Recent results on scattered spaces and MRD codes

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Outline

1. Linear sets, Scattered linear sets, and connections

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2. Maximum scattered linear sets in $\mathrm{PG}(1, \mathbb{F}_{q^n})$

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2. Maximum scattered linear sets in $\mathrm{PG}(1, \mathbb{F}_{q^n})$
3. Maximum r-scattered linear sets in $\mathrm{PG}(r, \mathbb{F}_{q^n})$

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3. Maximum r-scattered linear sets in $\mathrm{PG}(r, \mathbb{F}_{q^n})$
4. Rank-metric codes

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2. Maximum scattered linear sets in $\mathrm{PG}(1, \mathbb{F}_{q^n})$
3. Maximum r-scattered linear sets in $\mathrm{PG}(r, \mathbb{F}_{q^n})$
4. Rank-metric codes
5. Scattered sequences

Linear sets and scattered linear sets

Definition (Linear sets)

$\mathbb{U} \leq_q \; \mathbb{F}_{q^n}^{r+1}$, $\dim_q(\mathbb{U}) = k$

$$L(\mathbb{U}) = \{\langle u\rangle_{\mathbb{F}_{q^n}} \; : \; u \in \mathbb{U} \setminus \{\mathbf{0}\}\} \subset \mathrm{PG}(r, \mathbb{F}_{q^n})$$

$\mathbb{F}_q$-linear set of $\mathrm{PG}(r, \mathbb{F}_{q^n})$ of rank k

$$|L(\mathbb{U})| \leq \frac{q^k - 1}{q - 1}$$

$$k \leq \frac{n(r+1)}{2}$$

[Blokhuis and Lavrauw 2000]

Definition (Maximum scattered Linear sets in $\mathrm{PG}(r, q^n)$)

$$\dim_q(\mathbb{U}) = k \leftarrow \text{maximum possible}$$

$$|L(\mathbb{U})| = \frac{q^k-1}{q-1} \leftarrow \text{maximum possible}$$

Some applications

- *blocking sets*

  Ball,Blokhuis,Lavrauw, Lunardon, Polverino, Trombetti, Zhou...
- *two-intersection sets*

  Blokhuis, Lavrauw...
- *finite semifields*

  Cardinali, Polverino, Trombetti, Ebert, Marino, Lunardon...
- *translation caps*

  B., Giulietti, Marino, Polverino...
- *translation hyperovals*

  Durante, Trombetti, Zhou...

1. Scattered and Maximum scattered linear sets are not rare in general

2. One seeks for sets with a prescribed shape in general

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2. One seeks for sets with a prescribed shape in general

"Geometric" constructions (projection, intersection, ...)

"polynomial" constructions (description via polynomials over finite fields)

Maximum Scattered linear sets in $\mathrm{PG}(1, q^n)$

$f(X) = \sum_i a_i X^{q^i} \in \mathbb{F}_{q^n}[X]$

$$\mathbb{U} = \{(x, f(x)) \ : \ x \in \mathbb{F}_{q^n}\} \leq \mathbb{F}_{q^n}^2$$

Definition (Maximum scattered Linear sets in $\mathrm{PG}(1, q^n)$)

$\dim_q(\mathbb{U}) = n$, $|L(\mathbb{U})| = \frac{q^n-1}{q-1}$ $\Longrightarrow$ $L(\mathbb{U})$ is Maximum scattered, $f(X)$ scattered polynomial (Sheekey 2016)

Scattered Polynomials

Definition (B.-ZHOU; J. Alg. 2018)

$f(X) = \sum_i a_i X^{q^i} \in \mathbb{F}_{q^n}[X]$ Scattered of index t if

$$\mathbb{U} = \left\{ (x^{q^t}, f(x)) \ : \ x \in \mathbb{F}_{q^n} \right\}$$

maximum scattered

Scattered Polynomials

Definition (B.-ZHOU; J. Alg. 2018)

