A geometric reconstruction of linear sets

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Setting

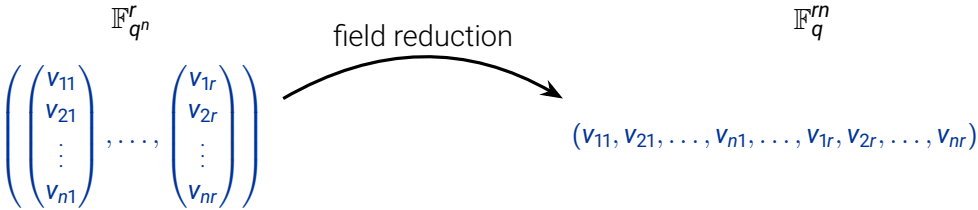
Vector spaces

$\mathbb{F}_{q^n}^r$

$\mathbb{F}_q^{rn}$

Setting

Vector spaces



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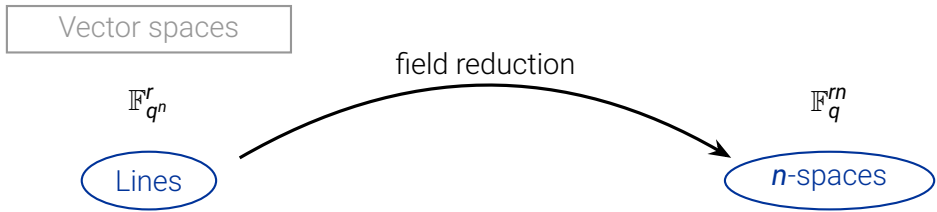
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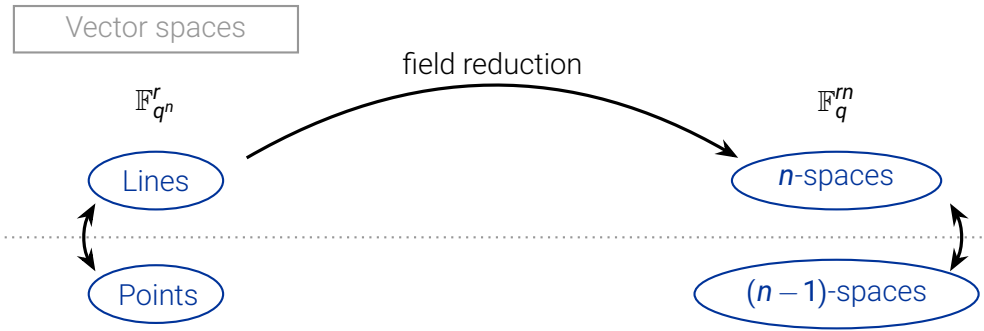
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$\mathrm{PG}(r-1, q^n)$
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Projective spaces

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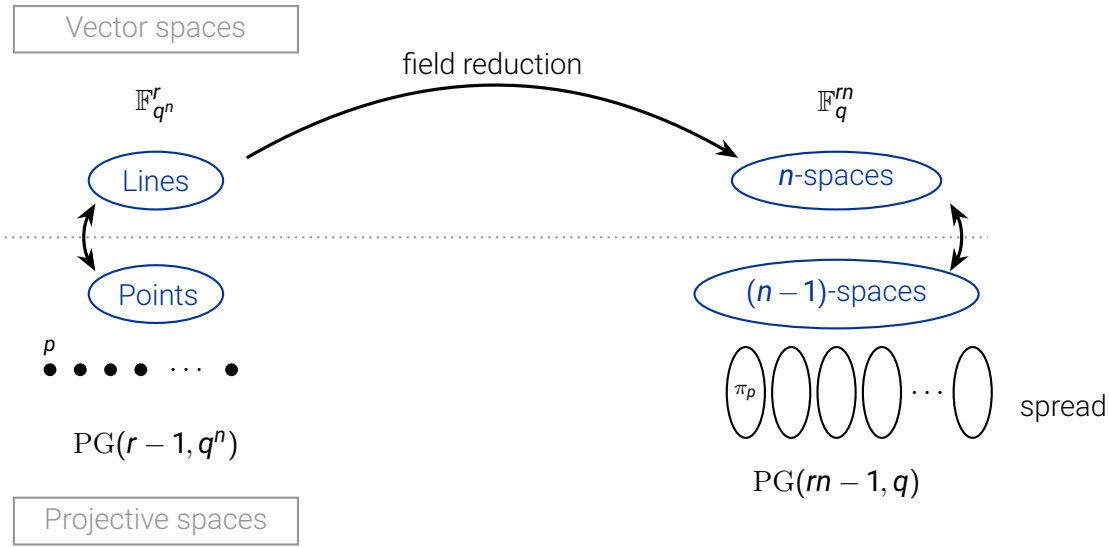
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Linear sets

