Ferrers Diagram Rank-Metric Codes

Alessandro Neri

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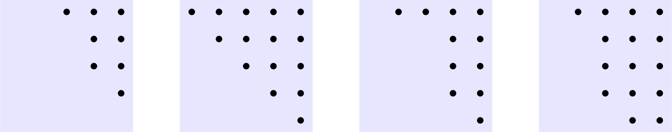

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Contents

Part 0 – Warm Up
Linear Spaces of Matrices: A Few Questions

Questions for the Audience

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It depends on the field $\mathbb{F}$!

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$$\frac{(n-r+1)(n-r+2)}{2}.$$

- (Etzion, Gorla, Ravagnani, Wachter Zeh 2016) if $|\mathbb{F}| \geq n-1$;
- (N., Stanojkovski 2023) for every $\mathbb{F}$.

Part I
(Ferrers Diagram) Rank-Metric Codes

Contents

Rank-Metric Codes

The **rank distance** d_{rk} on $\mathbb{F}^{n\times m}$ is defined by (Delsarte, 1978)

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N.B. We could also consider nonlinear rank-metric codes. But not in this talk.

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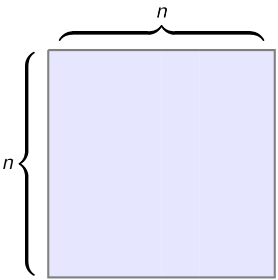

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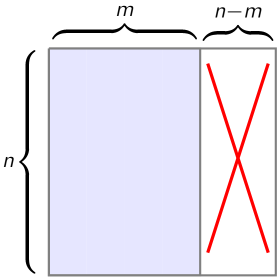
- Remove the last $n - m$ columns (**puncturing)** from an $[n \times n, n(n - (r + (n - m)) + 1), r + (n - m)]_{\mathbb{F}}$ MRD code.



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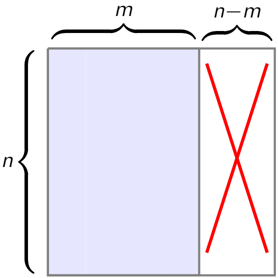

It is enough to do it in the **square case.**

Delsarte-Gabidulin Codes (I)

$$k \leq n(n-r+1).$$

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- $\mathbb{F} = \mathbb{F}_q$; • $\mathbb{L} = \mathbb{F}_{q^n}$; • $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{L}/\mathbb{F})$; • $\sigma(\beta) = \beta^q$.

Skew-Algebra Isomorphism

$$\bigoplus_{i=0}^{n-1} \mathbb{L} \cdot \sigma^i =: \mathbb{L}[\sigma] \cong \mathrm{End}_{\mathbb{F}}(\mathbb{L}) \cong \mathbb{F}^{n\times n}.$$

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$$\bigoplus_{i=0}^{n-1} \mathbb{L} \cdot x^{q^i} =: \mathcal{L}_{n,q}[x] \cong \mathrm{End}_{\mathbb{F}}(\mathbb{L}) \cong \mathbb{F}^{n\times n}.$$

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(Artin's Theorem of independence of characters) **(Linearized Polynomials)**

Delsarte-Gabidulin Codes (II)

Nullity-Degree Bound

If $P(\sigma) \in \mathbb{L}[\sigma]$ is nonzero, then

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Delsarte-Gabidulin construction: (Delsarte, 1978 - Gabidulin, 1985 - ...)

$$\mathbb{L}[\sigma]_{n-r} := \bigoplus_{i=0}^{n-r} \mathbb{L} \cdot \sigma^i$$

is (isomorphic to) an $[n \times n, n(n-r+1), r]_{\mathbb{F}}$ code.

Communication Through a Network

Question: Why do we care about rank-metric codes?

