Cryptography and its applications in the Italian scenario

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Of course, purely mathematical investigation of algorithms has to go along with research in computers science and engineering, otherwise bad protocols and bad implementations would jeopardize theoretical security.

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On the other hand, recent technological innovations stimulated new lines of research and determined advances in Cryptography in the last ten/fifteen years.

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- post-quantum cryptography

Already in the " Piano nazionale per la protezione cibernetica e la sicurezza informatica " of 2017, the Presidency of the Council of Ministers, in outlining the measures needed for a significant improvement in cybersecurity, invoked the world of research.
The plan envisioned the establishment of a National Cryptography Center, explicitly as-signing it four specific tasks: designing ciphers, creating a national encryption algorithm, developing a national blockchain, and providing security assessments.
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If on the one hand it is true that the national centre is still on paper, it is also true that the Italian cryptographic community, composed of many disciplinary and/or territorial entities, managed to join together to form a single national entity, the association De Componendis Cifris, that made itself available to the country.

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However, De Cifris, as an informal initiative, was born in 2017 and boasts a much larger community.
For instance, there are over 3000 members on their LinkedIn channel, including professors or researchers from 27 universities or research institutions such as CNR.
There are over 1000 members from the business world and almost as many students or recent graduates.

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- to bring the community together, mainly to assist institutions.

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They published a book containing summaries of 100 cryptography theses from various Italian universities and they also organized events for young people.

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The Virtual Meeting Centre, an online platform managed by De Cifris, facilitates interactions between job offers and requests, allowing members to upload their CVs and documents expressing their needs or preferences.

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To sum up the association aim to be a reliable point of reference for high-level national research and contribute concretely to the cryptographic community when called upon by the country's institutions.

Next conference is held on 25-26-27 September, 2024 at Bank of Italy's conference theater "Salone Margherita", Via Due Macelli 75 – 00187 Roma (Italy).
For more info, please check website https://www.decifris.it/eventi or contact cifris24@decifris.it.

Cutting blocking sets, saturating linear sets and rank-metric codes

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Definition of linear set

$$\Lambda = \mathrm{PG}(V) = \mathrm{PG}(r-1, q^n) \qquad V = V(r, \mathbb{F}_{q^n})$$

$L \subseteq \Lambda$ is an $\mathbb{F}_q$-linear set if

$$L = L_U = \{P = \langle \mathbf{u} \rangle_{q^n} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}$$

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$\dim_{\mathbb{F}_q} U = k \quad \Rightarrow L_U$ is an $\mathbb{F}_q$–linear set of Λ of *rank* k

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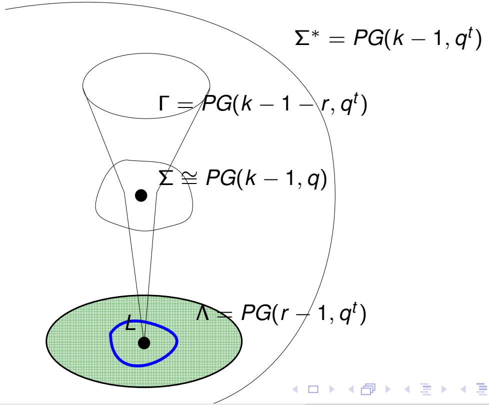
- Every projective subspace of Λ is an $\mathbb{F}_{q^n}$-linear set.
- Every subgeometry $\mathrm{PG}(s, q)$ of Λ $(s < r$ and $n > 1)$ is an $\mathbb{F}_q$-linear set.

Linear sets by projections

Theorem [Lunardon-Polverino 2004]

A linear set of a projective space Λ either is a subgeometry or is a projection of a subgeometry.

$rk_q\, L = k$



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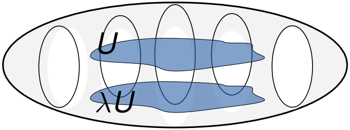

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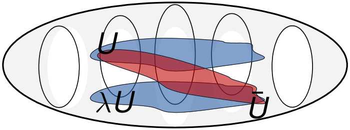

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An $\mathbb{F}_q$-linear set and the vector space defining it must be considered as coming in pair

Weight of a subspace

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If $|L_U| = \frac{q^k-1}{q-1}$, then L_U is said to be *scattered* (and U is said to be *scattered subspace*)

Blokhuis-Lavrauw 2000

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If L_U is a maximum scattered $\mathbb{F}_q$-linear set of $\mathrm{PG}(r-1,q^n)$, rn even, then L_U is a two-intersection set (wrt hyperplanes) with intersection numbers

$$\Theta_{\frac{rn}{2}-n-1}(q) = \frac{q^{\frac{rn}{2}-n}-1}{q-1} \qquad \Theta_{\frac{rn}{2}-n}(q) = \frac{q^{\frac{rn}{2}-n+1}-1}{q-1}$$

Linear sets and applications

- Blocking sets in finite projective spaces
- Two intersection sets in finite projective spaces
- Translation spreads of the Cayley Generalized Hexagon
- Translation ovoids of polar spaces
- Semifield flocks
- Finite semifields and finite semifield planes

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[M. Lavrauw: Scattered spaces in Galois Geometry, *Contemporary Developments in Finite Fields and Applications*, 2016, 195–216.]

