A geometric characterization of known maximum scattered linear sets of $\mathbf{PG}(1, q^n)$

(joint work G.G. Grimaldi, G. Longobardi and R. Trombetti)

OpeRa 2024 - Caserta Open Problems on Rank-Metric Codes

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February 16, 2024 — Università degli Studi di Napoli Federico II

Outline

Let

- $r, n, t \in \mathbb{Z}^+$ and $q = p^r$, p a prime number;
- $\mathbb{F}_{q^n}$ Galois field with q^n elements;
- $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$ projective space of dimension $r-1$ over $\mathbb{F}_{q^n}$.

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Theorem. [G. Lunardon, O. Polverino (2004)]

Every linear set is a subgeometry or projection of a canonical subgeometry.

Definition

A subset of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$ is called a **linear set** L_U if its points are defined by non-zero elements of an $\mathbb{F}_q$-subspaces U of W.

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- $|L_U| \leq \frac{q^t-1}{q-1} = q^{t-1} + q^{t-2} + \cdots + q + 1.$
- When the bound is attended, L_U is said to be a **scattered** linear set.

Definition

A **(canonical) subgeometry** Σ of $\Sigma^* = \mathrm{PG}(r-1, q^n) = \mathrm{PG}(W, q^n)$ is an $\mathbb{F}_q$-linear set of rank r and such that $\langle \Sigma \rangle = \Sigma^*$.

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- If $\Sigma \cong \mathrm{PG}(r-1, q)$ is a canonical subgeometry of Σ^*, there exists a semilinear collineation σ of Σ^* of order n such that $\Sigma = \mathrm{Fix}(\sigma)$.

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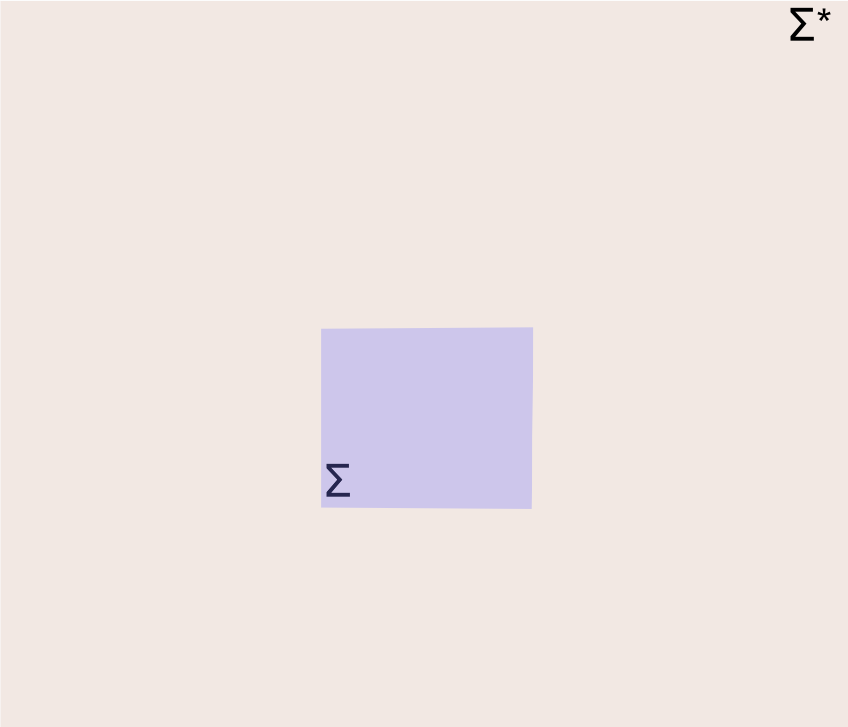
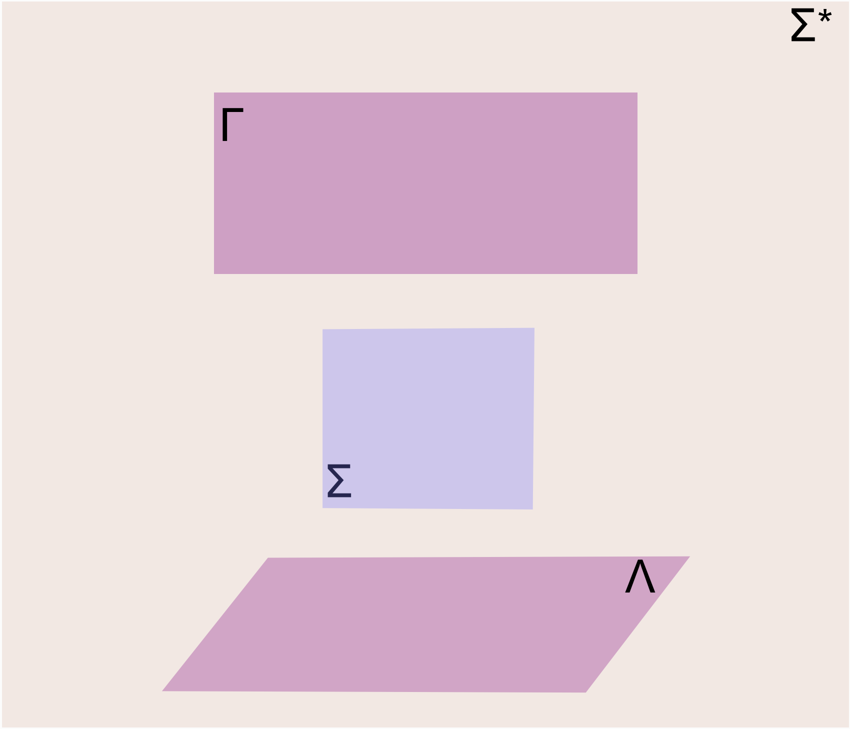
Then

$$L = p_{\Gamma,\Lambda}(\Sigma) = \{x \text{ is a point of } \Lambda \mid \exists y \in \Sigma \text{ such that } x = \langle \Gamma, y \rangle \cap \Lambda\}$$

is called **projection** of Σ from Γ to Λ (Λ and Γ the **center** and the **axis** of the projection, respectively).


