Algebraic* bounds for sum-rank-metric codes

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Sum-rank metric and the size of sum-rank-metric code

In general, **sum-rank-metric space** is

$$\mathbb{F}_q^{n_1 \times m_1} \times \cdots \times \mathbb{F}_q^{n_t \times m_t}$$

with **sum-rank distance** between $\mathrm{A} := (A_1, \ldots, A_t)$ and $\mathrm{B} := (B_1, \ldots, B_t)$:

$$\mathrm{srkd}(\mathrm{A},\mathrm{B}) = \sum_{i=1}^{t} \mathrm{rk}(A_i - B_i).$$

It is denoted by $\boxed{\mathrm{Mat}(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)}$, where $\mathbf{n} = [n_1, \ldots, n_t]$ and $\mathbf{m} = [m_1, \ldots, m_t]$.

The sum-rank distance can also be calculated as a rank of a block matrix:

$$\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\end{pmatrix}, 1\right) \longleftrightarrow \left(\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 1 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 1 & 1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\end{pmatrix}, 0\right)$$

$$\downarrow$$

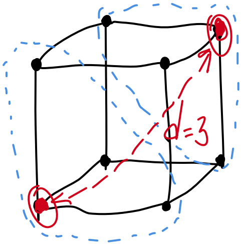
$$\operatorname{rk}\left(\begin{pmatrix}1 & 0 & 0 & & & & & & \\ 0 & 1 & 1 & & & & & & \\ 1 & 1 & 1 & & & & & & \\ & & & 1 & 0 & 0 & & & \\ & & & 1 & 0 & 1 & & & \\ & & & & & & 0 & 1 & \\ & & & & & & & & 1\end{pmatrix} \oplus \begin{pmatrix}0 & 0 & 0 & & & & & & \\ 1 & 0 & 0 & & & & & & \\ 1 & 0 & 1 & & & & & & \\ & & & 0 & 0 & 0 & & & \\ & & & 1 & 1 & 1 & & & \\ & & & & & & 0 & 1 & \\ & & & & & & & & 0\end{pmatrix}\right) = 6$$

A **sum-rank-metric code** $\mathcal{C}$ **with minimum distance** d is a subset of $\mathrm{Mat}(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ such that:

$$\min_{X,Y \in \mathcal{C}} \mathrm{srkd}(X, Y) = d.$$

NB! The code is non-linear in general.

Question: What is the maximal size of a sum-rank-metric code with minimum distance d?



Several upper bounds were introduced in

E. Byrne, H. Gluesing-Luerssen, and A. Ravagnani. Fundamental properties of sum-rank-metric codes. *IEEE Trans. Inf. Theory*, 67(10):6456–6475, 2021.

- Bounds *induced* by Singleton, Hamming, Plotkin, and Elias bounds from Hamming-metric case.
- *Singleton bound:* for j, δ such that $d-1=\sum_{i=1}^{j-1} n_i+\delta$ and $\delta \in [0, n_j-1]$,

$$|\mathcal{C}| \leq q^{\sum_{i=j}^{t} m_i n_i - m_j \delta}.$$

  In case of equality, $\mathcal{C}$ is an **MSRD code** (maximum sum-rank distance).
- Other bounds: Sphere-Packing, Projective Sphere-Packing, Total Distance.

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Sum-rank-metric graph and eigenvalue bounds