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Definition

*An $[n, k]_{q^m/q}$ **(rank-metric) code** is a k-dimensional $\mathbb{F}_{q^m}$-subspace of $V(n, q^m)$ endowed with the rank distance.*

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$$\mathrm{wt}_{\mathrm{rk}}(v) = \dim_{\mathbb{F}_q} \sigma^{rk}(v)$$

Generalized rank weights

Kurihara, Uyematsu, Matsumoto 2012, 2015

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Definition

Let $\mathcal{C}$ be an $[n,k]_{q^m/q}$ code and let $1 \leq j \leq k$. Then the ***j-th generalized rank weight*** *of $\mathcal{C}$ is*

$$d_{\mathrm{rk},j} = \min\left\{\mathrm{wt}_{\mathrm{rk}}(D) :\ \mathcal{D} \subseteq \mathcal{C}, \dim \mathcal{D} = j\right\}$$

Rank-metric codes

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Let $\mathcal{C}$ be an $[n, k, (d_1, \ldots, d_k)]_{q^m/q}$ code, with $d_i := d_{\mathrm{rk},i}(\mathcal{C})$.

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Theorem (Singleton Bound - Delsarte 1978)

Let $\mathcal{C}$ be an $[n,k,(d_1,\ldots,d_k)]_{q^m/q}$ code. Then

$$mk \leq \min\{m(n-d_1+1), n(m-d_1+1)\}. \tag{1}$$

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Proposition (U. Martínez-Peñas 2016)

Let $\mathcal{C}$ be an $[n,k,(d_1,\ldots,d_k)]_{q^m/q}$ code. Then for each $s \in \{1,\ldots,k\}$ we have

$$d_s \leq \min\left\{n-k+s, sm, \frac{m}{n}(n-k)+m(s-1)+1\right\}. \tag{2}$$

Rank-metric codes

Proposition (Kurihara, Matsumoto, Uyematsu 2015 - Ducoat, Kyureghyan 2015)

Let $\mathcal{C}$ be an $[n, k, (d_1, \ldots, d_k)]_{q^m/q}$ code and let $\mathcal{C}^\perp$ be its dual $[n, n-k, (d_1^\perp, \ldots, d_{n-k}^\perp)]_{q^m/q}$ code. Then

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1. $1 \leq d_1 < d_2 < \ldots < d_k \leq n$. ***(Monotonicity)***

The **dual code** $\mathcal{C}^\perp$ of $\mathcal{C}$ is the $[n, n-k]_{q^m/q}$ code given by

$$\mathcal{C}^\perp = \{u \in V(n, q^m) \,:\, uv^\top = 0, \text{ for every } v \in \mathcal{C}\}.$$

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1. $1 \leq d_1 < d_2 < \ldots < d_k \leq n$. ***(Monotonicity)***
2. $\{d_1, \ldots, d_k\} \cup \{n+1-d_1^\perp, \ldots, n+1-d_{n-k}^\perp\} = \{1, \ldots, n\}$. ***(Wei-type duality)***

Rank-metric codes

Definition

An $[n, k, (d_1, \ldots, d_k)]_{q^m/q}$ *code is* ***nondegenerate*** *if* $\sigma^{\mathrm{rk}}(\mathcal{C}) = \mathbb{F}_q^n$

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An $[n, k]_{q^m/q}$ *codes* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are* ***(linearly) equivalent*** *if there exist* $A \in \mathrm{GL}(n, q)$ *and* $a \in \mathbb{F}_{q^m}^*$ *such that* $\mathcal{C}_2 = a\mathcal{C}_1 A := \{avA \,:\, v \in \mathcal{C}_1\}$

q-systems (Randrianarisoa 2020)

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*An $[n,k,(d_1,\ldots,d_k)]_{q^m/q}$ **system** U is an $\mathbb{F}_q$-subspace of $V(k,q^m)$, with $\dim_{\mathbb{F}_q}(U)=n$ and such that $\langle U\rangle_{\mathbb{F}_{q^m}} = V(k,q^m)$. For each $i \in \{1,\ldots,k\}$, the parameter d_i is defined as*

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Definition

*Two $[n,k,(d_1,\ldots,d_k)]_{q^m/q}$ systems U_1, U_2 are **(linearly) equivalent** if there exists $A \in \mathrm{GL}(k,q^m)$ such that $U_2 = U_1 \cdot A := \{uA \,:\, u \in U_1\}$.*

Let $\mathfrak{U}(n,k,(d_1,\ldots,d_k))_{q^m/q}$ denote the set of equivalence classes $[U]$ of $[n,k,d]_{q^m/q}$ systems, and let $\mathfrak{C}(n,k,(d_1,\ldots,d_k))_{q^m/q}$ denote the set of equivalence classes $[\mathcal{C}]$ of nondegenerate $[n,k,d]_{q^m/q}$ codes. One can define the maps

$$\begin{array}{rccc} \Phi: & \mathfrak{C}(n,k,(d_1,\ldots,d_k))_{q^m/q} & \longrightarrow & \mathfrak{U}(n,k,(d_1,\ldots,d_k))_{q^m/q} \\ & [\mathrm{rowsp}(u_1^\top \mid \ldots \mid u_n^\top)] & \longmapsto & [\langle u_1,\ldots u_n\rangle_{\mathbb{F}_q}] \end{array},$$

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Theorem (Randrianarisoa 2020)

The maps Φ and Ψ are well-defined and they are one the inverse of each other. Hence, they define a one-to-one correspondence between equivalence classes of $[n,k,(d_1,\ldots,d_k)]_{q^m/q}$ codes and equivalence classes of $[n,k,(d_1,\ldots,d_k)]_{q^m/q}$ systems.

