Coloring Grassmann graphs using MRD codes

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Joint work with Francesco Pavese and Paolo Santonastaso

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GHENT UNIVERSITY

2 Basic Graph Theory Notions

Definition

A **finite simple graph** $\Gamma = \Gamma(V, E)$ is a pair, where V is a finite set of vertices, and $E \subset \binom{V}{2}$ is the set of edges.

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A **coloring** of the vertices of a graph is **valid** if there are no edges between any two vertices in the same color class.

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The **chromatic number** of a graph G, denoted $\chi(G)$, is the smallest positive integer k for which there exists a valid coloring of G using k colors.

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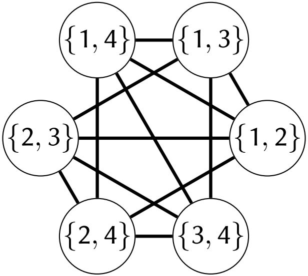

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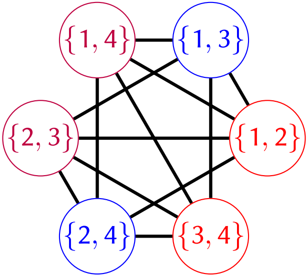

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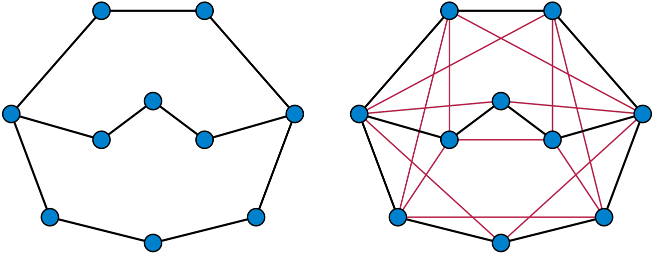
5 Power Graphs

Definition

The mth power $G^{(m)}$ of an undirected graph G is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in G is at most m.



G $\qquad$ G^2

6 Grassmann Graphs

Definition

The **Grassmann graph** $J_q(n,k,t)$ has as vertices the k-spaces of $V(n,q)$ and two vertices are adjacent if the corresponding k-spaces meet in a space of dimension at least t.

- $J_q(n,k,t)$ has order

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q-1)(q^2-1)\cdots(q^k-1)}.$$

- $J_q(n,k,t) \cong J_q(n,k,k-1)^{(k-t)}$, a power graph.
- A valid coloring of $J_q(n,k,t)$ is a partitioning of all k-spaces of $V(n,q)$ such that any two spaces in the same part meet in a space of dimension less than t.

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7 Chromatic Number of Grassmann Graphs

The chromatic number of the Grassmann graphs has been studied only when $t = k - 1$, often phrased in geometric terms.

1. Best-known general bounds on $\chi(J_q(n, k))$:

$$\begin{bmatrix} n - k + 1 \\ 1 \end{bmatrix}_q \leq \chi(J_q(n, k, k - 1)) \leq \begin{bmatrix} n \\ 1 \end{bmatrix}_q$$

2. When $k = 2$ and n is even, it is known that $\chi(J_q(n, 2, 1)) = \left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix}\right]_q$ for $q = 2, 3, 4, 8, 16$, and all q when $n = 2^m$. (This is equivalent to a long-studied question in finite geometry.)
3. It is known that $\chi(J_2(n, 2, 1)) = \left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix}\right]_2 + 3$ when n is odd

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8 New Results

Theorem (D'haeseleer, Pavese, Santonastaso, Taranchuk (2026 +))

Let q be a prime power and n, k be positive integers satisfying $n \geq 2k$. Then

$$\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q \leq \chi(J_q(n,k,t)) \leq (1+o(1))n^{k-t} \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q.$$

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9 Lower bound

Isomorphism using duality

$$J_q(n, k, t) \cong J_q(n, n-k, n-2k+t)$$

We may suppose that $n \geq 2k$.

