetilde{T}$, and let $g = \exp(\widetilde{g})$. Then g has order 2. By Lemma 3.1(i), $C_G(g)$ is connected (and it clearly has maximal torus T). The roots of $C_G(g)$ are precisely those roots of G which take integral values on $\widetilde{g}$: namely, $\pm(x_1 - x_2)$ and $\pm\frac{1}{3}(2x_3 - x_1 - x_2)$. Since these roots are orthogonal, the centralizer has type $A_1 \times A_1$. Also, the integral lattice is generated by the elements

$$e_1 = (1, -1, 0): \qquad (x_1 - x_2)(e_1) = 2 \qquad \tfrac{1}{3}(2x_3 - x_1 - x_2)(e_1) = 0$$

$$e_2 = (-1, -1, 2): \qquad (x_1 - x_2)(e_2) = 0 \qquad \tfrac{1}{3}(2x_3 - x_1 - x_2)(e_2) = 2$$

and $\frac{1}{2}(e_1+e_2)$. Upon comparing these with the integral lattices for $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ and $\mathrm{SO}(3)$, one sees that $C_G(g) \cong \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1)$.

Finally, by Lemma 3.1(ii), the number of elements in T conjugate to g is $|W|/4 = 3$. Since this accounts for all elements of order 2 in T, we see that all elements of order 2 in G are conjugate to g.

(ii) Assume that $A = \langle g_1, g_2, g_3\rangle$ and $A' = \langle g_1', g_2', g_3'\rangle$ are both isomorphic to $(C_2)^3$. (Possibly $A = A'$ with different bases.) We want to show that there exists $x{\in}G$ such that $xg_ix^{-1} = g_i'$ for each i.

Since g_1 is conjugate to g_1', we may assume that $g_1' = g_1$. Then

$$A, A' \subseteq C_{G_2}(g_1) = H_1 \cong \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1).$$

Upon inspection, we see that all noncentral elements of order 2 in H_1 are conjugate in H_1 to (i,i). So we may assume (upon conjugating A and A' by elements of H_1) that $g_2' = g_2 = (i,i)$.

We now have

$$A, A' \subseteq C_{G_2}(g_1, g_2) = H_2 = (H_1)^{(i,i)} = \langle S^1\times_{C_2}S^1, (j,j)\rangle.$$

Then g_3 and g_3' are both conjugate (in H_2) to (j,j). This shows that A is conjugate to A'; and also that

$$C_{G_2}(A) = (H_2)^{(j,j)} = A.$$

Also, $N_{G_2}(A)/A{\cong}\,\mathrm{GL}_3(\mathbb{F}_2)$, since we have just shown that any automorphism of A is an inner automorphism in G_2.

(iii) Fix a 2-stubborn subgroup $P{\subseteq}G_2$, and let ${}_2Z(P)$ be the 2-torsion subgroup of its center. If $\mathrm{rk}({}_2Z(P)) = 1$, say ${}_2Z(P) = \langle g\rangle$, then $N_G(P) = N_{C(g)}(P)$, and so P is also 2-stubborn in $C(g){\cong}\,\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1)$. By [21, Proposition 1.6], the 2-stubborn subgroups of $\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1)$ are precisely the groups of the form $P'\times_{C_2}P''$, where P', P'' are 2-stubborn in $\mathrm{Sp}(1)$. Also, the only 2-stubborn subgroups of $\mathrm{Sp}(1)$ are Q and N (just check the list of all 2-toral subgroups). Conversely, if $P = P'\times_{C_2}P''$ where P' and P'' are 2-stubborn in $\mathrm{Sp}(1)$, then $N_G(P)/P{\cong}N(P')/P'{\times}N(P'')/P''$, and so P is 2-stubborn in $G = G_2$. Note that the subgroups $N\times_{C_2}Q$ and $Q\times_{C_2}N$ are not conjugate in G_2 — since if they were it would have to be via an element in the centralizer $C_{G_2}(g)$.

Now assume that ${}_2Z(P) = \langle g_1, g_2\rangle$ has rank 2. Since $C(g_1)$ is connected, $\langle g_1, g_2\rangle$ is contained in a maximal torus, and is hence the 2-torsion in some maximal torus T. One now checks that $P{\subseteq}C_{G_2}(g_1, g_2) = T{\rtimes}C_2$. This group is normal in $N(P)$, and $N(P)/P$ has no nontrivial normal 2-subgroups. Hence $P = T\rtimes C_2$. Conversely, to check that this group is in fact 2-stubborn, note that $N(P)/P \cong W/C_2 \cong \Sigma_3$.

If $\mathrm{rk}({}_2Z(P)) \geq 3$, then by part (ii), $P \cong (C_2)^3$ and $N(P)/P \cong \mathrm{GL}_3(\mathbb{F}_2)$.

Now let $\alpha : G/P_i \to G/P_j$ be any G-equivariant map. Choose $x \in G$ such that $\alpha(P_i) = xP_j$; then $x^{-1}P_ix \subseteq P_j$. By inspection of the individual possibilities for (i,j), we see that $P_i \subseteq P_j$, and that $x^{-1}P_ix$ is conjugate by an element of $N(P_j)/P_j$ to P_i ($\subseteq P_j$). Hence, after replacing α by its composite with an automorphism of G/P_j, we may assume that $x^{-1}P_ix = P_i$. And in this case $x \in N(P_i)$, and α is the composite of an automorphism of G/P_i with the map induced by the inclusion $P_i \subseteq P_j$.

(iv) Take $g = \exp(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$. Then $C_{G_2}(g)$ has roots $\pm(x_i - x_j)$, and hence is isomorphic to $\mathrm{SU}(3)$. Also, since $(x \mapsto x^{-1}) \in W$ (note that this is in the Weyl group of $\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1) \subseteq G_2$), we see that complex conjugation on $\mathrm{SU}(3)$ is inner in G_2. Thus, there exists $a \in N_{G_2}(\langle g\rangle)$ such that $\mathrm{conj}(a)$ is complex conjugation on $\mathrm{SU}(3)$. Then $a^2 \in Z(\mathrm{SU}(3)) = \langle g\rangle$, and since $|g| = 3$ we may take a to have order 2. Thus, the normalizer of $\langle g\rangle$ is a semidirect product $\mathrm{SU}(3)\rtimes C_2 \subseteq G_2$.

Now let H be any other subgroup of G_2 isomorphic to $\mathrm{SU}(3)$, and let $g' \in Z(H) \cong C_3$ be a generator. We want to show that H is conjugate to the centralizer $C_{G_2}(g) \cong \mathrm{SU}(3)$ just constructed, and it will suffice to show that g' is conjugate to g. We may assume that $g' \in T$. Since $W \cong \Sigma_3 \times C_2$ acts on $\widetilde{T}$ by permuting coordinates and changing all signs, we see that g' must be conjugate either to $g = \exp(\frac{1}{3}, \frac{1}{3}, \frac{-2}{3})$, or to $\exp(\frac{1}{3}, \frac{-1}{3}, 0)$. And since the only roots in the centralizer of this last element are $\pm(2x_3 - x_1 - x_2)/3$, g' must be conjugate to g.

(v) Fix a 3-stubborn subgroup $P \in G_2$. We may assume that it is contained in the maximal 3-toral subgroup $N_3(T) = \langle T, A_1\rangle \subseteq \mathrm{SU}(3)$. Set $z = \mathrm{diag}(\zeta,\zeta,\zeta)$; then $z \in P$ since $C_{G_2}(P) = Z(P)$ [21, Lemma 1.5]. Also, $P \not\subseteq T$ (since $N(P)/P$ is finite and T is not 3-stubborn); and so (up to conjugation) we may assume that $A_1 \in P$. Furthermore, $P \supsetneqq \langle z, A_1\rangle$, since that subgroup is contained in a torus. By inspection, no cyclic subgroup of T strictly containing $\langle z\rangle$ is normalized by A_1, and so we must have $\langle z, A\rangle \subseteq P$ (the subgroup of 3-torsion in T). In particular, since $C_{G_2}(z) = \mathrm{SU}(3)$, $C_{G_2}(P\cap T) = C_{\mathrm{SU}(3)}(P\cap T) = T$.