$f(X) = \sum_i a_i X^{q^i} \in \mathbb{F}_{q^n}[X]$ Exceptional Scattered of index t if

$$\mathbb{U}_m = \left\{ (x^{q^t}, f(x)) \ : \ x \in \mathbb{F}_{q^{mn}} \right\}$$

maximum scattered for infinitely many m

Scattered Polynomials

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maximum scattered

Lemma

$$\mathbb{U} = \left\{(x^{q^t}, f(x)) \ : \ x \in \mathbb{F}_{q^n}\right\}$$

$L(\mathbb{U}) \subset \mathrm{PG}(1, q^n)$ *maximum scattered linear set* $\iff$

$$\mathcal{C}_f \ : \ \frac{f(X)Y^{q^t} - f(Y)X^{q^t}}{X^qY - XY^q} = 0 \subset \mathrm{PG}(2, q^n)$$

contains only points (x, y) *with* $\frac{y}{x} \in \mathbb{F}_q$

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$$\deg\left(\frac{g(X)Y^{q^t} - g(Y)X^{q^t}}{X^qY - XY^q}\right) << \deg\left(\frac{f(X)Y - f(Y)X}{X^qY - XY^q}\right)$$

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Theorem (B.-ZHOU; J. Alg. 2018)

- *INDEX* 0 $\Longrightarrow X^{q^k}$, $q > 5$
- *INDEX* 1 $\Longrightarrow$ X
  $bX + X^{q^2}$, $N_{\mathbb{F}_{q^n}/\mathbb{F}_q}(b) \neq 1$

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Tools

- *Estimation on the number and "type" of singular points*
- *Study the structure of the branches centered at singular points*

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(B.-MONTANUCCI, JCTA 2021)

Classification of exceptional scattered monic polynomial of INDEX $t = 2$ and partial classification for INDEX $t > 2$

Exceptional Scattered polynomials via group theory

Theorem (Ferraguti, Micheli, J. Alg. 2021)

f linearized nonmonomial polynomial
$d := \max\{\deg_q(f), t\}$ odd prime $\Longrightarrow$ *f not exceptional scattered*

Exceptional Scattered polynomials via group theory

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Theorem (B., Giulietti, Zini, 2022)

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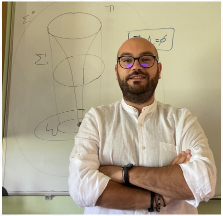
r-fat polynomials


Definition

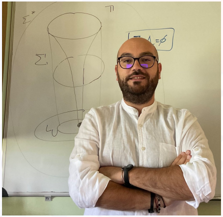
f linearized polynomial r-fat polynomial of index $t \in \{0, \dots, n-1\}$

$$\Updownarrow$$

$$\exists\ m_1, \dots, m_r \in \mathbb{F}_{q^n} :\ \dim_{\mathbb{F}_q} \ker(f(x) - m_i x^{q^t}) > 1$$

r-fat polynomials $\iff$ $U = \{(x^{q^t}, f(x)) : x \in \mathbb{F}_{q^n}\}$ *has r fat points*

r-fat polynomials

Theorem (B. Micheli, Zini, Zullo, JCTA 2021)

There exist no exceptional r-fat polynomials whenever $r > 0$

Maximum h-Scattered linear sets in $\mathrm{PG}(r, q^n)$

Definition (Linear sets)

$\mathbb{U} \leq_q \mathbb{F}_{q^n}^{r+1}$, $\dim_q(\mathbb{U}) = k$

$$L(\mathbb{U}) = \{\langle u\rangle_{\mathbb{F}_{q^n}} \ : \ u \in \mathbb{U} \setminus \{\mathbf{0}\}\} \subset \mathrm{PG}(r, \mathbb{F}_{q^n})$$

$\mathbb{F}_q$-linear set of $\mathrm{PG}(r, \mathbb{F}_{q^n})$ of rank k

Definition (Weight of a subspace)