Σ^*
Σ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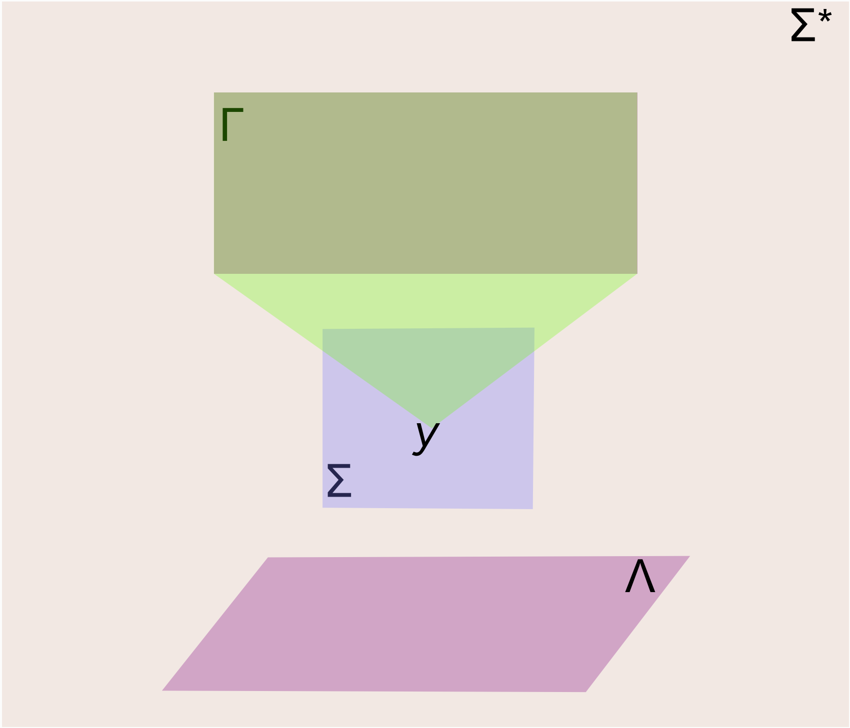
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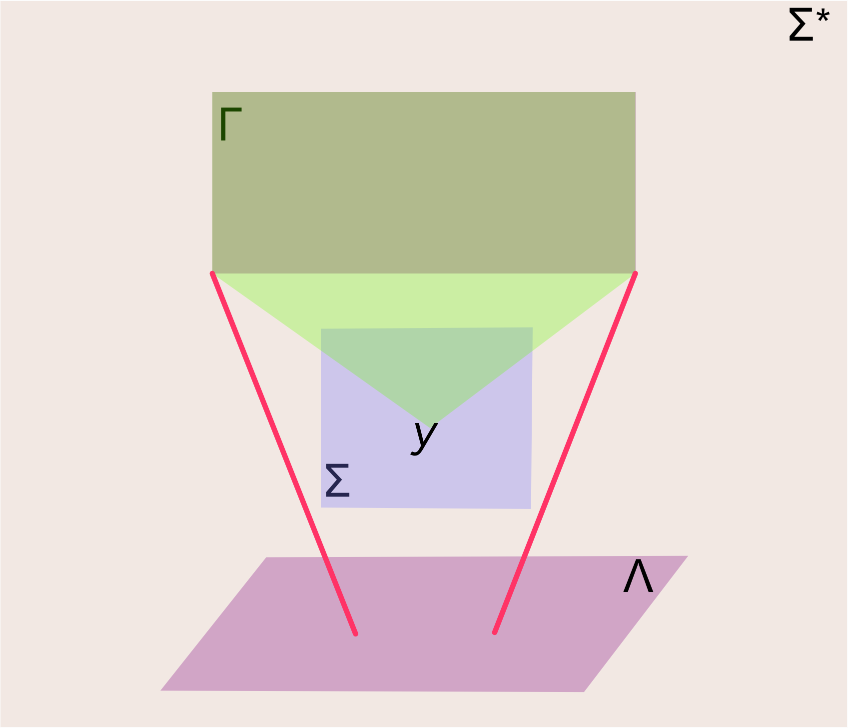
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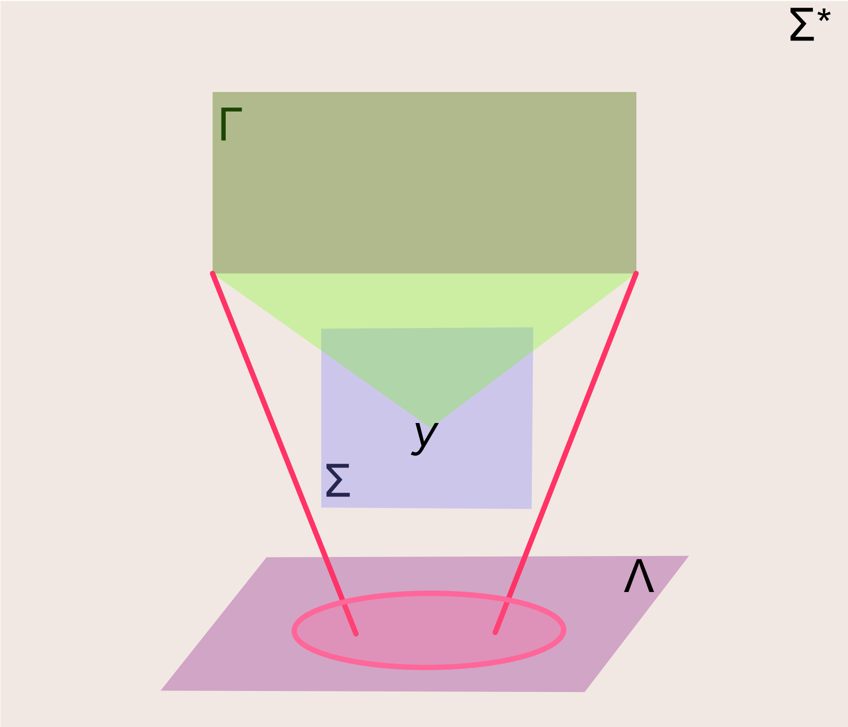

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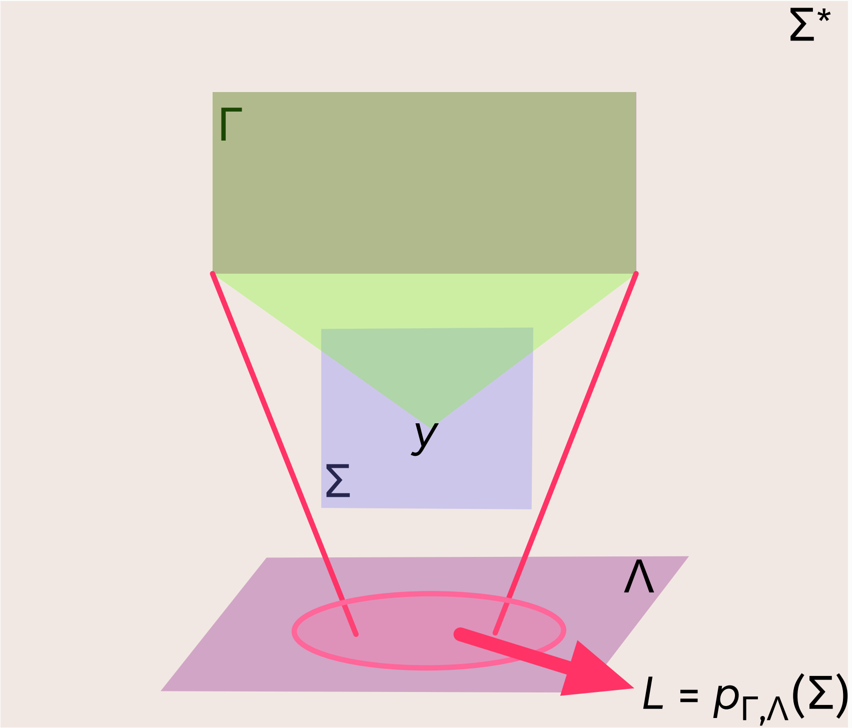

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- If $\mathrm{rk}P = n$, then P will be said to be an **imaginary** point of Σ^* with respect to Σ.

Definition

Let A, B, C, D be points of the line $\mathrm{PG}(1, q^n)$ with A, B, C distinct. The *cross-ratio* is defined as

$$(ABCD) = \frac{\begin{vmatrix} c_0 & a_0 \\ c_1 & a_1 \end{vmatrix} \begin{vmatrix} d_0 & b_0 \\ d_1 & b_1 \end{vmatrix}}{\begin{vmatrix} c_0 & b_0 \\ c_1 & b_1 \end{vmatrix} \begin{vmatrix} d_0 & a_0 \\ d_1 & a_1 \end{vmatrix}},$$

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Up to a suitable projectivity, any linear set L of rank $n(r-1)$ in $\mathrm{PG}(r-1, q^n)$ can be written in the following form

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then $\mathrm{Fix}(\sigma) = \Sigma$.

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Let L_f be a linear set of rank n on the projective line $\mathrm{PG}(1, q^n)$ with $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ with $m = \deg_q f(x)$ and $l = \mathrm{supp} f(x)$.