joint work with Aida Abiad and Alberto Ravagnani

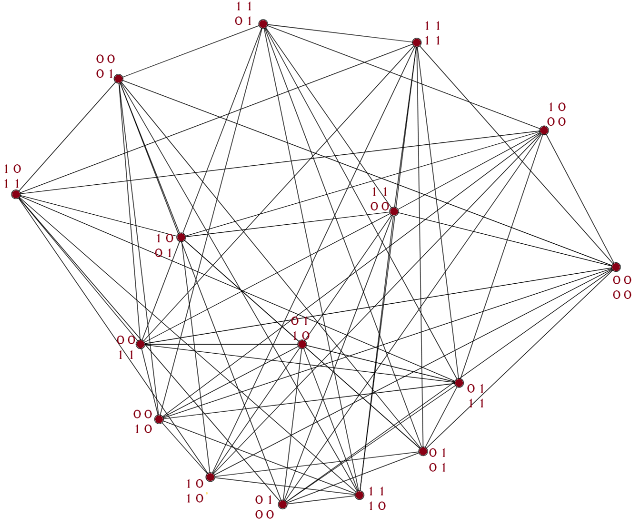
Sum-rank-metric graph $\Gamma := \Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$,
$\mathbf{n} = [n_1, \ldots, n_t]$, $\mathbf{m} = [m_1, \ldots, m_t]$:

- *vertices* of Γ = t-tuples of matrices from $\mathrm{Mat}(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$;
- $\mathrm{A} := (A_1, \ldots, A_t)$ and $\mathrm{B} := (B_1, \ldots, B_t)$ form an *edge* iff the sum-rank distance is 1:

$$\mathrm{srkd}(\mathrm{A},\mathrm{B}) = \sum_{i=1}^{t} \mathrm{rk}(A_i - B_i) = 1.$$

We assume $m_i \geq n_i$ and $m_1 \geq m_2 \geq \cdots \geq m_t$.

Sum-rank-metric graph



Sum-rank-metric graph $\Gamma := \Gamma(2, 2, \mathbb{F}_2)$:

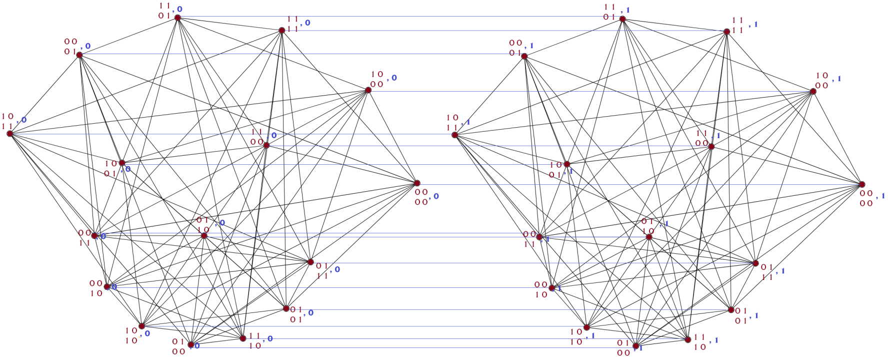
$V(\Gamma)$ = matrices 2×2 over $\mathbb{F}_2$.

$A \sim B$ if $\text{rk}(A - B) = 1$.

For $t = 1$ it is a **bilinear forms graph**.

Sum-rank-metric graph $\Gamma := \Gamma([2,1],[2,1],\mathbb{F}_2)$:

- vertices: (A_1, A_2), A_1 is size 2×2 over $\mathbb{F}_2$, $A_2 \in \{0,1\}$;
- edges: $(A_1, A_2) \sim (B_1, B_2)$ if $rk(A_1 - B_1) + rk(A_2 - B_2) = 1$.



Geodesic distance between A and B in Γ = sum-rank distance $\mathrm{srkd}(A, B)$.

For a graph G, its k-**independence number** α_k is the size of the largest set of vertices S such that distance between any $u, v \in S$ is more than k:

$$\min_{u,v \in S} \text{dist}_G(u, v) > k.$$

It is easy to see that α_{d-1} of $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ = the maximal size of a code in $\text{Mat}(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ with minimum distance d.

Question: What is an upper bound on α_{d-1} of the sum-rank-metric graph?

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A of a graph G.