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Evasive subspace (Bartoli, Csajbók, M., Trombetti 2021)

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*Let h, r be positive integers such that $h < k$. An $[n, k]_{q^m/q}$ system U is said to be an $(h, r)_q$-**evasive subspace** (or simply $(h, r)_q$-**evasive**) if $\langle U\rangle_{\mathbb{F}_{q^m}} = V(k, q^m)$ and $\dim_{\mathbb{F}_q}(U \cap H) \leq r$ for each $\mathbb{F}_{q^m}$-subspace H of $V(k, q^m)$ with $\dim_{\mathbb{F}_{q^m}}(H) = h$.*

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If U is a 1-scattered of $V(k,q^m)$, then $\dim_{\mathbb{F}_q} U \leq \left\lfloor \frac{km}{2} \right\rfloor$.

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If U is an h-scattered of $V(k,q^m)$, then either $\dim_{\mathbb{F}_q} U = k$ and U defines a subgeometry of $\mathrm{PG}(k-1,q^m)$ (and it is $(k-1)$-scattered) or $\dim_{\mathbb{F}_q}(U) \leq \left\lfloor \frac{km}{h+1} \right\rfloor$.

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When the previous equality is reached, then U is said be **maximum**.

Existence results of h-scattered subspaces

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Let k, m be positive integers and let q be a prime power. The following hold.

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Theorem (Zini, Zullo 2021)

Let $n := \frac{km}{h+1}$ and $m \geq h+3$. Let U be an $[n, k]_{q^m/q}$ system and let $\mathcal{C} \in \Psi([U])$ be any of its associated $[n, k]_{q^m/q}$ codes. Then, U is maximum h-scattered if and only if $\mathcal{C}$ is an MRD code.

Evasive subspace and rank-metric codes

Theorem (G.M., Neri, Trombetti 2023)

Let $\mathcal{C}$ be an $[n,k]_{q^m/q}$ code, and let $U \in \Phi([\mathcal{C}])$. Then, the following are equivalent.

1. *U is an $(h,r)_q$-evasive subspace.*
2. $\mathrm{d}_{\mathrm{rk},k-h}(\mathcal{C}) \geq n-r$.
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In particular, $\mathrm{d}_{\mathrm{rk},k-h}(\mathcal{C}) = n-r$ if and only if U is $(h,r)_q$-evasive but not $(h,r-1)_q$-evasive.

Linear cutting q-systems (Alfarano, Borello, Neri, Ravagnani 2022)

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Definition

*An $[n,k]_{q^m/q}$ system U is said to be t-**cutting** if for every $\mathbb{F}_{q^m}$-subspace H of $V(k,q^m)$ of codimension t we have $\langle H \cap U \rangle_{\mathbb{F}_{q^m}} = H$. When $t = 1$, we simply say that U is **(linear) cutting**.*

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Let $\mathcal{C}$ be an $[n,k]_{q^m/q}$ code, and let $U \in \Phi([\mathcal{C}])$ be any of the associated $[n,k]_{q^m/q}$ systems. Then, $\mathcal{C}$ is a minimal rank-metric code if and only if U is cutting.

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- *Let U be a cutting $[n,k]_{q^m/q}$ system, with $k \geq 2$. Then $n \geq m + k - 1$.*

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Proposition

- *Let U be a cutting $[n,k]_{q^m/q}$ system, with $k\geq 2$. Then $n \geq m+k-1$.*
- *If U is a scattered $[n,3]_{q^m/q}$ system with $n\geq m+2$, then U is cutting.*

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Theorem

Let $\mathcal{C}$ be an $[n,k]_{q^m/q}$ code, and let $U \in \Phi([\mathcal{C}])$ be any of the associated $[n,k]_{q^m/q}$ systems. Then, $\mathcal{C}$ is a minimal rank-metric code if and only if U is cutting.

Proposition

- *Let U be a cutting $[n,k]_{q^m/q}$ system, with $k \geq 2$. Then $n \geq m+k-1$.*
- *If U is a scattered $[n,3]_{q^m/q}$ system with $n \geq m+2$, then U is cutting.*

Questions:

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Questions:

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Linear cutting q-systems (Alfarano, Borello, Neri, Ravagnani 2022)

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Questions:

1. Can we generalize this result to larger value of $k > 3$?
2. Does the converse of this result hold?

Theorem (Bartoli-G.M.-Neri 2023)

Let U be an $[n,k]_{q^m/q}$ system. Then, U is $(k-2, n-m-1)_q$-evasive if and only if it is cutting.

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Let $\mathcal{C}$ be a nondegenerate $[n, k, (d_1, \ldots, d_k)]_{q^m/q}$ code. Then, $\mathcal{C}$ is minimal if and only if $d_2 \geq m+1$.

Evasive subspaces and cutting q-systems

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Corollary (Bartoli-G.M.-Neri 2023)

If $m < (k-1)^2$ then there are no cutting $[m+k-1,k]_{q^m/q}$ systems.

Cutting q-systems and rank-metric codes

Problem

Construct minimal rank-metric codes reaching the lower bound on the dimension of the associated q-system and with the maximum value for the corresponding values of generalized rank weights.

Shortest minimal $[n,3]_{q^m/q}$ code

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Minimal $[m+2,3]_{q^m/q}$ code

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  Examples of minimal $[m+3,3]_{q^m/q}$-codes for all $m \geq 4$ (they correspond to $(1,2)$-evasive subspaces)

Question

Are there examples of minimal $[m+2,3]_{q^m/q}$-codes for infinitely values of q and m?