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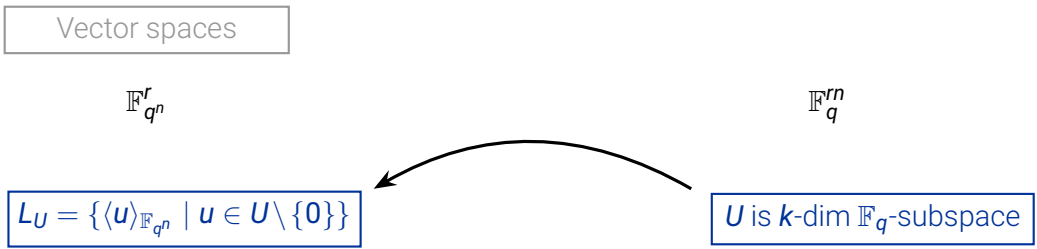
Linear sets



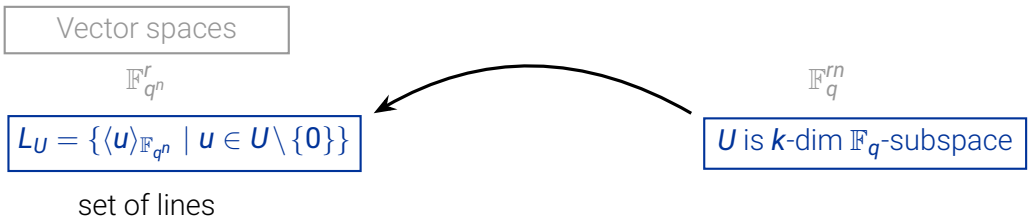
Definition

Let $V = \mathbb{F}_{q^n}^r$. Let U be a $\mathbb{F}_q$-subspace of V of dimension k. Then

$$L_U = \{\langle u \rangle_{\mathbb{F}_{q^n}} \mid u \in U \setminus \{0\}\}$$

is called an $\mathbb{F}_q$-*linear set* of rank k.

Linear sets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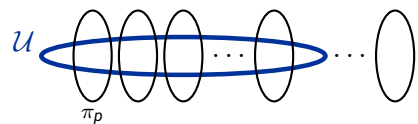

$\mathrm{PG}(rn-1, q)$

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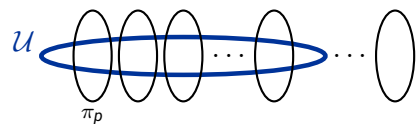

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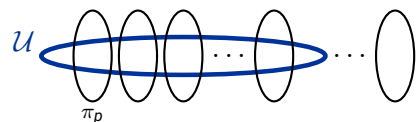

$\mathrm{PG}(rn-1, q)$

Motivation

Question

Suppose that a linear set L_U is given as a set of points in $\mathrm{PG}(r-1, q^n)$, can we reconstruct the subspace $\mathcal{U} \subseteq \mathrm{PG}(rn-1, q)$ generating L_U?

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Suppose that a linear set L_U is given as a set of points in $\mathrm{PG}(r-1, q^n)$, can we reconstruct the subspace $\mathcal{U} \subseteq \mathrm{PG}(rn-1, q)$ generating L_U?