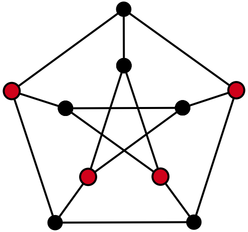
Ratio bound (Hoffman, 1974?): For a regular graph G, we have

$$\alpha_1 \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

The eigenvalues of Petersen graph (10 vertices) are

$$\mathbf{3}, 1, 1, 1, 1, 1, -2, -2, -2, -\mathbf{2}.$$

Then the Ratio bound is $\alpha_1 \leq 10 \cdot \frac{2}{3+2} = 4$, and it is tight:



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The following result which generalizes Hoffman's bound is introduced in:

A. Abiad, G. Coutinho, and M. A. Fiol. On the k-independence number of graphs. Discrete Math., 342(10):2875–2885, 2019.

Ratio-type bound: For a regular graph G and $p \in \mathbb{R}_{d-1}[x]$ let $W(p)$ be the largest element of the diagonal of $p(A)$. Then

$$\alpha_{d-1} \leq n \frac{W(p) - \min_{i \in [2,n]} p(\lambda_i)}{p(\lambda_1) - \min_{i \in [2,n]} p(\lambda_i)}.$$

Calculating the Ratio-type bound, $d = 3, 4$

How to obtain the best polynomial $p \in \mathbb{R}_{d-1}[x]$ for the bound?

For $d = 3$ **and** $d = 4$**,** the best polynomial for Ratio-type bound is known:

Ratio-type bound, $d = 3$ (Abiad, Coutinho, Fiol, 2019)

Let G be regular and $\theta_0 > \cdots > \theta_r$ be its distinct eigenvalues with $r \geq 2$ and $\theta_i \leq -1 < \theta_{i-1}$. Then

$$\alpha_2 \leq n\frac{\theta_0 + \theta_i\theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}.$$

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Ratio-type bound, $d = 4$ (Kavi, Newman, 2023)

Let G be a regular graph and $\theta_0 > \theta_1 > \cdots > \theta_r$ its distinct eigenvalues with $r \geq 3$ and $\theta_s \geq -\frac{\theta_0^2+\theta_0\theta_r-\Delta}{\theta_0(\theta_r+1)}$, where $\Delta = \max_{u\in V}\{(A^3)_{uu}\}$. Then

$$\alpha_3 \leq n\frac{\Delta - \theta_0(\theta_s + \theta_{s+1} + \theta_r) - \theta_s\theta_{s+1}\theta_r}{(\theta_0 - \theta_s)(\theta_0 - \theta_{s+1})(\theta_0 - \theta_r)}.$$

Calculating the Ratio-type bound, $d \geq 5$

A graph is ***d*-partially walk-regular** if for any $k \leq d$ the number of closed k-walks that start in u does not depend on the choice of u.

In general, the polynomial p can be obtained for any given d-partially walk-regular graph G using the Linear Program from:

M.A. Fiol, A new class of polynomials from the spectrum of a graph, and its application to bound the k-independence number. Linear Algebra Appl., 605:1–20, 2020.

Can these methods for calculating p be applied to sum-rank metric graph?

The Ratio-type bound is only applicable to **regular graphs**.

(Abiad, K, Ravagnani, 2023)
The sum-rank-metric graph $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ is regular.

$\Rightarrow$ We can apply the Ratio-type bound and calculate it for $d = 3, 4$.

A graph is **d-partially walk-regular** if for any $k \leq d$ the number of closed k-walks that start in u does not depend on the choice of u.

The sum-rank-metric graph $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ is d-partially walk-regular for any d.

$\Rightarrow$ We can use Fiol's LP to calculate the bound for $d \geq 5$.

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Let $\mathbf{n} = [n_1, \ldots, n_t]$, $\mathbf{m} = [m_1, \ldots, m_t]$.

