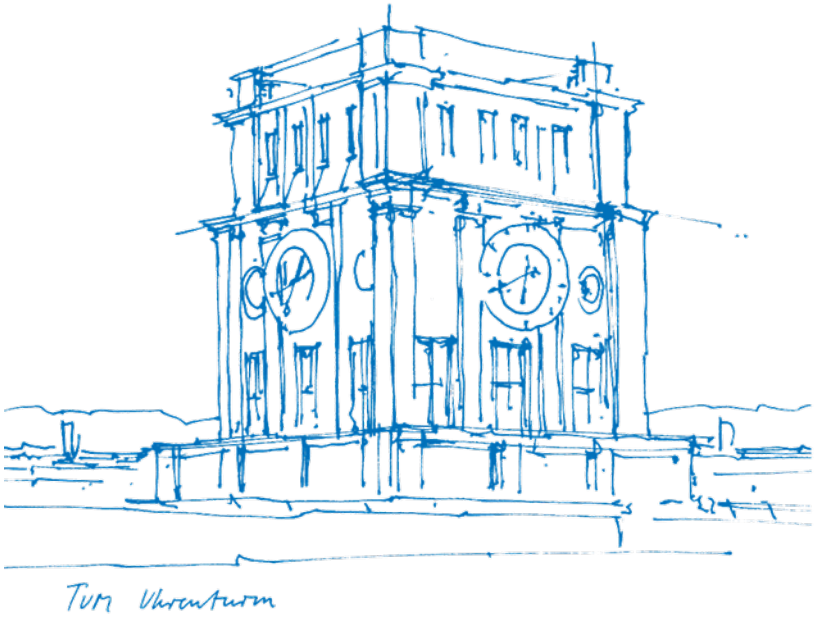
Polytopal characterization of irreducible Ferrers diagrams in the Etzion-Silberstein conjecture

Hugo Beeloo-Sauerbier Couvée
Technical University of Munich

Joint work with **Alessandro Neri**
(University of Naples Federico II)

OpeRa 2026 - Bordeaux
Open Problems on Rank-Metric Codes



Alessandro Neri

Etzion-Silberstein Conjecture

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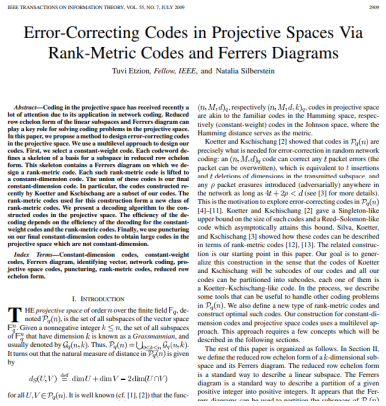
- Tuvi Etzion & Natalia Silberstein (2009)



IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 55, NO. 7, JULY 2009 2909

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Tuvi Etzion, *Fellow, IEEE*, and Natalia Silberstein

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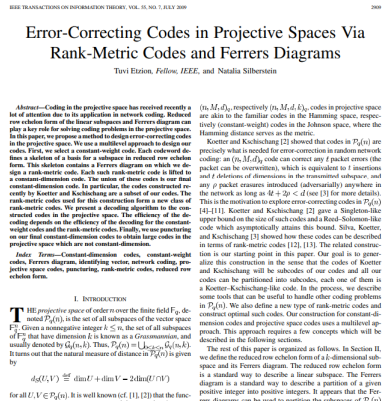

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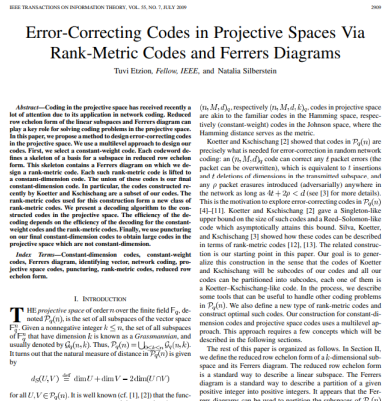

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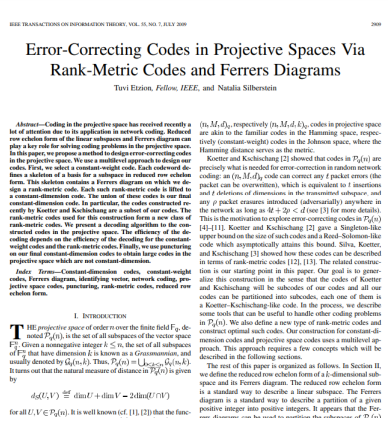

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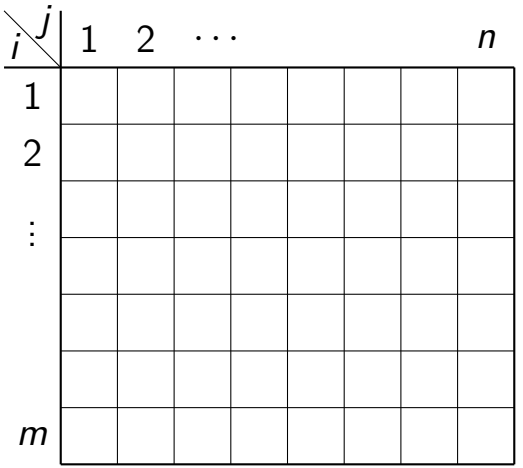

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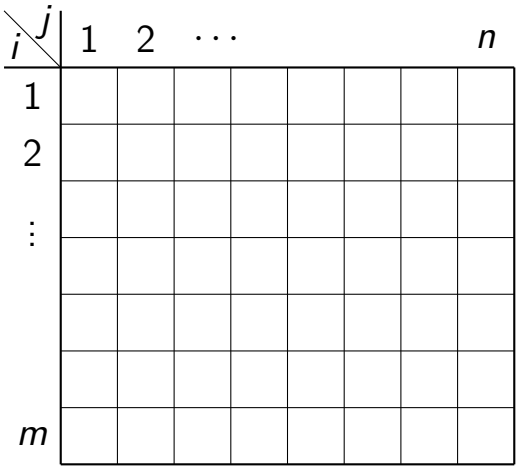

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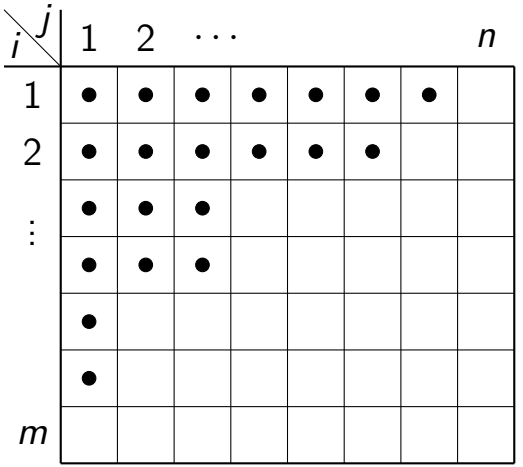

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  0 outside $\mathcal{D}$, any elements $\bullet \in \mathbb{F}_q$ inside $\mathcal{D}$

i \ j	1	2	$\cdots$					n
1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	0
2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	0	0
$\vdots$	$\bullet$	$\bullet$	$\bullet$	0	0	0	0	0
	$\bullet$	$\bullet$	$\bullet$	0	0	0	0	0
	$\bullet$	0	0	0	0	0	0	0
	$\bullet$	0	0	0	0	0	0	0
m	0	0	0	0	0	0	0	0

Singleton-like bound and conjecture

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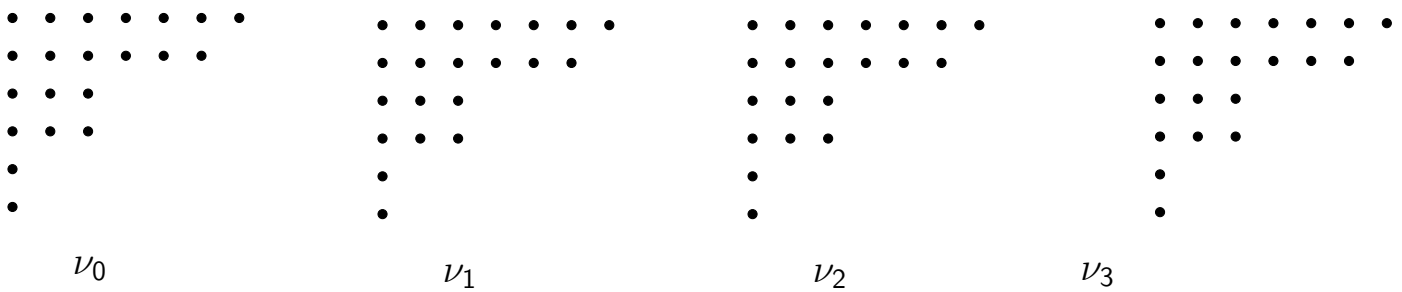

