How To (Not) Decode In the Rank Metric

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Joint work with

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15 February 2024

Contents

- Generic decoding: **Rank Syndrome Decoding Problem (RSDP)**

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- Focus on **combinatorial/geometric** attacks, might also help hybrid attacks
- Two potential approaches leading to *open problems on rank-metric codes*™

Should you study/care about Rank Syndrome Decoding Problem (RSDP)?

Decision tree:

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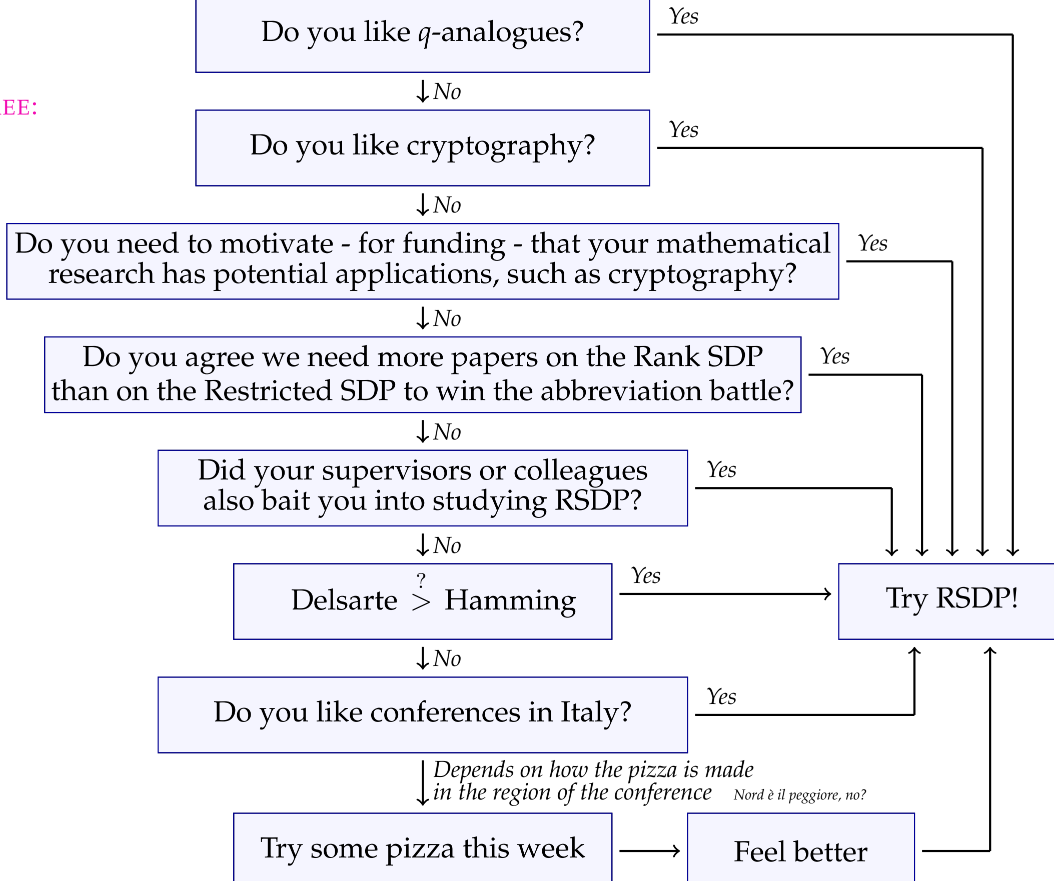

Part

Problem introduction

Definition (SDP)

Given a parity-check matrix $\mathbf{H}$ *of a code* $\mathcal{C}$, *syndrome vector* $\mathbf{s}$, *target weight* t, *find an error vector* $\mathbf{e}$ *such that*

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Slight abuse of notation: $\mathbf{e}$ for vector/matrix in $\mathbb{F}_{q^m}^n \cong \mathbb{F}_q^{m \times n} \cong \mathbb{F}_q^{mn}$

