Non-linear MRD codes from cones over exterior sets

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joint work with Nicola Durante and Giovanni Longobardi

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When this bound is achieved, $\mathcal{C}$ is called an $(m,n,q;d)$-**maximum rank distance code**, or $(m,n,q;d)$-**MRD code**.

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Two rank distance codes $\mathcal{C}, \mathcal{C}' \subseteq \mathbb{F}_q^{m\times n}$ are called **equivalent** if there exist $P \in GL(m,q), Q \in GL(n,q), R \in \mathbb{F}_q^{m\times n}$ and $\rho \in Aut(\mathbb{F}_q)$ s.t.

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The code $\mathcal{C}' \subseteq \mathbb{F}_q^{(m-u)\times n}$ obtained from $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ by deleting the last u rows, $1 \leq u \leq m-1$, is called a **punctured code** of $\mathcal{C}$. If $\mathcal{C}$ is an MRD code then $\mathcal{C}'$ is MRD as well.

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The set

$$\mathcal{L}_{n,q,\sigma} = \left\{ \sum_{i=0}^{n-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n} \right\}$$

with the usual sum, the scalar multiplication by an element of $\mathbb{F}_{q^n}$, and the map composition modulo $x^{q^n} - x$ is an algebra isomorphic to the algebra $\mathbb{E} = \mathrm{End}(\mathbb{F}_{q^n}, \mathbb{F}_q)$ of the endomorphisms of $\mathbb{F}_{q^n}$ seen as $\mathbb{F}_q$-vector space.

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$\mathcal{C}$ and $\mathcal{C}'$ are **equivalent** or **adjointly equivalent** if there exist $f, g, h \in \mathcal{L}_{n,q,\sigma}$, with f and g permutation polynomials, and $\rho \in \mathrm{Aut}(\mathbb{F}_q)$ s.t.

$$\mathcal{C}' = \{f \circ \alpha^\rho \circ g + h : \alpha \in \mathcal{C}\} \text{ or } \mathcal{C}' = \{f \circ \alpha^\rho \circ g + h : \alpha \in \mathcal{C}^\top\},$$

where the automorphism ρ acts only over the coefficients of a polynomial α.

Let V be a v-dimensional $\mathbb{F}_{q^n}$-vector space, let $\mathrm{PG}(v-1, q^n) = \mathrm{PG}(V, \mathbb{F}_{q^n})$ and let $U \subseteq V$ be an $\mathbb{F}_q$-vector space of V.

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$$\Omega = \left\{ \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \in \mathrm{PG}(v-1, q^n) : \mathbf{u} \in U \setminus \{\mathbf{0}\} \right\}$$

is an $\mathbb{F}_q$-**linear set of rank** $u = \dim_{\mathbb{F}_q} U$.

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We will consider the (canonical) subgeometry of $\mathrm{PG}(n-1,q^n)$

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A subspace S of $\mathrm{PG}(n-1, q^n)$ is called a **subspace of** Σ if

$$\dim_{\mathbb{F}_{q^n}} S = \dim_{\mathbb{F}_q}(S \cap \Sigma).$$

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$$\Phi : (x_1, \ldots, x_m) \in \mathbb{F}_{q^n}^m \mapsto (x_{1,0}, \ldots, x_{1,n-1}, \ldots, x_{m,0}, \ldots, x_{m,n-1}) \in \mathbb{F}_q^{mn},$$

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1. Φ maps h-dim. subspaces of $\mathbb{F}_{q^n}^m$ to hn-dim. subspaces of $\mathbb{F}_q^{mn}$.
2. Φ induces a map $\Phi' : \mathrm{PG}(m-1, q^n) \longrightarrow \mathrm{PG}(mn-1, q)$ sending $(h-1)$-dim. proj. subspaces to $(hn-1)$-dim. proj. subspaces.