$W \leq_{q^n} \mathbb{F}_{q^n}^{r+1}$

$$w_{\mathbb{U}}(W) = \dim_{\mathbb{F}_q}(\mathbb{U} \cap W)$$

Definition (h-scattered subspace)

$\mathbb{U}$ is said *h-scattered* if

$$\forall W \subset \mathbb{F}_{q^n}^{r+1}, \dim_{\mathbb{F}_{q^n}}(W) = h \Longrightarrow w_{\mathbb{U}}(W) \leq h$$

Maximum h-Scattered linear sets in $\mathrm{PG}(r, q^n)$

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1-scattered $\Longrightarrow$ scattered

$$k \leq \frac{n(r+1)}{h+1}$$

[Csajbók, Marino, Polverino, Zullo 2021]

Rank-metric codes

$$v = (v_1, \ldots, v_n) \in \mathbb{F}_{q^m}^n, \qquad u = (u_1, \ldots, u_n) \in \mathbb{F}_{q^m}^n$$

$$\mathrm{wt}_{\mathsf{rk}}(v) = \dim_{\mathbb{F}_q} \langle v_1, \ldots, v_n \rangle_{\mathbb{F}_q}$$ **rank weight** of v

$$\mathrm{d}_{\mathsf{rk}}(u, v) = \mathrm{wt}_{\mathsf{rk}}(u - v)$$ **rank distance**

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Definition (Delsarte 1978 - Gabidulin 1985)

An $[n, k]_{q^m/q}$ **(rank-metric) code** is a k-dimensional $\mathbb{F}_{q^m}$-subspace of $\mathbb{F}_{q^m}^n$ endowed with the rank distance.

Rank-metric codes

$$d(\mathcal{C}) := d = \min\{\mathrm{wt}_{\mathrm{rk}}(v) : \ v \in \mathcal{C}, v \neq 0\}$$

$\mathcal{C}$ *is an* $[n, k, d]_{q^m/q}$ *code*

Theorem (Singleton Bound - Delsarte 1978)

$$mk \leq \min\{m(n-d+1), n(m-d+1)\}$$

$mk = \min\{m(n-d+1), n(m-d+1)\} \Rightarrow \mathcal{C}$ *Maximum Rank Distance*

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Theorem (Zini, Zullo 2021)

$n := \frac{km}{h+1}$ L_U *linear set of rank* n *in* $\mathrm{PG}(k-1, q^m)$

L_U *maximum h-scattered* $\iff$ $\mathcal{C}_U^{\perp}$ *MRD*

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1. *Francesco's talk*
2. *Giuseppe's talk*

Maximum r-Scattered linear sets in $\mathrm{PG}(r, q^n)$

Consider the linear set

$$\mathbb{U} := \{(x^{q^t}, f_1(x), \ldots, f_r(x)) : x \in \mathbb{F}_{q^n}\}$$

where f_i are q-linearized

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When $L(\mathbb{U})$ is maximum r-scattered?

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Theorem

$L(\mathbb{U})$ is maximum r-scattered $\iff$

$$\det \begin{pmatrix} x_1^{q^t} & f_1(x_1) & \cdots & f_r(x_1) \\ x_2^{q^t} & f_1(x_2) & \cdots & f_r(x_2) \\ \vdots & \vdots & & \vdots \\ x_{r+1}^{q^t} & f_1(x_{r+1}) & \cdots & f_r(x_{r+1}) \end{pmatrix} = 0$$

has only solutions $(a_1, \ldots, a_{r+1})$ such that $a_1, \ldots, a_{r+1}$ are $\mathbb{F}_q$-dependent