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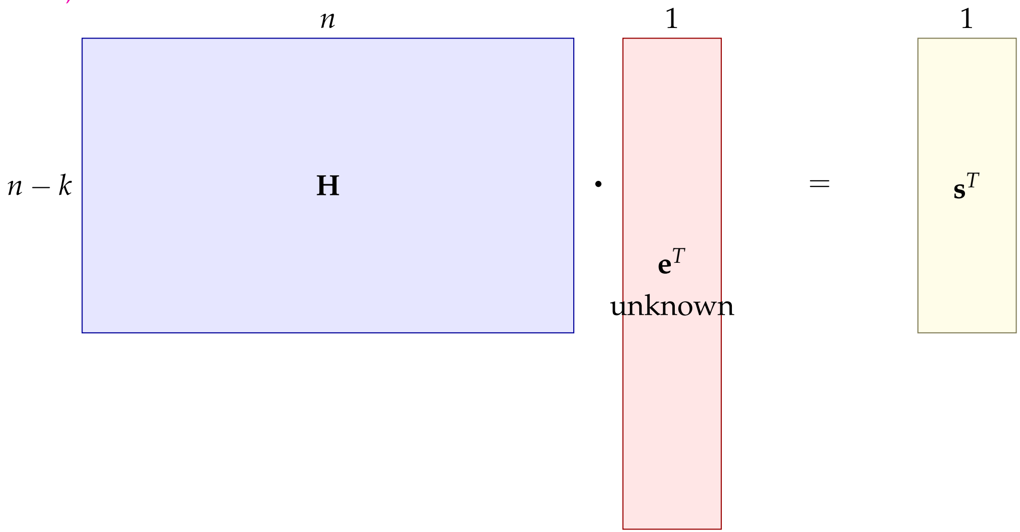
- $\boxed{\lim_{n\to\infty} \frac{1}{n^2} \log_q(complexity)}$ $\boxed{= \min\{M,1\}RT}$

Can we improve asymptotically?

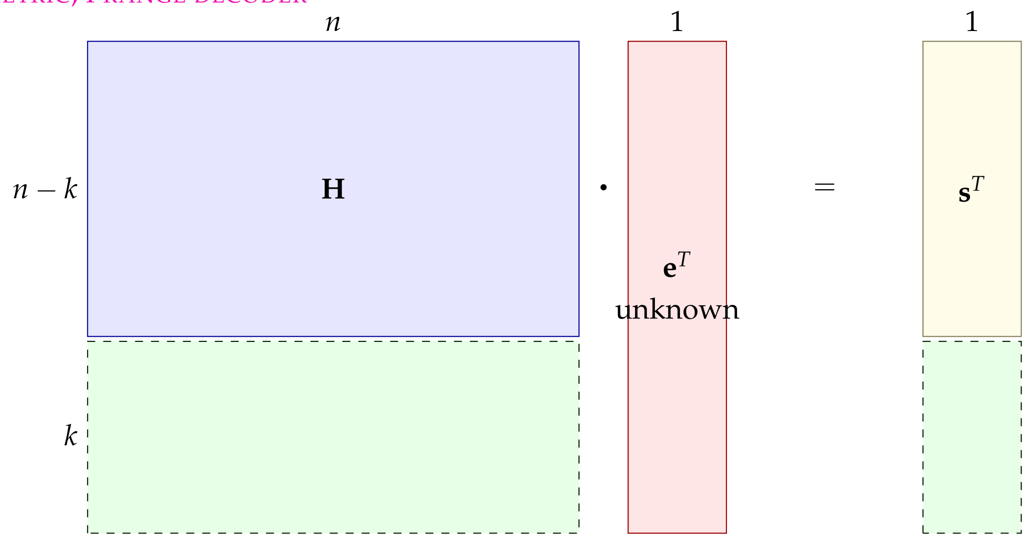
Part I

First approach

Hamming metric, Prange decoder



Hamming metric, Prange decoder



Idea: $\hat{\mathbf{e}}^T := \left(\frac{\mathbf{H}}{\textit{something}}\right)^{-1} \cdot \left(\frac{\mathbf{s}^T}{\textit{something else}}\right)$, try many times until $wt_H(\hat{\mathbf{e}}) \leq t \quad \rightarrow \quad \hat{\mathbf{e}} = \mathbf{e}$