If $P\cap T \lhd N(P)$, then $C_{G_2}(P\cap T) = T$ and $\langle T, P\rangle = N_3(T)$ are also normalized by $N(P)$. In particular, $N_{N_3(T)}(P)/P$ is normal in $N(P)/P$, is a 3-subgroup (cf. [21, Lemma A.3]), and must be trivial since P is 3-stubborn. And this is possible only if $P = N_3(T) = \langle T, A_1\rangle$.

Now assume that $P\cap T$ is not normal in $N(P)$. Choose $x \in N(P)$ such that $x(P\cap T)x^{-1} \neq P\cap T$, and set $Q = x(P\cap T)x^{-1}$ for short. Choose any $y \in Q \smallsetminus T$. Then $y \in A_1^{\pm 1}T$, and $[y, Q\cap T] = 1$ since Q is abelian. This implies that $Q\cap T \subseteq \langle z\rangle$; and hence (since Q has index 3 in P) that $P = \langle A, A_1\rangle$. And in this case, $P/\langle z\rangle \cong (C_3)^2$, and one easily checks that the induced homomorphism

$$N_{G_2}(P)/P \longrightarrow \mathrm{Aut}(P/\langle z\rangle) \cong \mathrm{GL}_2(\mathbb{F}_3)$$

is an isomorphism. $\square$

One way frequently used to describe G_2 directly is as the group of automorphisms of the Cayley numbers $\mathcal{C}$: a division algebra on $\mathbb{R}^8$ which contains the quaternions as a subalgebra $\mathbb{H} \subseteq \mathcal{C}$. Under this description, the subgroup $\mathrm{SU}(3)\subseteq G_2$ can be regarded as the group of automorphisms $\alpha\in \mathrm{Aut}(\mathcal{C})$ such that $\alpha(i) = i$ (such an α acts complex linearly on the remaining coordinates). This shows that $G_2/\mathrm{SU}(3)\cong S^6$: the set of unit vectors orthogonal to $1 \in \mathcal{C}$. Also, $\mathrm{Sp}(1)\times_{C_2} \mathrm{Sp}(1)\cong \mathrm{SO}(4)$ can be regarded as the group of those $\alpha\in \mathrm{Aut}(\mathcal{C})$ for which $\alpha(\mathbb{H}) = \mathbb{H}$. See [27, Appendix A.5] for more details.

Lemma 3.3. *We can identify $\widetilde{T}' = \widetilde{T}(F_4) \cong \mathbb{R}^4$ (with the usual inner product), such that the set of roots of F_4 is*

$$R' = \Big\{\pm x_i\ (1\le i\le 4),\ \pm x_i\pm x_j\ (1\le i<j\le 4),\ \tfrac{1}{2}(\pm x_1\pm x_2\pm x_3\pm x_4)\Big\} \subseteq (\widetilde{T}')^*;$$

and such that the integral lattice is

$$\Lambda' = \Big\{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 : \sum n_i \text{ even}\Big\}.$$

Also, $|W'| = 2^7{\cdot}3^2$, and W' contains all signed permutations of the coordinates in $\mathbb{R}^4$. Furthermore:

(i) Every element in F_4 is conjugate to its inverse.

(ii) If $\widetilde{g}$ and $\widetilde{g}'$ are two liftings to $\widetilde{T}'$ of an element $g\in T'$, then

$$\|\widetilde{g}\|^2 \equiv \|\widetilde{g}'\|^2 \pmod{\mathbb{Z}} \ \text{ if } |g| = 2,$$

and

$$\|\widetilde{g}\|^2 \equiv \|\widetilde{g}'\|^2 \ (\mathrm{mod}\ \tfrac{1}{3}\mathbb{Z}) \ \text{ if } |g| = 3.$$

Here, $\|\widetilde{g}\|$ denotes the norm of $\widetilde{g} \in \widetilde{T}' \cong \mathbb{R}^4$, and $|g|$ denotes the multiplicative order of $g \in T'$.

(iii) There are 2 conjugacy classes of elements of order 2 in F_4:

(I) $\|\widetilde{g}_I\|^2 \in \mathbb{Z}$, $C_{F_4}(g_I) \cong \mathrm{Spin}(9)$

(II) $\|\widetilde{g}_{II}\|^2 \in \mathbb{Z}+\frac{1}{2}$, $C_{F_4}(g_{II}) \cong \mathrm{Sp}(1)\times_{C_2} \mathrm{Sp}(3)$

(iv) There is a unique conjugacy class of subgroups $(C_2)^3\cong A\subseteq F_4$ with the property that all elements of order 2 in A are conjugate to each other in F_4. For any such A, all elements of order 2 in A have type (II); $C_{F_4}(A)\cong A\times \mathrm{SO}(3)$, and $N_{F_4}(A)/C_{F_4}(A) = \mathrm{Aut}(A)\cong \mathrm{GL}_3(\mathbb{F}_2)$.

(v) Set $g = \exp(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)\in T'$. Then $C_{F_4}(g)$ is the image of a map

$$\theta : \mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3) \hookrightarrow F_4$$

whose restriction to the maximal tori is given by

$$(\widetilde{\theta|T\times T})\,((x_1, x_2, x_3), (y_1, y_2, y_3)) = (x_1 + y_3, x_2 + y_3, x_3 + y_3, y_1 - y_2 = 2y_1 + y_3).$$

Also, $N_{F_4}(\langle g\rangle)$ *is a semidirect product* $(\mathrm{SU}(3)\times_{C_3}\mathrm{SU}(3))\rtimes C_2$*, where* C_2 *acts on each* $\mathrm{SU}(3)$ *factor by complex conjugation.*

PROOF. The description of the root system of F_4, and the order of its Weyl group, are given in Bourbaki [7, §VI.4.9]. The Weyl group is generated by reflections in the kernels of the roots, and hence contains all signed permutations of the 4 coordinates (consider the roots x_i and $x_i - x_j$). Since F_4 is centerfree, its integral lattice Λ' is just the group of elements in $\widetilde{T}'$ whose value on each root is an integer (cf. [1, Proposition 5.3]).

(i) As a special case of the above remarks, W' contains the automorphism $(t \mapsto t^{-1})$. Thus, g is conjugate to g^{-1} for all $g{\in}T'$; and hence for all $g{\in}F_4$.

(ii) This follows from the fact that if $x \in \frac{1}{2}\mathbb{Z}$, $y \in \frac{1}{3}\mathbb{Z}$, and $x \equiv x'$ and $y \equiv y'$ (mod $\mathbb{Z}$), then

$$x^2 \equiv (x')^2 \pmod{\mathbb{Z}} \quad \text{and} \quad y^2 \equiv (y')^2 \pmod{\tfrac{1}{3}\mathbb{Z}}$$

(iii) See [21, Proposition 6.12]. By Borel's theorem (Lemma 3.1), the centralizer of any element in F_4 is connected. By Borel & Siebenthal [6] (and a study of the root system for F_4), F_4 contains $\mathrm{Spin}(9)$ and $\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(3)$ as maximal connected subgroups. Hence, these groups must be the centralizers of their central elements g_I, g_{II} of order 2. Alternatively, this can be shown by taking the following explicit elements, and computing the roots of their centralizers:

$g_I = \exp(0,0,0,1)$: roots $\pm x_i$, $\pm x_i \pm x_j$

$g_{II} = \exp(\frac{1}{2},\frac{1}{2},0,0)$: roots $\pm x_3, \pm x_4, \pm x_1 \pm x_2, \pm x_3 \pm x_4, \frac{\pm(x_1-x_2)\pm x_3 \pm x_4}{2}$

A comparison of the orders of the Weyl groups ($W' = W(F_4)$ has order $2^7 \cdot 3^2$) shows that each maximal torus contains three elements conjugate to g_I and 12 conjugate to g_{II}. So this accounts for all elements of order 2. The norms of g_I and g_{II} are immediate from the explicit formulas above.