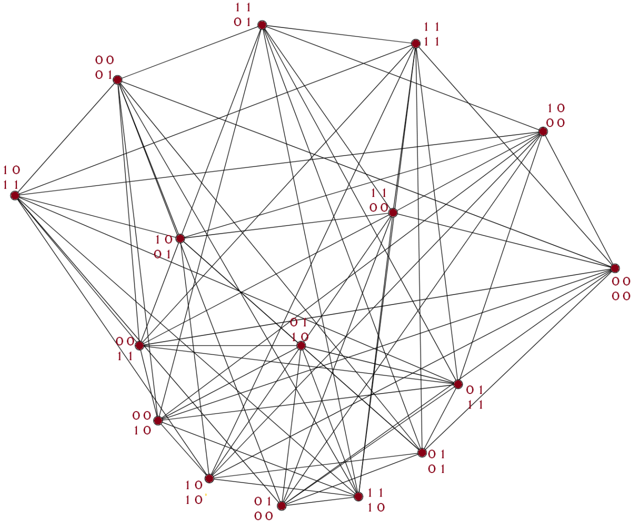
(Abiad, K, Ravagnani, 2023) The sum-rank-metric graph $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ is the Cartesian product of graphs $\Gamma(n_i, m_i, \mathbb{F}_q)$ for $i = 1, \ldots, t$.

The graph $\Gamma(n, m, \mathbb{F}_q)$ is a *bilinear forms graph*, with eigenvalues given by

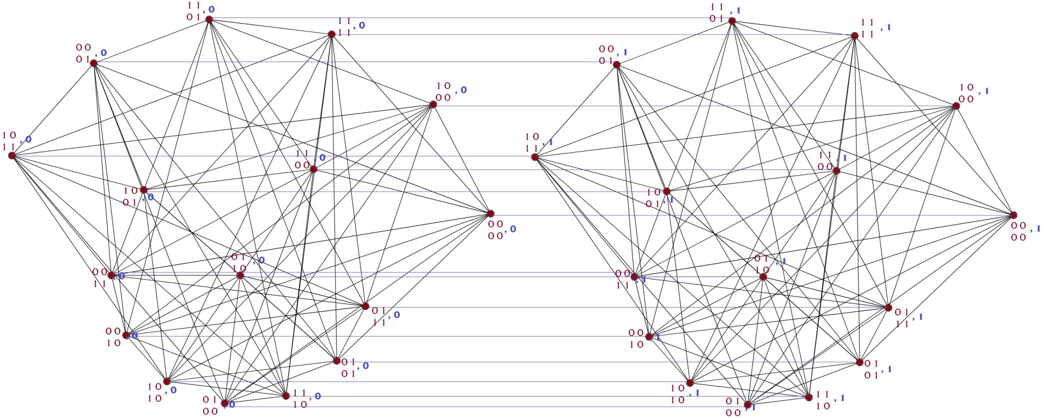
$$\theta_i = \frac{(q^{n-i} - 1)(q^m - q^i) - q^i + 1}{q - 1}, \quad i = 0, \ldots, n.$$

The eigenvalues of the Cartesian product are all possible sums of eigenvalues of the product's factors.

Bilinear forms graph, vertices are 2×2 matrices over $\mathbb{F}_2$:



Sum-rank-metric graph, each vertex is a 2×2 and 1×1 matrix over $\mathbb{F}_2$:



Let $\mathbf{n} = [n_1, \dots, n_t]$, $\mathbf{m} = [m_1, \dots, m_t]$.

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The Eigenvalue Formula

The graph $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ has the eigenvalues

$$\lambda_{(i_1,\dots,i_t)} = \sum_{j=1}^{t} \frac{(q^{n_j - i_j} - 1)(q^{m_j} - q^{i_j}) - q^{i_j} + 1}{q - 1}$$

with $i_j = 0, \dots, n_j$ for each $j \in [t]$.

Once all of the eigenvalues are calculated, one can obtain the list of distinct eigenvalues $\theta_0 > \cdots > \theta_N$.