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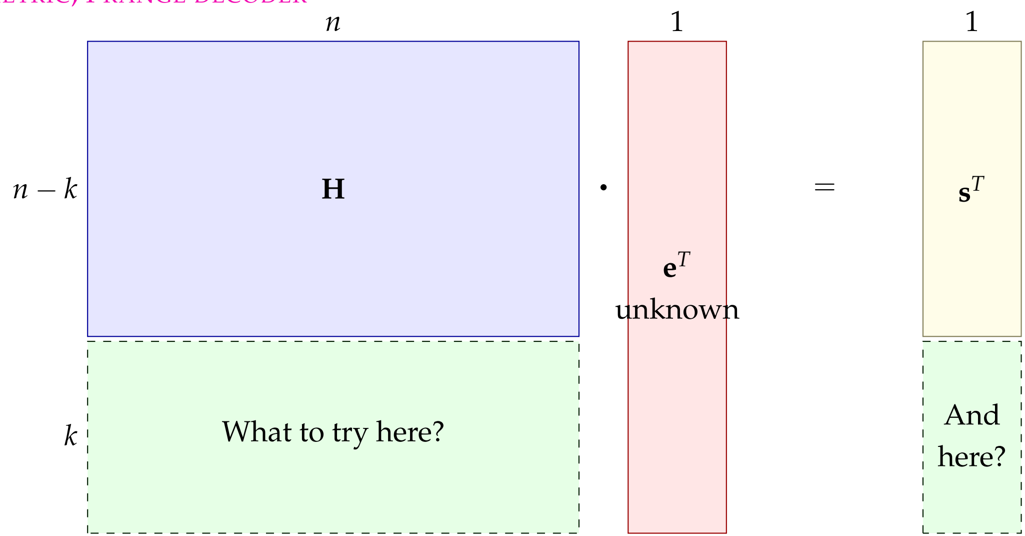

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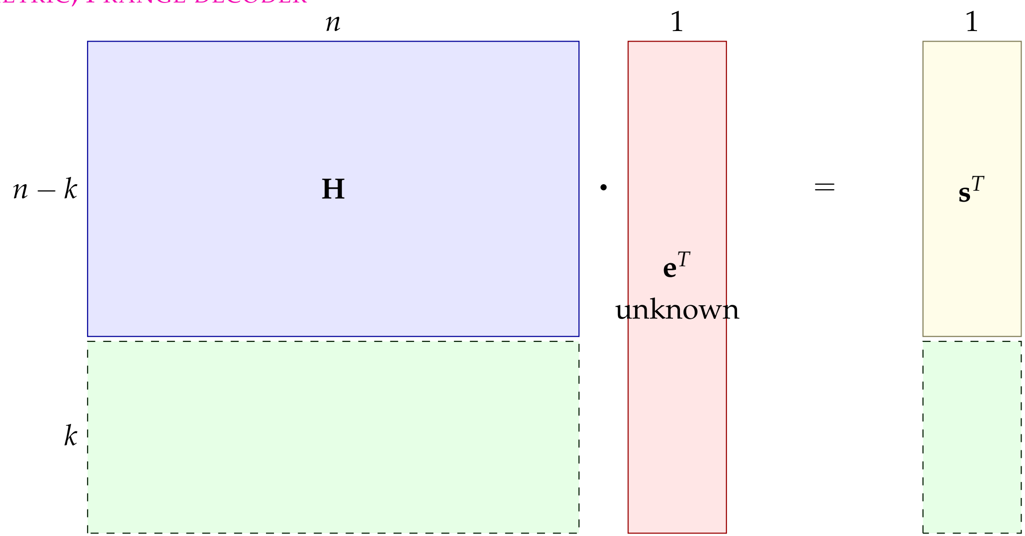

One answer: LEFT: rows that are orthogonal to many $\mathbf{e}$'s with $wt_H(\mathbf{e}) \leq t$, RIGHT: 0's.

Hamming metric, Prange decoder

$$
\begin{array}{cc}
 & n \\
n-k & \mathbf{H} \\
k & \begin{matrix} 1 & & & -v_{i_1}- & & \\ & 1 & & -v_{i_2}- & & \\ & & 1 & & & \\ & & & \vdots & & \\ & & & -v_{i_{k-1}}- & 1 & \\ & & & -v_{i_k}- & & 1 \end{matrix}
\end{array}
\cdot
\begin{array}{c} 1 \\ \mathbf{e}^T \\ \text{unknown} \end{array}
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$$