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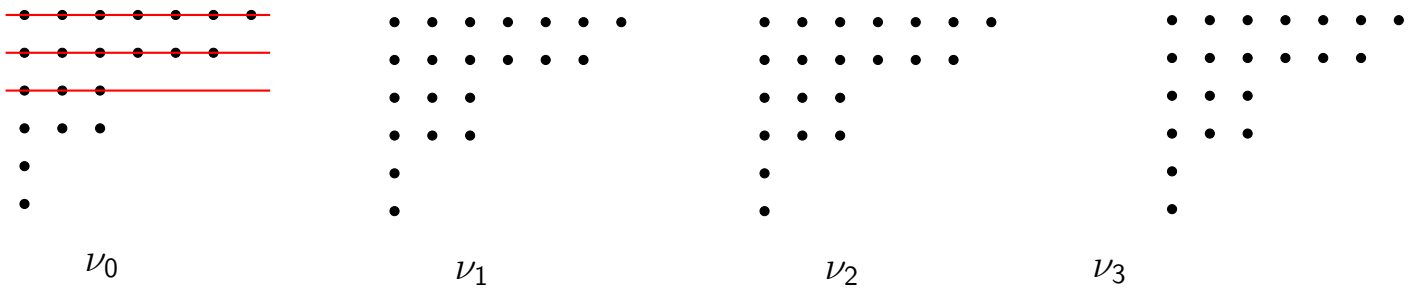

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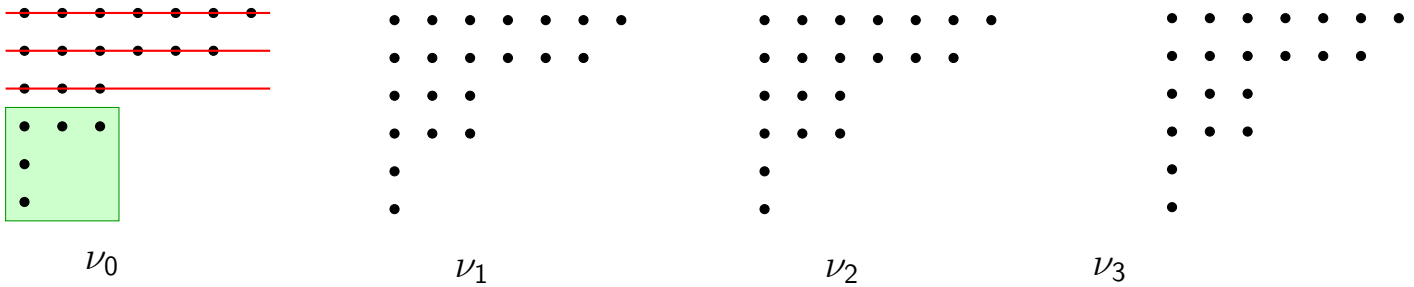

ν_0 ν_1 ν_2 ν_3

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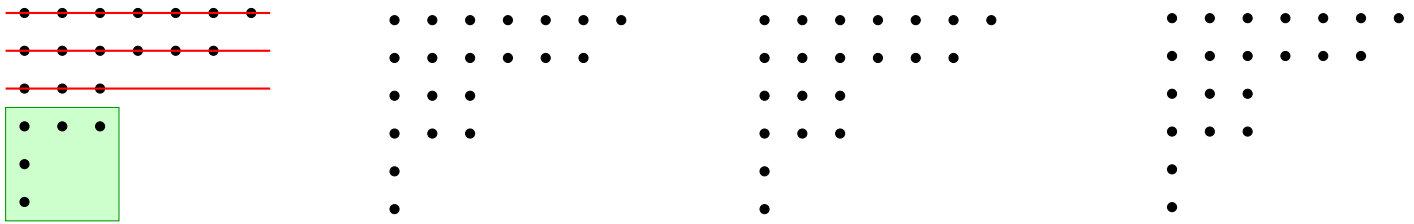

$\nu_0 = 5$ ν_1 ν_2 ν_3

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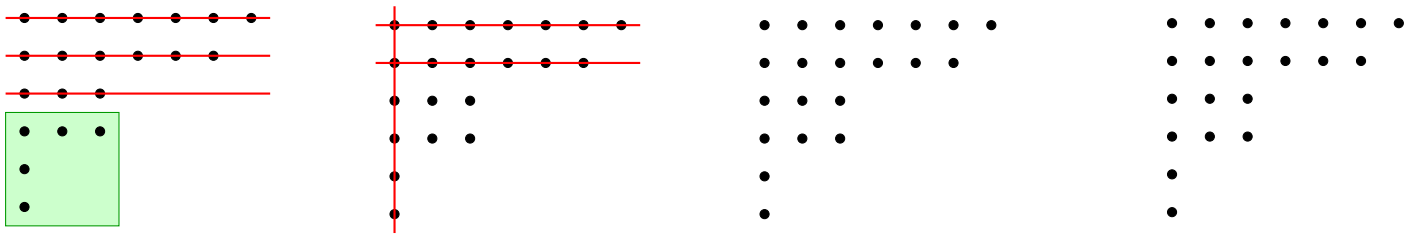
- **Singleton-like bound**: $k \leq \min\{\nu_i\}_{i=0}^{d-1}$

$$\nu_i := |\mathcal{D} \setminus \{\text{first } i \text{ columns and first } d-1-i \text{ rows}\}|$$

Ex.
$d = 4$



$\nu_0 = 5$ ν_1 ν_2 ν_3

Singleton-like bound and conjecture

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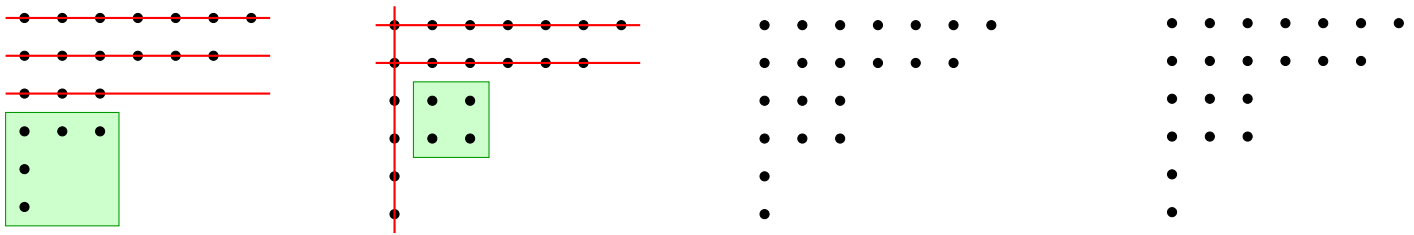

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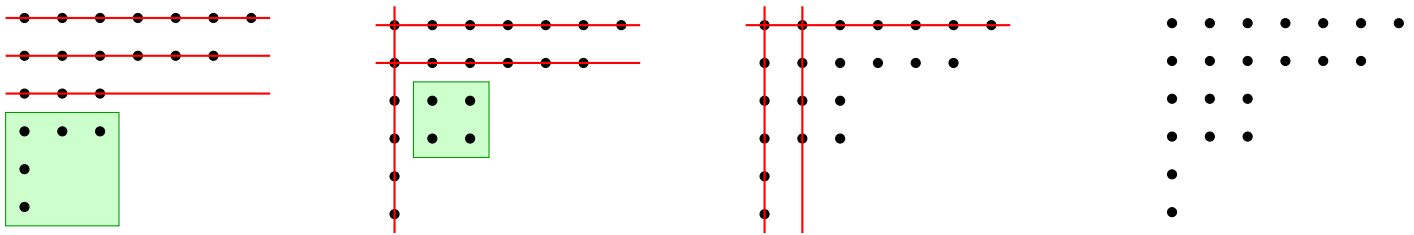