Monomial case

$f_i(x) = x^{q^{l_j}}$, $t = 0$, $l_1 < \ldots < l_r$

$$F = \det \begin{pmatrix} X_1 & X_1^{q^{l_1}} & \cdots & X_1^{q^{l_r}} \\ X_2 & X_2^{q^{l_1}} & \cdots & X_2^{q^{l_r}} \\ \vdots & \vdots & & \vdots \\ X_{r+1} & X_{r+1}^{q^{l_1}} & \cdots & X_{r+1}^{q^{l_r}} \end{pmatrix},$$

$$G = \det \begin{pmatrix} X_1 & X_1^{q} & \cdots & X_1^{q^{r}} \\ X_2 & X_2^{q} & \cdots & X_2^{q^{r}} \\ \vdots & \vdots & & \vdots \\ X_{r+1} & X_{r+1}^{q} & \cdots & X_{r+1}^{q^{r}} \end{pmatrix}$$

Theorem

$$\mathbb{U} := \{(x, x^{q^{l_1}}, \ldots, x^{q^{l_r}}) : x \in \mathbb{F}_{q^n}\}$$

$L(\mathbb{U})$ is maximum r-scattered if and only if $F/G = 0$ does not contain $\mathbb{F}_{q^n}$-rational points off $G = 0$.

Theorem (B.-Zhou JCTA, 2020)

$I = \{0, i_1, i_2, \ldots, i_r\}$, $0 < i_1 < \cdots < i_r$ *NOT an arithmetic progression.*

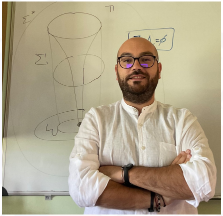
(a) $i_2 - i_0 \neq 2(i_1 - i_0)$;

(b) $i_2 - i_0 = 2(i_1 - i_0)$, $k > 3$ *and* $q \geq 7$;

(c) $i_2 - i_0 = 2(i_1 - i_0)$, $k > 3$, $q = 3, 4, 5$ *and* $i_1 - i_0 > 1$;

(d) $i_2 - i_0 = 2(i_1 - i_0)$, $k > 3$ *and* $q = 2$ *with* $i_1 - i_0 > 2$.

$\exists N$ *such that* $L(\mathbb{U})$ *is not maximum* r*-scattered if* $n > N$

Theorem (B.-Zini-Zullo, IEEE 2023)

$$\mathbb{U} := \{(x^{q^t}, f_1(x), \ldots, f_r(x)) : x \in \mathbb{F}_{q^n}\}$$

$L(\mathbb{U})$ *maximum r-scattered*

- *If* $t = 0$, *and* $(q, \deg_q(f_1(x))) \notin \{(2,2),(2,4),(3,2),(4,2),(5,2)\}$ *then* f_i *is a monomial*
- *If* $t > 0$ *and* $\deg(f_i(x)) > \max\{q^t, \deg(f_1(x))\}$ *for each* $i = 2, \ldots, r$, *then* $f_1(x)$ *is exceptional scattered of index* t.

Tools

$$F(X_1,\dots,X_n,T) = F_0(X_1,\dots,X_n) + F_1(X_1,\dots,X_n)T + \dots + F_d(X_1,\dots,X_n)T^d$$

$$\mathcal{S} : F(X_1,\dots,X_n,T) = 0$$

$F_0(X_1,\dots,X_n)$ has an absolutely irreducible factor non-repeated and $\mathbb{F}_q$-rational

$\Downarrow$

$\mathcal{S}$ contains an absolutely irreducible $\mathbb{F}_q$-rational component

$\mathcal{H}$ hypersurface

$\mathcal{H} \cap \mathcal{S}$ contains an absolutely irreducible non-repeated $\mathbb{F}_q$-rational component

$\Downarrow$

$\mathcal{S}$ contains an absolutely irreducible $\mathbb{F}_q$-rational component

Specific types of RD codes: Minimal RD codes

Definition

$L_{\mathbb{U}}$ linear set of rank n in $\mathrm{PG}(k-1, q^m)$ t-**cutting** $\iff$ $\forall$ $\mathbb{F}_{q^m}$-subspace $H \subset \mathbb{F}_{q^m}^k$ of codimension t $\langle H \cap \mathbb{U} \rangle_{\mathbb{F}_{q^m}} = H$