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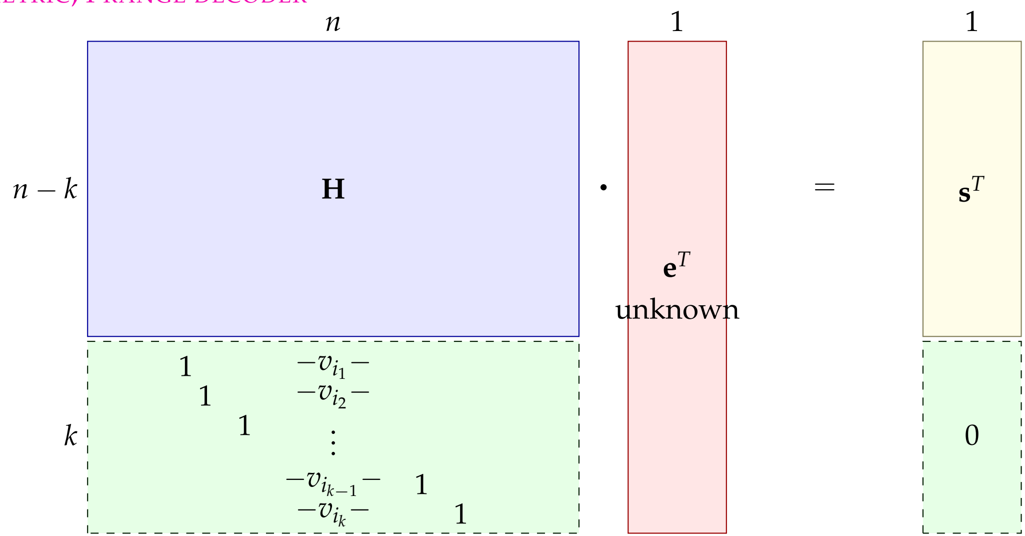
Try space spanned by k standard basis vectors v_i

HAMMING METRIC, PRANGE DECODER



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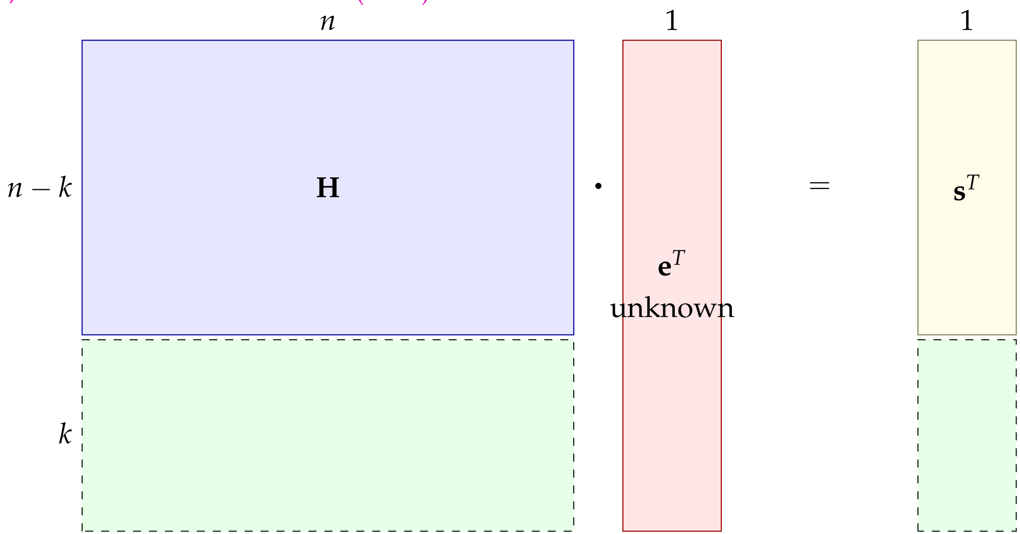
Try space spanned by k standard basis vectors $v_i \rightarrow \binom{n}{k}$ choices, $\binom{n-t}{k}$ successful if all $i \in \mathrm{Support}(\mathbf{e})^c$

Rank metric, Gaborit-Ruatta-Schrek (GRS)



LEFT: rows that are orthogonal to many $\mathbf{e}$'s with $wt_R(\mathbf{e}) \leq t$, RIGHT: 0's.

Rank metric, Gaborit-Ruatta-Schrek (GRS)

$$\begin{array}{cc} & n \\ n-k & \mathbf{H} \\ k & \begin{matrix} -v_1- \\ -v_2- \\ \vdots \\ -v_{k-1}- \\ -v_k- \end{matrix} \end{array} \cdot \begin{array}{c} 1 \\ \mathbf{e}^T \\ \text{unknown} \end{array} = \begin{array}{c} 1 \\ \mathbf{s}^T \\ 0 \end{array}$$