Minimal $[m+2,3]_{q^m/q}$-code

Theorem (Lia-Longobardi-G.M.-Trombetti, submitted)

Let $V = \mathbb{F}_{q^m}^3$, $q = p^e$ *and* $m \geq 5$ *odd. Consider the* $(m+2)$*-dimensional subspace*

$$\mathcal{U}_\sigma = \{(x, x^\sigma + a, x^{\sigma^2} + b) : x \in \mathbb{F}_{q^m}, a, b \in \mathbb{F}_q\}$$

of V, *where* $\sigma : x \in \mathbb{F}_{q^m} \longrightarrow x^{q^s} \in \mathbb{F}_{q^m}$, $1 \leq s \leq m-1$ *and* $(s, m) = 1$. *If*

- *i*) $(q-1, m) = 1$,
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$$Q(X) = X^{\sigma^2+1} - X^{\sigma+1} - X^\sigma + X \in \mathbb{F}_q[X]$$

has not roots in $\mathbb{F}_{q^m} \setminus \mathbb{F}_q$,

then $\mathcal{U}_\sigma$ *is scattered.*

Minimal $[m+2,3]_{q^m/q}$-code

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Trivial cases:

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  iii) $\to X^5 + X^3 + X^2 + X = X(X+1)(X^3 + X^2 + 1)$ has no roots in $\mathbb{F}_{2^m} \setminus \mathbb{F}_2$. This is equivalent to say that the polynomial $X^3 + X^2 + 1$ has no roots in $\mathbb{F}_{2^m}$. Since the latter polynomial has degree 3, has no roots in $\mathbb{F}_2$ and $(m, 3) = 1$, it also has no roots in $\mathbb{F}_{2^m}$.

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Trivial cases:

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  iii) $\rightarrow X^5+X^3+X^2+X = X(X+1)(X^3+X^2+1)$ has no roots in $\mathbb{F}_{2^m}\setminus\mathbb{F}_2$. This is equivalent to say that the polynomial X^3+X^2+1 has no roots in $\mathbb{F}_{2^m}$. Since the latter polynomial has degree 3, has no roots in $\mathbb{F}_2$ and $(m,3)=1$, it also has no roots in $\mathbb{F}_{2^m}$.
- $q=3$, $s=1$ and $(m,12)=1$
  iii) $\rightarrow X^3+2X^2+2X+2$ and X^4+X^3+2X+1, have no roots in $\mathbb{F}_{3^m}$. Since the latter polynomials have degrees 3 and 4, they have no roots in $\mathbb{F}_3$ and $(m,12)=1$, they also have no roots in $\mathbb{F}_{3^m}$.

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Sketch of the proof

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Consider the 2-dimensional $\mathbb{F}_{q^m}$-vector subspaces of $\mathbb{F}_{q^m}^3$ containing the vector $(0,0,1)$, they have equation

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and define $\mathcal{Z}_{\lambda,\sigma} = \mathcal{U}_\sigma \cap \ell_\lambda$ with $\lambda \in \mathbb{F}_{q^m} \cup \{\infty\}$. Since $\mathcal{U}_\sigma \cap \langle v \rangle_{\mathbb{F}_{q^m}} = \mathcal{Z}_{\lambda,\sigma} \cap \langle v \rangle_{\mathbb{F}_{q^m}}$ for some $\lambda \in \mathbb{F}_q \cup \{\infty\}$, then

$\mathcal{U}_\sigma$ is a scattered subspace if and only if $\mathcal{Z}_{\lambda,\sigma}$ is scattered as well for any $\lambda \in \mathbb{F}_{q^m} \cup \{\infty\}$.

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Note that

$$\mathcal{U}_\sigma = \mathcal{W}_\sigma \oplus \mathcal{Z}_{\infty,\sigma},$$

where $\mathcal{W}_\sigma = \{(x, x^\sigma, x^{\sigma^2}) : x \in \mathbb{F}_{q^m}\}$ and $\mathcal{Z}_{\infty,\sigma} = \langle (0,0,1), (0,1,0) \rangle_{\mathbb{F}_q}$.

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Let $\lambda \in \mathbb{F}_{q^m}$ and $\bar{v} = (\bar{x}, \bar{x}^\sigma + \bar{a}, x^{\sigma^2} + \bar{b}) \in \mathcal{Z}_{\lambda,\sigma}$ with $\bar{x} \in \mathbb{F}^*_{q^m}$, $\bar{a}, \bar{b} \in \mathbb{F}_q$ and, hence, $\bar{x}^\sigma + \bar{a} = \lambda \bar{x}$.

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The property of being scattered for $\mathcal{Z}_{\lambda,\sigma}$ is equivalent to require that the number of triples $(y, y^\sigma + a, y^{\sigma^2} + b)$ with $y \in \mathbb{F}_{q^m}$, $a, b \in \mathbb{F}_q$ such that

$$\begin{cases} \lambda y = y^\sigma + a \\ \frac{y}{\bar{x}}(\bar{x}^\sigma s + \bar{a}) = y^\sigma + a \\ \frac{y}{\bar{x}}(\bar{x}^{\sigma^2} + \bar{b}) = y^{\sigma^2} + b \end{cases}$$

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is at most q. By using the second and third equation, then $\mathcal{Z}_{\lambda,\sigma}$ is scattered if and only if for any $\bar{v} \in \mathcal{Z}_{\lambda,\sigma}$, the previous System is satisfied by at most q values $y \in \mathbb{F}_{q^m}$.

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From the previous system we get that if the equation

$$y^{\sigma^2} - (1+\lambda)y^{\sigma} + \lambda y = 0$$

has at most q solutions then the subspace $\mathcal{Z}_{\lambda,\sigma}$ of V is scattered.

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By [McGuire-Sheekey, Csajbók-G.M.-Polverino-Zullo 2019], this polynomial has exactly q^2 solutions in $\mathbb{F}_{q^m}$ if and only if the m-th power of the matrix

$$A_\lambda = \begin{pmatrix} 0 & -\lambda \\ 1 & 1+\lambda \end{pmatrix}, \tag{3}$$

is equal to the identity matrix I_2.