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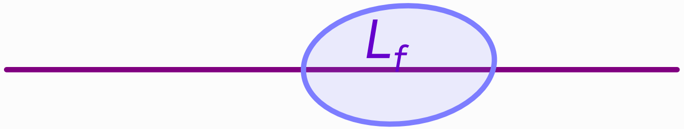
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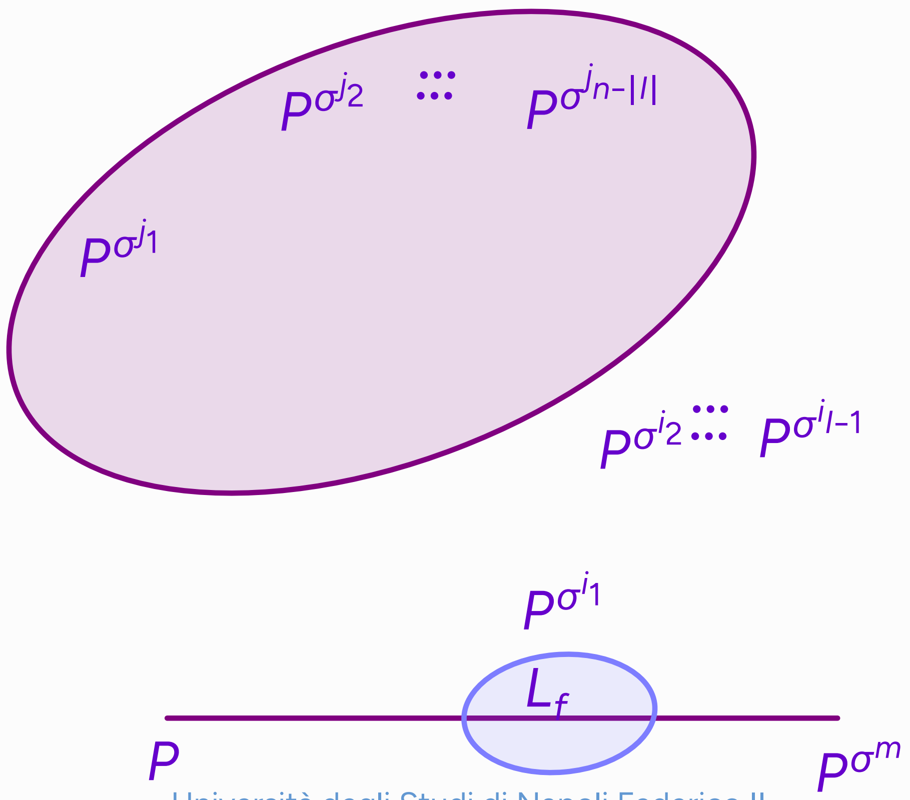
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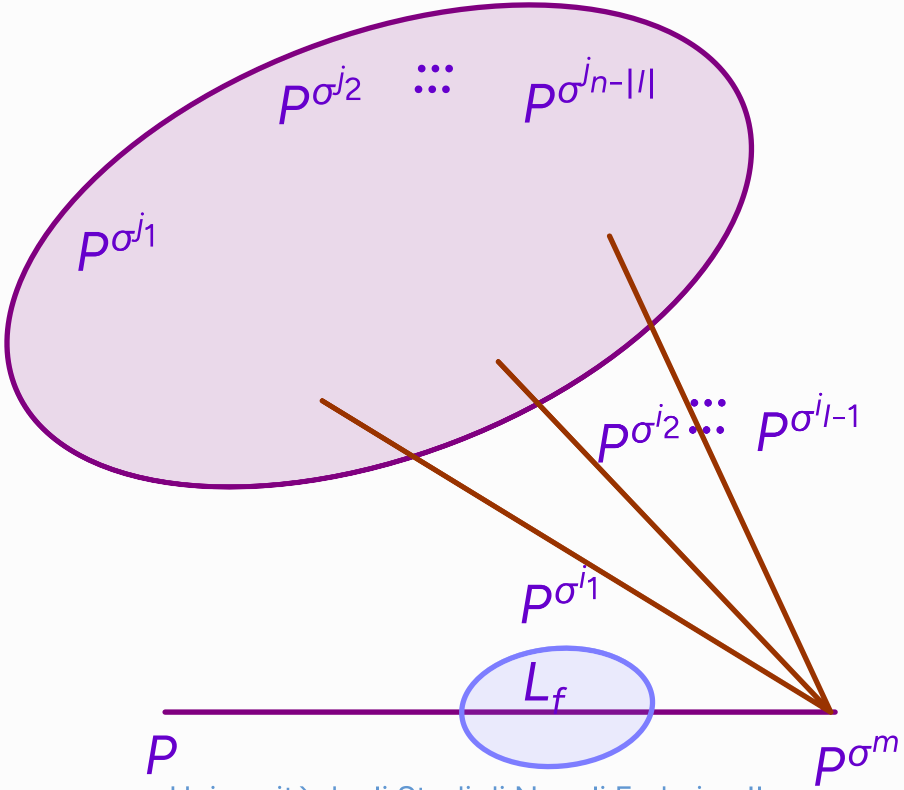
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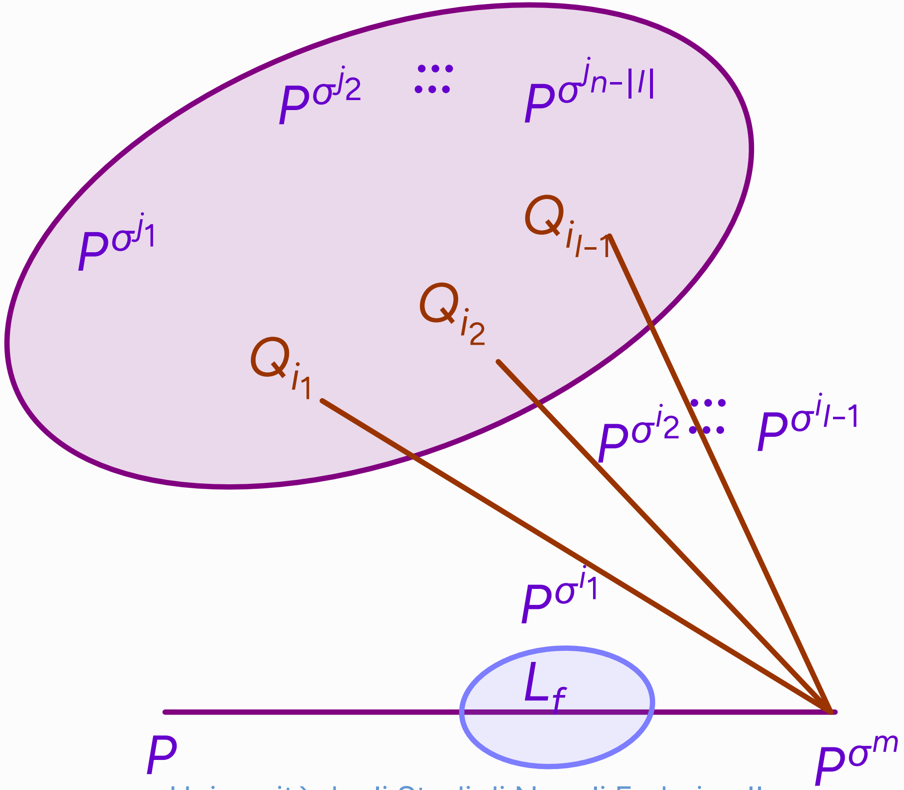
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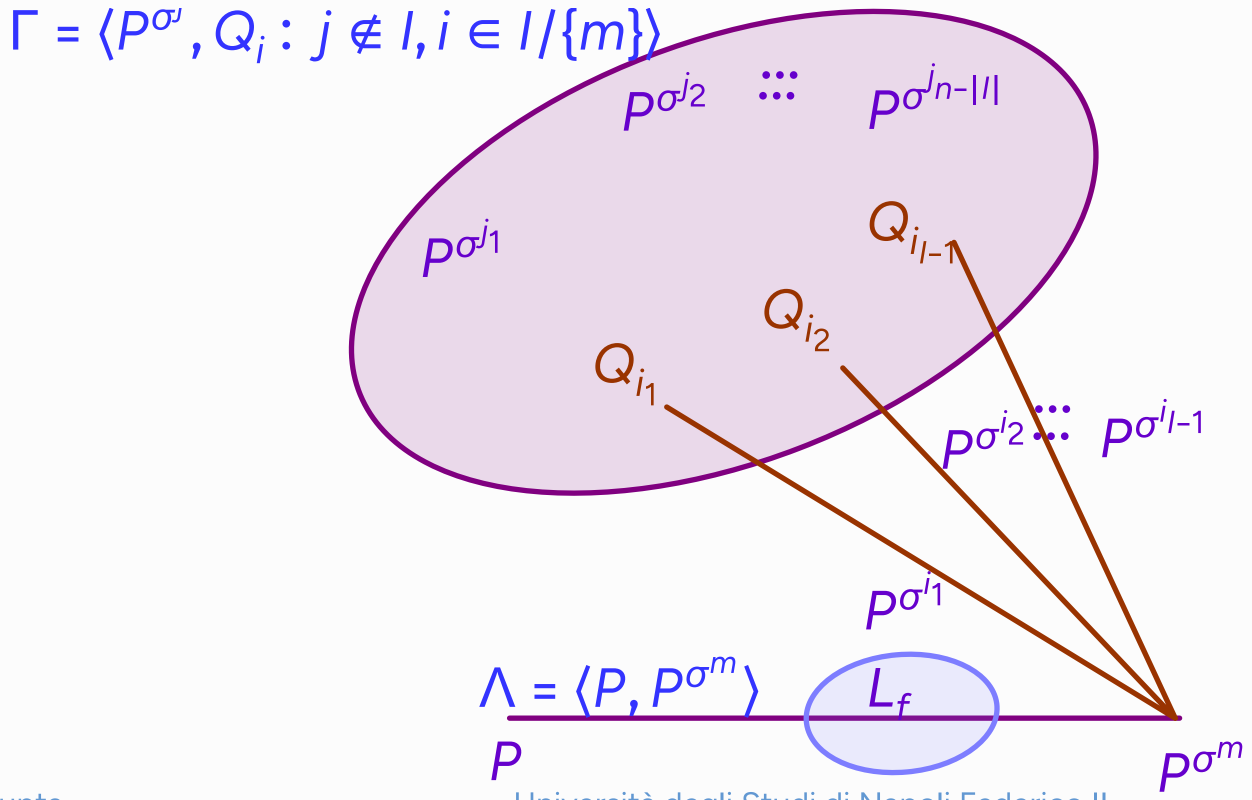