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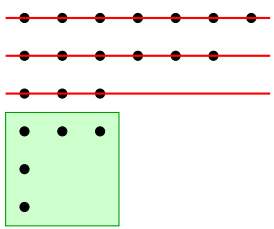

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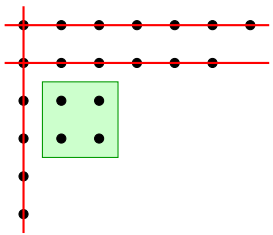
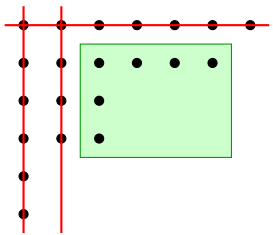
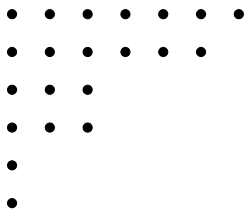


$\nu_1 = 4$



$\nu_2 = 6$



ν_3

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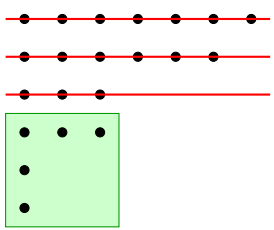

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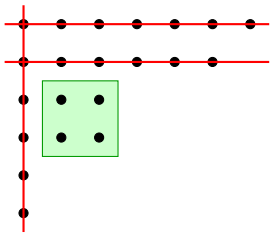
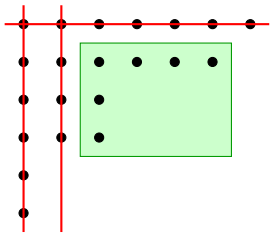
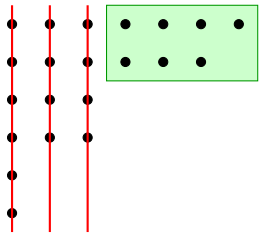


$\nu_1 = 4$



$\nu_2 = 6$



$\nu_3 = 7 \quad \rightarrow \quad \min\{\nu_i\} = 4$

Singleton-like bound and conjecture

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- Codes attaining bound: **Maximal Ferrers Diagram (MFD)** codes

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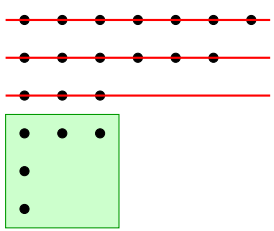

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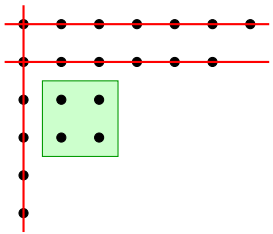
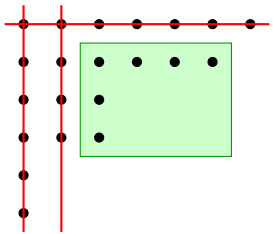
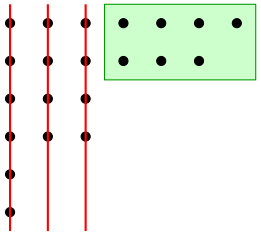


$\nu_1 = 4$



$\nu_2 = 6$



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- Codes attaining bound: **Maximal Ferrers Diagram (MFD)** codes
- **ES-conjecture**: for all $(q, \mathcal{D}, d)$ there exists an MFD code (only some cases solved)

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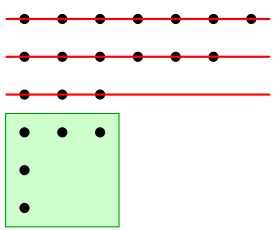

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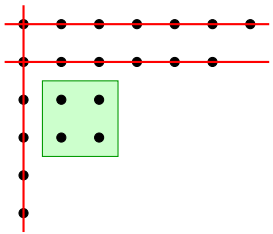

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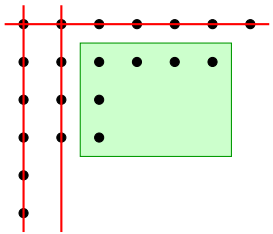

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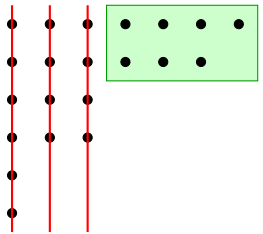

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Obtaining new MFD codes from old

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- Consider diagrams $\mathcal{D}$ and $\mathcal{D}' = \mathcal{D} \cup \{p\}$

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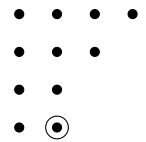
Ex. $d = 3$



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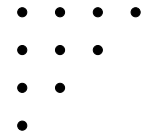

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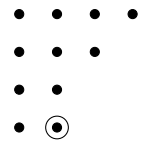

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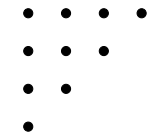

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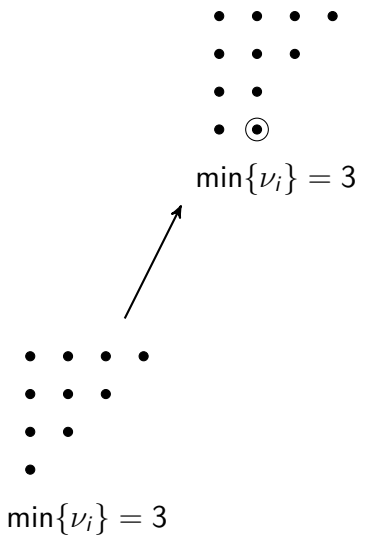

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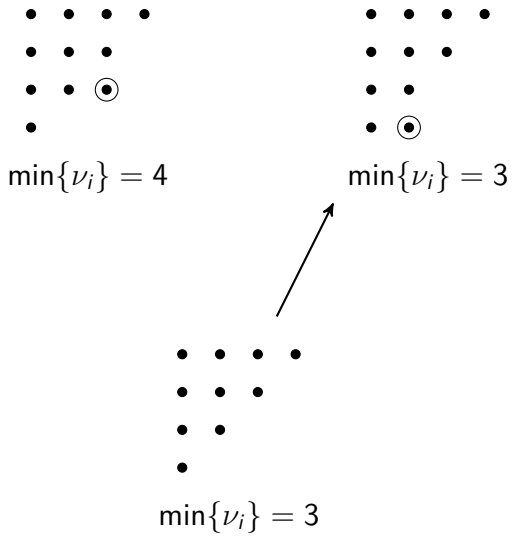

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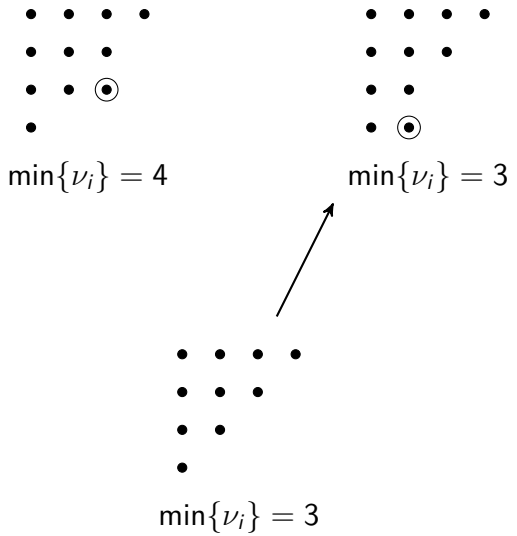