No.

- There can be more than one subspace that is generating the same linear set.
- Even the rank of the linear set is not uniquely defined.

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Question

Given linear set L_U as a set of points in $\mathrm{PG}(r-1, q^n)$. What is the 'minimal' information we need to reconstruct the subspace $\mathcal{U} \subseteq \mathrm{PG}(rn-1, q)$ generating L_U?

- This is more difficult and more interesting!

For a point in $L_U = \{p \mid \mathcal{U} \cap \pi_p \neq \emptyset\}$ we define **the weight** of p as

$$\mathrm{wt}(p) = \dim(\mathcal{U} \cap \pi_p) + 1.$$

In vector spaces, there is a vector line $\langle u \rangle_{\mathbb{F}_{q^n}}$ corresponding to p. The weight is

$$\mathrm{wt}(\langle u \rangle_{\mathbb{F}_{q^n}}) = \dim_{\mathbb{F}_q}(\langle u \rangle_{\mathbb{F}_{q^n}} \cap U).$$

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Proposition (O. Polverino)[3]

Let L be a linear set of rank k in $\mathrm{PG}(r-1, q^n)$. Let N_i denote the number of points in L of weight i. Then the following relations hold:

$$|L| \leqslant q^{k-1} + \ldots + q + 1;$$
$$|L| = N_1 + \ldots + N_n;$$
$$N_1 + (q+1)N_2 + \ldots + (q^{n-1} + \ldots + q + 1)N_n = q^{k-1} + \ldots + q + 1.$$

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Example

Let $V = \mathbb{F}_{q^4}^2$ and define $\mathbb{F}_q$-subspaces $U_1 = \mathbb{F}_{q^2}^2$ and $U_2 = \{(x, y) \mid x \in \mathbb{F}_{q^2}, y \in \mathbb{F}_q\}$. Then

- $\dim_{\mathbb{F}_q}(U_1) = 4$, $\dim_{\mathbb{F}_q}(U_2) = 3$,
- $L_{U_1} = L_{U_2}$ is projectively equivalent to $\mathrm{PG}(1, q^2)$ in $\mathrm{PG}(1, q^4)$.

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Example

Let $V = \mathbb{F}_{q^3}^2$ and let W be a 2-dim $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^3}$. Define $U_1 = W \times W$ and $U_2 = \{(x, y) \mid x \in \mathbb{F}_{q^3}, y \in \mathbb{F}_q\}$. Then

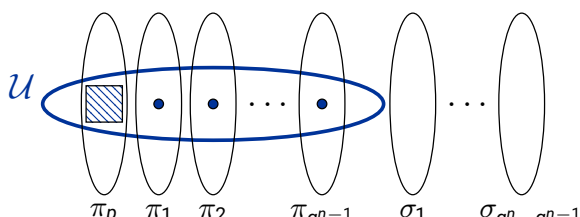
- $\dim_{\mathbb{F}_q}(U_1) = 4 = \dim_{\mathbb{F}_q}(U_2)$,
- $L_{U_1} = L_{U_2}$ is projectively equivalent to $\mathrm{PG}(1, q^3)$ embedded in $\mathrm{PG}(1, q^6)$.

Note that all points in L_{U_1} have weight at most 2 and in L_{U_2} the point $\langle 0, 1\rangle_{\mathbb{F}_{q^3}}$ has weight 3.

Clubs in $\mathrm{PG}(1, q^n)$

A linear set L_U of rank k is called a **club** if there is a point $p \in L_U$ of maximal weight, i.e. $\mathrm{wt}(p) = k - 1$. We will consider clubs of maximum rank $k = n$.

- Any other point in L_U has weight 1.
- $|L_U| = q^{n-1} + 1$.