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L_f
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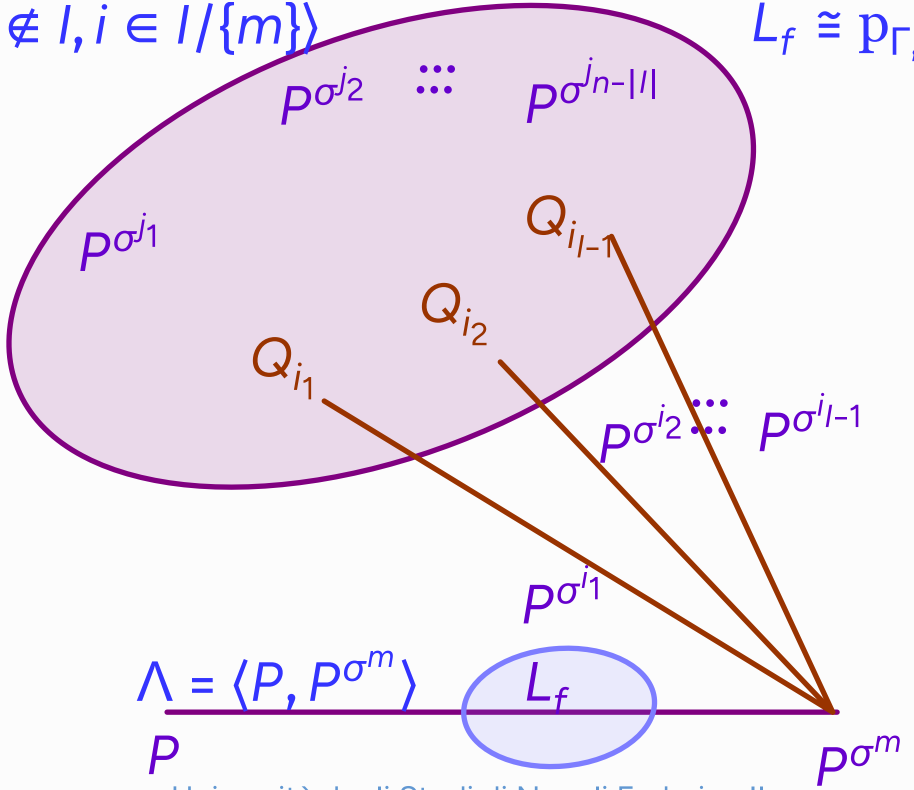

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3. If f is a permutation polynomial then $\Sigma \cap \langle P, P^{\sigma^j}, Q_i \mid j \notin I \text{ and } i \in I/\{m\}\rangle = \emptyset$.

Theorem [G.G. Grimaldi, G., G. Longobardi, Trombetti, In Preparation]

Let L_F be a linear set of rank $n(r-1)$ of $\mathrm{PG}(r-1,q^n)$ with $F(\mathbf{x})$, a multivariate polynomial. Let $\Sigma^* = \mathrm{PG}(n(r-1)-1,q^n)$ and Σ be the subgeometry of Σ^*. Then, there exist

1. $r-1$ imaginary points $P_0, P_1, \ldots, P_{r-2}$ of Σ^* (wrt Σ),
2. an $(n(r-1)-(r+1))$-dimensional subspace Γ of Σ^* fulfilling
   - *i)* $P_k^{\sigma^{j_k}} \in \Gamma$ for any $j_k \notin I_k, j_k \neq 0$ and $k \in \{0,\ldots,r-2\}$,
   - *ii)* any line
   $$\langle P_\ell^{\sigma^{j_\ell}}, P_\ell^{\sigma^{m_\ell}} \rangle$$
   with $j_\ell \in I_\ell / \{m_\ell\}$ for $\ell \in \{0,1,\ldots,r-2\}$ meets Γ,
   - *iii)* if $r > 2$, any line
   $$\langle P_i^{\sigma^{m_i}}, P_{r-2}^{\sigma^{m_{r-2}}} \rangle$$
   with $i \in \{0,1,\ldots,r-3\}$ meets Γ.
3. an $(r-1)$-dimensional subspace Λ of Σ^* through the points P_i, $i \in \{0,\ldots,r-2\}$,

such that $\Gamma \cap \Sigma = \Gamma \cap \Lambda = \emptyset$ and $\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ is equivalent to L_F.

Lemma [G.G. Grimaldi, G., G. Longobardi, Trombetti, In Preparation]

- Let Σ be a subgeometry of $\Sigma^* = \mathrm{PG}(n(r-1)-1, q^n)$ and σ denote a semilinear collineation of order n of Σ^* such that $\mathrm{Fix}(\sigma) = \Sigma$.
- Let $P_0, P_1, \ldots, P_{r-2}$ and $R_0, R_1, \ldots, R_{r-2}$ imaginary points such that $\mathcal{L}_{P_i} = \mathcal{L}_{R_i}$ and $\langle \mathcal{L}_{P_0}, \mathcal{L}_{P_1}, \ldots, \mathcal{L}_{P_{r-2}} \rangle = \Sigma^*$.

Then there exists a collineation $\varphi \in \mathrm{LAut}(\Sigma)$ such that $\varphi(P_i) = R_i$ for $i \in \{0, 1, \ldots, r-2\}$.

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Then, the group $\mathrm{LAut}(\Sigma)$ acts $(r-1)$-transitively on $r-1$ independent points $P_0, P_1, \dots, P_{r-2}$ such that $\langle \mathcal{L}_{P_0}, \mathcal{L}_{P_1}, \dots, \mathcal{L}_{P_{r-2}} \rangle = \Sigma^*$.