$t = 1 \implies L_{\mathbb{U}}$ is **(linear) cutting (strong blocking set)**

Theorem

$L_{\mathbb{U}}$ linear set cutting $\iff$ $\mathcal{C}_{\mathbb{U}}$ minimal rank-metric code

Alfarano, Borello, Neri, Ravagnani 2022

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Alfarano, Borello, Neri, Ravagnani 2022

Definition

$[n, k]_{q^m/q}$ code $\mathcal{C}$

$v \in \mathcal{C}$ **minimal codeword** $\iff$ $\forall$ $v' \in \mathcal{C}$, $\sigma^{\mathrm{rk}}(v') \subseteq \sigma^{\mathrm{rk}}(v)$ implies $v' = \alpha v$ for some $\alpha \in \mathbb{F}_{q^m}$

$\mathcal{C}$ is **minimal** if all its codewords are minimal

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Alfarano, Borello, Neri, Ravagnani 2022

See Giuseppe's talk

Specific types of RD codes: Covering RD codes

Definition (Saturating Set)

$S \subset \mathrm{PG}(k-1, q^m)$ ρ-saturating $\iff$ $\forall P \in \mathrm{PG}(k-1, q^m)$ $P \in \langle P_1, \ldots, P_{\rho+1} \rangle$ $P_1, \ldots, P_{\rho+1} \in S$

Theorem

ρ-saturating sets $\iff$ $(\rho + 1)$-covering codes

Specific types of RD codes: Covering RD codes

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Theorem

ρ-saturating sets $\iff$ $(\rho+1)$-covering codes

What if S is a linear set? [Bonini, Borello, and Byrne]

1. construction of saturating linear sets meeting the lower bound
2. link with other properties (scatteredness)
3. When the bound is not tight?

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Indecomposable scattered sequences

  

Definition (B., Marino,Neri, Vicino 2022)

$i_1, i_2, \ldots, i_m \in \mathbb{N}$
$f_1, \ldots, f_s$ linearized q-polynomials over $\mathbb{F}_{q^n}$ in $\underline{X} = (X_1, \ldots, X_m)$

$\mathcal{F} := (f_1, \ldots, f_s)$ is $(\mathcal{I}; h)_{q^n}$-**scattered sequence** of order m

$$\Updownarrow$$

$$\mathbb{U}_{\mathcal{F}} := \left\{ \begin{array}{c} (x_1^{q^{i_1}}, \ldots, x_m^{q^{i_m}}, f_1(\underline{x}), \ldots, f_s(\underline{x})) : \\ \underline{x} = (x_1, \ldots, x_m) \in \mathbb{F}_{q^n}^m \end{array} \right\} \subseteq_{\mathbb{F}_q} \mathbb{F}_{q^n}^{s+m}$$

is maximum h-scattered in $\mathbb{F}_{q^n}^{s+m}$,

Indecomposable scattered sequences

Definition (B. Marino, Neri, Vicino 2022)

$i_1, i_2, \dots, i_m \in \mathbb{N}$

$f_1, \dots, f_s$ linearized q-polynomials over $\mathbb{F}_{q^n}$ in $\underline{X} = (X_1, \dots, X_m)$

$\mathcal{F} := (f_1, \dots, f_s)$ is $(\mathcal{I}; h)_{q^n}$**-scattered sequence** of order m

$$\Updownarrow$$

$$\mathbb{U}_{\mathcal{F}} := \left\{ \begin{array}{c} (x_1^{q^{i_1}}, \dots, x_m^{q^{i_m}}, f_1(\underline{x}), \dots, f_s(\underline{x})) : \\ \underline{x} = (x_1, \dots, x_m) \in \mathbb{F}_{q^n}^m \end{array} \right\} \subseteq_{\mathbb{F}_q} \mathbb{F}_{q^n}^{s+m}$$

is maximum h-scattered in $\mathbb{F}_{q^n}^{s+m}$,

$f(X)$ scattered polynomial of index t $\iff$ $\{(x^{q^t}, f(x)) : x \in \mathbb{F}_{q^n}\} \subseteq_{\mathbb{F}_q} \mathbb{F}_{q^n}^2$ scattered