The **Segre variety** $\mathcal{S}_{m-1,n-1}$ of $\mathrm{PG}(\mathbb{F}_q^{m\times n}) = \mathrm{PG}(mn-1, q)$, $m \leq n$, is the set of all points $\langle M \rangle \in \mathrm{PG}(\mathbb{F}_q^{m\times n})$ s.t. $\mathrm{rk} M = 1$.

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Theorem (N. Durante, G.G.G., G. Longobardi - 202x)

Let $\mathcal{A} \subseteq \mathrm{PG}(r-1,q)$ such that $\langle \mathcal{A}\rangle = \mathrm{PG}(r-1,q)$. Let $\mathcal{E} \subseteq \mathrm{PG}(r-1,q)$ be an exterior set with respect to $\Omega_h(\mathcal{A})$, $0 \leq h \leq r-1$. Then

$$|\mathcal{E}| \leq \frac{q^{r-h-1}-1}{q-1}.$$

Given M, N two sets of points of $\mathrm{PG}(r-1, q)$, with $M \cap N = \emptyset$, we will denote by $\mathcal{K}(M, N)$ the cone with vertex M and base N.

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Corollary (N. Durante, G.G.G., G. Longobardi - 202x)

Let $\mathcal{A} \subseteq \mathrm{PG}(r-1,q)$ such that $\langle \mathcal{A} \rangle = \mathrm{PG}(t-1,q)$, $1 \leq t < r$, and let $\mathcal{E} \subseteq \mathrm{PG}(r-1,q)$ be an exterior set with respect to $\Omega_h(\mathcal{A})$, $0 \leq h \leq t-1$. Then $\mathcal{E}$ is contained in a cone $\mathcal{K} = \mathcal{K}(S_{r-t-1}, \bar{\mathcal{E}})$ with base $\bar{\mathcal{E}} = \mathcal{E} \cap \langle \mathcal{A} \rangle$ and vertex an $(r-t-1)$-dimensional subspace S_{r-t-1} complementary with $\langle \mathcal{A} \rangle$. Moreover,

$$|\mathcal{E}| \leq \frac{q^{r-h-1}-1}{q-1}.$$

Given M, N two sets of points of $\mathrm{PG}(r-1,q)$, with $M \cap N = \emptyset$, we will denote by $\mathcal{K}(M,N)$ the cone with vertex M and base N.

Corollary (N. Durante, G.G.G., G. Longobardi - 202x)

Let $\mathcal{A} \subseteq \mathrm{PG}(r-1,q)$ such that $\langle \mathcal{A} \rangle = \mathrm{PG}(t-1,q)$, $1 \leq t < r$, and let $\mathcal{E} \subseteq \mathrm{PG}(r-1,q)$ be an exterior set with respect to $\Omega_h(\mathcal{A})$, $0 \leq h \leq t-1$. Then $\mathcal{E}$ is contained in a cone $\mathcal{K} = \mathcal{K}(S_{r-t-1}, \bar{\mathcal{E}})$ with base $\bar{\mathcal{E}} = \mathcal{E} \cap \langle \mathcal{A} \rangle$ and vertex an $(r-t-1)$-dimensional subspace S_{r-t-1} complementary with $\langle \mathcal{A} \rangle$. Moreover,

$$|\mathcal{E}| \leq \frac{q^{r-h-1}-1}{q-1}.$$

When this bound is achieved the set $\mathcal{E}$ will be called **maximum exterior set**.

The image of $\Omega_h(\Sigma_{m,n}) \subseteq \mathrm{PG}(m-1, q^n)$ under the field reduction is the h-secant variety $\Omega_h(\mathcal{S}_{m-1,n-1})$ of the points whose representative matrices in $\mathbb{F}_q^{m \times n}$ have rank at most $h+1$.