Theorem [G.G. Grimaldi, G., G. Longobardi, Trombetti, In Preparation]

- Let $r, n \geq 2$, $I_j \subseteq \{1, \dots, n-1\}$, with $j \in \{0, 1, \dots, r-2\}$.
- Let Σ be a canonical subgeometry of Σ^* and consider Γ and Λ subspaces of Σ^* with dimensions $(n(r-1) - (r+1))$ and $(r-1)$, respectively, such that $\Gamma \cap \Sigma = \emptyset = \Lambda \cap \Gamma$.
- Let $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma) \subseteq \Lambda$ be a linear set of rank $n(r-1)$ associated with an $(r-1, n(r-1)-2)$-evasive $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^n}^r$.
- If there exist $r-1$ points $P_0, P_1, \dots, P_{r-2}$ such that Γ is spanned by points defined above.

Then,

1. $P_0, P_1, \dots, P_{r-2}$ are imaginary points (w.r.t. Σ), and
2. $L \cong L_F$, where $\mathrm{supp} f_j(x_j) = I_j$ and $m_j = \deg_q f_j(x_j)$, $j = 0, \dots, r-2$.

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- $f_1(x) = x^{q^s}, 1 \leq s \leq n-1, \gcd(s, n) = 1$, [Blokhuis, Lavrauw, 2000].

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- $f_{3,n}(x) = \delta x^{q^s} + x^{q^{s+n/2}}, n \in \{6, 8\}, \gcd(s, n/2) = 1$, $\mathrm{N}_{q^n/q^{n/2}}(\delta) \notin \{0, 1\}$, for some conditions on δ and q, [Csajbók, Marino, Polverino, Zanella, 2018].

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- $f_4(x) = x^q + x^{q^3} + \zeta x^{q^5}$ where $\zeta \in \mathbb{F}_{q^6}^*$ such that $\zeta^2 + \zeta = 1$; ([Csajbók, Marino, Zullo, 2018] q odd, for $q \equiv 0, \pm 1 \pmod 5$), [Marino, Montanucci, Zullo, 2020] for the remaining congruences of q). [Bartoli, Longobardi, Marino, Timpanella, 2024] q even and some additional codintions.

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Theorem [Csajbok and Zanella, 2016]

Let Σ be a canonical subgeometry of $\mathrm{PG}(n-1, q^n)$, $q > 2$, $n \geq 3$. Assume that Γ and Λ are an $(n-3)$-subspace and a line of $\mathrm{PG}(n-1, q^n)$, respectively, such that $\Sigma \cap \Gamma = \Lambda \cap \Gamma = \emptyset$. Then the following assertions are equivalent:

1. The set $p_{\Gamma,\Lambda}(\Sigma)$ is a scattered $\mathbb{F}_q$- linear set of pseudoregulus type;
2. A generator σ exists of the subgroup of $\mathrm{P\Gamma L}(n, q^n)$ fixing Σ pointwise, such that $\dim(\Gamma \cap \Gamma^\sigma) = n-4$; furthermore Γ is not contained in the span of any hyperplane of Σ;
3. There exists a point P and a generator σ of the subgroup of $\mathrm{P\Gamma L}(n, q^n)$ fixing Σ pointwise, such that $\langle P, P^\sigma, \cdots, P^{\sigma^{n-1}} \rangle = \mathrm{PG}(n-1, q^n)$, and

$$\Gamma = \langle P, P^\sigma, \cdots, P^{\sigma^{n-3}} \rangle.$$

Zanella and Zullo,2020

- Let Γ be a subspace of $\mathrm{PG}(n-1,q^n)$, n odd of dimension $n-3 \geq 2$ and Σ a canonical subgeometry of $\mathrm{PG}(n-1,q^n)$ such that $\Gamma \cap \Sigma = \emptyset$.
- Assume that a generator σ of the subgroup of $P\Gamma L(n,q^n)$ exists, fixing Σ pointwise, such that $\mathrm{intn}_\sigma(\Gamma) = 2$

Then, there exists a point $R \in \mathrm{PG}(n-1,q^n)$ such that

$$R^{\sigma^2}, R^{\sigma^3}, \ldots, R^{\sigma^{n-2}} \in \Gamma.$$

Furthermore,

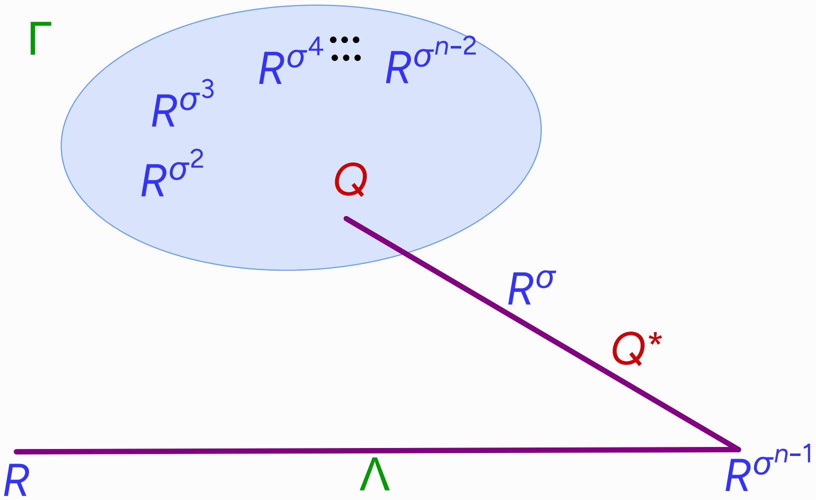
- Assume that $\langle R^{\sigma}, R^{\sigma^{n-1}} \rangle$ and Γ meet in a point Q and $R^{\sigma} \neq Q \neq R^{\sigma^{n-1}}$.
- Let Q^* be the point such that the pair $\{R^{\sigma}, R^{\sigma^{n-1}}\}$ separates $\{Q, Q^*\}$ harmonically.
- Such Q^* is defined by the property that there are two representative vectors v_0 and v_1 for R^{σ} and $R^{\sigma^{n-1}}$, respectively, such that $\langle v_0 + v_1 \rangle_{\mathbb{F}_{q^n}} = Q$, $\langle v_0 - v_1 \rangle_{\mathbb{F}_{q^n}} = Q^*$.

Under these assumptions the linear set $L = p_{\Gamma,\Lambda}(\Sigma)$, with Λ a line disjoint from Γ, is a maximum scattered linear set of LP-type if and only if

$$\Sigma \cap \langle R, R^{\sigma^2}, R^{\sigma^3}, \dots, R^{\sigma^{n-2}}, Q^* \rangle = \emptyset.$$

$$L_{f_1} = \{\langle (x, \eta x^{q^s} + x^{q^{(n-1)s}})\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}\}$$

$$\Gamma = \begin{cases} x_0 = 0 \\ x_s(n-1) = -\delta x_s. \end{cases} \quad \text{and } \Lambda = x_i = 0, i \in \{s, \dots, s(n-2)\}$$



- Q^* be the point such that the pair $\{R^\sigma, R^{\sigma^{n-1}}\}$ separates $\{Q, Q^*\}$ harmonically.
- Two representative vectors v_0 and v_1 for R^σ and $R^{\sigma^{n-1}}$ respectively.
- $\langle v_0 + v_1\rangle_{\mathbb{F}_{q^n}} = Q$,
  $\langle v_0 - v_1\rangle_{\mathbb{F}_{q^n}} = Q^*$.
- $L = p_{\Gamma,\Lambda}(\Sigma)$ is a maximum scattered linear set of LP-type if and only if

$$\Sigma \cap \langle R, R^{\sigma^2}, R^{\sigma^3}, \dots, R^{\sigma^{n-2}}, Q^*\rangle = \emptyset.$$

Theorem [G.G. Grimaldi, G., G. Longobardi, Trombetti, In Preparation]

Let Γ be a solid, Λ a line and $\Sigma \cong \mathrm{PG}(5, q)$ a canonical subgeometry of $\mathrm{PG}(5, q^6)$ such that $\Gamma \cap \Lambda = \emptyset = \Gamma \cap \Sigma$ and let $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ be a maximum scattered linear set of Λ. Assume there exists a point $P = \langle v \rangle_{\mathbb{F}_{q^6}}$ such that

$$\Gamma = \langle P^{\sigma^i}, Q : i \notin \{0, 2, 5\} \rangle$$

with $Q \in \langle P^{\sigma^2}, P^{\sigma^5} \rangle$. Then the linear set L is equivalent to $L_{3,6} = \{(x, x^{q^2} + \delta x^{q^5}) : x \in \mathbb{F}^*_{q^6}\}$ if and only if the equation

$$Y^2 - (\mathrm{Tr}_{q^3/q}(\gamma) - 1)Y +_{q^3/q} (\gamma) = 0$$

admits two solutions in $\mathbb{F}_q$ where $\gamma = (Q, P^{\sigma^5}, P^{\sigma^2}, Q^{\sigma^3})$.