For $d = 3$:

Let θ_0 be the largest eigenvalue and $\theta_i \leq -1 < \theta_{i-1}$. Then

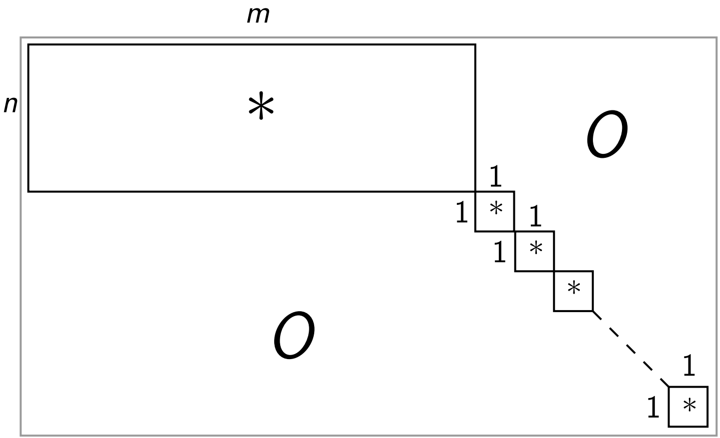
$$\alpha_2 \leq n\frac{\theta_0 + \theta_i\theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}.$$

- Use THE EIGENVALUE FORMULA to calculate θ_j.
- Find the specific eigenvalues θ_0, θ_i, θ_{i-1} requested by the bound.

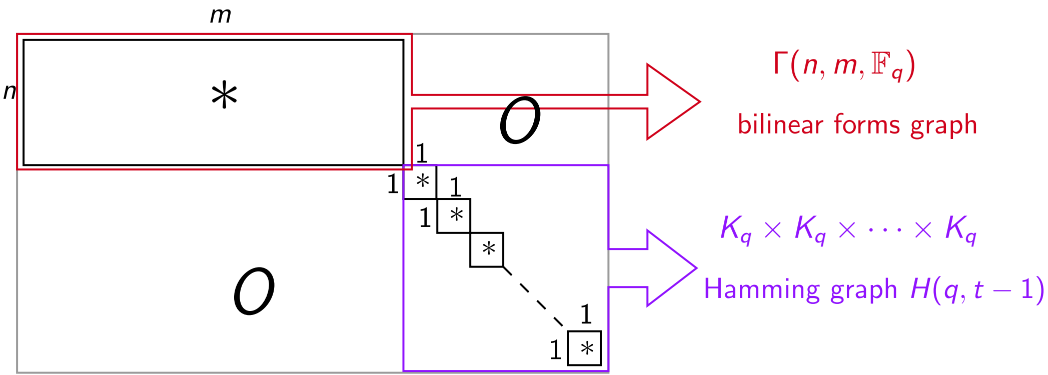
Similarly for $d \geq 4$, we calculate all the eigenvalues from the formula and use them to obtain the bound (using LP for $d \geq 5$).

Applying the bound to a family of sum-rank-metric graphs

Consider matrix space with one $n \times m$ matrix (block) and $t-1$ matrices 1×1:



Example



m
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$\Gamma(n, m, \mathbb{F}_q)$
bilinear forms graph
$K_q \times K_q \times \cdots \times K_q$
Hamming graph $H(q, t-1)$

$$\alpha_2 \leq q^{mn+t-1} \frac{\theta_0 + \theta_i \theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}.$$

For the family of matrices with blocks $n \times m$, $1 \times 1, \ldots, 1 \times 1$, the three eigenvalues can be found explicitly:

$$\theta_0 = \frac{(q^n - 1)(q^m - 1)}{q - 1} + (t-1)(q-1),$$

$$\theta_i = -1 - (t - 1 \mod q), \quad \theta_{i-1} = q - 1 - (t - 1 \mod q).$$

The bound on α_2 can be calculated from q, t, n, m explicitly.