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Theorem

Let $\mathcal{E} \subseteq \mathrm{PG}(m-1, q^n)$ be an exterior set with respect to $\Omega_h(\Sigma_{m,n})$ and denote by $\mathcal{E}'$ the image of $\mathcal{E}$ under the field reduction. Then, the set

$$\mathcal{C} = \{\rho M : \langle M\rangle_{\mathbb{F}_q} \in \mathcal{E}', \rho \in \mathbb{F}_q\}$$

is an $(m, n, q; h+2)$-RD code closed under $\mathbb{F}_q$-multiplication. In addition, if $\mathcal{E}$ is maximum then $\mathcal{C}$ is MRD.

Let A and B be two distinct points of $\mathrm{PG}(d, q^n)$, $d \geq 2$, $\mathcal{S}_A$ and $\mathcal{S}_B$ be the stars of lines (pencils of lines if d $= 2$) through A and B, respectively.

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Theorem (G. Donati, N. Durante - 2018)

Any C_F^σ-set is projectively equivalent to the set

$$\mathcal{X} = \{A, B\} \cup \bigcup_{a \in \mathbb{F}_q^*} \mathcal{X}_a,$$

- $A = (0, \ldots, 0, 1)$, $B = (1, 0, \ldots, 0)$ *vertices of $\mathcal{X}$*
- $\mathcal{X}_a = \{(1, t, t^{\sigma+1}, \ldots, t^{\sigma^{d-1}+\ldots+\sigma+1}) : \mathrm{N}_{q^n/q}(t) = a\}$ *components of $\mathcal{X}$*

Any component $\mathcal{X}_a$ is a scattered $\mathbb{F}_q$-linear set of rank n. In particular, $\mathcal{X}_a \cong \mathrm{PG}(n-1, q)$.

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$$AB = \{A, B\} \cup \bigcup_{a \in \mathbb{F}_q^*} J_a$$

where any J_a is a scattered $\mathbb{F}_q$-linear set of rank n defined by

$$J_a = \{(1, 0, \ldots, 0, (-1)^{d+1} t) : \mathrm{N}_{q^n/q}(t) = a\}.$$

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Note that

$$\mathcal{X}_1 = \{(x, x^\sigma, \dots, x^{\sigma^d}) : x \in \mathbb{F}_{q^n}^*\} \cong \mathrm{PG}(n-1, q)$$

and let $\Pi \cong \mathrm{PG}(d, q)$ be a subgeometry of $\mathcal{X}_1$.

Theorem (G. Donati, N. Durante - 2018)

For any $T \subseteq \mathbb{F}_q^*$, $1 \in T$, *the set*

$$\mathcal{E} = \left(\mathcal{X} \setminus \bigcup_{a \in T} \mathcal{X}_a \right) \cup \bigcup_{a \in T} J_a$$

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Corollary (G. Donati, N. Durante - 2018)

To the set $\mathcal{E}$ *corresponds a* $(d+1, n, q; d)$*-MRD code* $\mathcal{C}$, *with* $q > 2, n \geq 3$ *and* $2 \leq d \leq n-1$.

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$d = n-1, n \geq 3 \Longrightarrow$ A. Siciliano, N. Durante (2018)

Our construction:

Let $\Sigma \cong \mathrm{PG}(n-1, q)$ be a canonical subgeometry of $\mathrm{PG}(n-1, q^n)$ and consider a $(k-3)$-subspace Λ^* disjoint from Σ and Λ an $(n-k+1)$-subspace of $\mathrm{PG}(n-1, q^n)$ disjoint from Λ^*.

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Definition (G. Lunardon - 2017)

Let $\Gamma = p_{\Lambda^*,\Lambda}(\Sigma)$ be an $\mathbb{F}_q$-linear set of rank n. It is called $(n-k+1)$-**embedding of** Σ if any $(n-k+1)$-subspace of Σ is disjoint from Λ^*.