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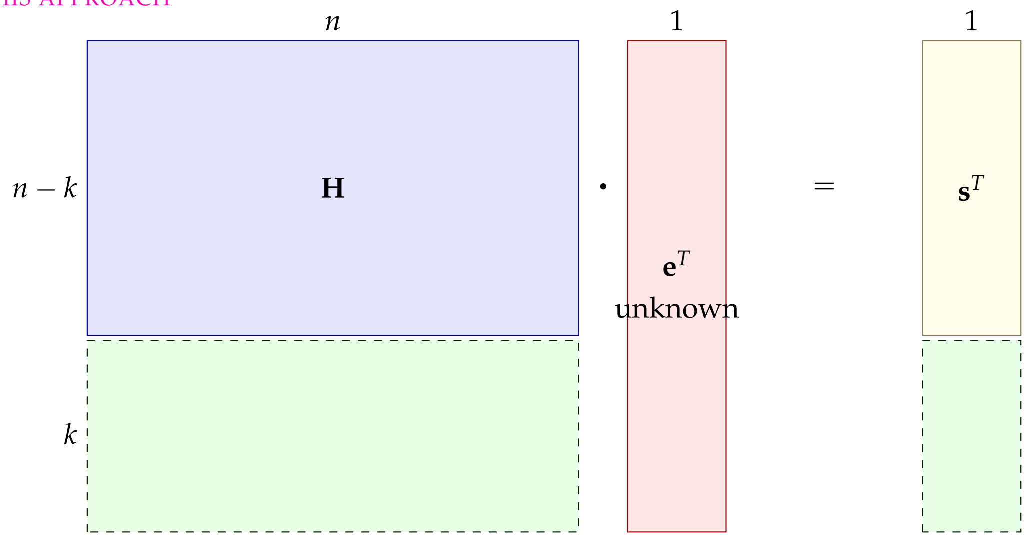
Try space spanned by k indep. vectors $v_i \in \mathbb{F}_q^n \quad \rightarrow \quad \begin{bmatrix} n \\ k \end{bmatrix}_q$ choices, $\begin{bmatrix} n-t \\ k \end{bmatrix}_q$ successful if all $v_i \in \mathrm{RowSpan}_{\mathbb{F}_q}(\mathbf{e})^\perp$

Bounds to this approach



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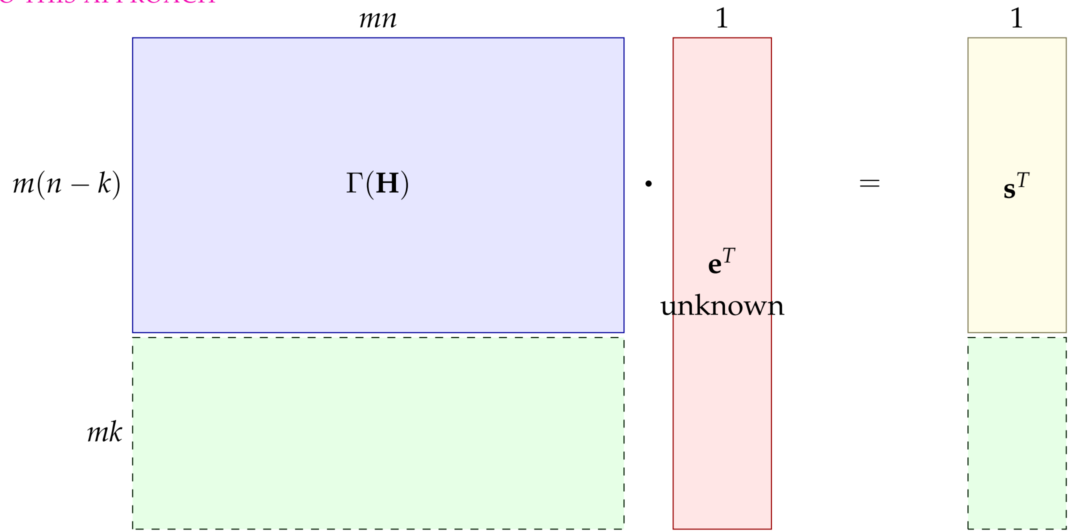