(iv) Assume first that $A = \langle g_1, g_2, g_3\rangle \cong (C_2)^3$ is such that all elements of order 2 in A have type (I). Then $A{\subseteq}C_{F_4}(g_1){\cong}\mathrm{Spin}(9)$. Since $\mathrm{Spin}(9)$ is simply connected, $C_{F_4}(g_1,g_2)$ is again connected by Lemma 3.1(i). Since any maximal torus of $C_{F_4}(g_1,g_2)$ which contains g_3 also contains g_1 and g_2, A must be contained in a maximal torus of F_4. And this contradicts the fact that each maximal torus contains only 3 elements of type (I).

Now assume that $A = \langle g_1, g_2, g_3\rangle$ and $A' = \langle g_1', g_2', g_3'\rangle$ are both isomorphic to $(C_2)^3$, and that all elements of order 2 in A, A' have type (II). (Possibly $A = A'$ with different bases.) Since g_1 is conjugate to g_1', we may conjugate A' to arrange that $g_1' = g_1$. Then $A, A'{\subseteq}C_{F_4}(g_1) = H_1{\cong}\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(3)$. Upon inspection, we see that all noncentral elements in H_1 of type (II) are conjugate in H_1 to

$$h_1 = (i, \mathrm{diag}(i,i,i)) \qquad \text{or} \qquad h_2 = (1, \mathrm{diag}(-1,1,1)).$$

And $g_1 {\cdot} h_2$ does not have type (II) (where $g_1 = (-1, I) = (1,\text{-I})$). Thus, g_2, g_2' are both conjugate to h_1, and we may assume (by conjugating by appropriate elements of H_1) that $g_2' = g_2 = h_1$. We now have

$$A, A' \subseteq C_{F_4}(g_1, g_2) = H_2 \cong C_{H_1}(h_1) \cong \langle S^1 \times_{C_2} \mathrm{U}(3), (j, j \cdot I) \rangle.$$

Finally, we see that g_3, g_3' must both be conjugate (in H_2) to (j, jI). This shows that A is conjugate to A'; and that

$$C_{F_4}(A) \cong (H_2)^{(j,jI)} \cong A \times \mathrm{SO}(3).$$

Also, $N_{F_4}(A)/C_{F_4}(A) \cong \mathrm{GL}_3(\mathbb{F}_2)$, since we have just seen that any automorphism of A is an inner automorphism in F_4.

(v) The centralizer $C_{F_4}(g)$ is connected by Lemma 3.1(i), and it contains T' as a maximal torus. The roots of $C_{F_4}(g)$ are precisely those roots of F_4 which take integer values on $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$: namely,

$$\pm(x_i - x_j)(1 \le i < j \le 3), \pm x_4, \pm\frac{1}{2}(x_1 + x_2 + x_3 \pm x_4). \tag{1}$$

The homomorphism $\widetilde{\theta} : \widetilde{T}(\mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3)) \to \widetilde{T}'$, defined by setting

$$\widetilde{\theta}((x_1, x_2, x_3), (y_1, y_2, y_3)) = (x_1 + y_3, x_2 + y_3, x_3 + y_3, y_1 - y_2),$$

is an isomorphism of vector spaces and integral lattices, and its dual sends the roots in (1) bijectively to the roots of $\mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3)$. Since an isomorphism between root systems and integral lattices induces an isomorphism between compact connected Lie groups (cf. [8, p.40, Prop. 15]), $\widetilde{\theta}$ extends to an isomorphism

$$\theta : \mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3) \xrightarrow{\ \cong\ } C_{F_4}(g) \subseteq F_4.$$

By point (i), g is conjugate to g^{-1} in F_4. In particular, there exists $a \in N_{F_4}(\langle g \rangle)$ such that $\mathrm{conj}(a)$ is complex conjugation on both $\mathrm{SU}(3)$ factors in the centralizer. Then $a^2 \in Z(C_{F_4}(\langle g \rangle)) = \langle g \rangle$, and (since $|g| = 3$) we can take a of order 2. This shows that $N_{F_4}(\langle g \rangle)$ is a semidirect product $(\mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3)) \rtimes C_2$; and finishes the proof of the lemma. $\square$

Note, in part (iii) above, that $\theta(\mathrm{SU}(3) \times 1)$ is the factor which contains long roots of F_4: in keeping with the notation in [3, Example 2.11]. As noted in [3], there is a subgroup G_2 of F_4 such that $\theta(\mathrm{SU}(3) \times 1) \subseteq G_2 \subseteq F_4$. One way to see this is to consider the elements of order 2 in the maximal torus of $\theta(1 \times \mathrm{SU}(3))$: namely, $g_1 = \exp(0, 0, 0, 1)$ and $g_2, g_1 g_2 = \exp(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$. We have seen that $C_{F_4}(g_1) \cong \mathrm{Spin}(9)$, and so $\theta(\mathrm{SU}(3) \times 1) \subseteq C_{F_4}(g_1, g_2) \cong \mathrm{Spin}(8)$. If we regard G_2 as the group of automorphisms of the Cayley numbers, then the induced inclusion $G_2 \subseteq \mathrm{SO}(8)$ lifts to $\mathrm{Spin}(8)$ (G_2 being simply connected), and its restriction to $\mathrm{SU}(3) \subseteq G_2$ is the composite $\mathrm{SU}(3) \subseteq \mathrm{U}(3) \subseteq \mathrm{U}(4) \subseteq \mathrm{SO}(8)$. This is isomorphic to the above inclusion of $\theta(\mathrm{SU}(3) \times 1)$ into $C(g_1, g_2)$; and so this inclusion factors through G_2.

The following result now completely classifies homotopy classes of maps from BG_2 to BF_4.

EXAMPLE 3.4. [3, Example 2.11] *Set* $G = G_2$ *and* $G' = F_4$. *Use the inclusions* $\mathrm{SU}(3) \subseteq G_2$ *and* $\mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3) \subseteq F_4$ *defined above to identify*

$$\widetilde{T} = \widetilde{T}(G_2) = \widetilde{T}(\mathrm{SU}(3)) \quad \text{and} \quad \widetilde{T}' = \widetilde{T}(F_4) = \widetilde{T}(\mathrm{SU}(3)) \times \widetilde{T}(\mathrm{SU}(3)).$$

Under this identification, define $\phi = \phi_{k,m} : T \to T'$ *(any* $k, m \in \mathbb{Z}$*) by setting*

$$\widetilde{\phi}(x) = \widetilde{\phi}_{k,m}(x) = (kx, mx) \quad (\text{for any } x \in \widetilde{T}(G_2) = \widetilde{T}(\mathrm{SU}(3)))$$

Then

(i) ϕ *extends to a map* $(f_{k,m})^{\hat{}}_2 : BG_2 \to (BF_4)^{\hat{}}_2$ *if and only if either* $k = m = 0$*, or* k *is odd and* $m \in \{0, \pm k, 2k\}$*. The extension is unique if* $m = 0$ *or* $m = -k$*; otherwise there are exactly two distinct homotopy classes of maps which extend* ϕ*.*