$\mathrm{PG}(2n-1, q)$

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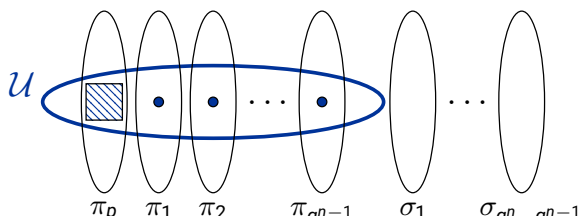


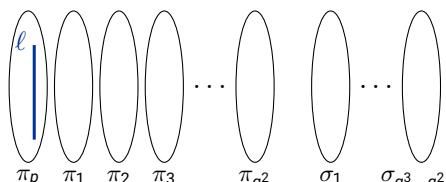
$\mathrm{PG}(2n-1, q)$

Let $L_U = \{p, p_1, \ldots, p_{q^n-1}\}$ be a club of maximum rank n in $\mathrm{PG}(1, q^n)$, where the point p of highest weight and the hyperplane $\pi_p \cap \mathcal{U}$ are given. Then we can reconstruct the subspace $\mathcal{U}$.

Reconstruction for clubs of rank 3 in $\mathrm{PG}(1, q^3)$

Given the set of points of $L_U = \{p, p_1, \ldots, p_{q^2}\}$ and the line $\ell = \mathcal{U} \cap \pi_p$.

$\mathrm{PG}(5, q)$



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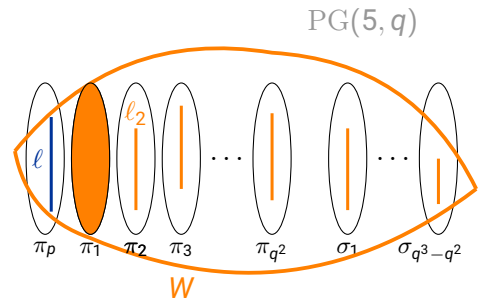

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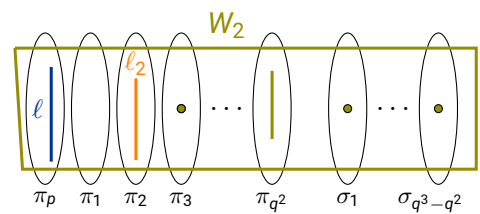

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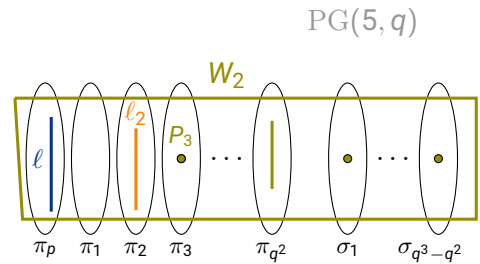
$$q^3 + q^2 + q + 1 = (q+1)(q+1) + (q^3 - q)$$

W_2 meets q spread elements π_i (different from π_p) in a line and $q^3 - q$ in a point.



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- WLOG $\pi_3 \cap W_2$ is a point P_3. Then $\langle \ell, P_3 \rangle$ is the subspace $\mathcal{U}$ generating L_U.

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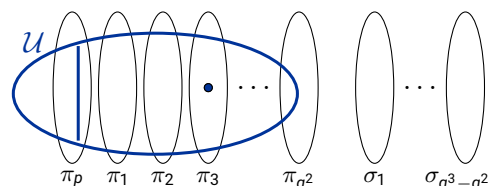

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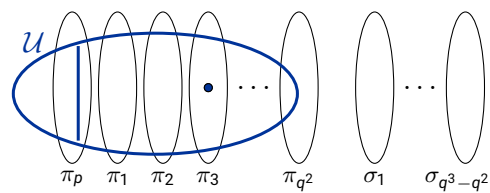


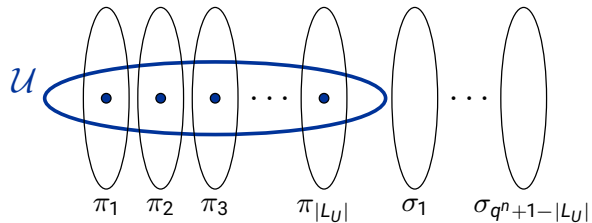
Remark

For any line in π_p there exists a subspace $\mathcal{U}$ generating the same linear set.