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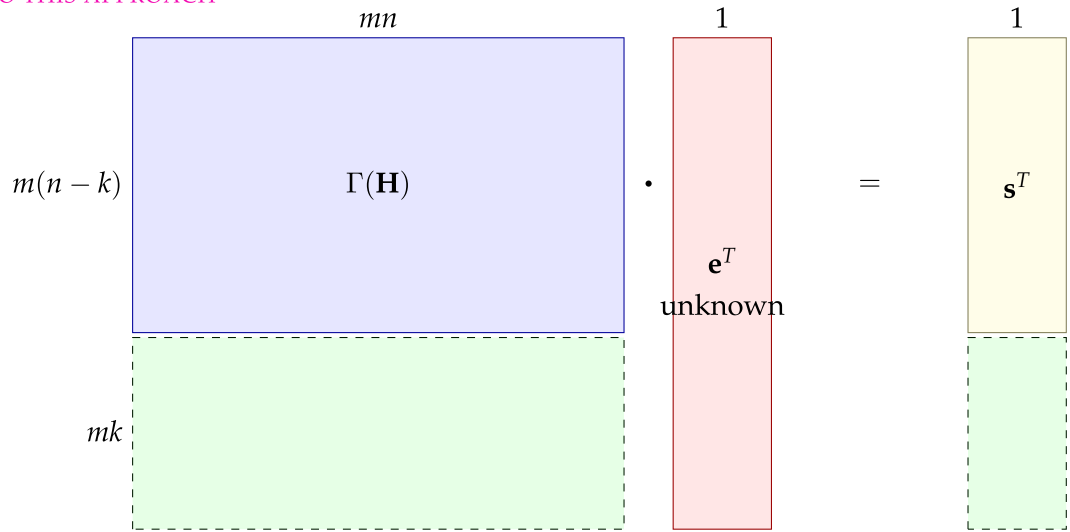

For generality: transform into $\mathbb{F}_q$-linear problem using $\mathbb{F}_q$-basis Γ of $\mathbb{F}_{q^m}$

(Additional step: can use parity-check matrix of $\mathcal{C} + \langle \mathbf{e} \rangle$ instead of $\mathcal{C}$)

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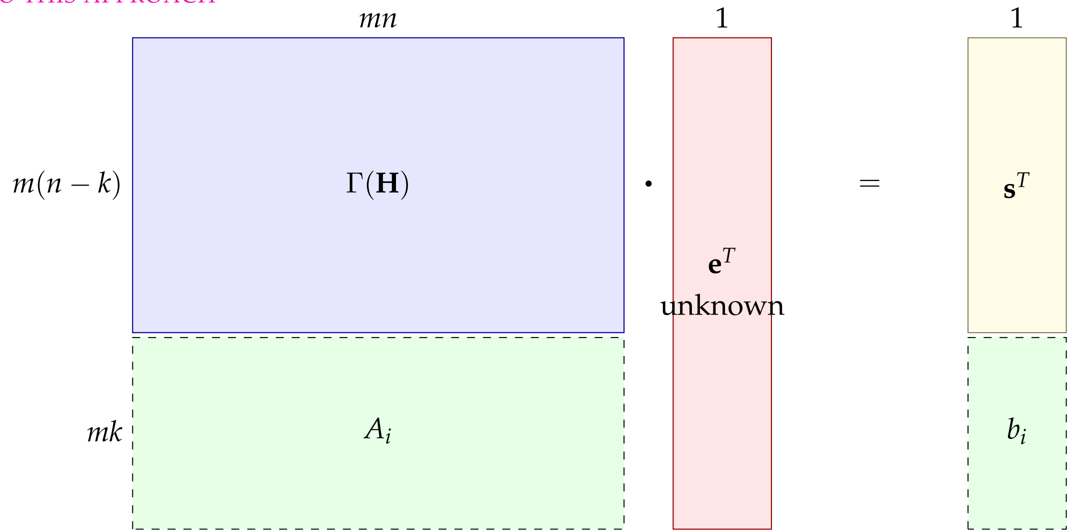
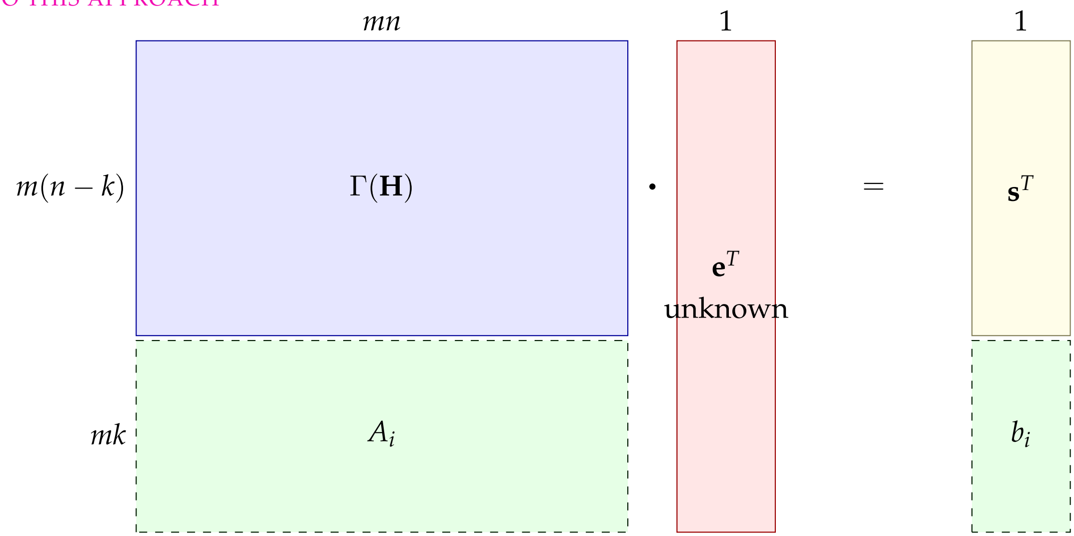