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If λ is not a root of the polynomial

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then the subspace $\mathcal{Z}_{\lambda,\sigma}$ of V is scattered.

Minimal $[m+2,3]_{q^m/q}$-code

Theorem (Lia-Longobardi-G.M.-Trombetti, submitted)

Let $V=\mathbb{F}_{q^m}^3$, $q=p^e$ and $m\geq 5$ odd. Consider the $(m+2)$-dimensional subspace

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of V, where $\sigma : x\in\mathbb{F}_{q^m}\longrightarrow x^{q^s}\in\mathbb{F}_{q^m}$, $1\leq s\leq m-1$ and $(s,m)=1$. If

i) $(q-1,m)=1$,

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Question

When $Q(X)$ *has no roots in* $\mathbb{F}_{q^m}\setminus\mathbb{F}_q$*?*

Minimal $[m+2,3]_{q^m/q}$-code

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$$Q(X) = X^{\sigma^2+1} - X^{\sigma+1} - X^{\sigma} + X = Q_1(X)Q_2(X),$$

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If $(r, m) = 1$ (where r is the degree of the splitting field over $\mathbb{F}_q$ of the polynomial

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Recall that a polynomial $f(X)$ of degree n with coefficients over a field $\mathbb{F}$ has splitting field $\mathbb{K}$ of degree at most $n!$ over $\mathbb{F}$.

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Corollary (Lia-Longobardi-G.M.-Trombetti, submitted)

Let $V = \mathbb{F}_{q^m}^3$, $q = p^e$ *and* $m \geq 5$ *odd. Consider the* $(m+2)$*-dimensional subspace*

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- *i)* $(q-1, m) = 1$,
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- *iii)* $(m, (q^{2s} - q^s + 1)!) = 1$

then $\mathcal{U}_\sigma$ *is scattered.*

Question

When the polynomial

$$Q(X) = X^{\sigma^2+1} - X^{\sigma+1} - X^{\sigma} + X = Q_1(X)Q_2(X),$$

has no roots in $\mathbb{F}_{q^m} \setminus \mathbb{F}_q$*?*

Linearized & projective polynomials

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$$C_L = \begin{pmatrix} 0 & 0 & \dots & 0 & -\frac{\alpha_0}{\alpha_d} \\ 1 & 0 & \dots & 0 & -\frac{\alpha_1}{\alpha_d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{\alpha_{d-1}}{\alpha_d} \end{pmatrix}.$$

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Consider the matrix

$$A_L = C_L C_L^{\sigma} \dots C_L^{\sigma^{m-1}}.$$

Linearized & projective polynomials

Linearized & projective polynomials

THEOREM. [McGuire-Sheekey 2019]

The number of roots of P_L in $\mathbb{F}_{q^m}$ equals

$$\sum_{\lambda\in\mathbb{F}_q} \frac{q^{n_\lambda}-1}{q-1},$$

where n_λ is the dimension of the eigenspace of A_L corresponding to the eigenvalue λ. The number of roots of $L(X)$ in $\mathbb{F}_{q^m}$ is equal to q^{n_1}, i.e., to the size of the eigenspace of A_L corresponding to the eigenvalue 1.

Linearized & projective polynomials

Theorem. [McGuire-Sheekey 2019]

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Let $L(X) = \alpha_0 x + \alpha_1 x^\sigma + \alpha_2 x^{\sigma^2}$ with $\alpha_i \in \mathbb{F}^*_{q^m}$. Then putting $u = \frac{\alpha_0^\sigma \alpha_2}{\alpha_1^{\sigma+1}}$, one has

$$A_L = \begin{pmatrix} 0 & -\alpha_0/\alpha_2 \\ 1 & -\alpha_1/\alpha_2 \end{pmatrix}^{1+\sigma+\ldots+\sigma^{m-1}} = \mathrm{N}_{q^m/q}(\alpha_1/\alpha_2) \begin{pmatrix} -u^{\sigma^{-1}} G^\sigma_{m-2} & -(\alpha_0/\alpha_1) G^\sigma_{m-1} \\ (\alpha_2/\alpha_1)^{\sigma^{-1}} G_{m-1} & G_m \end{pmatrix},$$

$$G_0 = 1, \quad G_1 = -1, \quad G_k + G^\sigma_{k-1} + u G^{\sigma^2}_{k-2} = 0.$$

The polynomial $X^{\sigma^2+1} - X^{\sigma+1} - X^{\sigma} + X$

Theorem (Lia-Longobardi-G.M.-Trombetti, submitted)

Let $m \geq 5$ *be an odd integer,* $\sigma : x \in \mathbb{F}_{q^m} \longmapsto x^{q^s} \in \mathbb{F}_{q^m}$ *with* $(m, s) = 1$ *and consider the projective polynomials*

$$P_\gamma(X) = X^{\sigma+1} - \gamma X + \gamma \in \mathbb{F}_q[X]$$

with $\gamma \in \mathbb{F}_q$*. The polynomial*

$$Q(X) = X^{\sigma^2+1} - X^{\sigma+1} - X^{\sigma} + X \in \mathbb{F}_q[X]$$

has exactly the elements of $\mathbb{F}_q$ *as roots in* $\mathbb{F}_{q^m}$ *if and only if the set*

$$\{x \in \mathbb{F}_{q^m} \mid P_\gamma(x) = 0 \text{ for some } \gamma \in \mathbb{F}_q\}$$

of zeros of the polynomials P_γ *has size at most* q*, namely if and only if* $G_{m-1}(\gamma) \neq 0$ *for any* $\gamma \in \mathbb{F}_q^*$*.*

The case $m = 7$

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Theorem (Lia-Longobardi-G.M.-Trombetti, submitted)

Let $q = p^e$ *and*

$$\mathcal{U}_\sigma = \{(x, x^\sigma + a, x^{\sigma^2} + b) : x \in \mathbb{F}_{q^7}, a, b \in \mathbb{F}_q\} \subset \mathbb{F}_{q^7}^3,$$

where $\sigma : x \in \mathbb{F}_{q^7} \longrightarrow x^{q^s} \in \mathbb{F}_{q^7}$, $s \in \{1, \dots, 6\}$. *Then,*

- *for* $p = 2, 3, 5$, $\mathcal{U}_\sigma$ *is scattered if* $3 \nmid e$.
- *for* $p > 7$, $\mathcal{U}_\sigma$ *is scattered if* $7/18(1/3 + \sqrt{-3})$ *is not a cube in* $\mathbb{F}_q(\sqrt{-3})$.