Scattered linear sets in $\mathrm{PG}(1, q^n)$

A linear set L_U is called **scattered** if any point in L_U has weight 1. We consider scattered linear sets of maximum rank n.

- $|L_U| = q^{n-1} + q^{n-2} + \ldots + q + 1$.



$\mathrm{PG}(2n-1, q)$

Let $L_U = \{p, p_1, \ldots, p_{|L_U|}\}$ be a scattered linear set of maximum rank n in $\mathrm{PG}(1, q^n)$. Suppose that an $(n-2)$-dimensional subspace $Z \subset \mathcal{U}$ is given. Then we can reconstruct the subspace $\mathcal{U}$.

Reconstruction for scattered linear sets of rank 3 in $\mathrm{PG}(1, q^3)$

Given the set of points of a scattered linear set $L_U = \{p_1, \ldots, p_{|L_U|}\}$ and a line $\ell \subset \mathcal{U}$.

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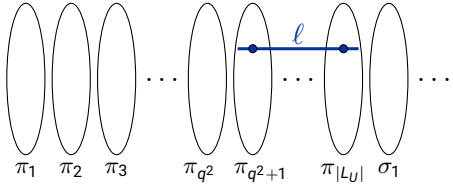

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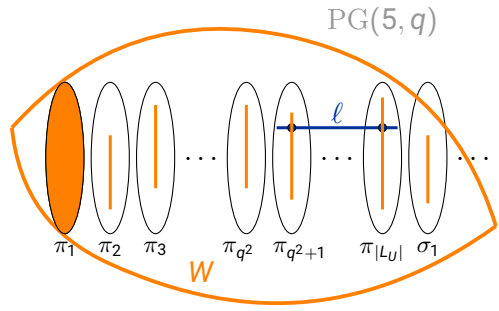

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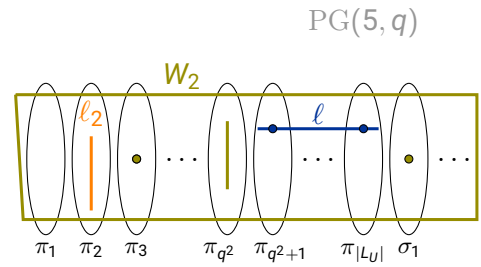

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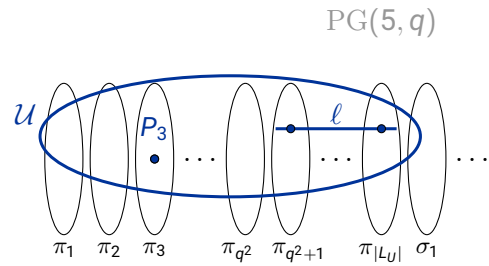
  W_2 meets $q+1$ spread elements in a line and $q^3 - q$ in a point.
- WLOG $\pi_3 \cap W_2$ is a point P_3. Then $\langle \ell, P_3 \rangle$ is the subspace $\mathcal{U}$ generating L_U. $(q \geqslant 3)$



Remarks

Let L be a scattered linear set in $\mathrm{PG}(1, q^3)$. Let $\Pi_L = \{\pi_1, \dots \pi_{|L|}\}$ be the set of spread elements in $\mathrm{PG}(5, q)$ corresponding to the points of L.

Proposition (G. Van de Voorde, M. Lavrauw)[4]

Through two points of a scattered linear set L of rank 3 in $\mathrm{PG}(1, q^3)$, there are exactly 2 sublines intersecting all elements of Π_L.

[4] *On linear sets on a projective line.* Designs, Codes and Cryptography **56**, 89-104 (2010).

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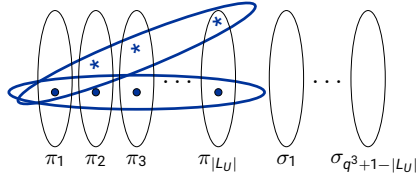
Hence, for any point $P \in \mathcal{U}$, we can construct two different subspaces generating the scattered linear set.