(ii) ϕ *extends to a map* $(f_{k,m})^{\hat{}}_3 : BG_2 \to (BF_4)^{\hat{}}_3$ *if and only if each of* k, m *is* 0 *or prime to* 3*; and the extension is unique up to homotopy.*

(iii) ϕ *extends to a map* $f_{k,m} : BG_2 \to BF_4$ *if and only if either* $k = m = 0$*, or* $(k, 6) = 1$ *and* $m \in \{0, \pm k, 2k\}$*. The extension is unique if* $m = 0$ *or* $m = -k$*. If* $m = k$ *or* $m = 2k$*, then there are exactly two distinct homotopy classes of maps which extend* ϕ*.*

Furthermore, for any $f : BG_2 \to BF_4$*,* $f|BT \simeq B\phi_{k,m}$ *for some* k *and* m*.*

PROOF. We first check that the $\phi_{k,m}$ are admissible. By construction, they are equivariant with respect to the diagonal inclusion

$$W(\mathrm{SU}(3)) \xrightarrow{\ \Delta\ } W(\mathrm{SU}(3)) \times W(\mathrm{SU}(3)) \subseteq W'.$$

Also, $W = W(\mathrm{SU}(3)) \times \langle w \rangle$, where $w(t) = t^{-1}$ for all $t \in T$. Choose $w' \in W'$ such that $w'(t) = t^{-1}$ for all $t \in T'$ (Lemma 3.3(v)). Then the $\phi_{k,m}$ are equivariant with respect to the homomorphism

$$\Delta \times (w \mapsto w') : W \longrightarrow W';$$

and hence are admissible.

Now let ϕ be any nontrivial admissible homomorphism from G_2 to F_4. If ϕ is equivariant with respect to the diagonal inclusion $\Delta : \Sigma_3 \to \Sigma_3 \times \Sigma_3$ as above, then (by Schur's lemma) ϕ must be equal to $\phi_{k,m}$ for some k, m. By [3, Theorem 2.21], there is some homomorphism $\bar{\phi} : W \to W'$ such that ϕ is $\bar{\phi}$-equivariant, and $\bar{\phi}$ is injective since $\widetilde{\phi}$ is. We will show that $\bar{\phi}$ can be chosen so $\bar{\phi}|\Sigma_3$ is conjugate in W' to Δ, and hence that ϕ is conjugate by an element of W' to some $\phi_{k,m}$.

To see this, fix elements $a, b \in W(\mathrm{SU}(3)) \subseteq W$ such that $|a| = 3$ and $|b| = 2$ (so $bab^{-1} = a^{-1}$). Write elements in W' as matrices with respect to the standard basis for $\widetilde{T}' \cong \mathbb{R}^4$; then

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

generate the first Σ_3 factor, and

$$A_2 = \begin{pmatrix} \frac12 & -\frac12 & -\frac12 & -\frac12 \\ -\frac12 & \frac12 & -\frac12 & -\frac12 \\ -\frac12 & -\frac12 & \frac12 & -\frac12 \\ \frac12 & \frac12 & \frac12 & -\frac12 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

generate the second. Note that A_1 acts via rotation by $2\pi/3$ on the plane $Pl_1 = \{(x,y,-x-y,0)\}$ and fixes $Pl_2 = \{(x,x,x,y)\} = Pl_1^{\perp}$, while A_2 rotates Pl_2 and fixes Pl_1.

Since $\langle A_1, A_2\rangle$ is a Sylow 3-subgroup of W', we may assume that $\bar\phi(a)$ is one of the elements A_1, A_2, or A_1A_2. If $\bar\phi(a) = A_1$, then $\mathrm{Im}(\widetilde\phi) = Pl_1$, since this is the unique A_1-invariant subplane in $\widetilde{T}'$ upon which A_1 acts by rotation. The relation $bab^{-1} = a^{-1}$ forces $\bar\phi(b)$ to leave both Pl_1 and Pl_2 invariant, and to act on Pl_1 via a reflection. Define $\bar\phi' : W \to W'$ by setting $\bar\phi'(a) = A_1A_2$; $\bar\phi'(w) = w'$ (these are again the elements which act via $(t \mapsto t^{-1})$); and $\bar\phi'(b) = \bar\phi(b)$ or $\bar\phi(b)\cdot B_2$ according to which one acts as a reflection on Pl_2. This is a well defined homomorphism — $\langle a,b\rangle \cong \Sigma_3$ acts on each Pl_i via the standard 2-dimensional representation — and ϕ is $\bar\phi'$-equivariant since $\mathrm{Im}(\phi) = Pl_1$.

The same argument applies if $\bar\phi(a) = A_2$, and shows that $\bar\phi$ can always be chosen such that $\bar\phi(a) = A_1A_2$. Consider the homomorphism $\mu : W' \to \mathrm{GL}_4(\mathbb{F}_2)$ induced by the action of W' on $\Lambda'/2\Lambda'$, with respect to the basis

$$\big\{(2,0,0,0),\ (1,1,1,1),\ (1,-1,0,0),\ (0,1,-1,0)\big\}$$

of the integral lattice $\Lambda' \subseteq \widetilde{T}'$. It is not hard to see that $\mathrm{Ker}(\mu) = \{\pm I\}$, and that

$$\mathrm{Im}(\mu) = \left\{ \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \ :\ X, Z \in \mathrm{GL}_2(\mathbb{F}_2),\ Y \in \mathrm{M}_2(\mathbb{F}_2) \right\}.$$

Also, $\mu(A_1) = \left(\begin{smallmatrix} I & 0 \\ 0 & \alpha \end{smallmatrix}\right)$, $\mu(B_1) = \left(\begin{smallmatrix} I & 0 \\ 0 & \beta \end{smallmatrix}\right)$, $\mu(A_2) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & I \end{smallmatrix}\right)$, and $\mu(B_2) = \left(\begin{smallmatrix} \beta & 0 \\ 0 & I \end{smallmatrix}\right)$; where $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\beta = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence, since any element of order 2 in $\mathrm{GL}_2(\mathbb{F}_2)$ is conjugate to β (conjugate by an element of $\langle\alpha\rangle$), we may assume that

$$\mu(\bar\phi(a)) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \mu(\bar\phi(b)) = \begin{pmatrix} \beta & X \\ 0 & \beta \end{pmatrix}$$

for some X. The relations $b^2 = 1$ and $bab = a^{-1}$ imply that $X = 0$ or β. Since $\left(\begin{smallmatrix} I & \alpha \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} \beta & \beta \\ 0 & \beta \end{smallmatrix}\right)\left(\begin{smallmatrix} I & \alpha \\ 0 & I \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta \end{smallmatrix}\right)$, we may assume that $X = 0$; i.e., that $\bar\phi(b) = \pm B_1B_2$. And if we set $U = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in W'$, then $[U, A_1A_2] = I$ and $UB_1B_2U^{-1} = -B_1B_2$; and this finishes the proof that $\bar\phi|\Sigma_3$ is conjugate to the diagonal inclusion.

It remains to determine which of the $\phi_{k,m}$ extend to maps from $BG_2 \to (BF_4)^\wedge_p$ (for $p = 2,3$), and to count the number of homotopy classes of such maps. As before, we identify

$$\mathrm{SU}(3) = C_{G_2}\Big(\exp(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{-2}{3})\Big) \subseteq G_2.$$

Fix an embedding

$$\theta : \mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3) \longrightarrow F_4 :$$

where $\mathrm{Im}(\theta) = C_{F_4}\big(\exp(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)\big)$ and (by Lemma 3.3(v))

$$\widetilde{\theta|T}\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) = (x_1 + y_3, x_2 + y_3, x_3 + y_3, y_1 - y_2).$$

Using this, we get the explicit formula

$$\widetilde{\phi_{k,m}}(x_1, x_2, x_3) = (kx_1 + mx_3, kx_2 + mx_3, kx_3 + mx_3, mx_1 - mx_2). \tag{1}$$

Recall (Lemma 1.3) that ϕ is *regular* if and only if $\mathrm{Im}(\widetilde{\phi})$ is not contained in the kernel of any root of F_4. Using this last criterion, one checks that ϕ is regular unless either $k = 0$, or $m = 0$, or $k = -m$, or $k = 2m$.