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Theorem (N. Durante, G.G.G., G. Longobardi - 202x)

Let $\Sigma \cong \mathrm{PG}(n-1,q)$ be a canonical subgeometry of $\mathrm{PG}(n-1,q^n)$ and let Λ^* and Λ be subspaces of $\mathrm{PG}(n-1,q^n)$ of dimension $k-3$ and $n-k+1$, respectively, such that $\Lambda^* \cap \Sigma = \emptyset = \Lambda^* \cap \Lambda$. Let $\Gamma = p_{\Lambda^*,\Lambda}(\Sigma)$ be an $(n-k+1)$-embedding of Σ. Let $\mathcal{E} \subseteq \Lambda$ be a (maximum) exterior set with respect to $\Omega_{n-k-1}(\Gamma)$. Then $\mathcal{K} = \mathcal{K}(\Lambda^*, \mathcal{E})$ is a (maximum) exterior set with respect to $\Omega_{n-k-1}(\Sigma)$.

In $\mathrm{PG}(n-1, q^n)$, let

$$\Lambda^* : X_0 = X_1 = \ldots = X_{n-k+1} = 0$$

$$\Lambda : X_{n-k+2} = X_{n-k+3} = \ldots = X_{n-1} = 0$$

be disjoint subspaces of dimension $k-3$ and $n-k+1$, respectively.

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$$\Gamma = p_{\Lambda^*,\Lambda}(\Sigma) = \{(\alpha, \alpha^{\sigma}, \ldots, \alpha^{\sigma^{n-k+1}}, 0, \ldots, 0) : \alpha \in \mathbb{F}_{q^n}^*\}.$$

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Let $\mathcal{X} = \bigcup_{a \in \mathbb{F}_q^*} \mathcal{X}_a \cup \{A, B\}$ be a C_F^σ-set of Λ where

$$A = (\underbrace{0, \ldots, 0}_{n-k+1}, 1, \ldots, 0), \quad B = (1, 0, \ldots, 0),$$

$$\mathcal{X}_a = \{(1, t, t^\sigma, \ldots, t^{\sigma^{n-k}+\ldots+\sigma+1}, 0, \ldots, 0) : \mathrm{N}_{q^n/q}(t) = a\}.$$

For any $T \subseteq \mathbb{F}_q^*$, $1 \in T$, the set

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where

$$J_a = \left\{ (1, \underbrace{0, \ldots, 0}_{n-k}, (-1)^{n-k} t, 0, \ldots, 0) : \mathrm{N}_{q^n/q}(t) = a \right\}$$

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Corollary (N. Durante, G.G.G., G. Longobardi - 202x)

For any $T \subseteq \mathbb{F}_q^*$, $1 \in T$, the set $\mathcal{K} = \mathcal{K}(\Lambda^*, \mathcal{E})$ is a maximum exterior set with respect to $\Omega_{n-k-1}(\Sigma)$. Then the set

$$\mathcal{C}_{\sigma,T} = \{\rho M : \langle M \rangle_{\mathbb{F}_q} \in \mathcal{K}, \rho \in \mathbb{F}_q\}$$

is an $(n, n, q; d = n - k + 1)$-MRD code.

Known non-linear MRD codes:

- A. Cossidente, G. Marino, F. Pavese (2016)
- S. Siciliano, N. Durante (2018)
- G. Donati, N. Durante (2018): it is the punctured code $\mathcal{C}'_{\sigma,T} \subseteq \mathbb{F}_q^{(n-k+2)\times n}$ obtained from $\mathcal{C}_{\sigma,T}$ by deleting the last $(k-2)$ rows.
- K. Otal, F. Özbudak (2018)

The code $\mathcal{C}_{\sigma,T}$ in terms of σ-linearized polynomials is given by the union of the sets

$$\left\{\sum_{i=0}^{d}\lambda\alpha^{\sigma^i}\xi^{\frac{\sigma^i-1}{\sigma-1}}x^{\sigma^i}+\sum_{i=d+1}^{n-1}\beta_i x^{\sigma^i} : \lambda,\alpha,\beta_i\in\mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\xi)\in\mathbb{F}_q^*\setminus T\right\}$$