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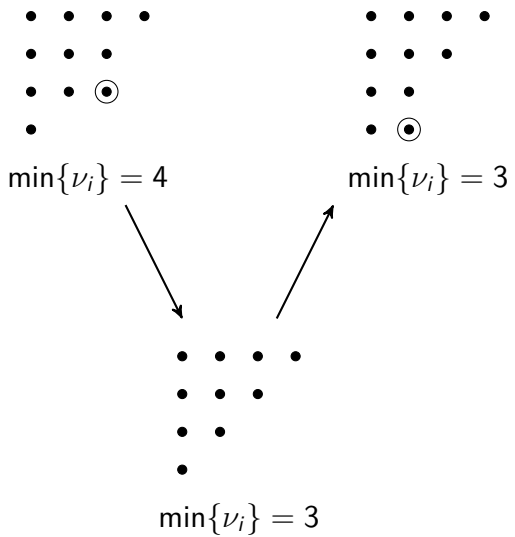

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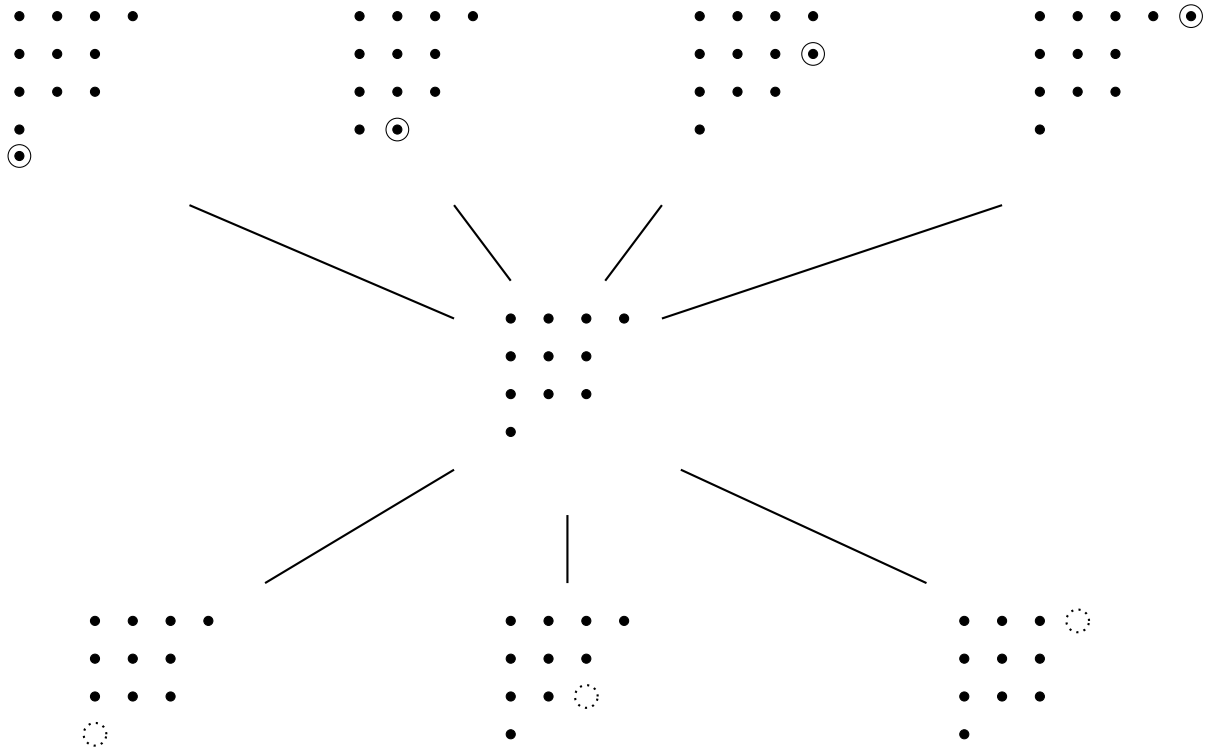
- New subgoal: **classify/construct** all irreducible diagrams

Young lattice: all diagrams with inclusion relation



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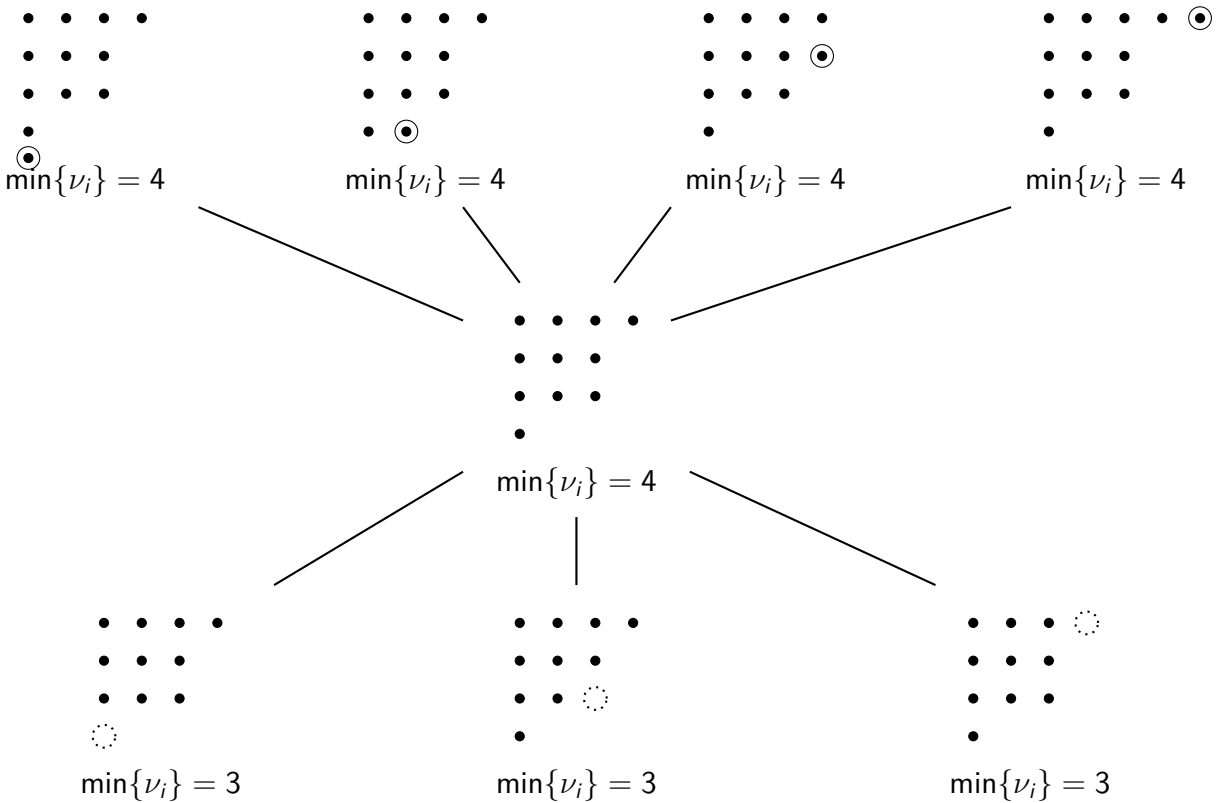
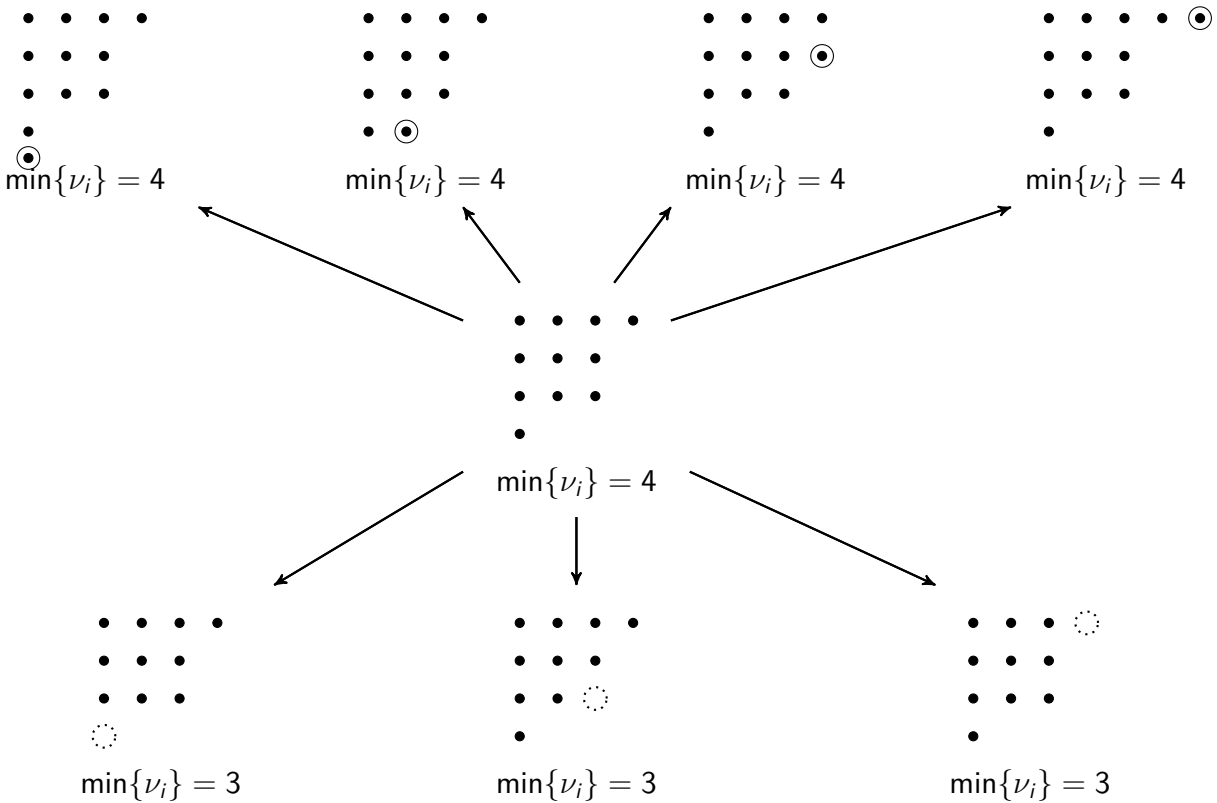