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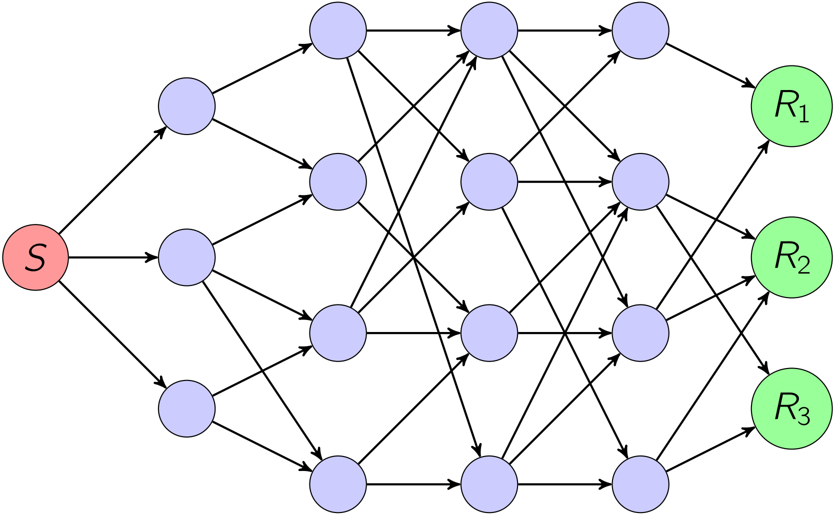
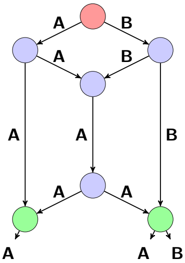


Routing vs. Network Coding

Routing:

- Nodes can only forward packets
- If two packets are received, one has to be discarded



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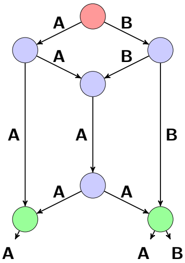
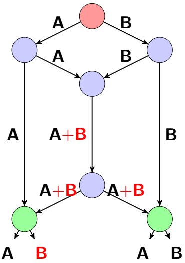


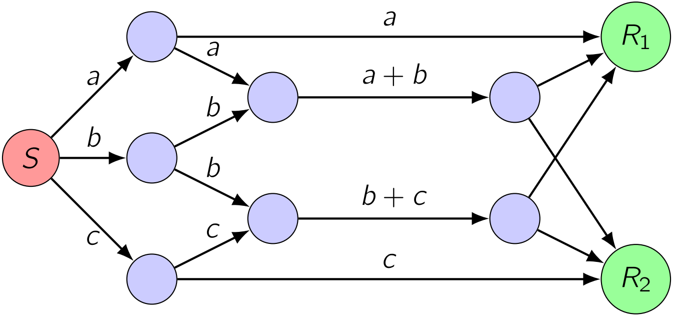
Network coding:

- Nodes can forward linear combinations

$\Longrightarrow$ higher throughput achievable!



Example



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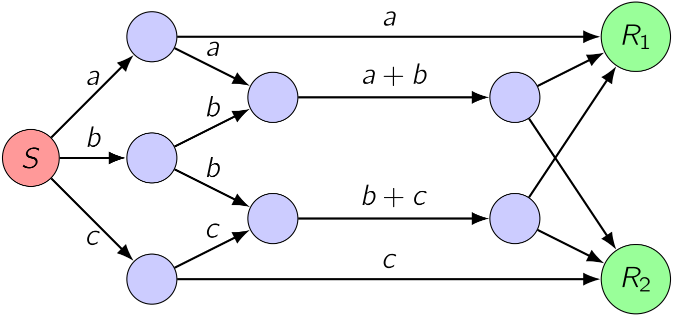

Receiver R_1 and Receiver R_2 get, respectively, the following packets:

$$\begin{pmatrix} a \\ a+b \\ b+c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \begin{pmatrix} a+b \\ b+c \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

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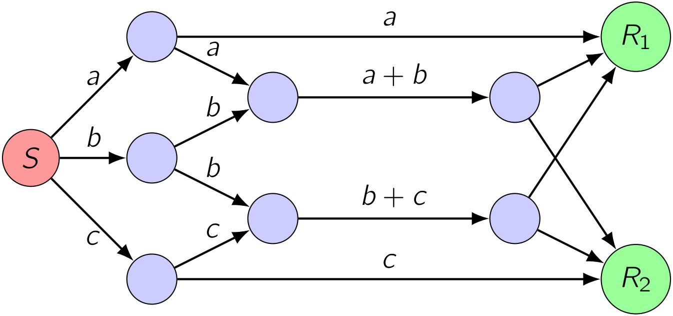