Test set $\mathcal{T} = \{(A_i, b_i) \mid i \in \mathcal{I}\}$.

Bounds to this approach



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Lemma

$$\boxed{N_{\mathcal{T}} \geq \frac{|\{\mathbf{e} \mid wt(\mathbf{e}) = t\}|}{\max_{(A,b)\in\mathcal{T}} |\{\mathbf{e} \mid wt(\mathbf{e}) = t,\ A\mathbf{e}^T = b\}|}}$$

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Open Problem

v.1: Let $A \in \mathbb{F}_q^{mk \times mn}$, $b \in \mathbb{F}_q^{mk}$. Give an upper-bound on the maximum cardinality of

$$\{\mathbf{e} \mid wt_R(\mathbf{e}) = t,\ A\mathbf{e}^T = b\}.$$

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Open Problem

v.2: Let $\mathcal{S} = \{x \mid Ax = b\} \subset \mathbb{F}_q^{mn}$ *be a translated subspace/coset with* $|\mathcal{S}| = q^{m(n-k)}$*. Give an upper-bound on the maximum cardinality of*

$$\mathcal{S} \ \cap\ \{\mathbf{e} \mid wt_R(\mathbf{e}) = t\}.$$

BOUNDS TO THIS APPROACH

Test set $\mathcal{T} = \{(A_i, b_i) \mid i \in \mathcal{I}\}$. Expected number of pairs to try is $N_{\mathcal{T}}(\mathbf{e}) := \dfrac{|\mathcal{T}|}{|\{(A_i, b_i) \in \mathcal{T} \mid A_i\mathbf{e}^T = b_i\}|}$
Average of $N_{\mathcal{T}}(\mathbf{e})$ over all $\mathbf{e}$'s of weight t: $N_{\mathcal{T}}$ = expected run-time/complexity.

Lemma

$$\boxed{N_{\mathcal{T}} \geq \frac{|\{\mathbf{e} \mid wt(\mathbf{e}) = t\}|}{\max_{(A,b)\in\mathcal{T}} |\{\mathbf{e} \mid wt(\mathbf{e}) = t,\, A\mathbf{e}^T = b\}|}}$$

Open Problem

v.3: Let $\mathcal{S} = \mathcal{D}\,(+v) \subset \mathbb{F}_q^{m\times n}$ be a (translated) matrix code with $|\mathcal{S}| = q^{m(n-k)}$. Give an upper-bound on

$$W_t(\mathcal{S}) := |\{X \in \mathcal{S} \mid wt_R(X) = t\}|,$$

in terms of q, m, n, k, t that holds for all $\mathcal{S}$.