Theorem [G.G. Grimaldi, G., G. Longobardi, Trombetti, In Preparation]

Let Γ be a 5-dimensional subspace, Λ a line and $\Sigma \cong \mathrm{PG}(7,q)$ a canonical subgeometry of $\mathrm{PG}(7,q^8)$ such that $\Gamma \cap \Lambda = \emptyset = \Gamma \cap \Sigma$ and let $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ be a maximum scattered linear set of Λ. If there exists a point $P = \langle v \rangle_{\mathbb{F}_{q^8}}$ such that

$$\Gamma = \langle P^{\sigma^i}, Q : \ i \notin \{0, s, s+4\}, \ (s,4) = 1 \rangle$$

with $Q \in \langle P^{\sigma^s}, P^{\sigma^{s+4}} \rangle$, then $L \cong L_{3,8}$ with $f_{3,8}(x) = x^{q^s} + \delta x^{q^{s+4}}$, $\delta \in \mathbb{F}_{q^8}$.
Moreover, assume $q \leq 11$ or $q \geq 1039891$ odd. Then, the linear set L is equivalent to $L_{3,8}$ if the pair $\{P^{\sigma^s}, P^{\sigma^{s+4}}\}$ separates $\{Q, Q^{\sigma^4}\}$ harmonically.

Theorem[G.G. Grimaldi, G., G. Longobardi, Trombetti, In Preparation]

Let Γ be a solid, Λ a line and $\Sigma \cong \mathrm{PG}(5,q)$ a canonical subgeometry of $\mathrm{PG}(5,q^6)$ such that $\Gamma \cap \Lambda = \emptyset = \Gamma \cap \Lambda$ and let $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ be a maximum scattered linear set of Λ.
If there exists a point $P = \langle v \rangle_{\mathbb{F}_{q^6}}$ such that

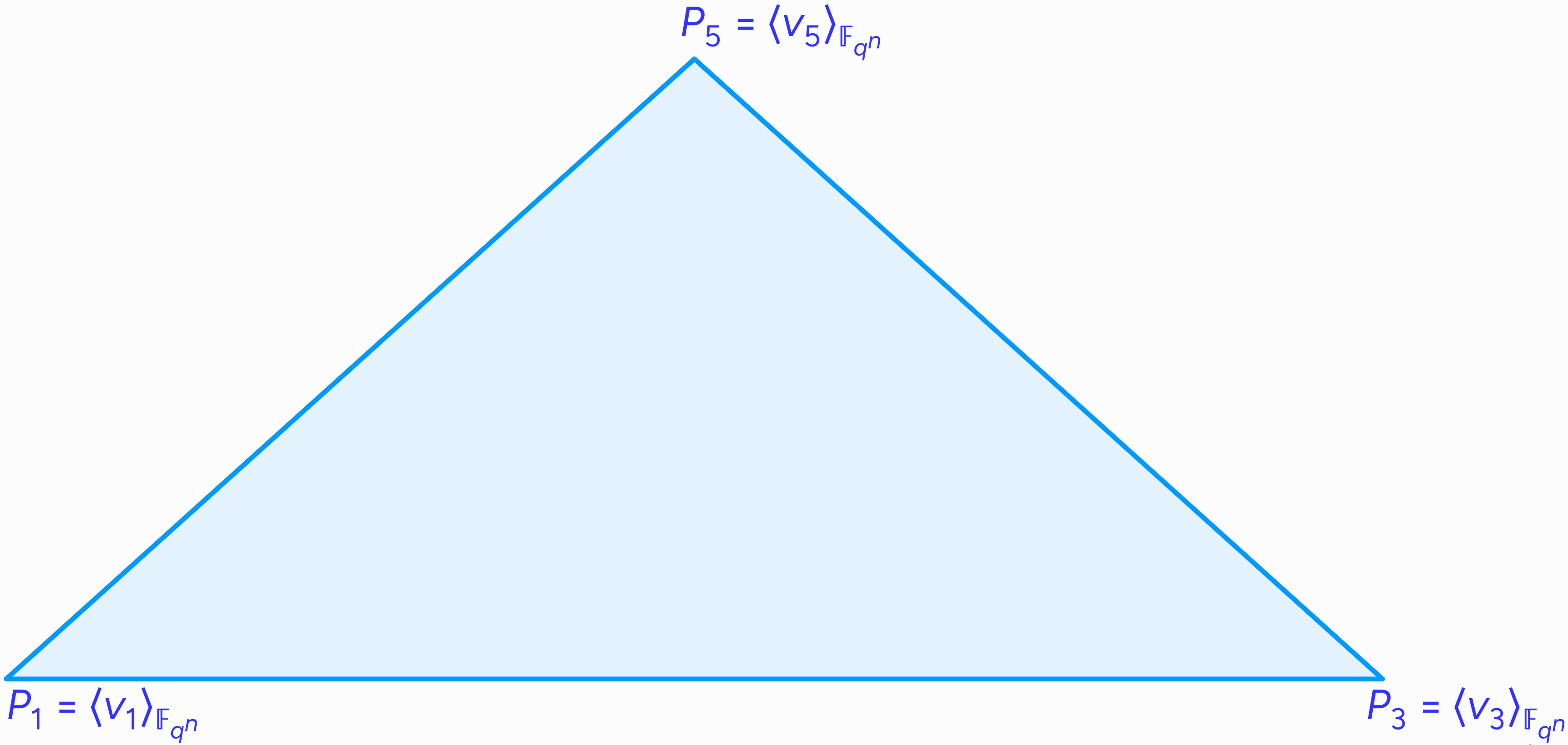
$$\Gamma = \langle P^{\sigma^2}, P^{\sigma^4}, Q, R \rangle$$