Young **digraph**: all diagrams with **reduction** relation



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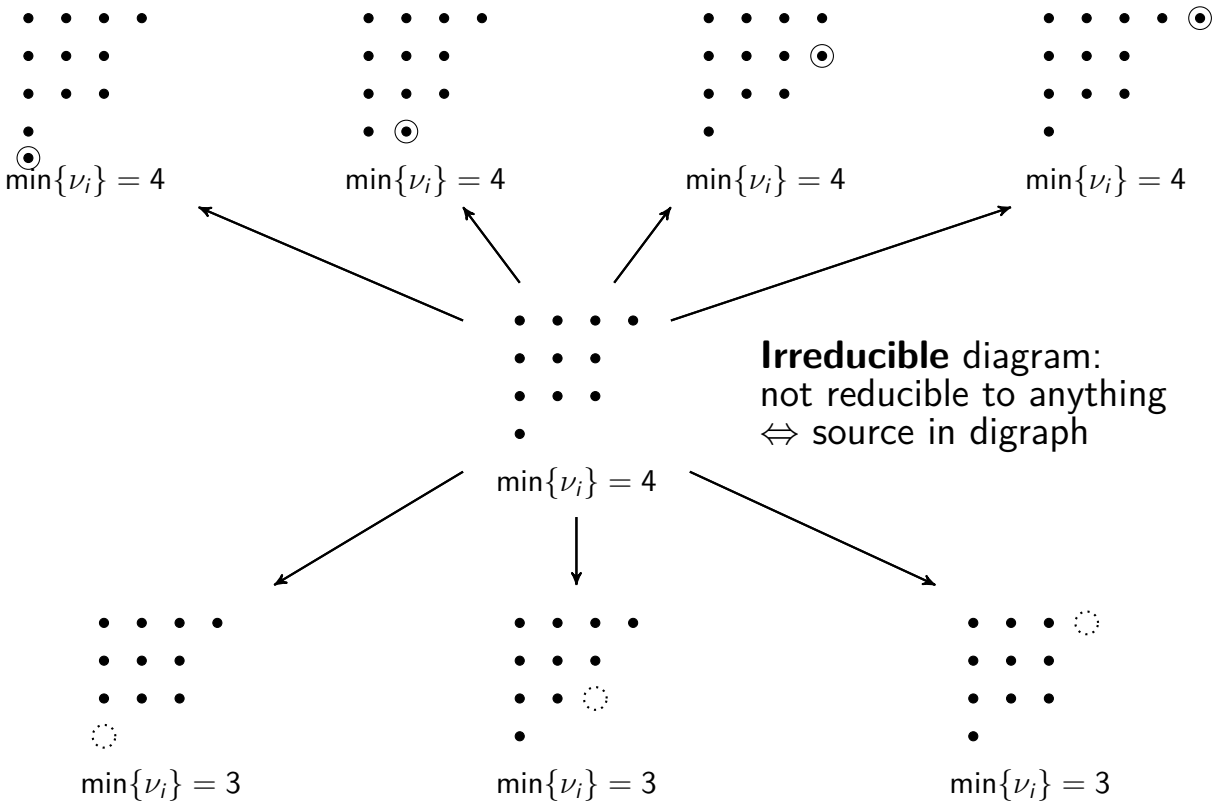

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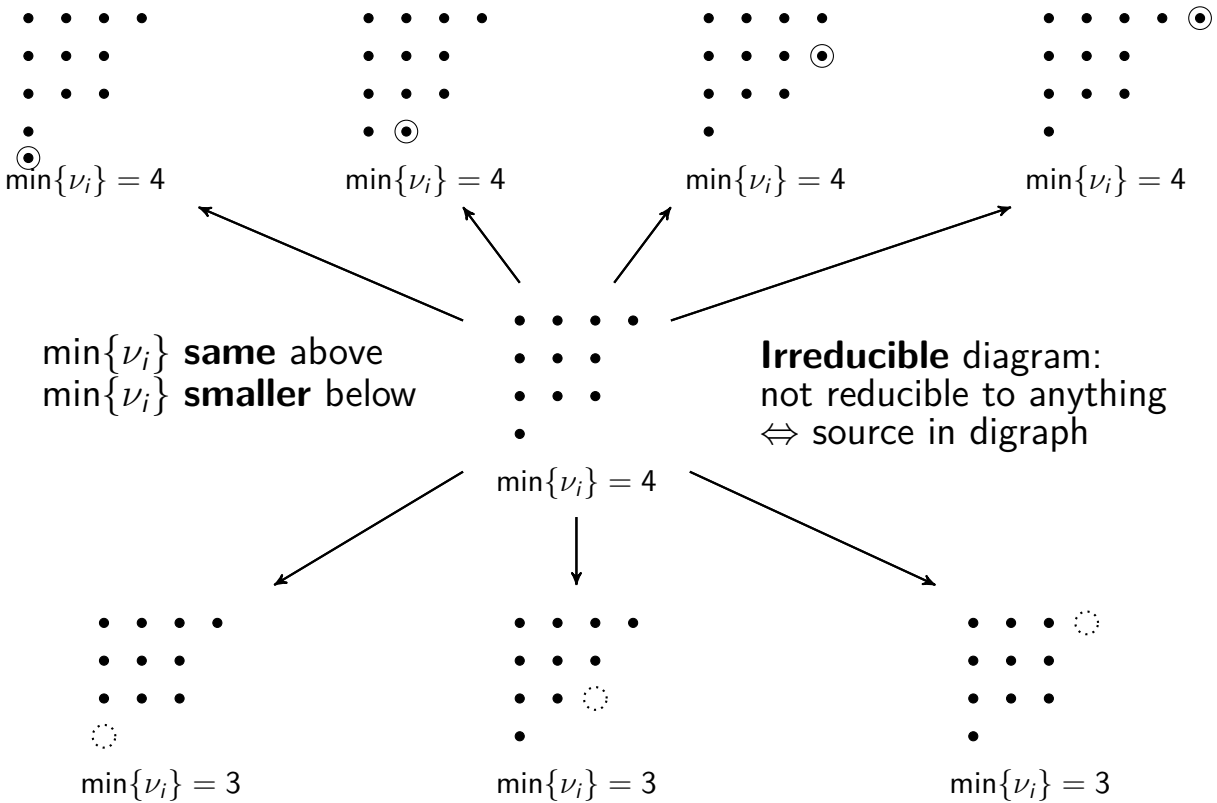