<u>$p = 3$:</u> If $\phi = \phi_{k,m}$ is trivial, then any extension $f : BG_2 \to (BF_4)\hat{_3}$ of $B\phi_{k,m}$ is null homotopic (cf. [21, Theorem 3.11]).

Now fix $(k, m) \neq (0, 0)$, and assume that $\phi = \phi_{k,m}$ lifts to a map $f : BG_2 \to (BF_4)\hat{_3}$. We first show that each of k, m must be 0 or prime to 3. If k and m are both multiples of 3, then all elements of order 3 in T lie in $\mathrm{Ker}(\phi)$, and this contradicts Proposition 1.8. If one of k or m is prime to 3 and the other a nonzero multiple of 3, then ϕ is regular by the above remarks. So there is only one possible induced map $\bar{\phi} : W \to W'$, and by the remarks at the beginning of the proof it must be Δ.

Again consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \qquad \text{and} \qquad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

of order 3 in $\mathrm{SU}(3) \subseteq G_2$. These are conjugate in $\mathrm{SU}(3)$; and, in fact, every element in $\langle T, A_1 \rangle \smallsetminus T$ is conjugate in $\mathrm{SU}(3)$ to A. From the previous paragraph, we see that $\bar{\phi}(A_1 T) = \theta(A_1, A_1) \cdot T'$. Hence, for any $\mathcal{R}_p$-invariant representation $\rho : N_3(T) = \langle T, A_1 \rangle \to F_4$, $\rho(A_1) \equiv \theta(A_1, A_1) \pmod{T'}$. Thus $\rho(A_1)$ is conjugate to (A, A); but since ρ is $\mathcal{R}_p$-invariant it is also conjugate to $\rho(A) = (1, A^{\pm 1})$ or $(A^{\pm 1}, 1)$. And a norm computation, using Lemma 3.3(ii), shows that $(A, A) = \exp(-\frac{1}{3}, 0, -\frac{2}{3}, -\frac{1}{3})$ cannot be conjugate in F_4 to $(A, 1) = \exp(0, \frac{1}{3}, -\frac{1}{3}, 0)$ or $(1, A) = \exp(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$.

<u>Construction of $\mathcal{R}_3$-invariant representations:</u> Now assume that k and m each is 0 or prime to 3. Extend $\phi = \phi_{k,m}$ to

$$\rho : N_3(T) = \langle T, A_1 \rangle \longrightarrow \mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3) \subseteq F_4$$

by setting

$$\rho(A_1) = \begin{cases} \theta(1, A_1) & \text{if } k = 0 \\ \theta(A_1, 1) & \text{if } m = 0 \\ \theta(A_1, A_1) & \text{otherwise.} \end{cases}$$

This clearly gives a well defined homomorphism, and we claim that it is $\mathcal{R}_3$-invariant.

We must show, for any pair $P, P' \subseteq N_3(T)$ of 3-stubborn subgroups of G_2, and any $x \in G_2$ such that $xPx^{-1} \subseteq P'$, that the maps $\rho|(xPx^{-1})$ and $\mathrm{conj}(x) \circ (\rho|P)$ are conjugate in F_4. Consider the 3-stubborn subgroups of G_2 listed in Lemma 3.2(v). For any subgroup $P'' \subseteq \langle T, A_1 \rangle$ isomorphic to $\langle A, A_1 \rangle$, $P \cap T = {}_3T$ (the 3-torsion subgroup); and hence P'' is conjugate to $\langle A, A_1 \rangle$ by an element of T. Hence, we need only consider the cases where $x \in N(T)$ (where the maps are conjugate since ϕ is admissible); or where $P = \langle A, A_1 \rangle$ and $x \in N(P)$. And in this latter case, $\rho|(xPx^{-1})$ and $\mathrm{conj}(x) \circ (\rho|P)$ are conjugate in $\mathrm{Im}(\theta)$, since they have the same character in each SU(3)-factor.

Uniqueness of the $\mathcal{R}_3$-invariant representations: Assume that $\rho' : \langle T, A_1 \rangle \to F_4$ is another $\mathcal{R}_3$-invariant representation. If ϕ is regular, then

$$\rho'(A_1) \in \rho(A_1) \cdot C_{F_4}(T) = \rho(A_1) \cdot T'.$$

Also, $\rho(A_1) = \theta(A_1, A_1)$ in this case, and every element in $\rho(A_1)T'$ is conjugate by an element of T' to $\rho(A_1)$. So ρ and ρ' are conjugate.

Now assume that ϕ is not regular. Consider the elements $g_1 = \exp(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$ and $g_2 = \exp(\frac{1}{3}, -\frac{1}{3}, 1, \frac{1}{3})$, where in both cases $C_{F_4}(g_i) \cong \mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3)$. If $m = 0$ or $k = 0$, then $\phi(T)$ is the maximal torus in one of the factors of $C_{F_4}(g_1)$, while if $k = -m$ or $k = 2m$ then $\phi(T)$ is the maximal torus in one of the factors of $C_{F_4}(g_2)$. Fix $\eta : \mathrm{SU}(3) \times_{C_3} \mathrm{SU}(3) \to F_4$ such that $\mathrm{Im}(\phi) = \eta(T \times 1)$. Note that (since g_1 and g_2 are conjugate by an element of W'), η is conjugate in F_4 to θ, except possibly with the factors reversed. Now, $C_{F_4}(\phi(T)) = \eta(T \times_{C_3} \mathrm{SU}(3))$, and

$$\mathrm{Im}(\rho), \mathrm{Im}(\rho') \subseteq \eta(\langle T, A_1 \rangle \times_{C_3} \mathrm{SU}(3)).$$

And after conjugating, we may assume that $\mathrm{Im}(\rho), \mathrm{Im}(\rho') \subseteq \eta(\langle T, A_1 \rangle \times_{C_3} T)$.

Write $\rho(A_1) = \eta(x, y)$ and $\rho'(A_1) = \eta(x', y')$, where $x, x' \in A_1 T$, and $y, y' \in T$. Since $y^3 = (y')^3 = 1$, we may assume that $y, y' \in \langle A \rangle$. Since A and A^{-1} are conjugate in SU(3), we may assume that $y, y' \in \{1, A\}$. And since any element in $A_1 T$ is conjugate to A_1, we may assume $x, x' = A_1$. Finally, $\rho(A_1)$ and $\rho'(A_1)$ must be conjugate in F_4 — since both are conjugate to $\phi(A)$ — and hence $\eta(A, y)$ must be conjugate to $\eta(A, y')$. And since η is conjugate in F_4 to θ (except possibly with the factors switched), formula (1) can be used to show that $y = y'$, and hence that $\rho = \rho'$.

Existence and uniqueness of maps: It remains to check that the appropriate higher limits vanish. By Lemma 3.2(v) again, G_2 contains up to conjugacy two 3-stubborn subgroups: $P_1 = \langle T, A_1 \rangle$ and $P_2 = \langle A, A_1 \rangle$. Also, $N(P_1)/P_1$ has order prime to 3, and $3^2 \nmid |N(P_2)/P_2|$.