Large cliques

The set of all k-spaces through a fixed t-space gives a maximal clique of size $\begin{bmatrix} n-t \\ k-t \end{bmatrix}$ in $J_q(n, k, t)$. Hence

$$\begin{bmatrix} n-t \\ k-t \end{bmatrix} \leq \chi(J_q(n, k, t))$$

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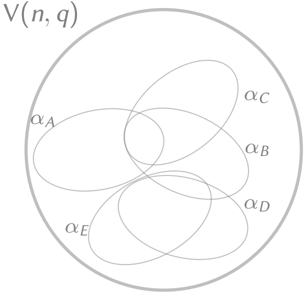
10 Upper bound via MRD Codes

Equivalence between k-spaces in $V(n, q)$ and full rank $k \times n$ matrices over $\mathbb{F}_q$ in reduced row echelon form.

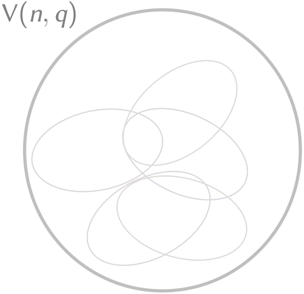
$$M = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix} \longleftrightarrow \mathrm{Row}(M)$$

Thus, coloring subspaces is equivalent to coloring full rank $k \times n$ matrices which are in RREF.

11 Coloring of the subspaces



Coloring of the subspaces disjoint to $\langle [0_k \mid I_k X] \rangle$



Since $|\mathcal{C}| = q^{(n-k)t}$, we colored the subspaces disjoint to $\langle [0_k \mid I_k X] \rangle$ with $\frac{q^{(n-k)k}}{q^{(n-k)t}} = q^{(n-k)(k-t)}$ colors.

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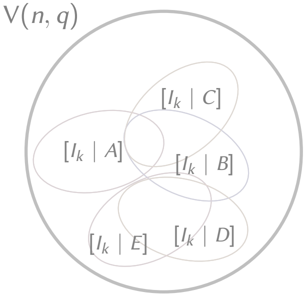

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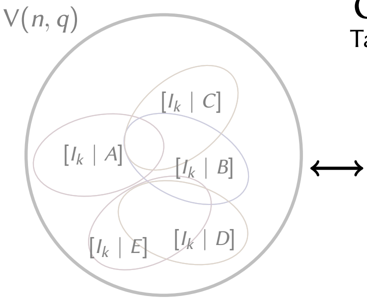
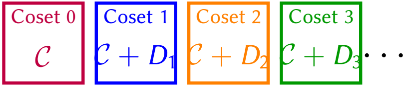


Coloring of the matrices.

Take an MRD code $\mathcal{C}$ in $M_{k\times n-k}(\mathbb{F}_q)$ with $d(\mathcal{C}) = m - t + 1$



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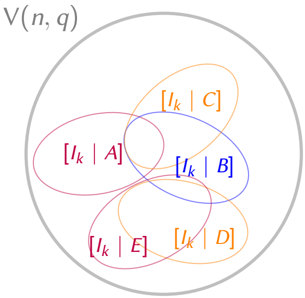

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Coset 0	Coset 1	Coset 2	Coset 3	
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Coloring of the other subspaces?

$$M = \begin{bmatrix} \mathbf{1} & 2 & \mathbf{0} & 1 & \mathbf{0} & 0 & 2 \\ \mathbf{0} & 0 & \mathbf{1} & 2 & \mathbf{0} & 3 & 1 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{1} & 0 & 2 \end{bmatrix} \longleftrightarrow \mathrm{Row}(M)$$

Coloring of the other subspaces?

Associate with each RREF matrix M the k-subset $M_{id} \subset \{1, 2 \ldots, n\}$ which records the positions of the leading 1's.

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Hence,

$$M_{id} = \{1, 3, 5\} \subset \{1, 2, \ldots, 7\}.$$

We color all matrices with the same M_{id} using the MRD coloring idea.

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