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where $\sigma : x \in \mathbb{F}_{q^5} \longrightarrow x^{q^s} \in \mathbb{F}_{q^5}$, $s \in \{1, 2, 3, 4\}$. Then,

- *for $p = 2$ even, $\mathcal{U}_\sigma$ is maximum scattered if e is an odd positive integer.*
- *for p odd, $\mathcal{U}_\sigma$ is maximum scattered if $q \equiv 2, 3 \pmod 5$.*

Intersection characters of $\mathcal{L}_{\mathcal{U}}$

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Theorem (Lia-Longobardi-G.M.-Trombetti, submitted)

Let $L_U \subseteq \mathrm{PG}(2, q^m)$ be a linear set of rank $m + 2$, $m \geq 5$. If U contains an m-dimensional 2-scattered subspace W, then the linear set $L_{\mathcal{U}}$ has exactly three characters $\{q + 1, q^2 + q + 1, q^3 + q^2 + q + 1\}$ with respect to the lines.

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Remark

- *$d = m - 2$ is the maximum possible value for the minimum distance of an $[m + 2, 3]_{q^m/q}$ rank metric code for every $m \geq 3$.*
- *$d_2 = m + 1$ and this is the largest possible value of an $[m + 2, 3]_{q^m/q}$ rank metric code*

Open problems

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Open Problem 3. Compare examples of maximum scattered linear sets in $\mathrm{PG}(2,q^5)$ found by Bartoli, Csajbók, G.M., Trombetti with those belonging to our infinite family.

Cutting q-systems and rank-metric codes

Problem

Construct minimal rank-metric codes reaching the lower bound on the dimension of the associated q-system and with the maximum value for the corresponding values of generalized rank weights.

$(k, m) \in \{(4, 3), (4, 4)\}$

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Proposition (Bartoli-G.M.-Neri 2023)

Let $u \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$, the $\mathbb{F}_q$-subspace

$$U = \{(\alpha_0 + \alpha_1 u, \alpha_2 + \alpha_3 u, \alpha_4 + \alpha_5 u, \alpha_6 + \alpha_7 u) \;:\; \alpha_i \in \mathbb{F}_q\}$$

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Theorem (Bartoli-G.M.-Neri 2023)

If $q = 2^h$ and $h \equiv 1 \pmod 2$, then U is $(2,3)$-evasive, hence U is cutting and it produces a <u>minimal</u> $[8,4,3]_{q^4/q}$ MRD code.

$k = m = 4$

Code	is MRD?	$d_{rk,1}$	$d_{rk,2}$	$d_{rk,3}$	$d_{rk,4}$	is minimal?
$\mathcal{C}$	yes	3	5	7	8	yes
$\mathcal{D}_1 \oplus \mathcal{D}_2$	yes	3	4	7	8	no

Table: The table recaps the properties and the generalized rank weights of the code $\mathcal{C}$ compared to those of $[8,4]_{q^4/q}$ MRD codes obtained as direct sum of two $[4,2]_{q^4/q}$ MRD codes.

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Proposition

The dual code $\mathcal{C}^\perp$ is also an $[8,4,(3,5,7,8)]_{q^4/q}$ code. Also, $\mathcal{C}^\perp$ is equivalent to $\mathcal{C}$.

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Open Problem 3. Generalize the construction of the $[8, 4]_{q^m/q}$ q-system U in order to obtain more general short minimal rank-metric codes.

Covering problem [Cohen, Honkala, Litsyn, Lobstein 1997]

Given a vector space over a finite fied, a metric, and a positive integer ρ, what is the smallest number of sheres of radius ρ that can be placed in such a way that every vector in the space is contained in at least of them?

Covering radius

Covering radius

$\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ a linear code, $d_* : \mathbb{F}_{q^m}^n \times \mathbb{F}_{q^m}^n \to \mathbb{R}_{\geq 0}$ a distance

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Definition

The covering radius of $\mathcal{C}$ w.r.t. d_* is the integer

$$\rho_*(\mathcal{C}) := \max_{v \in \mathbb{F}_{q^m}^n} \min_{c \in \mathcal{C}} d_*(v, c) = \min \left\{ \rho : \bigcup_{c \in \mathcal{C}} B_*(c, \rho) = \mathbb{F}_{q^n} \right\}.$$

The metrics considered are

- the Hamming metric: $d_H(v, w) := |\{i : v_i \neq w_i\}|$
  [Brualdi-Pless-Wilson 1989, Cohen-Honkala-Litsyn-Lobstein 1997, Davydov-Östergard 2000, Davydov-Giulietti-Marcugini-Pambianco 2011, etc.]
- the rank metric $d_{\mathrm{rk}}(v, w) := \dim_{\mathbb{F}_q} \langle v_1 - w_1, \ldots, v_n - w_n \rangle_{\mathbb{F}_q}$
  [Byrne-Ravagnani 2020, Gadouleau 2009]