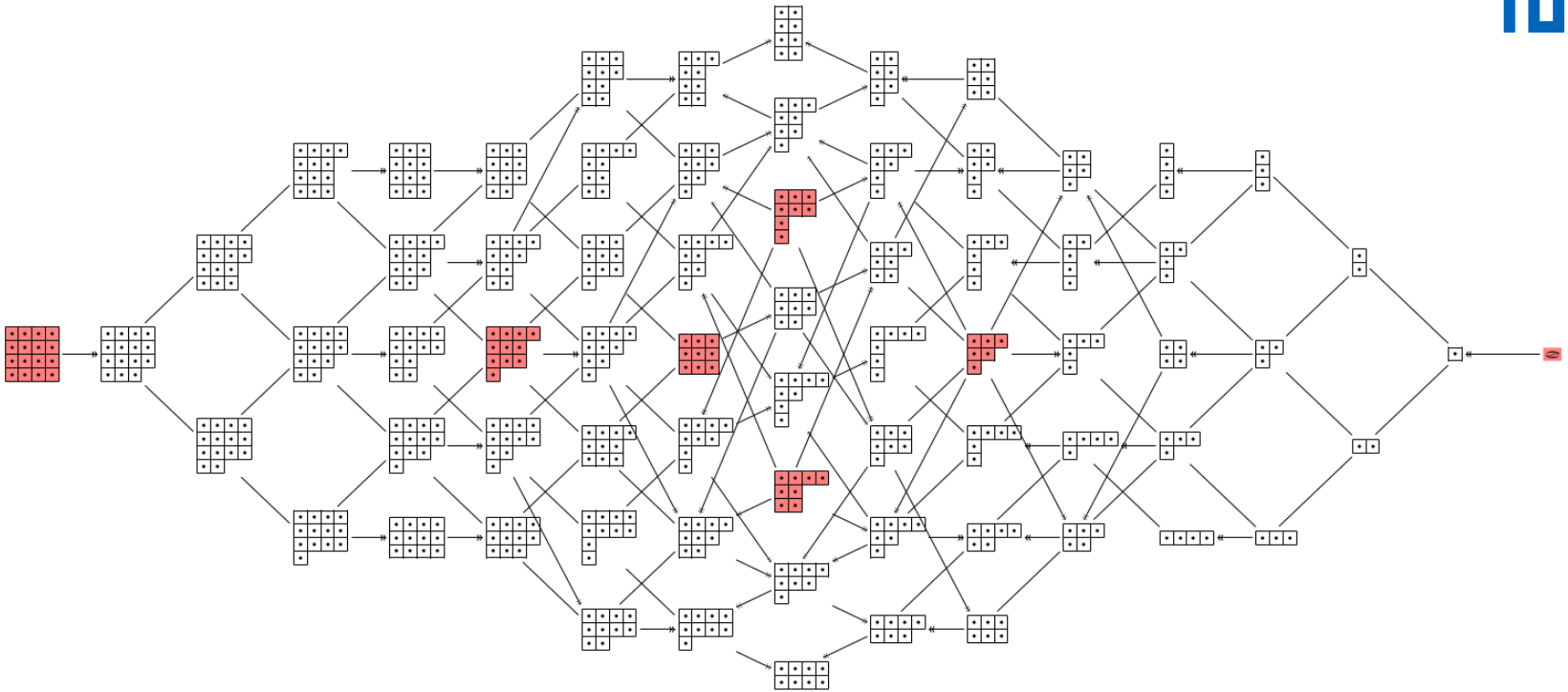

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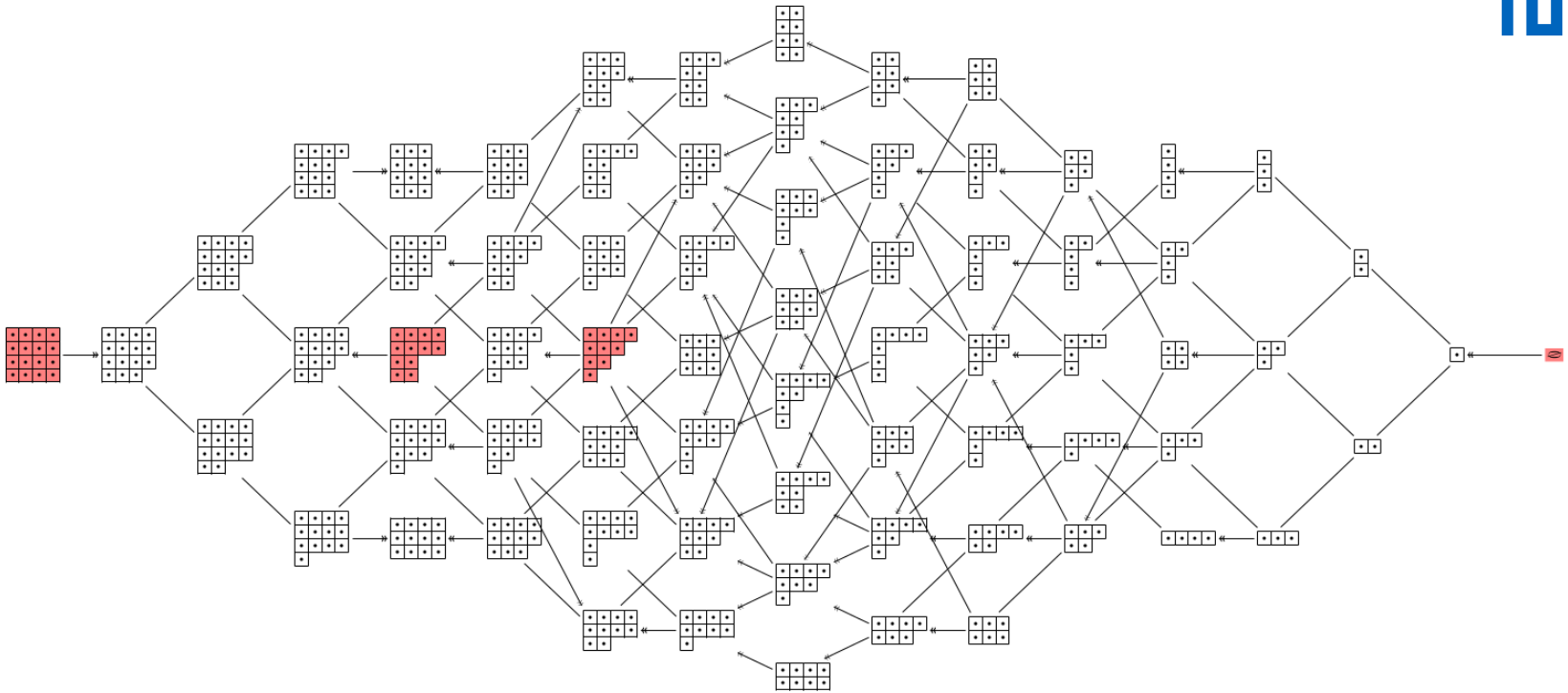

$d = 4$

Main result

Fix field size q, min. distance d

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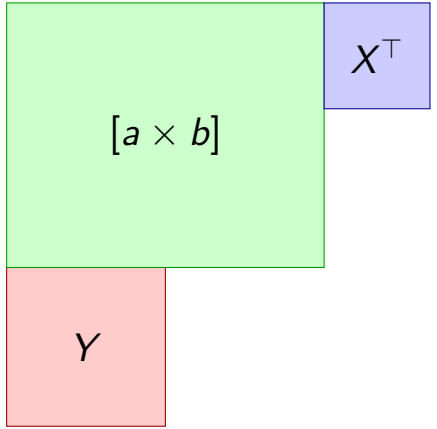
Suppose Ferrers diagram $\mathcal{D}$ contains $a \times b$ rectangle with $a, b \geq d - 1$