Sheekey, B.-Zhou

Indecomposable scattered sequences

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f is a $(\{t\}, 1)$-scattered sequence of order 1

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$$\mathbb{U} := \{(x^{q^t}, f_1(x), \ldots, f_r(x)) : x \in \mathbb{F}_{q^n}\}$$ maximum r-scattered $\iff$ $f_1, \ldots, f_r$ $(\{t\}, r)$-scattered sequence of order 1

B.-Zhou, B.-Zini-Zullo

Indecomposable scattered sequences

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is maximum h-scattered in $\mathbb{F}_{q^n}^{s+m}$,

Definition

$(\mathcal{I}; h)_{q^n}$-**scattered sequence** $\mathcal{F} := (f_1, \ldots, f_s)$ of order m is **indecomposable** if it not the direct sum of two smaller scattered sequences

Indecomposable scattered sequences

$$\mathbb{U} := \left\{ \left(x, y, x^q + y^{q^2}, x^{q^2} + y^q + y^{q^2} \right) \;:\; x, y \in \mathbb{F}_{q^4} \right\} \subset \mathbb{F}_{q^4}^4$$

indecomposable $((0,0),1)_{q^4}$-scattered sequence
of order larger than one for $q = 2^{2s+1}$

Indecomposable scattered sequences

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indecomposable $((0,0),1)_{q^4}$-scattered sequence
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Definition

$\alpha, \beta, \gamma \in \mathbb{F}_{q^n}^*$, and $I \neq J \in \mathbb{N}$, $I, J < n-1$

$$\mathbb{U}_{\alpha,\beta,\gamma}^{I,J,n} := \left\{ \left(x, y, x^{q^I} + \alpha y^{q^J}, x^{q^J} + \beta y^{q^I} + \gamma y^{q^J} \right) \; : \; x, y \in \mathbb{F}_{q^n} \right\}.$$

Indecomposable scattered sequences

Theorem

$\gcd(I, J, n) = 1$

$$P^{I,J}_{\alpha,\beta,\gamma}(X) := \begin{cases} X^{q^{J-I}+1} + \gamma X - \alpha\beta, & \text{if } I < J, \\ X^{q^{I-J}+1} + \gamma X^{q^{I-J}} - \alpha\beta, & \text{if } I > J, \end{cases} \tag{1}$$

$P^{I,J}_{\alpha,\beta,\gamma}(X)$ *no roots in* $\mathbb{F}_{q^n} \implies$ $\mathbb{U}^{I,J,n}_{\alpha,\beta,\gamma}$ *exceptional scattered*
$\mathbb{U}^{I,J,n}_{\alpha,\beta,\gamma}$ $(2, 2\max\{I, J\})_q$*-evasive*

Corollary

$P^{I,J}_{\alpha,\beta,\gamma}(X)$ *no roots in* $\mathbb{F}_{q^n}$
$\max\{I, J\} \leq (n-1)/2$
$\implies \mathbb{U}^{I,J,n}_{\alpha,\beta,\gamma}$ *indecomposable cutting exceptional scattered*

Link with algebraic varieties

- $f(x)$ is scattered $\iff$ $rank \begin{pmatrix} x & f(x) \\ y & f(y) \end{pmatrix} = 2$ unless $x = \lambda y$ with $\lambda \in \mathbb{F}_q$

$$\frac{Xf(Y) - Yf(X)}{X^qY - XY^q} = 0 \rightarrow \text{plane curve}$$

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- $(x^{q^I} + \alpha y^{q^J}, x^{q^J} + \beta y^{q^I} + \gamma y^{q^J})$ is $((0,0),1)_{q^4}$-scattered sequence

$$\iff rank \begin{pmatrix} x & y & x^{q^I} + \alpha y^{q^J} & x^{q^J} + \beta y^{q^I} + \gamma y^{q^J} \\ z & t & z^{q^I} + \alpha t^{q^J} & z^{q^J} + \beta t^{q^I} + \gamma t^{q^J} \end{pmatrix} = 2$$