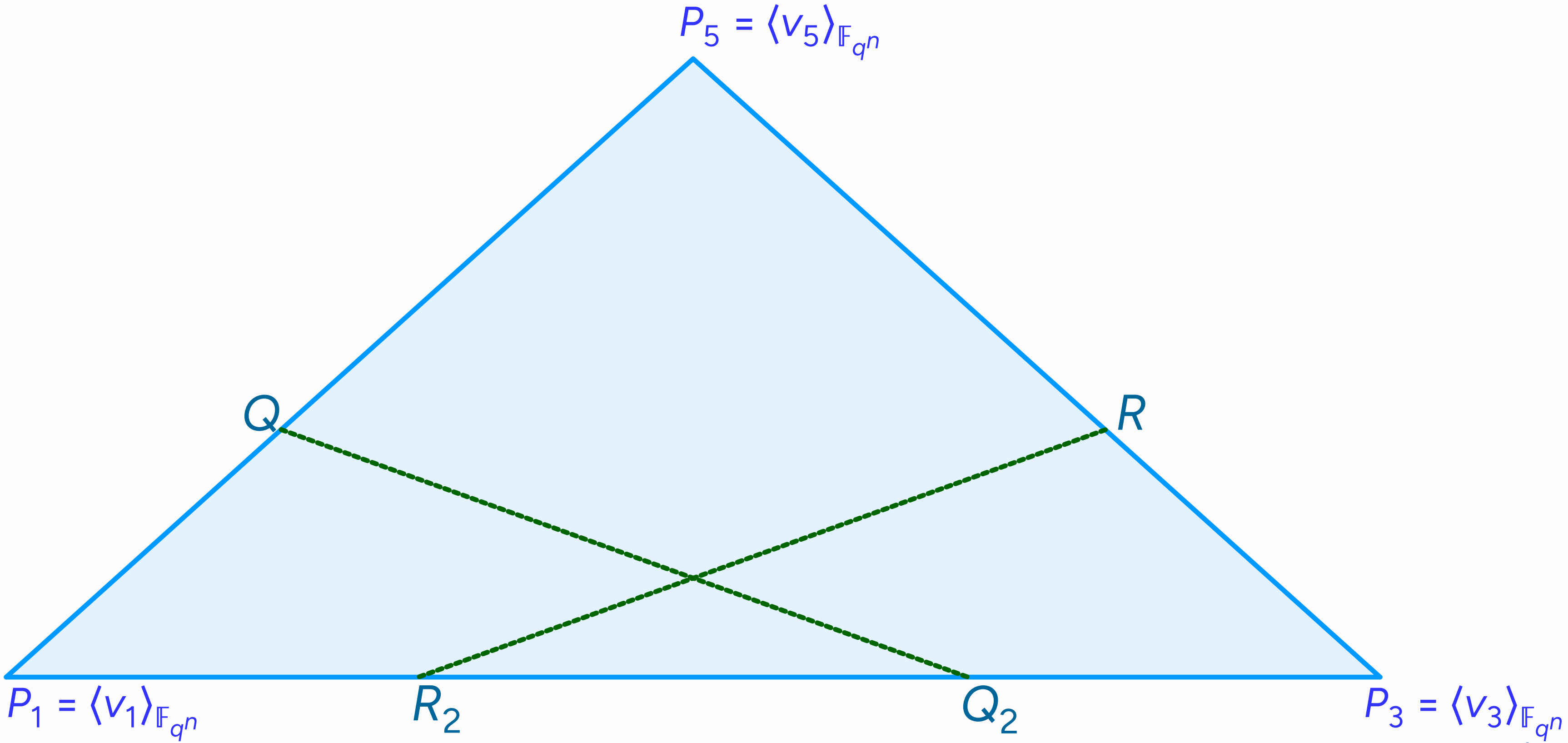
with $P^{\sigma^i} \notin \Gamma, i \in \{0,1,3,5\}$, $Q \in \langle P^{\sigma}, P^{\sigma^5} \rangle$ and $R \in \langle P^{\sigma^3}, P^{\sigma^5} \rangle$, then $L \cong L_4$ with $f_4(x) = x^q + bx^{q^3} + cx^{q^5} \in \mathbb{F}_{q^n}[x]$. Moreover, the linear set $L \cong L_{f_4}$ if and only if

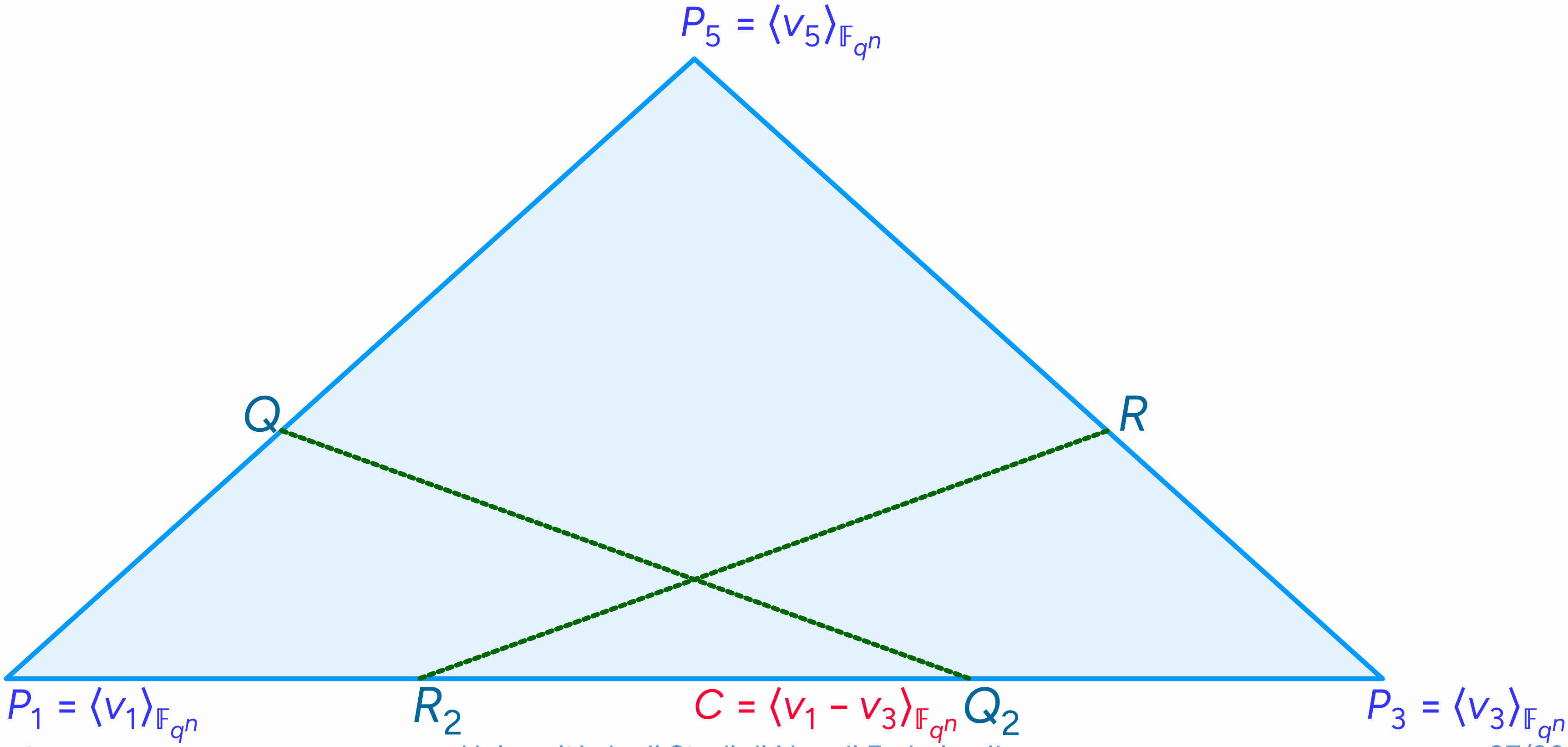
i) the point $C = \langle v_1 - v_3 \rangle_{\mathbb{F}_{q^6}} \in \Gamma$,

ii) the cross-ratio $(P^{\sigma}, P^{\sigma^3}, Q^{\sigma^2}, C) \in \mathbb{F}_{q^2}$ and

iii) the points C, P^{σ^5} and $\langle Q, Q^{\sigma^2} \rangle \cap \langle R, R^{\sigma^2} \rangle$ are collinear.