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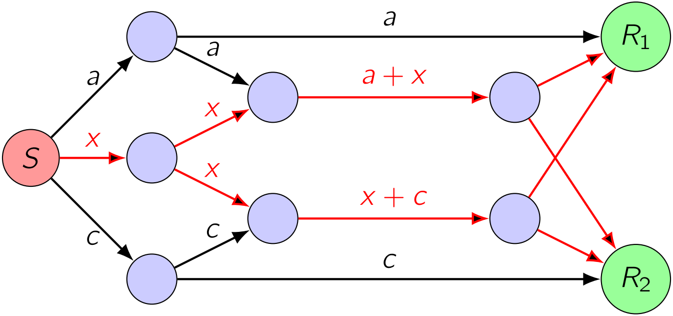
N.B. What is sent and what is received have **the same** 3**-dim'l rowspace**

Example: An Error Occurs



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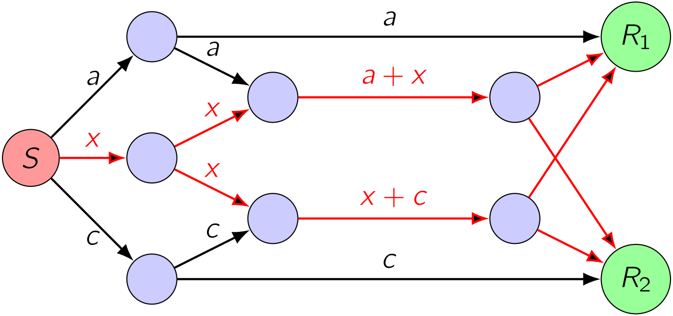

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N.B. The two rowspaces still share a 2-**dim'l subspace.**

Subspace Codes

(Koetter, Kschischang, 2008)

A **(constant dimension) subspace code** is a subset of the Grassmannian $\mathcal{G}_{\mathbb{F}}(n, n+m)$ endowed with the

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Natural notion of $\mathbb{F}$-linearity

Cell Decomposition of the Grassmannian

The above embedding can be generalized.

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- Every $\mathcal{U} \in \mathcal{G}_{\mathbb{F}}(n, n+m)$ has a unique Reduced Row Echelon Form. This gives a set of pivot positions.
- The space $\mathcal{G}_{\mathbb{F}}(n, n+m)$ can be partitioned into $\binom{n+m}{n}$ **Schubert cells** according to the n pivot positions of the subspaces.
- To each set of n pivot positions, we can associate a **Ferrers diagram**[1], and a **Ferrers diagram matrix space**[1]!

[1] yet to be defined

Pivot Positions $\longleftrightarrow$ Ferrers Diagrams

$n = m = 5;$ pivots $P = \{2, 5, 6, 7, 9\}$

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$n = m = 5;$ pivots $P = \{2, 5, 6, 7, 9\}$

$$\mathcal{G}_{\mathbb{F}}^{P}(5,10) = \left\{ \mathbf{rs} \begin{pmatrix} 0 & 1 & a_{1,2} & a_{1,3} & 0 & 0 & 0 & a_{1,4} & 0 & a_{1,5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_{2,4} & 0 & a_{2,5} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & a_{3,4} & 0 & a_{3,5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{4,4} & 0 & a_{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{5,5} \end{pmatrix} : a_{i,j} \in \mathbb{F} \right\}.$$

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$$P \longleftrightarrow \mathcal{D}_P = \begin{matrix} \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \\ & & \bullet & \bullet \\ & & \bullet & \bullet \\ & & & \bullet \end{matrix}$$

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Isometry (Etzion, Silberstein 2009)

$$(\mathcal{G}_{\mathbb{F}}^{P}(n, 2n), \mathrm{d_I}) \cong (\mathbb{F}^{\mathcal{D}_P}, \mathrm{d_{rk}})$$

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- Natural notion of $\mathbb{F}$-linearity
- Multilevel construction for subspace codes (Etzion, Silberstein 2009)

Ferrers Diagrams

A **Ferrers diagram** $\mathcal{D}$ of order n is a subset of $[n]^2$ s.t.:

- $\mathcal{D} \neq \emptyset$;
- If $(i, j) \in \mathcal{D}$, then $(i, j') \in \mathcal{D}$ for every $j' \in \{j, \ldots, n\}$;
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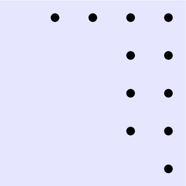
Example

$$\begin{aligned}\mathcal{D} = \{&(1,2),(1,3),(1,4),(1,5),\\ &(2,4),(2,5),(3,4),(3,5),\\ &(4,4),(4,5),(5,5)\}\end{aligned}$$



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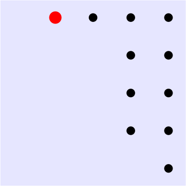

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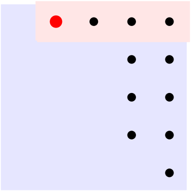

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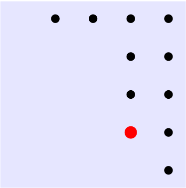

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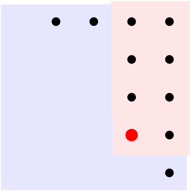

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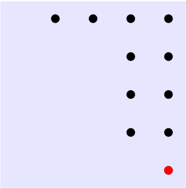

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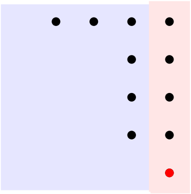

Representations of Ferrers Diagrams

Subset of $[n]^2$

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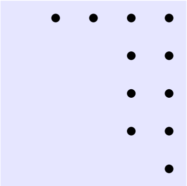
Graphical Repr.



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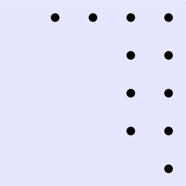

Vector of Columns

$$\mathcal{D} = (0,1,1,4,5)$$

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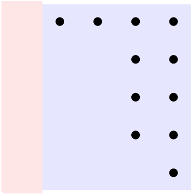

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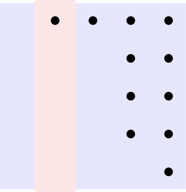

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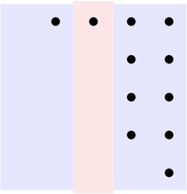

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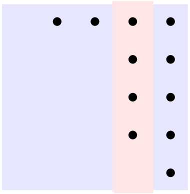

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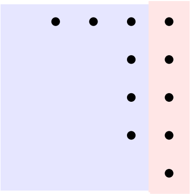

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Ferrers Diagram Matrix Spaces

- (Finite) field $\mathbb{F}$.
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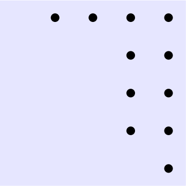

$$\mathbb{F}^{\mathcal{D}} = \left\{ \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & 0 & 0 & a_{2,4} & a_{2,5} \\ 0 & 0 & 0 & a_{3,4} & a_{3,5} \\ 0 & 0 & 0 & a_{4,4} & a_{4,5} \\ 0 & 0 & 0 & 0 & a_{5,5} \end{pmatrix} \, : \, a_{i,j} \in \mathbb{F} \right\}.$$

Ferrers Diagram Rank-Metric Codes

The rank distance d_{rk} on $\mathbb{F}^{n\times n}$ is defined by

$$\mathrm{d}_{\mathrm{rk}}(X,Y) := \mathrm{rk}(X-Y), \quad X,Y \in \mathbb{F}^{n\times n}.$$

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A $[\mathcal{D}, k, r]_{\mathbb{F}}$ **Ferrers diagram rank-metric code** $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}^{\mathcal{D}}$ endowed with the rank distance. The **minimum rank distance** r is equal to the **minimum rank**

$$r = \min\{\mathrm{rk}(A) \mid A \in \mathcal{C} \setminus \{0\}\}.$$

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N.B. We could also consider nonlinear Ferrers diagram rank-metric codes. But not in this talk.