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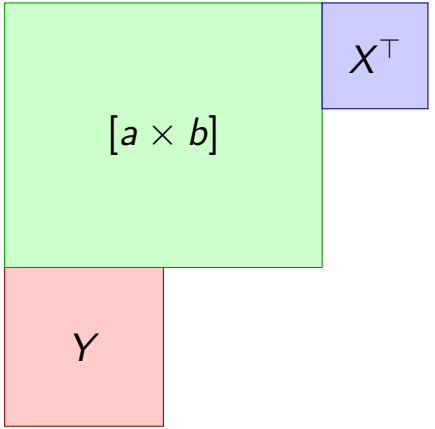
Irreducible & of this form $\rightarrow$ $\nu_0 = \nu_{d-1} = \min\{\nu_i\}$



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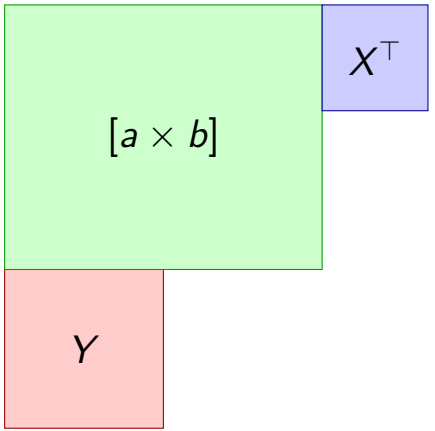

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$\mathcal{D} = [a \times b] \cup X^\top \cup Y$ *is irreducible* ***iff*** X *and* Y *satisfy* (c_i *is* i*-th column*)

- $|Y| - |X| = (b - a)(d - 1)$;
- *For all* $j \in \{1, 2, \ldots, d - 2\}$: $j(b - a + d - 1 - j) + \sum_{i=1}^{j} c_{d-i}(X) - \sum_{i=1}^{j} c_i(Y) \geq 0$

Main result

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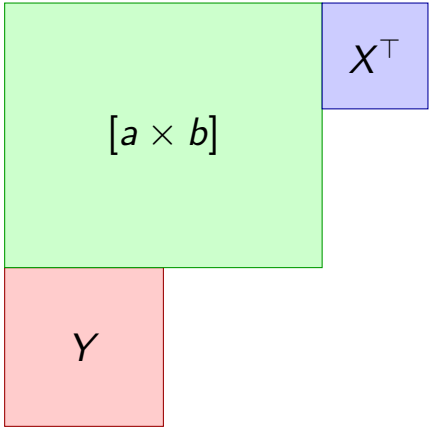

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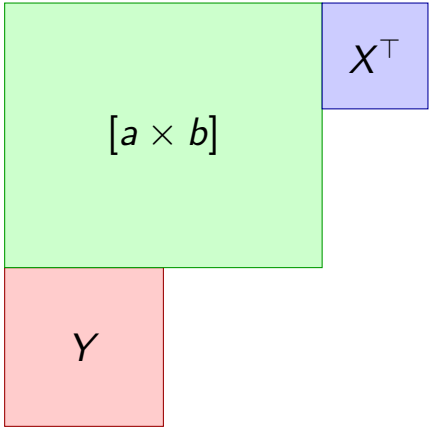

Proposition (B., Neri, '26+)

Finitely many integer solutions to linear (in)equalities in $c_j(X)$ and $c_i(Y)$.

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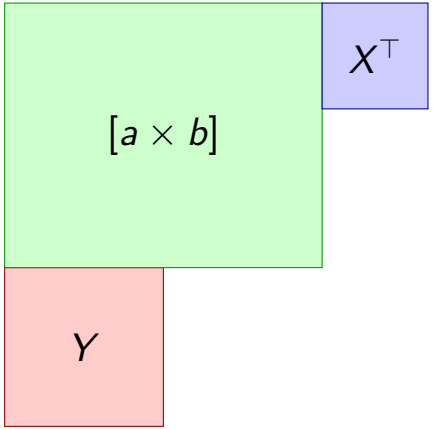

Proposition (B., Neri, '26+)

Finitely many integer solutions to linear (in)equalities in $c_j(X)$ and $c_i(Y)$.

→ Finite polytope characterizing all pairs (X, Y) of irreducible diagrams

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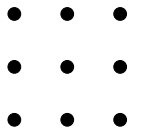
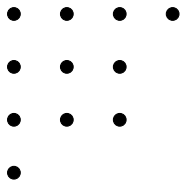
Polytope of irreducibles

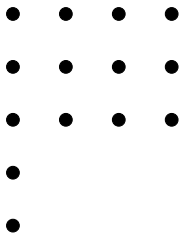
$d = 3$
$\min\{a, b\} = 3$



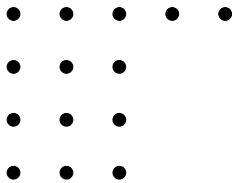
$\mathcal{D}_1$
$b - a = 0$
$c_1(Y) = 0$
$c_1(X) = 0$
Solved



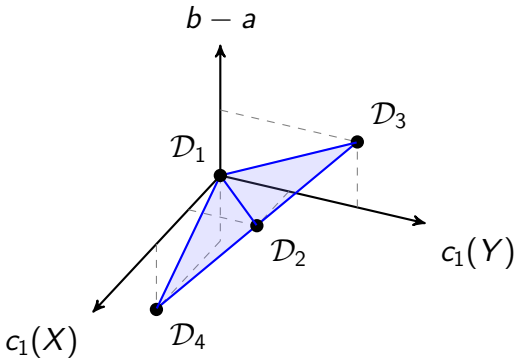
$\mathcal{D}_2$
$b - a = 0$
$c_1(Y) = 1$
$c_1(X) = 1$
Solved



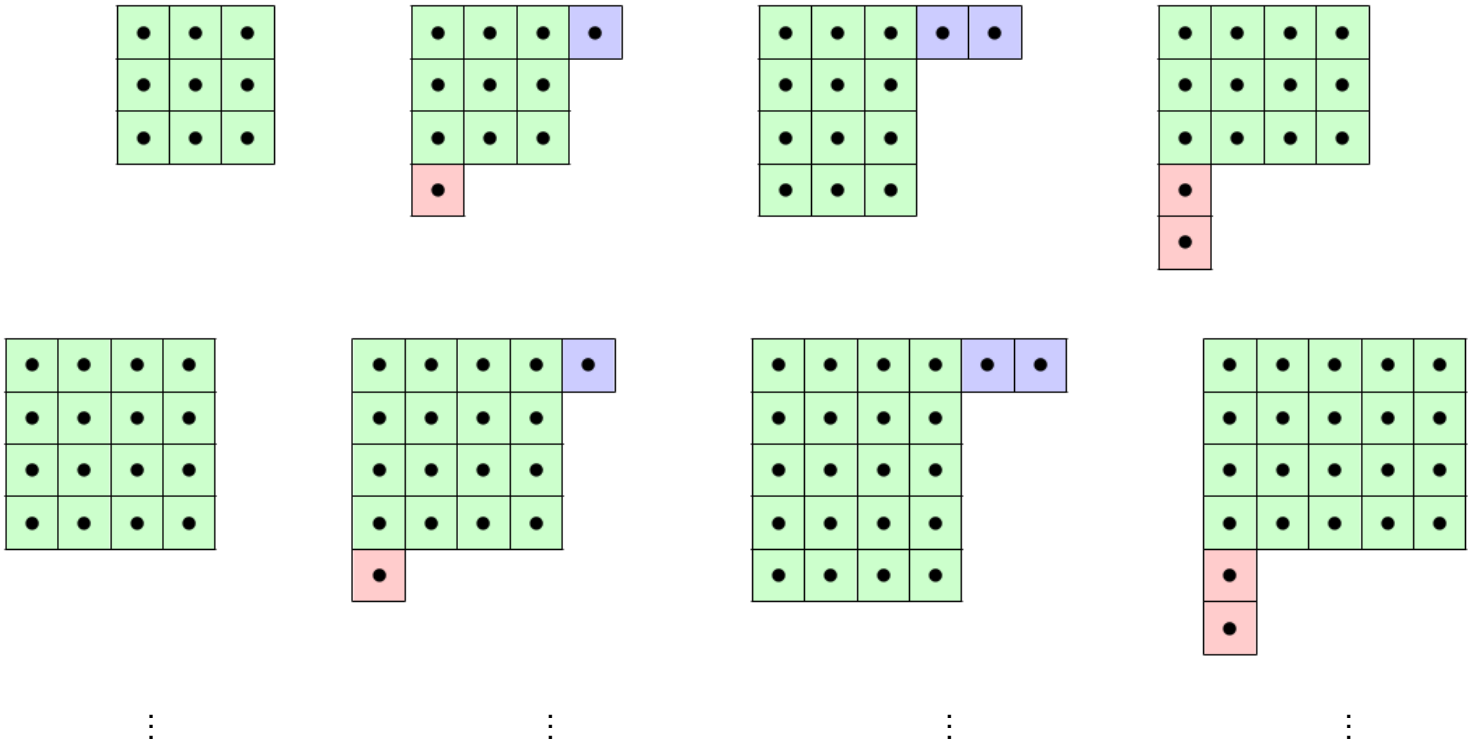
$\mathcal{D}_3$
$b - a = 1$
$c_1(Y) = 2$
$c_1(X) = 0$
Unsolved!