$P_5 = \langle v_5 \rangle_{\mathbb{F}_{q^n}}$
$P_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$
$P_3 = \langle v_3 \rangle_{\mathbb{F}_{q^n}}$


$P_5 = \langle v_5 \rangle_{\mathbb{F}_{q^n}}$
Q
R
$P_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$
R_2
Q_2
$P_3 = \langle v_3 \rangle_{\mathbb{F}_{q^n}}$


$P_5 = \langle v_5 \rangle_{\mathbb{F}_{q^n}}$
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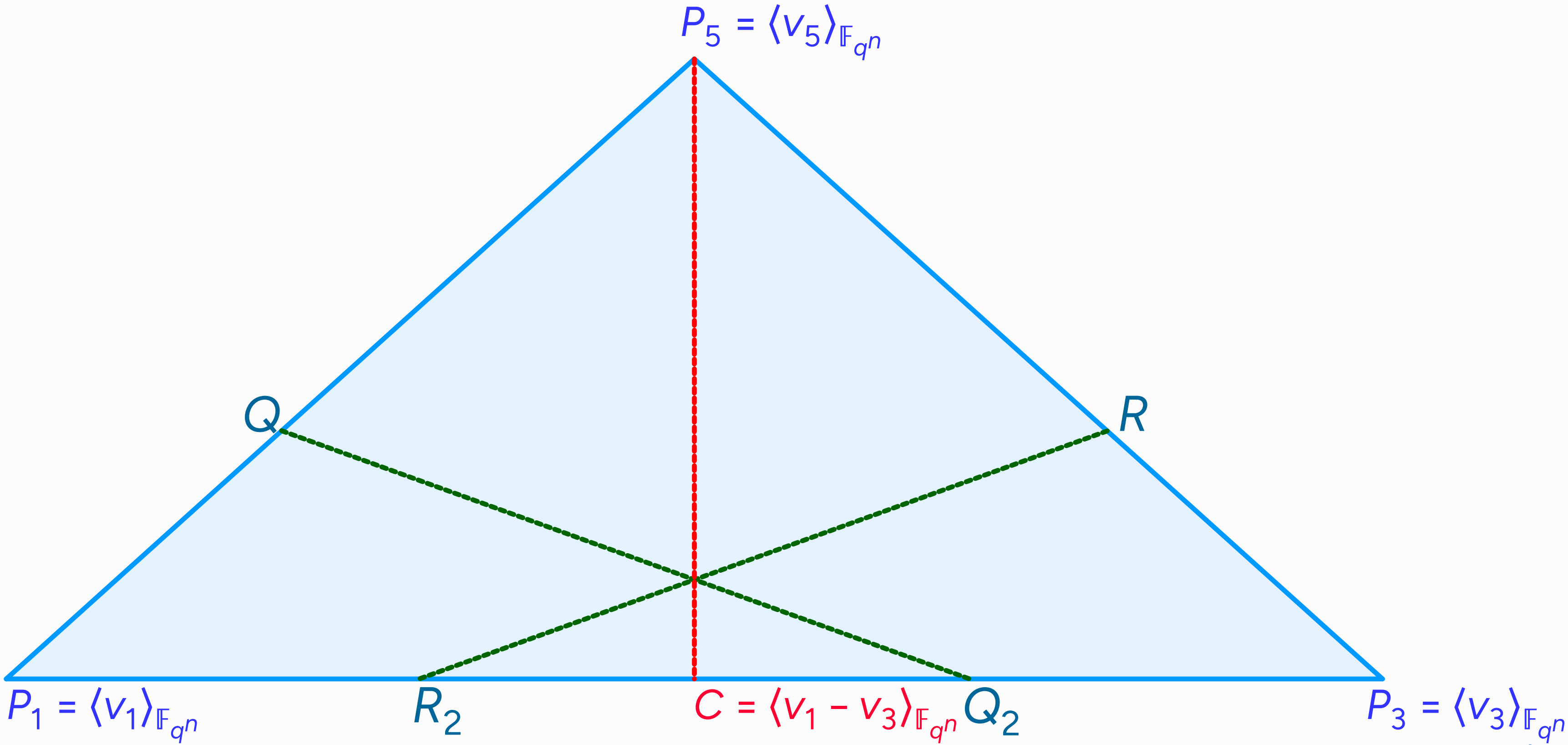
R
$P_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$
R_2
$C = \langle v_1 - v_3 \rangle_{\mathbb{F}_{q^n}}$
Q_2
$P_3 = \langle v_3 \rangle_{\mathbb{F}_{q^n}}$


$P_5 = \langle v_5 \rangle_{\mathbb{F}_{q^n}}$
Q
R
$P_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$
R_2
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Q_2
$P_3 = \langle v_3 \rangle_{\mathbb{F}_{q^n}}$

Theorem [G.G. Grimaldi, G., G. Longobardi, Trombetti, In Preparation]

Let Γ be an $(n-3)$-dimensional subspace, $n = 2t$. Let Λ be a line and $\Sigma \cong \mathrm{PG}(n-1, q)$ be a canonical subgeometry of $\mathrm{PG}(n-1, q^n)$ such that $\Gamma \cap \Lambda = \emptyset = \Gamma \cap \Sigma$ and let $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ be a maximum scattered linear set of Λ.

If there exists a point $P = \langle v \rangle_{\mathbb{F}_{q^n}}$ such that

$$\Gamma = \langle P_i, Q, R, S \ : \ i \notin \{0, 1, t-1, t+1, 2t-1\} \rangle$$

with $Q \in P_1P_{2t-1}$, $R \in P_{t-1}P_{2t-1}$ and $S \in P_{t+1}P_{2t-1}$ then $L \cong L_f$ with $f(x) = x^q + ax^{q^{t-1}} + bx^{q^{t+1}} + cx^{q^{2t-1}} \in \mathbb{F}_{q^n}[x]$ for some $a, b, c \in \mathbb{F}_{q^n}$. Moreover, $L \cong L_{f_6}$ if and only if the pairs $\{P_1, P_{t-1}\}$ separates $\{P_1 + P_{t-1}, X\}$ harmonically where $X = QR \cap P_1P_{t-1}$ and there exists $h \in \{z \in \mathbb{F}_{q^n} / \mathbb{F}_{q^t} \ : \ \mathrm{N}_{q^n/q^t}(z) = -1\}$ such that

(*i*) the cross-ratio $(P_1, P_{2t-1}, P_1 + P_{2t-1}, Q) = -h^{1-q^{2t-1}}$

(*iii*) the cross-ratio $(P_1, P_{t+1}, R_2, Y) = h^{1+q^2}$ where $Y = QS \cap P_1P_{t+1}$.

Corollary [G.G. Grimaldi, G., G. Longobardi, Trombetti, In Preparation]

Let Γ be an $(n-3)$-dimensional subspace, $n = 2t$. Let Λ be a line and $\Sigma \cong \mathrm{PG}(n-1, q)$ be a canonical subgeometry of $\mathrm{PG}(n-1, q^n)$ such that $\Gamma \cap \Lambda = \emptyset = \Gamma \cap \Sigma$ and let $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ be a maximum scattered linear set of Λ.
If there exists a point $P = \langle v \rangle_{\mathbb{F}_{q^n}}$ such that

$$\Gamma = \langle P_i, Q, R, S \;:\; i \notin \{0, 1, t-1, t+1, 2t-1\} \rangle$$

with $Q \in P_1P_{2t-1}, R \in P_{t-1}P_{2t-1}$ and $S \in P_{t+1}P_{2t-1}$ then $L \cong L_f$ with $f(x) = x^q + ax^{q^{t-1}} + bx^{q^{t+1}} + cx^{q^{2t-1}} \in \mathbb{F}_{q^n}[x]$ for some $a, b, c \in \mathbb{F}_{q^n}$. Moreover, $L \cong L'_{f_5}$ if and only if

(*i*) the pairs $\{P_1, P_{t-1}\}$ separates $\{P_1 + P_{t-1}, X\}$ harmonically where $X = QR \cap P_1P_{t-1}$.
(*ii*) the pairs $\{P_1, P_{2t-1}\}$ and $\{P_1 + P_{2t-1}, Q\}$ harmonically.
(*iii*) the pairs $\{P_1, P_{t+1}\}$ separates a $\{R_2, Y\}$ harmonically where $Y = QS \cap P_1P_{t+1}$.

That’s all for today!