$\mathrm{PG}(5, q)$

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Linear sets of complementary weight in $\mathrm{PG}(1, q^n)$

A linear set L_U of maximum rank n in $\mathrm{PG}(1, q^n)$ is of **complementary weight** if there are two points $p, q \in L$ such that $\mathrm{wt}(p) + \mathrm{wt}(q) = n$.

[5] *Linear sets on the projective line with complementary weights*. Discrete Mathematics **345** (2022).

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Proposition (V. Napolitano, O. Polverino, P. Santonastaso, F. Zullo) [5]

Let L be a $\mathbb{F}_q$-linear set of maximum rank n in $\mathrm{PG}(1, q^n)$ for which there exist a point of weight one and two distinct points $P, Q \in L$ such that $\mathrm{wt}(P) = s, \mathrm{wt}(Q) = t$ and $s + t = n$. Then

$$q^{n-1} + 1 \leq |L| \leq q^{n-1} + \ldots + q^{\max\{s,t\}} - q^{\min\{s,t\}-1} - \ldots - q + 1.$$

Note that if $|L| = q^{n-1} + 1$, then L is a club. Hence, we assume $|L| > q^{n-1} + 1$.

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Reconstruction for linear sets of complementary weight in $\mathrm{PG}(1, q^4)$

Let L be a linear set of $\mathrm{PG}(1, q^4)$ of rank 4 such that there are two distinct points $p, q \in L$ such that $\mathrm{wt}(p) + \mathrm{wt}(q) = 4$.

- If there is a point of weight 3, then L is a club.
- Assume that $\mathrm{wt}(p) = \mathrm{wt}(q) = 2$.

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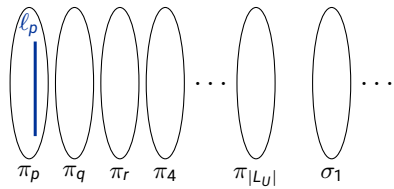
$\mathrm{PG}(7, q)$



Case 1: $\exists r \in L_U \setminus \{p, q\}$ s.t. $\mathrm{wt}(r) = 2$.
Given the line $\ell_p = \mathcal{U} \cap \pi_p$, we can reconstruct the subspace $\mathcal{U}$ generating L_U.

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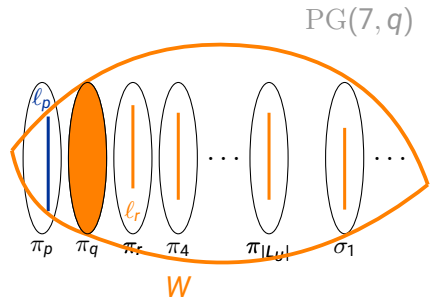
- Define $W = \langle \ell_p, \pi_q \rangle$.
- Then $\dim(W) = 5$ and $\mathcal{U} \subset W$.



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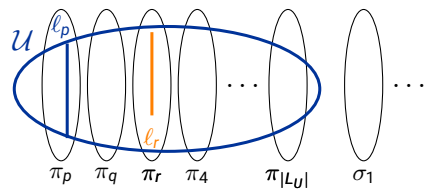

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- $W \cap \pi_r = \ell_r = \mathcal{U} \cap \pi_r$, so $\mathcal{U}$ is given by $\langle \ell_p, \ell_r \rangle$.

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Let L be a linear set of $\mathrm{PG}(1, q^4)$ of rank 4 and let $p, q \in L$ s.t. $\mathrm{wt}(p) = \mathrm{wt}(q) = 2$.