The main problem is to determine the centralizer of $\rho(P_2)$. Set $z = \mathrm{diag}(\zeta,\zeta,\zeta) \in \mathrm{SU}(3) \subseteq G_2$. If $3 \nmid k+m$, then by (1),

$$\rho(z) = \rho(\exp(\tfrac{1}{3},\tfrac{1}{3},-\tfrac{2}{3})) = \exp(\pm(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},1));$$

and $C_{F_4}(\rho(P_2)) = C_{\mathrm{Im}(\theta)}(\rho(P_2))$. If $3|k+m$, then $\rho(z) = 1$, and $C_{F_4}(\rho(P_2)) = C_{C_{F_4}(\rho(A))}(\rho(A_1))$, where

$$\rho(A) = \rho(\exp(\tfrac{1}{3},-\tfrac{1}{3},0)) = \exp(\tfrac{k}{3},\tfrac{-k}{3},0,\tfrac{2m}{3})$$

is conjugate to $\exp(\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$. In either case, $C_{F_4}(\rho(P_2))$ is the centralizer of a 3-subgroup of $\mathrm{SU}(3)\times_{C_3}\mathrm{SU}(3)$, and hence is either connected or has 3 components. In particular, the action of $N(P_2)/P_2$ on

$$\Pi_1^\rho(G/P_2) \cong \pi_0(C_{F_4}(\rho(P_2)))$$

has order at most 2. By Corollary 1.11, $\varprojlim^i(\Pi_1^\rho) = 0$ for all $i \geq 1$, and $\varprojlim^i(\Pi_n^\rho) = 0$ for all $i,n \geq 2$. By Theorem 1.9, ρ (and hence ϕ) extends to a unique homotopy class of maps $f : BG_2 \to (BF_4)\hat{}_3$.

<u>$p=2$:</u> By [21, Theorem 3.11], the trivial homomorphism $\phi_{0,0}$ extends to a unique (null homotopic) map. So we restrict attention to the case $\phi = \phi_{k,m}$ for $(k,m) \neq (0,0)$.

Assume that ϕ extends to an $\mathcal{R}_p$-invariant representation ρ. Note that $N_{\mathrm{SU}(3)}(T)\smallsetminus T \subseteq N_{G_2}(T)\smallsetminus T$ contains elements of order 2, which are conjugate to any given element of order 2 in T (Lemma 3.2(i)). So by Proposition 1.8, for each $g\in T = T(G_2)$ of order 2, $\phi(g) \neq 1$. Also, $\phi(g)$ must have type (II) in F_4 by Lemmas 3.2(i,ii) and 3.3(iv). This is the case if and only if k is odd: since by formula (1), $\phi(g)$ has type (I) if k is even and m is odd, and $\phi(g) = 1$ if k and m are both even.

Assume now that k is odd.

<u>Uniqueness of the $\mathcal{R}_2$-invariant representation:</u> Fix some $\phi = \phi_{k,m}$, and assume that $\rho,\rho' : N_2(T) \to F_4$ are two $\mathcal{R}_2$-invariant representations which extend ϕ. We must show that they are conjugate.

Fix $g \in T$ of order 2, set $h = \phi(g)$, and identify

$$C_{G_2}(g) = \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1), \quad N_2(T) = N\times_{C_2}N, \quad \text{and} \quad C_{G_2}(h) = \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(3).$$

The images of ρ and ρ' lie in $C_{F_4}(h)$; and ρ and ρ' lift to homomorphisms

$$\widetilde{\rho},\widetilde{\rho}' : N\times N = \langle S^1, j\rangle \times \langle S^1, j\rangle \longrightarrow \mathrm{Sp}(1)\times \mathrm{Sp}(3).$$

The easiest way to check this last step is to take the pullbacks of $\mathrm{Sp}(1)\times\mathrm{Sp}(3)$ along ρ and ρ', and check that they both must be isomorphic to $N\times N$.

Since ρ is $\mathcal{R}_2$-invariant, there is some $x \in F_4$ such that $x\rho(z,j)x^{-1} = \rho(z,i)$ for all $z \in S^1$. In particular, $x \in C_{F_4}(h)$, and hence lifts to $\widetilde{x} \in \mathrm{Sp}(1)\times\mathrm{Sp}(3)$ such that $\widetilde{x}\widetilde{\rho}(z,j)\widetilde{x}^{-1} = \pm\widetilde{\rho}(z,i)$ for all z. The sign ($\pm$) must be constant (by continuity); and since $(1,i)$ is conjugate to $(1,-i)$ in $N\times N$ we may assume that

$\widetilde{x}\widetilde{\rho}(z,j)\widetilde{x}^{-1} = \widetilde{\rho}(z,i)$ for all z. In particular, $\rho(z,j)$ is conjugate to $\rho'(z,j)$; and in fact $\rho(z,z'j)$ is conjugate to $\rho'(z,z'j)$ for all $z,z' \in S^1$ (since $z'j$ is conjugate in N to j). The same argument applies to all other cosets of T in $N_2(T)$. Thus, $\widetilde{\rho}$ and $\widetilde{\rho}'$ have the same characters in $\mathrm{Sp}(1)$ and $\mathrm{Sp}(3)$, and hence are conjugate. And hence ρ and ρ' are conjugate.

Existence and uniqueness of maps: Before going further, we must specify more precisely some of the identifications already used. Set $g = \exp(\frac{1}{2}, -\frac{1}{2}, 0) \in G_2$. Then

$$\phi_{k,m}(g) = \exp(\tfrac{k}{2}, -\tfrac{k}{2}, 0, m) = \begin{cases} h_1 = \exp(\frac{1}{2}, -\frac{1}{2}, 0, 0) & \text{if } m \text{ is even} \\ h_2 = \exp(\frac{1}{2}, -\frac{1}{2}, 0, 1) & \text{if } m \text{ is odd.} \end{cases}$$

By Lemmas 3.2(i) and 3.3(iii), we know that the centralizers of these elements are the images of embeddings

$$\sigma : \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1) \to G_2 \qquad \text{and} \qquad \tau_1, \tau_2 : \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(3) \to F_4.$$

So if we choose $N_2(T) \subseteq \mathrm{Im}(\sigma)$, it then follows that $\mathrm{Im}(\rho) \subseteq \mathrm{Im}(\tau_i)$ (where $i = 1, 2$ depending on the parity of m).

Let $S \subseteq \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1)$ and $S' \subseteq \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(3)$ denote the standard maximal tori. Upon checking the roots of these centralizers, we see that σ and the τ_i can be chosen to satisfy the following formulas:

$$\begin{aligned} (\widetilde{\sigma|S})(x,y) &= (x+y, -x+y, -2y) \\ (\widetilde{\tau_1|S'})(x_1; x_2, x_3, x_4) &= (x_1+x_2, -x_1+x_2, -x_3-x_4, -x_3+x_4) \\ (\widetilde{\tau_2|S'})(x_1; x_2, x_3, x_4) &= (x_1+x_2, x_1-x_2, -x_3+x_4, -x_3-x_4). \end{aligned} \tag{2}$$

For example, if m is even, then the roots of $C_{F_4}(h_1)$ are $\pm x_1 \pm x_2$, $\pm x_3 \pm x_4$, $\pm x_3$, $\pm x_4$, $\pm\frac{1}{2}(x_1+x_2\pm x_3\pm x_4)$ (i.e., the roots of F_4 whose value on $(\frac{1}{2}, -\frac{k}{2}, 0, 0)$ is integral). And these are the roots of $\mathrm{Im}(\tau_1)$.

The idea of the proof is now to push all computations into the centralizers $C_{G_2}(g)$ and $C_{F_4}(\phi(g))$. So we start by identifying the composite $(\tau_1|S')^{-1} \circ \phi \circ (\sigma|S)$. This splits into two cases, depending on whether m is even or odd.