unless $(x, y) = \lambda(z, t)$ with $\lambda \in \mathbb{F}_q$

$$\begin{cases} xt - zy = 0 \\ x(z^{q^I} + \alpha t^{q^J}) - z(x^{q^I} + \alpha y^{q^J}) = 0 \\ x(z^{q^J} + \beta t^{q^I} + \gamma t^{q^J}) - z(x^{q^J} + \beta y^{q^I} + \gamma y^{q^J}) = 0 \end{cases} \rightarrow \text{space curve}$$

Scattered sequences: new constructions and infinite families

 

Definition

$I, J, n, m \in \mathbb{N},\ I < J < n,\ m \geq 3 \qquad \alpha_1, \ldots, \alpha_m \in \mathbb{F}_{q^n}^*$

$$U_{\mathbf{A}}^{I,J} := \{(x_1, \ldots, x_m, x_1^{q^I} + \alpha_2 x_2^{q^J}, \ldots, x_{m-1}^{q^I} + \alpha_m x_m^{q^J}, x_m^{q^I} + \alpha_1 x_1^{q^J}) : x_1, \ldots, x_m \in \mathbb{F}_{q^n}\},$$

where $\mathbf{A} := (\alpha_1, \ldots, \alpha_m)$

 

Theorem (B.-Giannoni-Marino 2023)

Then the set $U_{\mathbf{A}}^{I,J}$ is exceptional scattered and indecomposable... often

See Alessandro's talk

Maximum 2-scattered linear set

Main open problem about maximum h-scattered in $V(r, q^n)$

Do they exist for every admissible values of r, n and $h \geq 2$?

Maximum 2-scattered linear set

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Do they exist for every admissible values of r, n and $h \geq 2$?

$2 \mid rn \Longrightarrow \exists$ *maximum 1-scattered linear sets in* $V(r, q^n)$

Ball, Blokhuis, Lavrauw, 2000

Blokhuis, Lavrauw, 2000

Csajbók, Marino, Polverino, Zullo, 2017

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What about $h = 2$?

1. $r \equiv 0 \pmod 3$ *[Csajbók, Marino, Polverino and Zullo, 2021]*
2. $r = 4$ *and* $n = 3$

Our contribution


$$U := \{(x, y, x^{q^2} + y^q, x^q + y^{q^3}) : x, y \in \mathbb{F}_{q^6}, Tr_{q^6/q^2}(x) = Tr_{q^6/q^2}(y) = 0\}.$$

Proposition

1. *U is 1-scattered*
2. *$W \leq \mathbb{F}_{q^6}^4$ q^2-rational* $\Longrightarrow$ $\dim_{\mathbb{F}_q}(W \cap U)$ *even*
3. *$W \leq \mathbb{F}_{q^6}^4$ q^2-rational hyperplane* $\Longrightarrow$ $\dim_{\mathbb{F}_q}(W' \cap U) \leq 2$ $\forall W' \leq W$ *of dimension 2 and not q^2-rational*

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Theorem

U is 2-scattered

Our contribution


$$U := \{(x, y, x^{q^2} + y^q, x^q + y^{q^3}) : x, y \in \mathbb{F}_{q^6}, Tr_{q^6/q^2}(x) = Tr_{q^6/q^2}(y) = 0\}.$$

Theorem

U is 2-scattered

1. Case by case analysys
2. Polynomial systems
3. No curves nor varieties :-(
4. $\Longrightarrow \exists$ 2-scattered $V(r, q^6)$ for $r = 4$ and $r \geq 6$

Open questions, future directions

1. Determine whether U is 2-saturating
2. Construct h-scattered linear sets using sequences
3. Find the automorphism groups
4. Find a lower bound on the number of inequivalent examples
5. Generalize U

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THANK YOU

FOR YOUR ATTENTION