Case 2: In $L_U \setminus \{p, q\}$ any point has has weight 1.
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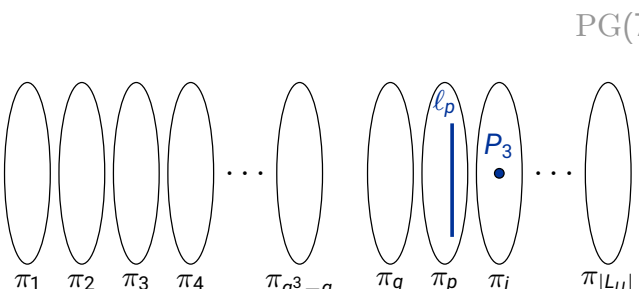

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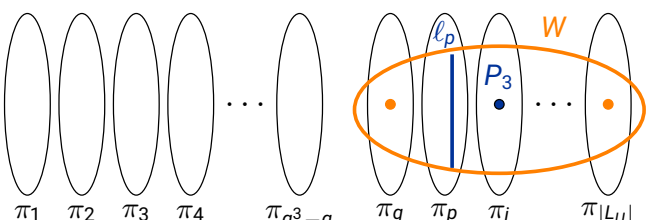

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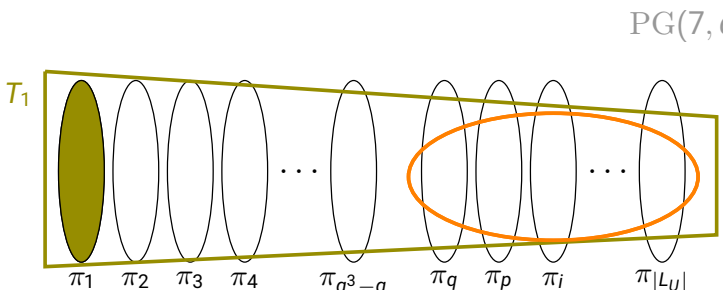

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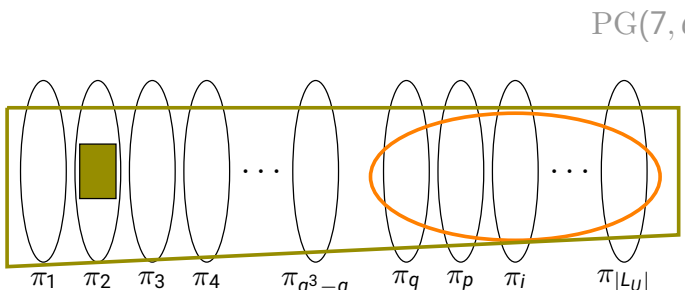

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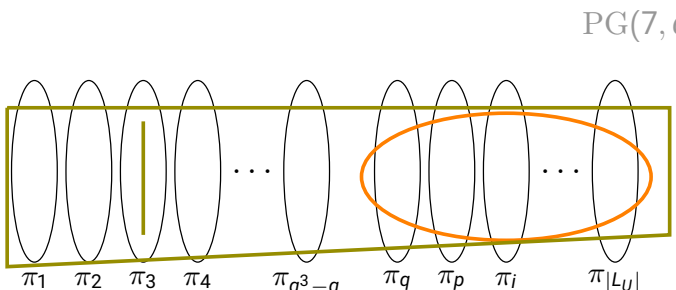

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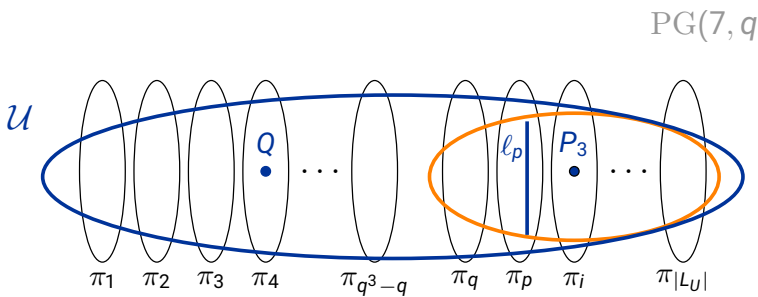

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- ...
- $T_j \cap \pi_k = Q$
- $\mathcal{U} = \langle W, Q \rangle$

Future work

Questions

- Given U. Can we construct the other subspaces U' such that $L_U = L_{U'}$?
- Can we do this for other/more general types of linear sets?
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Thank you for listening!