Covering radius

Covering radius

Definition

$\mathcal{S} \subseteq \mathrm{PG}(k-1, q^m)$ is $(\rho - 1)$-saturating if

- *for each point Q of $\mathrm{PG}(k-1, q^m)$ there exist ρ points $P_1, P_2, \ldots, P_\rho \in \mathcal{S}$ s.t.*

$$Q \in \langle P_1, \ldots, P_\rho \rangle_{\mathbb{F}_{q^m}};$$

- *ρ is the smallest value with this property.*

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Definition

An $[n,k]_{q^m/q}$ system $\mathcal{U}$ of $V(k, q^m)$ is a rank ρ-saturating system if $L_{\mathcal{U}}$ is a $(\rho-1)$-saturating set in $\mathrm{PG}(k-1, q^m)$

Covering radius

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Let $\mathcal{U}$ be an $[n,k]_{q^m/q}$ system associated with a code $\mathcal{C}$. Then $\mathcal{U}$ is rank-ρ-saturating if and only if $\rho_{rk}(\mathcal{C}^{\perp}) = \rho$

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Corollary (Bonini-Borello-Byrne 2023)

Let $\mathcal{C}$ be an $[n, k]_{q^m/q}$ generalized Gabidulin code and let $\mathcal{U}$ be an $[n, k]_{q^m/q}$ system associated with $\mathcal{C}$. Then $\mathcal{U}$ is a rank-k-saturating system.

Covering problem

Classical covering problem: given n and ρ estimate the least number of spheres of radius ρ such that the union of the balls of radius ρ covers the ambient vector space of dimension n.

Covering problem

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In terms of rank ρ-saturating system one may ask to find the least value of n such that an $[n, k]_{q^m/q}$ rank ρ-saturating system exists, for fixed k and ρ.

How small?

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$$s_{q^m/q}(k,\rho) := \min\left\{\dim_{\mathbb{F}_q}\mathcal{U} \colon \mathcal{U} \text{ is a rank } \rho\text{-saturating system in } V(k,q^m)\right\}$$

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Theorem (Gadouleau-Yan 2008, Bonini-Borello-Byrne 2023)

Let $\mathcal{U}$ be a rank ρ-saturating system in $V(k,q^m)$. Then

$$\begin{bmatrix} \dim_{\mathbb{F}_q}\mathcal{U} \\ \rho \end{bmatrix}_q \geq q^{m(k-\rho)}$$

In particular,

$$s_{q^m/q}(k,\rho) \geq \begin{cases} \left\lceil \frac{mk}{\rho} \right\rceil - m + \rho & \text{if } q > 2, \\ \left\lceil \frac{mk-1}{\rho} \right\rceil - m + \rho & \text{if } q = 2, \rho > 1, \\ m(k-1)+1 & \text{if } q = 2, \rho = 1. \end{cases}$$

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Some upper bounds [Bonini-Borello-Byrne 2023]

$$\begin{aligned} s_{q^m/q}(k,\rho) &\leq m(k-\rho)+\rho \\ s_{q^m/q}(k,(s-1)t+1) &\leq tk-(t-1)((s-1)t+1) \end{aligned}$$

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and the equality holds:

$$\begin{aligned} s_{q^m/q}(k,1) &= m(k-1)+1 \\ s_{q^m/q}(k,k) &= k \\ s_{q^{2r}/q}(2r,2r-1) &= 2r+1 \end{aligned}$$

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Theorem (Bartoli-Borello-G.M., 202?)

If $k = \rho t$ *for some integer* $t \geq 1$

$$s_{q^m/q}(\rho t, \rho) = m(t-1) + \rho.$$

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Proof.

$$\mathcal{U} := \{(x_1, x_1^q, \ldots, x_1^{q^{\rho-1}}, \ldots, x_{t-1}, x_{t-1}^q, \ldots, x_{t-1}^{q^{\rho-1}}, a_1, \ldots, a_\rho) : x_i \in \mathbb{F}_{q^m}, a_j \in \mathbb{F}_q\}$$

is rank ρ-saturating. □

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Theorem (Bonini-Borello-Byrne 2023)

Let $\mathcal{U}$ be an $[n,k]_{q^m/q}$ system. If $\mathcal{U}$ is cutting, then it is a rank-$(k-1)$-saturating $[n,k]_{q^{m(k-1)}/q}$ system.

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Since, from [Alfarano-Borello-Neri-Ravagnani 2022],

- If $\mathcal{U}$ is a linear cutting $[n,k]_{q^m/q}$ system then $\dim_{\mathbb{F}_q}\mathcal{U} \geq m+k-1$
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Corollary

For every $m,k \geq 2$,

$$k+m-1 \leq s_{q^{m(k-1)}/q}(k,k-1) \leq l_{q^m/q}(k) \leq 2k+m-2,$$

where $l_{q^m/q}(k)$ is the minimum $\mathbb{F}_q$-dimension of a linear cutting blocking set in $\mathbb{F}_{q^m}^k$.

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Alfarano-Borello-Neri-Ravagnani 2022

Bartoli-Csajbók-M.-Trombetti 2021

Lia-Longobardi-M.-Trombetti 202?

$\Downarrow$

$s_{q^{2r}/q}(3,2) = r+2,$ for $r \neq 3,5 \mod 6$ and $r \geq 4$,

$s_{q^{2r}/q}(3,2) = r+2,$ for $\gcd(r,(q^{2s}-q^s+1)!) = 1$, r odd, $1 \leq s \leq r$, $\gcd(r,s) = 1$,

$s_{q^{10}/q}(3,2) = 7,$ for $q = p^{15h+s}$, $p \in \{2,3\}$, $\gcd(s,15) = 1$ and for $q = 5^{15h+1}$,

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Theorem (Bonini-Borello-Byrne 2023)

For all positive integers m, k, k', $\rho \in [\min\{k, m\}]$, $\rho' \in [\min\{k', m\}]$.