By analyzing the bounds, we can derive conditions under which Ratio-type bound performs better than Singleton bound:

(Abiad, K, Ravagnani, 2023) Let $\mathrm{Mat}(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ be a matrix space with $\mathbf{n} = [n, 1, \ldots, 1]$ and $\mathbf{m} = [m, 1, \ldots, 1]$ for some t. Then for a code $\mathcal{C}$ with minimum distance $d = 3$ the Ratio-type bound performs better than the Singleton bound if

$$t > \begin{cases} 1 + q^m, & n = 1, \\ \frac{q^{2m} - q^{m+1} - q^m + 2q - 1}{q-1}, & n = 2, \\ \frac{q^{2m+1} - q^{2m} - q^{m+n} + q^m + q^n + q^2 - 3q + 1}{(q-1)^2}, & n > 2. \end{cases}$$

These conditions extend to other known bounds on code size:

(Abiad, K, Ravagnani, 2023) Let $\mathrm{Mat}(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ be a matrix space with $\mathbf{n} = [n, 1, \ldots, 1]$ and $\mathbf{m} = [m, 1, \ldots, 1]$ for some t. Then for a code $\mathcal{C}$ with minimum distance $d = 3$ Ratio-type bound performs better than Singleton bound, Sphere-Packing bound, Total Distance bound, Induced Singleton, Hamming, and Plotkin bounds, if

$$t > \begin{cases} 1 + q^m, & n = 1, \\ \frac{q^{2m} - q^{m+1} - q^m + 2q - 1}{q-1}, & n = 2, \\ \frac{q^{2m+1} - q^{2m} - q^{m+n} + q^m + q^n + q^2 - 3q + 1}{(q-1)^2}, & n > 2. \end{cases}$$

Example: application to MSRD codes

A **MSRD code** (maximum sum-rank distance) is a code of the size that achieves Singleton bound.

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$$t \leq \begin{cases} 1 + q^m, & n = 1, \\ \frac{q^{2m} - q^{m+1} - q^m + 2q - 1}{q-1}, & n = 2, \\ \frac{q^{2m+1} - q^{2m} - q^{m+n} + q^m + q^n + q^2 - 3q + 1}{(q-1)^2}, & > 2. \end{cases}$$

Non-existence of MSRD codes

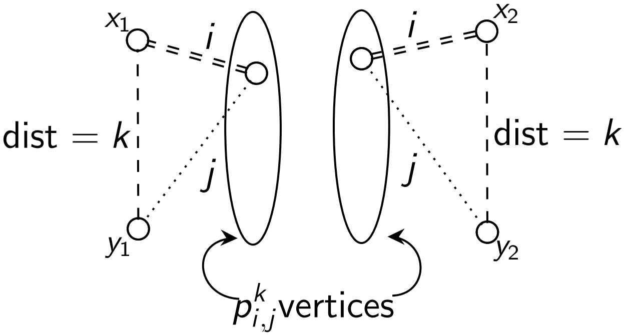
Sum-rank-metric graphs on $N < 50000$ vertices for which an MSRD code cannot exist.