Case 1. k odd and m even: In this case, (1) and (2) yield the formula

$$((\tau_1|S')^{-1} \circ \phi \circ (\sigma|S))^{\sim}(x,y) = (kx; ky - 2my, ky + m(y-x), ky + m(x+y)). \tag{3}$$

Case 1a. $m \neq 0, 2k$: Consider the subgroup $P = \sigma(Q\times_{C_2}N) \subseteq G_2$ (see Lemma 3.2(iii)). There is $x \in N(P)$ such that $x(i,1)x^{-1} = (j,1)$ and x centralizes $\sigma(1\times S^1)$. Hence, since ρ is $\mathcal{R}_p$-invariant, $\rho\sigma(j,1)$ must lie in and be conjugate to $\rho\sigma(i,1)$ in

$$\begin{aligned} C_{F_4}(\phi\sigma(1\times S^1)) &= \tau_1\Big(C_{\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(3)}(\exp(\mathbb{R}\cdot(0; k-2m, k+m, k+m)))\Big) \\ &= \tau_1\big(\mathrm{Sp}(1)\times_{C_2}(\mathrm{U}(1)\times\mathrm{U}(2))\big). \end{aligned}$$

(Recall that the centralizer of any subtorus is connected.) By (3),

$$\tau_1^{-1}\rho\sigma(i,1)=\tau_1^{-1}\phi\sigma(i,1)=\exp(\tfrac{k}{4};0,-\tfrac{m}{4},\tfrac{m}{4})=\left(\pm i,\begin{pmatrix}1&0&0\\0&\pm1&0\\0&0&\pm1\end{pmatrix}\right)$$

and so $\tau_1^{-1}\rho\sigma(j,1)$ must have the form

$$\tau_1^{-1}\rho\sigma(j,1)=\left(z,\begin{pmatrix}1&0&0\\0&\pm1&0\\0&0&\pm1\end{pmatrix}\right)$$

for some $z\in\mathrm{Sp}(1)$ conjugate to i. In addition (since ρ is a homomorphism) this element must act on

$$\tau_1^{-1}\rho\sigma(S^1\times 1)=\exp(\mathbb{R}\cdot(k;0,-m,m))$$

via $(z\mapsto z^{-1})$; and this is impossible.

Case 1b. $m=2k$: In this case, (3) takes the form

$$((\tau_1|S')^{-1}\circ\phi\circ(\sigma|S))\tilde{}\,(x,y)=(kx;-3ky,3ky-2kx,3ky+2kx)\,. \tag{3a}$$

We extend this to a homomorphism $\bar\rho:N\times_{C_2}N\to\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(3)$ by setting

$$\bar\rho(z,1)=\left(z^k,\begin{pmatrix}1&0&0\\0&z^{-2k}&0\\0&0&z^{2k}\end{pmatrix}\right)\qquad \bar\rho(1,z)=\left(1,\begin{pmatrix}z^{-3k}&0&0\\0&z^{3k}&0\\0&0&z^{3k}\end{pmatrix}\right)$$

$$\bar\rho(j,1)=\left(j,\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right)\qquad \bar\rho(1,j)=\left(1,\begin{pmatrix}-j&0&0\\0&0&j\\0&j&0\end{pmatrix}\right).$$

and set $\rho=\tau_1\circ\bar\rho:N\times_{C_2}N\to F_4$. Note in particular that

$$\bar\rho(i,1)=\left(\pm i,\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right)\qquad\text{and}\qquad \bar\rho(1,i)=\left(1,\begin{pmatrix}\pm i&0&0\\0&\mp i&0\\0&0&\mp i\end{pmatrix}\right).$$

We claim that the composite

$$\rho=\tau_1\circ\bar\rho:N_2(T)=N\times_{C_2}N\longrightarrow F_4$$

is $\mathcal{R}_2$-invariant. To show this, it suffices using Lemma 3.2(iii) to check that for any of the subgroups $P_i\subseteq N\times_{C_2}N$ listed there ($i=1,\dots,6$), and any $x\in N(P_i)$, the homomorphisms $\tau_1\circ\rho|P_i$ and $\tau_1\circ(\rho|P_i)\circ\mathrm{conj}(x)$ are conjugate in F_4.

The case $P_6\cong(C_2)^3$ follows from Lemma 3.3(iv), once one has checked that all elements of order 2 are sent to elements of type (II) in F_4. To see this, note that for any $1\neq x\in P_6$, $\rho(x)$ is conjugate in $\mathrm{Im}(\tau_1)$ to $\phi\sigma(i,i)$ or $\phi\sigma(1,-1)$, both of which have type (II).

For $i=1,\dots,4$, this is straightforward, and the homomorphisms are in fact always conjugate in $\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(3)$ (i.e., before composing with τ_1). In the case $P_1=N\times_{C_2}N$, there is nothing to prove ($N(P_1)/P_1=1$). In the next two cases,

$P_2 = N\times_{C_2}Q$ and $P_3 = Q\times_{C_2}N$, the arguments can be greatly simplified by conjugating elements of Sp(3) with the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} j & 0 & 0 \\ 0 & i/\sqrt{2} & i/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

respectively. Note also that the centralizer in U(2) of the matrix $\left(\begin{smallmatrix} 0 & j \\ -j & 0 \end{smallmatrix}\right)$ is SU(2). The last case, $P_4 = Q\times_{C_2}Q$, follows from the two previous ones.

It remains to consider the case $P_5 = \langle T, \gamma\rangle$, where we write $\gamma = \sigma(j,j)$ for short. Fix any $x \in N(P_5) = N(T)$. Since ϕ is admissible, there is $y \in N(T') \subseteq F_4$ such that $\rho(xax^{-1}) = y\rho(a)y^{-1}$ for all $a \in T$. Also, $\rho(\gamma)$, $y\rho(\gamma)y^{-1}$, and $\rho(x\gamma x^{-1})$ all have the same conjugation action on $\rho(T)$, and hence lie in the same coset of $C_{F_4}(\rho(T)) = T'$. From the formula $\rho(\gamma) = \tau_1(j, j{\cdot}I)$, we see that any two elements in $\rho(\gamma)T'$ are conjugate by an element of T'. And this shows that $\mathrm{conj}(y') \circ \rho = \rho \circ \mathrm{conj}(x)$ (on $P_5 = \langle T, \gamma\rangle$) for some $y' \in T'y$.

The higher limits $\varprojlim^i(\Pi_n^\rho)$ can now be computed using Theorem 1.10 (and the formula for Π_n^ρ in 1.9); together with the computation $\Lambda^*(\mathrm{GL}_3(\mathbb{F}_2); (\mathbb{F}_2)^3) = 0$ of [21, Proposition 6.3]. The results are summarized in the following table:

P	$N(P)/P$	$C_{F_4}(\rho(P))$	$\Lambda^0(NP/P; \Pi_1^\rho(-))$	$\Lambda^1(NP/P; \Pi_1^\rho(-))$
$N\times_{C_2}N$	1	$(C_2)^2$	$(\mathbb{Z}/2)^2$	0
$N\times_{C_2}Q$	$1\times\Sigma_3$	$(C_2)^2$	0	0
$Q\times_{C_2}N$	$\Sigma_3\times 1$	$(C_2)^3$	0	$\mathbb{Z}/2$
$Q\times_{C_2}Q$	$\Sigma_3\times\Sigma_3$	$(C_2)^3$	0	0
$T\rtimes C_2$	Σ_3	${}_2(T')\cong (C_2)^4$	0	$(\mathbb{Z}/2)^2$
$(C_2)^3$	$\mathrm{GL}_3(\mathbb{F}_2)$	$(C_2)^3\times \mathrm{SO}(3)$	0	$0.$

Note for example that whenever there is $P' \supsetneqq P$ such that $C(\rho(P')) = C(\rho(P))$, then $(N(P) \cap P')/P$ acts trivially on all homotopy groups of $BC(\rho(P))$, and hence $\Lambda^*(NP/P; \Pi_n^\rho(G/P)) = 0$ for all n by Theorem 1.10(iii).