(a) *If* $\rho < \min\{k, m\}$, *then* $s_{q^m/q}(k, \rho + 1) \leq s_{q^m/q}(k, \rho)$.

(b) $s_{q^m/q}(k, \rho) < s_{q^m/q}(k + 1, \rho)$.

(c) *If* $\rho < m$, *then* $s_{q^m/q}(k + 1, \rho + 1) \leq s_{q^m/q}(k, \rho) + 1$.

(d) *If* $\rho + \rho' \leq \min\{k + k', m\}$, $s_{q^m/q}(k + k', \rho + \rho') \leq s_{q^m/q}(k, \rho) + s_{q^m/q}(k', \rho')$.

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Theorem (Bartoli-Borello-G.M., 202?)

Let $m \geq h+1$. If $\mathcal{U}$ is an h-scattered $\mathbb{F}_q$-subspace of $V(k, q^m)$ of $\mathbb{F}_q$-dimension (at least) $\left\lfloor \frac{m(k-1)}{h+1} \right\rfloor + 1$, then $\mathcal{U}$ is rank ρ-saturating, with $\rho \leq h+1$.

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Proof.

Let $v \notin \mathcal{U}$ and project $\mathcal{U}$ from v to a hyperplane Γ of $V(k, q^m)$ not containing v. The projection $\overline{\mathcal{U}}$ is a subspace of Γ of $\mathbb{F}_q$-dimension $\left\lfloor \frac{m(k-1)}{h+1} \right\rfloor + 1$ which is not h-scattered. Then there exists an $\mathbb{F}_{q^m}$-subspace M of Γ of $\mathbb{F}_{q^m}$-dimension h such that $\dim_{\mathbb{F}_q}(M \cap \overline{\mathcal{U}}) \geq h+1$. Let $N = \langle v, M \rangle_{\mathbb{F}_{q^m}}$ and $u_1, \ldots, u_{h+1} \in N \cap \mathcal{U}$ be $h+1$ linearly independent vectors over $\mathbb{F}_q$.
$N = \langle u_1, \ldots, u_{h+1} \rangle_{\mathbb{F}_{q^m}} \Rightarrow v \in N$ is $(h+1)$-saturated by $\mathcal{U}$. □

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Corollary

The 2-maximum $\mathbb{F}_q$-scattered subspace of $V(4,q^6)$, $q = 2^{2h+1}$, $h \geq 1$ found by Bartoli-Giannoni-Ghiandoni-G.M. is rank ρ-saturating, with $\rho \leq 3$. MAGMA computations show that it is rank 2-saturating at least when $q = 2$.

How small?

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Open Problem Show that the Bartoli-Giannoni-Ghiandoni-G.M. example is rank 2-saturating for each $q = 2^{2h+1}$, with $h > 1$

How small?

Corollary (Bartoli-Borello-G.M., 202?)

Let $m \geq 4$ be an even integer. For $q > 2$, if $r = 3$ and $m < 12$ or $r > 3$ odd, then

$$\frac{mr}{2} - 2 \leq s_{q^m/q}\left(\frac{r(m-2)}{2}, m-2\right) \leq \frac{mr}{2} - 1.$$

For $q = 2$, the same holds if $r = 3$ and $m < 10$ or $r > 3$ odd.

Corollary (Bartoli-Borello-G.M., 202?)

If mk is even, then

$$\left\lceil \frac{m(k-2)}{2} \right\rceil + 2 \leq s_{q^m/q}(k,2) \leq \left\lfloor \frac{m(k-1)}{2} \right\rfloor + 1 = \left\lceil \frac{m(k-2)}{2} \right\rceil + 2 + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

In particular $s_{q^2/q}(k,2) = k$. Moreover,

$$s_{q^5/q}(3,2) \in \{5,6\}$$

for $q = p^t$ with $p \in \{2,3,5\}$ and

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We know

$$s_{q^4/q}(3,2) \geq 4.$$

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Theorem (Bartoli-Borello-G.M., 202?)

If q is even and large enough, then

$$s_{q^4/q}(3,2) = 5 > 4.$$

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Sketch of the proof.

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Remark

Let L_U be a linear set in $\mathrm{PG}(k-1, q^m)$, H a hyperplane and P a point not belonging to L_U nor to H. If the projection of L_U from P to H is scattered, then the point is not 1-saturated, because otherwise in the projection we would find a point of weight at least 2.

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A rank 2-saturating U of rank 4 in $V(3,q^4)$, up to $\mathrm{GL}(3,q^4)$-equivalence, has one of these forms:

1) $U = \left\{(x, x^q, x^{q^2}) : x \in \mathbb{F}_{q^4}\right\}$;

2) $U_\alpha = \left\{(x, x^q + \alpha x^{q^2}, x^{q^3}) : x \in \mathbb{F}_{q^4}\right\}$, with $\alpha \in \mathbb{F}^*_{q^4}$;

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4) $U_{\alpha,\beta} = \left\{(x, x^q + \alpha x^{q^3}, x^{q^2} + \beta x^{q^3}) : x \in \mathbb{F}_{q^4}\right\}$ with $\alpha, \beta \in \mathbb{F}^*_{q^4}$;

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In each case, we get a point of the plane not contained in L_U through which does not pass any secant line to L_U.

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- Can we generalize this framework to other metrics (e.g. the sum-rank metric)?

Thank you for your attention!