t	q	**n**	**m**	d	N	RT_{d-1}	iS_d	iH_d	iE_d	S_d	SP_d	PSP_d
3	2	[3, 2, 1]	[3, 2, 2]	3	32768	494	4096	6096	15362	512	528	528
4	2	[2, 2, 2, 1]	[2, 2, 2, 1]	4	8192	98	256	744	407	128	282	204
5	2	[2, 2, 1, 1, 1]	[2, 2, 2, 2, 1]	4	8192	107	256	744	407	128	315	292
5	2	[2, 2, 2, 1, 1]	[2, 2, 2, 1, 1]	4	16384	193	1024	2621	1419	256	546	409
5	2	[2, 2, 2, 1, 1]	[2, 2, 2, 2, 1]	4	32768	338	1024	2621	1419	512	1024	819
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 2, 2, 2, 1]	4	8192	119	256	744	407	128	356	512
6	2	[2, 2, 1, 1, 1, 1]	[2, 2, 2, 1, 1, 1]	4	8192	123	1024	2621	1419	128	327	292
6	2	[2, 2, 1, 1, 1, 1]	[2, 2, 2, 2, 1, 1]	4	16384	212	1024	2621	1419	256	606	585
6	2	[2, 2, 1, 1, 1, 1]	[2, 2, 2, 2, 2, 1]	4	32768	371	1024	2621	1419	512	1129	1170
6	2	[2, 2, 2, 1, 1, 1]	[2, 2, 2, 1, 1, 1]	4	32768	378	4096	9362	5026	512	1057	819
7	2	[2, 1, . . . , 1]	[2, 1, . . . , 1]	4	1024	30	1024	1024	1024	32	64	64
7	2	[2, 1, . . . , 1]	[2, . . . , 2, 1, 1]	4	16384	235	1024	2621	1419	256	682	1024
7	2	[2, 1, . . . , 1]	[2, . . . , 2, 1]	4	32768	397	1024	2621	1419	512	1260	2048
7	2	[2, 2, 1, . . . , 1]	[2, 2, 2, 1, . . . , 1]	4	16384	246	4096	9362	5026	256	630	585
7	2	[2, 2, 1, . . . , 1]	[2, . . . , 2, 1, 1, 1]	4	32768	422	4096	9362	5026	512	1170	1170
8	2	[2, 1, . . . , 1]	[2, 1, . . . , 1]	4	2048	57	2048	2048	2048	64	120	128
8	2	[2, 1, . . . , 1]	[2, . . . , 2, 1, 1, 1]	4	32768	459	4096	9362	5026	512	1310	2048
8	2	[2, 2, 1, . . . , 1]	[2, 2, 2, 1, . . . , 1]	4	32768	467	16384	32768	18037	512	1213	1170
9	2	[2, 1, . . . , 1]	[2, 1, . . . , 1]	4	4096	107	4096	4096	4096	128	227	256
10	2	[2, 1, . . . , 1]	[2, 1, . . . , 1]	4	8192	204	8192	8192	8192	256	431	512
11	2	[2, 1, . . . , 1]	[2, 1, . . . , 1]	4	16384	384	16384	16384	16384	512	819	1024
12	2	[2, 1, . . . , 1]	[2, 1, . . . , 1]	4	32768	738	32768	32768	32768	1024	1560	2048

Delsarte's LP approach for sum-rank-metric codes

joint work with Aida Abiad, Alexander Gavrilyuk, and Ilia Ponomarenko

The graph G is **distance-regular** if for any two vertices x, y at distance k from each other the number of vertices at distance i from x and at distance j from y is a constant $p^k_{i,j}$ that does not depend on the choice of x, y.



$\mathcal{A} = (X, \mathcal{R})$ is a **symmetric association scheme** on set X with relations $\mathcal{R} = \{R_0, \ldots, R_D\}$ that form a partition of $X \times X$ such that:

- R_0 consists of all $(x, x) \in X$ for $x \in X$.
- $(x, y) \in R_i$ means $(y, x) \in R_i$ for any R_i, x, y.
- If $(x, y) \in R_k$, then the number of z such that $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant $p^k_{i,j}$ that does not depend on the choice of x, y.

If G is a distance-regular graph, then $(V(G), \mathcal{R})$ is a symmetric association scheme with relations:

$$(x, y) \in R_i \Leftrightarrow x \text{ and } y \text{ are at distance } i \text{ from each other.}$$

It is well-known that *bilinear forms graphs are distance-regular*.
A symmetric association scheme defined on a bilinear forms graph is called a **bilinear scheme**.

Sum-rank-metric schemes?

When an association scheme is defined, one can use *Delsarte's LP* to upper bound the size of the code with given minimum distance.

$\Rightarrow$ We can use Delsarte's LP bound if the graph is distance-regular.

Is sum-rank-metric graph distance-regular?

(Abiad, K, Ravagnani, 2023) A sum-rank-graph on $t \geq 2$ blocks is distance-regular if and only if all of the blocks are of size $1 \times m$ for some positive integer m.

Hence *sum-rank-graph is not distance-regular in general.*

But can we still apply Delsarte's LP bound?

When an association scheme is defined, one can use *Delsarte's LP* to upper bound the size of the code with given minimum distance.

$\Rightarrow$ We can use Delsarte's LP bound if the graph is distance-regular.

Is sum-rank-metric graph distance-regular?

(Abiad, K, Ravagnani, 2023) A sum-rank-graph on $t \geq 2$ blocks is distance-regular if and only if all of the blocks are of size $1 \times m$ for some positive integer m.