In all other cases ($n > 1$ or $i > 1$), $\Lambda^i(N(P)/P; \Pi_n^\rho(G/P)) = 0$. Since the Λ^i are the higher limits of the quotient functors of a certain filtration of Π_n^ρ (see Theorem 1.10(i)), these computations show that $\varprojlim^1(\Pi_1^\rho) = \mathbb{Z}/2$, and that $\varprojlim^i(\Pi_n^\rho) = 0$ for all $(i,n) \neq (1,1)$. It follows that ρ (and hence $\phi_{k,2k}$) extends to exactly two homotopy classes of maps $(f_{k,2k})\hat{_2}, (f'_{k,2k})\hat{_2} : BG_2 \to (BF_4)\hat{_2}$. This last step, the existence in this situation of exactly two extensions, follows from the arguments in Wojtkowiak [29], even though it is not stated explicitly there.

Case 1c. $m = 0$: In this case, formula (3) simplifies to give

$$((\tau_1|S')^{-1} \circ \phi \circ (\sigma|T))^{\sim}(x,y) = (kx; ky, ky, ky). \tag{3b}$$

In other words, $\tau_1^{-1}\phi\sigma(z,w) = (z^k, w^k{\cdot}I)$ for any $(z,w)\in S^1\times_{C_2}S^1$. This can now be extended to a homomorphism $\rho : \sigma(N\times_{C_2}N) \to F_4$ ($N = \langle S^1, j\rangle$) by setting $\rho\sigma(j,1) = \tau_1(j, I)$ and $\rho\sigma(1,j) = \tau_1(1, j{\cdot}I)$.

We check that ρ is $\mathcal{R}_2$-invariant, by again referring to the list of 2-stubborn subgroups of G_2 in Lemma 3.2(iii). Invariance with respect to conjugation and inclusion of the subgroups $N{\times_{C_2}}Q$, $Q{\times_{C_2}}N$, and $Q{\times_{C_2}}Q$ is easily checked (and holds within $\mathrm{Im}(\tau_1){\cong}\,\mathrm{Sp}(1){\times_{C_2}}\,\mathrm{Sp}(3)$). Invariance for the subgroup $T{\rtimes}C_2$ follows automatically from the fact that ϕ is admissible. And invariance for the subgroup $(C_2)^3{\subseteq}G_2$ follows from Lemma 3.3(iv) (uniqueness of $(C_2)^3{\subseteq}F_4$) — after checking that all elements of order 2 in $N{\times_{C_2}}N$ are sent to elements of type (II) in F_4.

The higher limits $\varprojlim^i(\Pi_n^\rho)$ can again be computed using Theorem 1.10 (and 1.9); together with the computation $\Lambda^*(\mathrm{GL}_3(\mathbb{F}_2);(\mathbb{F}_2)^3)=0$ of [21, Proposition 6.3]. This time, we get the following table:

P	$N(P)/P$	$C_{F_4}(\rho(P))$	$\Lambda^0(NP/P;\Pi_1^\rho(-))$	$\Lambda^1(NP/P;\Pi_1^\rho(-))$
$N{\times_{C_2}}N$	1	$\mathrm{O}(3)$	$\mathbb{Z}/2$	0
$N{\times_{C_2}}Q$	$1\times\Sigma_3$	$\mathrm{O}(3)$	0	0
$Q{\times_{C_2}}N$	$\Sigma_3\times 1$	$\mathrm{O}(3)$	0	0
$Q{\times_{C_2}}Q$	$\Sigma_3\times\Sigma_3$	$\mathrm{O}(3)$	0	0
$T\rtimes C_2$	Σ_3	$(C_2)^2{\times}\,\mathrm{SO}(3)$	0	$\mathbb{Z}/2$
$(C_2)^3$	$\mathrm{GL}_3(\mathbb{F}_2)$	$(C_2)^3{\times}\,\mathrm{SO}(3)$	0	$0.$

Also, $\Lambda^i(N(P)/P;\Pi_n^\rho(G/P))=0$ whenever $i>1$, or $n>1$ and $i>0$. So in this case, $\varprojlim^0(\Pi_n^\rho)\cong\pi_n(B\,\mathrm{SO}(3))$ for all n, and $\varprojlim^i(\Pi_n^\rho)=0$ whenever $i>0$. Thus, by Theorem 1.9, ρ, and hence $\phi_{k,2k}$, extend to a unique homotopy classe of map $(f_{k,0})\hat{_2}:BG_2\to(BF_4)\hat{_2}$.

<u>Case 2. k odd and m odd:</u> These cases can be handled by the same straightforward procedure as that carried out in Case 1. But it is much simpler to use Case 1 directly, together with the relation

$$\phi_{k,m}\circ\epsilon=\phi_{-k+2m,k+m}$$

given in [3, Example 2.11]. Here, ϵ denotes the restriction to the maximal torus $T\subseteq G_2$ of the "exceptional isogeny" $\Phi:(BG_2)\hat{_2}\to(BG_2)\hat{_2}$ constructed by Friedlander [16]. Since Φ is a homotopy equivalence, we can define (for any odd k):

$$(f_{k,k})\hat{_2}=\Phi^{-1}\circ(f_{k,2k})\hat{_2}\qquad\text{and}\qquad(f_{k,-k})\hat{_2}=\Phi\circ(f_{-k,0})\hat{_2}.$$

And the same argument shows that $(f_{k,-k})\hat{_2}$ is unique, and that $\phi_{k,k}$ extends to exactly two maps.

Conversely, if k and m are odd and $(f_{k,m})\hat{_2}$ is defined, then $(f_{-k+2m,k+m})\hat{_2}=\Phi\circ(f_{k,m})\hat{_2}$ is also defined. So either $k+m=0$ or $k+m=2(-k+2m)$ by Case 1. And these relations imply that $m=\pm k$. $\square$

Among the maps $f_{k,m}:BG_2\to BF_4$ constructed above, $f_{1,0}$ is (aside from $f_{0,0}$) the only one which is induced by a homomorphism between the (real) compact Lie groups. See the remarks before Example 3.4 on embedding of G_2

as a subgroup of F_4. It is not hard, using Lemmas 3.2(ii) and 3.3(iv), to show that any two such subgroups are conjugate.

There are irreducible complex representations V^{27} of G_2 and W^{26} of F_4 (with dimensions as given by the superscripts). It is not hard to check that $V|T \cong (W|\phi_{1,1}(T)) \oplus \epsilon$, where ϵ denotes the trivial 1-dimensional representation (and that this relation holds for no other $\phi_{k,m}$). This shows in particular that there is no homomorphism of type $(1,1)$: since $W|\phi_{1,1}(T)$ is not the weight system (restriction to T) of any G_2-representation. In characteristic 7, the corresponding representation $V(\overline{\mathbb{F}_7})$ of $G_2(\overline{\mathbb{F}_7})$ contains a 1-dimensional fixed subspace, and Testerman's embedding $\sigma : G_2(\overline{\mathbb{F}_7}) \to F_4(\overline{\mathbb{F}_7})$ in [32] is characterized by the property that $W(\overline{\mathbb{F}_7})|\sigma(G_2(\overline{\mathbb{F}_7})) \cong V(\overline{\mathbb{F}_7})/\overline{\mathbb{F}_7}$. Hence the map

$$\sigma_* : (BG_2)\hat{_p} \simeq BG_2(\overline{\mathbb{F}_7})\hat{_p} \longrightarrow BF_4(\overline{\mathbb{F}_7})\hat{_p} \simeq (BF_4)\hat{_p},$$

defined for any $p \neq 7$ using the equivalences of Friedlander and Mislin [31, Theorem 1.4], has type $(1,1)$.

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