Part II
The Etzion-Silberstein Conjecture

Contents

A Singleton-like Bound

Question: Let $\mathcal{D} = (c_1, \ldots, c_n)$ be a Ferrers diagram, $2 \leq r \leq n$. Find

$$\kappa_{\mathbb{F}}(\mathcal{D}, r) = \max\{k \in \mathbb{N} : \exists \text{ an } [\mathcal{D}, k, r]_{\mathbb{F}} \text{ code }\}.$$

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$$\kappa_{\mathbb{F}}(\mathcal{D}, r) \leq \nu_{\min}(\mathcal{D}, r) := \min_{0 \leq j < r} \nu_j(\mathcal{D}, r)$$

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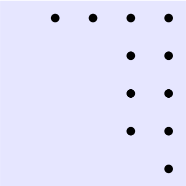
Example:

$r = 3$



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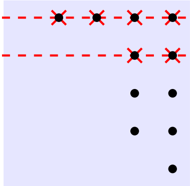

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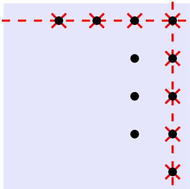

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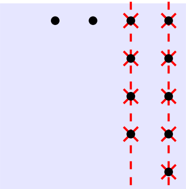
$$\kappa_{\mathbb{F}}(\mathcal{D}, r) \leq \nu_{\min}(\mathcal{D}, r) := \min_{0 \leq j < r} \nu_j(\mathcal{D}, r)$$

$$= \min_{0 \leq j < r} |\mathcal{D}| - |(\{ \text{ dots in first } r - j - 1 \text{ rows}\} \cup \{ \text{ dots in last } j \text{ col's}\})|$$

Example:

$r = 3$



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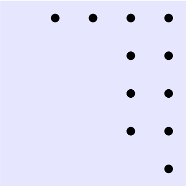

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Two constructive proofs for some special cases

(1) **Subcodes of MRD codes**: (Etzion, Silberstein, 2009) (Etzion, Gorla, Ravagnani, Wachter-Zeh, 2016) (Gorla, Ravagnani, 2017) (Antrobus, Gluesing-Luerssen, 2019) (Liu, Chang, Feng, 2019)

(2) **MDS-constructible Ferrers diagrams** (for fields large enough): (Etzion, Gorla, Ravagnani, Wachter-Zeh, 2016)

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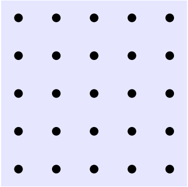

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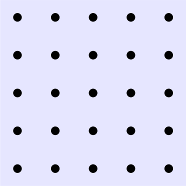

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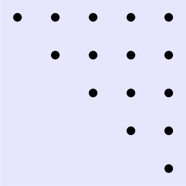

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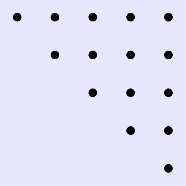

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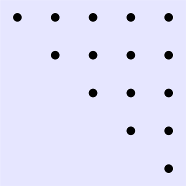

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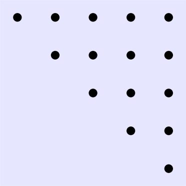

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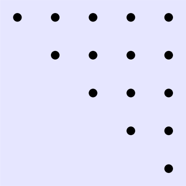

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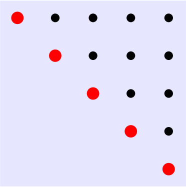

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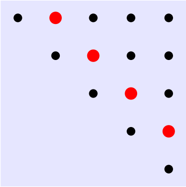

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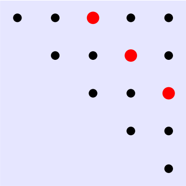

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Formal Definition: The pair $(\mathcal{D}, r)$ is **MDS-constructible** if

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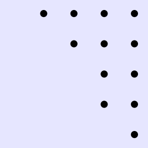

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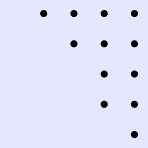

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$$\begin{aligned} \nu_{\mathsf{min}}(\mathcal{D}, 3) &= \min\{5, 4, 3\} &&= 3 \\ \nu_{\mathrm{MDS}}(\mathcal{D}, 3) &= 0 + 2 + 1 + 0 + 0 &&= 3 \end{aligned}$$

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Theorem (Etzion, Gorla, Ravagnani, Wachter-Zeh 2016)