$$\left\{\lambda\alpha x+(-1)^{d+1}\lambda\alpha^{\sigma}\eta x^{\sigma^d}+\sum_{i=d+1}^{n-1}\beta_i x^{\sigma^i} : \lambda,\alpha,\beta_i\in\mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\eta)\in T\right\}$$

$$\left\{\alpha x^{\sigma^d}+\sum_{i=d+1}^{n-1}\beta_i x^{\sigma^i} : \alpha,\beta_i\in\mathbb{F}_{q^n}\right\}\cup\left\{\alpha x+\sum_{i=d+1}^{n-1}\beta_i x^{\sigma^i} : \alpha,\beta_i\in\mathbb{F}_{q^n}\right\}$$

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Note that the code $\mathcal{C}_{\sigma,T}$ is closed under $\mathbb{F}_{q^n}$-multiplication.

Let $1 \leq k \leq n$, the set

$$\mathcal{G}_{k,\sigma} = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n} \right\}$$

is a linear $(n, n, q; n-k+1)$-MRD code called **generalized Gabidulin code**.

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$$\mathcal{U} = \left\{ \alpha x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha, \beta_i \in \mathbb{F}_{q^n} \right\} = \left\{ f \circ x^{\sigma^d} : f \in \mathcal{G}_{k-1,\sigma} \right\}$$

$$\mathcal{V} = \left\{ \alpha x + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha, \beta_i \in \mathbb{F}_{q^n} \right\} = \left\{ f \circ x^{\sigma^{d+1}} : f \in \mathcal{G}_{k-1,\sigma} \right\}$$

contained in $\mathcal{C}_{\sigma,T}$ are equivalent to $\mathcal{G}_{k-1,\sigma}$.

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$$\mathcal{C}^{(1)}_{n,k,\sigma,I} = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\alpha_0) \in I \right\}$$

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then $\mathcal{C}_{n,k,\sigma,I} = \mathcal{C}^{(1)}_{n,k,\sigma,I} \cup \mathcal{C}^{(2)}_{n,k,\sigma,I}$ is an $(n, n, q; n-k+1)$-MRD code.

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Corollary (K. Otal, F. Özbudak - 2018)

1. *If $q = 2$ or $I \in \{\emptyset, \{0\}, \mathbb{F}_q^*, \mathbb{F}_q\}$, then $\mathcal{C}_{n,k,\sigma,I}$ is equivalent to a generalized Gabidulin code.*
2. *If $q > 2$ and $I \notin \{\emptyset, \{0\}, \mathbb{F}_q^*, \mathbb{F}_q\}$, then $\mathcal{C}_{n,k,\sigma,I}$ is not an affine code (i.e., not a translated version of an additively closed code)*

Theorem (N. Durante, G.G.G., G. Longobardi - 202x)

If $q = 2$ or $T = \mathbb{F}_q^*$ and $I \in \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, then the codes of type $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are both equivalent to a $\mathcal{G}_{k,\sigma}$.

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The non-linear $(n, n, q; n-k+1)$-MRD code $\mathcal{C}_{n,k,\sigma,I}$ contains the set

$$\mathcal{W} = \left\{ \sum_{i=1}^{k-1} \gamma_i x^{\sigma^i} : \gamma_i \in \mathbb{F}_{q^n} \right\} = \left\{ f \circ x^{\sigma} : f \in \mathcal{G}_{k-1,\sigma} \right\}.$$

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Theorem (N. Durante, G.G.G., G. Longobardi - 202x)

Let $I \notin \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, the code $\mathcal{C}_{n,k,\sigma,I}$ contains a unique subspace equivalent to $\mathcal{G}_{k-1,\sigma}$.

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Let $I \notin \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, $1 \in T \subseteq \mathbb{F}_q^*$ and $\sigma, \tau \in \mathrm{Aut}(\mathbb{F}_{q^n})$. Then the codes of type $\mathcal{C}_{n,k,\tau,I}$ and $\mathcal{C}_{\sigma,T}$ are neither equivalent nor adjointly equivalent.

Thanks for your attention!