Hence *sum-rank-graph is not distance-regular in general*.

But can we still apply Delsarte's LP bound?

Given two association schemes $\mathcal{A}_i = (X_i, \mathcal{R}_i)$ with $D_i + 1$ relations R^i_j for $j = 0, \dots, D_i$, $i = 1, 2$, the **tensor product** $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the association scheme $(X_1 \times X_2, \mathcal{R})$ such that:

- $\mathcal{R} = \{R_{0,0}, R_{0,1}, \dots, R_{0,D_2}, R_{1,0}, \dots, R_{D_1,D_2}\}$;
- If $(x_1, y_1) \in R^1_i$ and $(x_2, y_2) \in R^2_j$, then $((x_1, x_2), (y_1, y_2)) \in R_{i,j}$.

(Abiad, Gavrilyuk, K, Ponomarenko, 2024++) If the graph G is a sum-rank-metric graph which is a Cartesian product of bilinear forms graphs $G_1, \dots, G_t$, then its association scheme is contained in the tensor product of bilinear schemes corresponding to $G_1, \dots, G_t$.

$\Rightarrow$ We can *define an association scheme for a sum-rank-metric graph* G and apply Delsarte's LP bound.

Bound comparison: computational results

bold = best performing bound;
underlined = cases when Ratio-type bound outperforms coding bounds.

t	q	**n**	**m**	d	$\|V\|$	Ratio-type	Delsarte LP	iS_d	iH_d	iE_d	S_d	SP_d	PSP_d
2	2	[2, 2]	[2, 2]	3	256	11	**10**	16	19	34	16	13	13
3	2	[2, 2, 1]	[2, 2, 1]	3	512	25	**20**	64	64	151	32	25	25
3	2	[2, 2, 1]	[2, 2, 1]	4	512	10	**6**	16	64	27	8	25	18
3	2	[2, 2, 1]	[2, 2, 2]	3	1024	38	**34**	64	64	151	64	46	46
3	2	[2, 2, 1]	[2, 2, 2]	4	1024	15	**8**	16	64	27	16	46	36
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	3	512	28	**24**	64	64	151	32	30	30
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	4	512	11	**6**	16	64	27	8	30	32
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	3	1024	44	**42**	64	64	151	64	53	53
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	4	1024	18	**10**	16	64	27	16	53	64
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	3	1024	46	**40**	256	215	529	64	48	48
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	4	1024	19	**12**	64	215	119	16	48	36
5	2	[2, 1, 1, 1, 1]	[2, 1, 1, 1, 1]	5	256	5	**2**	16	26	19	4	4	3
5	2	[2, 1, 1, 1, 1]	[3, 1, 1, 1, 1]	5	1024	8	**2**	64	336	240	4	6	3
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	3	1024	56	**49**	256	215	529	64	56	56
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	4	1024	22	**13**	64	215	119	16	56	64
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	4	512	16	**12**	256	512	407	16	34	32
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	5	512	8	**4**	64	77	99	8	6	5
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	5	1024	11	**6**	64	77	99	8	9	8
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	6	1024	7	**2**	16	77	14	4	9	3

There is no example with $|V| \leq 1024$ and $t \leq 7$ when Delsarte's LP is strictly outperformed.

Conclusion and future research

- ? Calculation of Ratio-type bound: solutions for graphs which are not partially walk-regular; obtaining the polynomial p for $d \geq 5$.
- ? Can the Delsarte's LP approach be applied to other metrics? (In case the respective graph is not distance-regular.)

Thank you for your attention!

The talk is based on:

Abiad, A., Khramova, A.P., Ravagnani A.
Eigenvalue bounds for sum-rank-metric codes. *IEEE Transactions in Information Theory.*
https://doi.org/10.1109/TIT.2023.3339808

Abiad, A., Gavrilyuk A., Khramova, A.P., Ponomarenko I.
The linear programming bound for sum-rank-metric codes.
Work in progress (coming soon!)