$\mathcal{D}_4$
$b - a = -1$
$c_1(Y) = 0$
$c_1(X) = 2$
Unsolved!



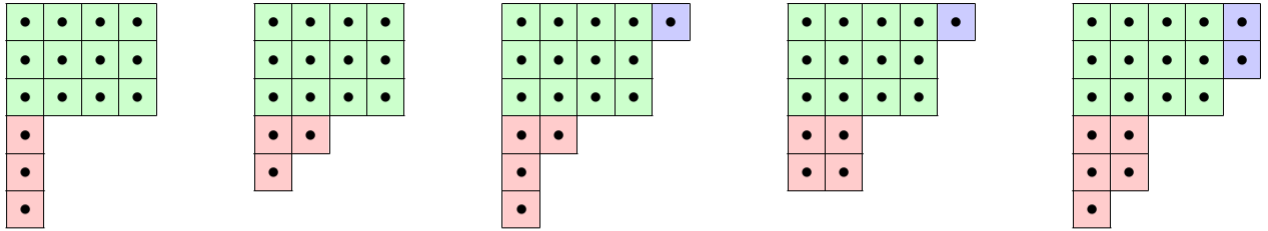
(X, Y)'s only depend on difference $b - a$! $\rightarrow$ arbitrarily large diagrams



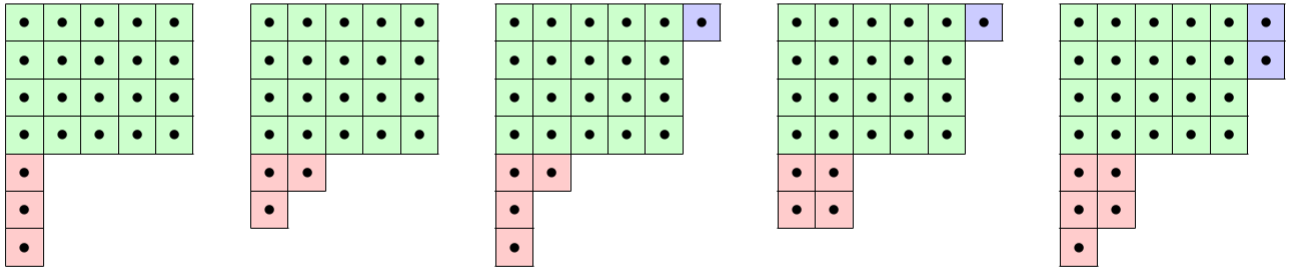
Irreducible diagrams for $d = 3$, varying $b - a$

(X, Y)'s only depend on difference $b - a$! $\rightarrow$ arbitrarily large diagrams

$\mathfrak{Irr}_4^{(3,4)}$



$\mathfrak{Irr}_4^{(4,5)}$

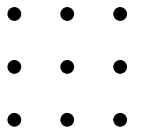


$\mathfrak{P}_4^{(a,a+1)}(\mathbb{Z})$ $(3,0,0,0)$ $(2,1,0,0)$ $(3,1,1,0)$ $(2,2,1,0)$ $(3,2,1,1)$.

Irreducible diagrams for $d = 4$, fixed $b - a = 1$

Polytope of irreducibles

$d = 3$
$\min\{a, b\} = 3$



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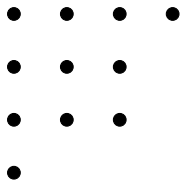
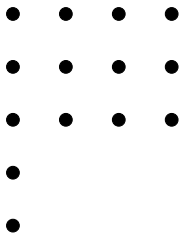


$\mathcal{D}_2$
$b - a = 0$
$c_1(Y) = 1$
$c_1(X) = 1$
Solved



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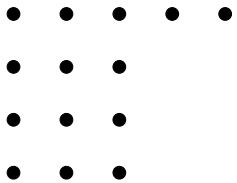

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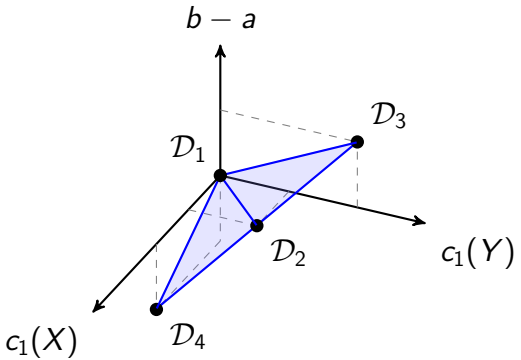


ES conjecture for $d = 3$

Theorem 6.6. Let $\mathbb{F}$ be a finite field. The following are equivalent.

(1) The Etzion-Silberstein conjecture holds for $d = 3$ over any Ferrers diagram $\mathcal{D}$.

(2) For every $n \geq 3$ there exists an $[n \times (n-1), n(n-3), 3]_{\mathbb{F}}$ MRD code $\mathcal{C}$ such that its puncturing on one row is contained in an $[(n-1) \times (n-1), (n-1)(n-2), 2]_{\mathbb{F}}$ MRD code.

you could solve it! pls

Main result 2

For fixed q, d and $\min\{a, b\}$, the polytope classifying all pairs (X, Y) of irreducible diagrams has exactly $\mathbf{3^{d-2}}$ vertices, that are all **integral!** (9 pages to prove).

$(x_1, \dots, x_{d-2}, y_1, \dots, y_{d-2}, z) \in \mathbb{R}^{2d-3}$ satisfying the constraints

$$\begin{cases} x_j - x_{j+1} \geq 0 & j \in [d-3] \\ y_j - y_{j+1} \geq 0 & j \in [d-3] \\ x_j, y_j \geq 0 & j \in [d-2] \\ j(z+d-1-j) + \sum_{i=0}^{j-2} x_{d-2-i} - \sum_{i=1}^{j} y_i \geq 0 & j \in [d-2] \\ \sum_{i=1}^{d-2} y_i - \sum_{i=1}^{d-2} x_i = z(d-1) \end{cases}.$$

Open: shape of polytope

For $3 \leq d \leq 7$, polytope classifying all pairs (X, Y) of irreducible diagrams has isomorphic combinatorial structure (vertices, edges, faces, etc.) as **product of $d-2$ triangles**!

Conjecture: also for all $d > 7$?

Open: counting irreducible diagrams

d	$\lvert\mathfrak{P}_d(\mathbb{Z})\rvert$
3	4
4	22
5	155
6	1301
7	12330
8	127275

Table 1: For $d \leq 8$: the number of integer points $\lvert\mathfrak{P}_d(\mathbb{Z})\rvert$.

??

$$\Delta = b - a$$

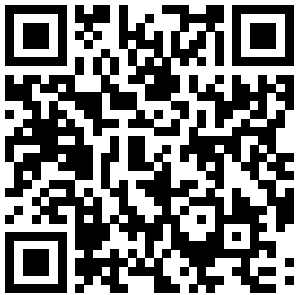
$d \setminus \Delta$	-4	-3	-2	-1	0	1	2	3	4	Total
3				1	2	1				4
4			2	5	8	5	2			22
5		5	17	34	43	34	17	5		155
6	16	66	159	257	305	257	159	66	16	1301

???

$\rightarrow$ Ehrhart theory?

Thanks for listening!

Link to (almost ready) paper
on personal website:



(On arXiv next week)