If $(\mathcal{D}, r)$ is MDS-constructible and $|\mathbb{F}| \geq \max_i |\mathcal{D} \cap \Delta_i^n| - 1$, then

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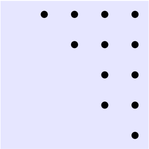

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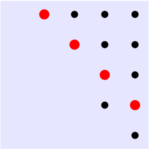

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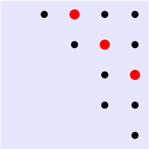

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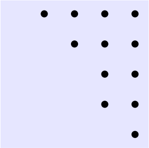

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Part III
Recent Results on the ES Conjecture

Joint work with Mima Stanojkovski

Check out our preprint!

A. Neri. M. Stanojkovski. "A proof of the Etzion-Silberstein conjecture for monotone and MDS-constructible Ferrers diagrams", arXiv:2306.16407, 2023.

SCAN ME

Contents

Our Result in a Nutshell

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If $(\mathcal{D}, r)$ is MDS-constructible, then

$$\kappa_{\mathbb{F}}(\mathcal{D}, r) = \nu_{\min}(\mathcal{D}, r).$$

- Get rid of the field condition. **Completely different construction!**
- First crucial idea: **embed** Ferrers diagrams matrix spaces in **larger matrix spaces**. (Lemmino)
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- Third crucial idea: Use **flag** associated to **nilpotent endomorphism** $\sigma - \mathrm{id}$ when $n = p^m$. (Theorem 2)

Achtung: The ES Conjecture is still **widely open**

The End

Thank you! Dankjewel!

Grazie!

Check out our preprint!

A. Neri. M. Stanojkovski. "A proof of the Etzion-Silberstein conjecture for monotone and MDS-constructible Ferrers diagrams", arXiv:2306.16407, 2023.

SCAN ME

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Achtung: Need a **smart choice** of an $\mathbb{F}$-basis $\mathcal{B}$ of $\mathbb{L}$!

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Smart choice: $\mathcal{B} = (\beta_1, \dots, \beta_n)$ compatible with $\mathcal{F}$: $\langle \beta_1, \dots, \beta_i \rangle_{\mathbb{F}} = \mathcal{F}_i$.

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Strictly monotone Ferrers Diagram:
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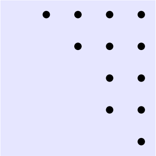

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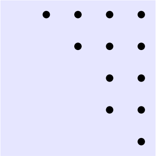

Theorem (N., Stanojkovski, 2023)

Let $\mathcal{D} = (c_1, \ldots, c_n)$ be a **strictly monotone** Ferrers diagram of order $n = p^m$. Then for any $\mathcal{F}$-compatible basis $\mathcal{B}$ of $\mathbb{L}$ we have

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Let $\mathcal{D} = (c_1, \dots, c_n)$ be a **strictly monotone** Ferrers diagram of order $n = p^m$. Then, for every $d \in \{2, \dots n\}$,

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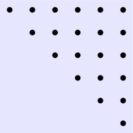
- $\mathbb{F} = \mathbb{F}_2$. $n = 6$, $r = 4$, $\mathcal{D} = \mathcal{T}_6$.
- $\nu_{\mathsf{min}}(\mathcal{D}, 4) = 6$.



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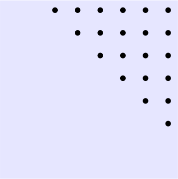

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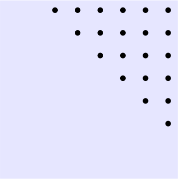

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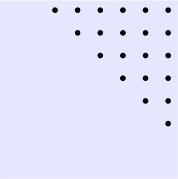

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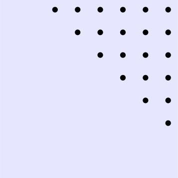

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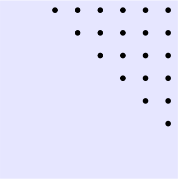

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The $[\mathcal{D}', 6, 4]_{\mathbb{F}_2}$ code representation in $\mathbb{F}_{2^8}[\sigma; \mathcal{D}']$ is

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$$\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1&0&1&0\\ 0&0&0&1&1&1&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

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