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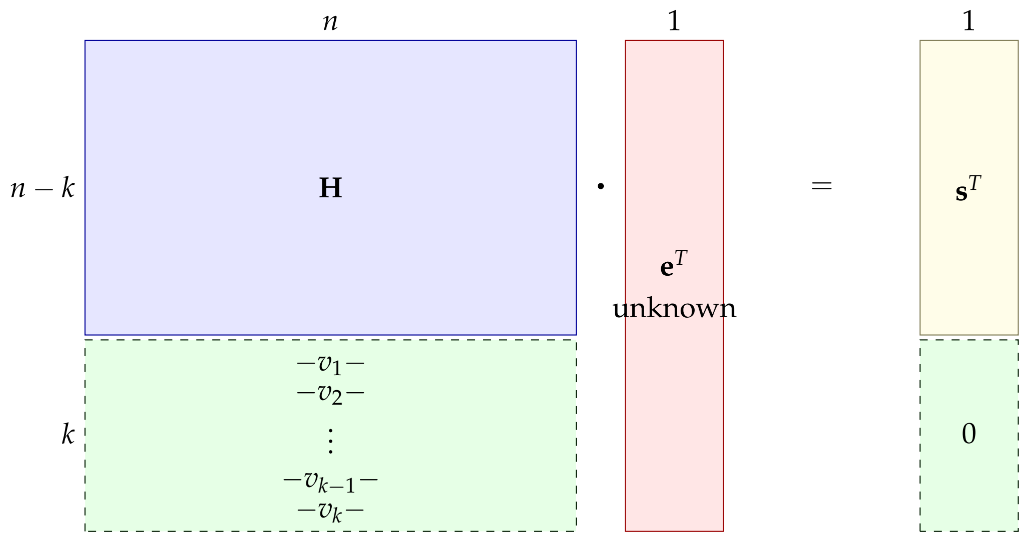
Any upper bound on $W_t(\mathcal{S})$ will imply a **lower bound on the complexity** possible with this approach.

Part II

Second Approach

GRS-like decoder



Try space spanned by k indep. vectors $v_i \in \mathbb{F}_q^n \quad \rightarrow \quad \begin{bmatrix} n \\ k \end{bmatrix}_q$ choices, $\begin{bmatrix} n-t \\ k \end{bmatrix}_q$ successful if $\mathrm{Span}(v_i) \subseteq \mathrm{RowSpan}_{\mathbb{F}_q}(\mathbf{e})^\perp$

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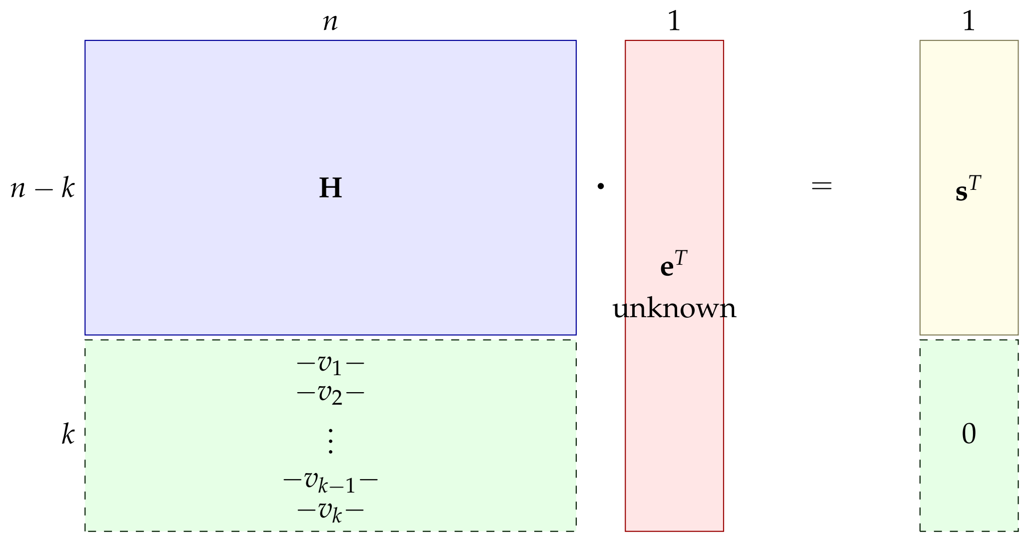

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Equivalent: choosing an $(n-k)$-dim space $W = \mathrm{Span}(v_i)^\perp$, successful if $W \supseteq \mathrm{RowSpan}_{\mathbb{F}_q}(\mathbf{e})$

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Idea: learn from each failed solution $\rightarrow$ requires **magic!**

Open Problem (Magic[1] Step)

[1] Official terminology by V. Weger (2023)

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   If not, and MagicStep outputs "*trivial*", go back to 1.
   If not, and MagicStep outputs "*non-trivial*", let $U := \mathrm{RowSpan}(\mathbf{x})$.
      Go back to 1. but narrow down the search to only spaces W with $W \cap U$ non-trivial.

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For some trivial magic steps (slow & large p or fast & small p), we get GRS up to polynomial factor.

Can we do better?

Thank you for listening! and for your solutions to:

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Open Problem (Weight Upper-Bound)

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